The Numerical Approximation of Blow-Up Times for Fractional Reaction-Diffusion Equations

Mariam Khachatryan, Erkan Nane, Hans-Werner van Wyk

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Abstract

This paper investigates the numerical estimation of blow-up phenomena of the space fractional reaction-diffusion equation

$$\partial_t u + (-\Delta)^{\alpha/2} u = f(u), \quad x \in \Omega, t > 0$$

with non-negative initial value and exterior Dirichlet boundary conditions. First, we consider the full discretization of the fractional equation using the already existing novel and accurate finite difference method for the fractional operator. Next, we implement an auxiliary function H to detect the blow-up. This auxiliary function approach was used in [8] for the above equation with Laplacian, $\alpha = 2$. The numerical blow-up times are computed for the fractional reaction-diffusion equation with the reaction term $f(u) = u^2$ and $f(u) = e^u$. Convergence results are proven. Moreover, the numerical blow-up time computed for the fractional reaction-diffusion equation with $\alpha \to 2$ is compared with the numerical blow-up time for the classical reaction-diffusion equation with $\alpha = 2$, and consistent results are obtained.

1 Introduction

Reaction-diffusion equations model quantities that experience local changes in concentration and spread out in space. Applications range from chemical and biological processes [23, 22] to medicine, genetics [16, 11], physics, chemistry, social science, finance [2, 3], and weather prediction. For systems exhibiting anomalous diffusion [4, 28, 12], the diffusive spread is commonly modeled by a non-local fractional Laplace operator $-(-\Delta)^{\alpha/2}$ with fractional exponent $\alpha \in (0, 2]$, giving rise to the space-fractional partial differential equation

$$\partial_t u + (-\Delta)^{\alpha/2} u = f(u), \tag{1}$$

where $f(u)$ is a non-negative reaction term. Depending on the spatial domain, the initial and boundary conditions, as well as the form of the reaction term f, it is possible for the solution u of Equation (1) to blow up at a finite time $T < \infty$, i.e. for u to diverge to infinity as $t \to T^-$.

Finite-time blow-up has long been studied theoretically [19] as the obverse of global existence. Its presence indicates an explosive growth and points toward the limitation of the reaction equation to model the underlying physical system at the blow-up time. Theoretical investigations have mostly focused on the existence of finite blow-up times [19], estimates for their onset [24], and growth rates of the solution nearby [1]. The critical exponents for blow-ups of the reaction-diffusion equation with anomalous diffusion has been studied initially in Nagasawa and Sirao [25]

$$\begin{aligned} \partial_t u + (-\Delta)^{\alpha/2} u &= \lambda u^p, \quad x \in \mathbb{R}^d, t > 0 \\ u(x, 0) &= u_0(x) \end{aligned} \tag{2}$$

where $\alpha \in (0, 2]$ and λ is a nonnegative bounded function on $\mathbb{R}^d$. The Equation (2) with classical Laplace operator, $\alpha = 2$ case, has been studied initially by Fujita [19]. He has shown that

$$\begin{cases} \text{for any initial condition no global solution exists} & \text{if} \quad p < 1 + 2/d \\ \text{for small initial data global solutions exist} & \text{if} \quad p > 1 + 2/d \end{cases}$$

The proof for the critical case $p = 1 + 2/d$ which results in blow-up solutions can be found in [20], [21] and [30]. Nagasawa and Sirao [25] established the Fujita's result above for Equation (2) with $1 + 2/d$ replaced with $1 + \alpha/d$. They also worked on existence and blow-up criteria for general infinitesimall operators of a linear nonnegative contraction semigroups on the space of bounded measurable function on $\mathbb{R}^d$. Blow-up rates for solution of Equation (2) was established by [18] in the case $\lambda \in \{1, -1\}$, and [17] in the case $\lambda = 1$.

While closed-form expressions for blow-up times are available in a few simple cases [5], they generally require numerical estimation. A fundamental challenge arising in computing reliable approximations include the difficulty in resolving the solution near the blow-up. In [29], the authors show that for a fixed temporal step size $\tau > 0$, explicit time-stepping methods may result in spurious approximations of the solution beyond the blow-up time, while the use of implicit methods may lead to systems with no unique solutions. These observations suggest the use of adaptive time-stepping schemes, the earliest of which was developed by Nakagawa et al in [26]. They used the reciprocal of the absolute maximum of the approximate solution to adjust the step size.

In this work we adopt another approach, developed in [7, 8, 9, 10] for the classical Laplacian, in which the numerical criterion for blow-up is based on a scaling argument. It allows for the use of a non-adaptive forward Euler discretization and the resulting approximate blow-up time has been proved to converge at the rate $O(\tau)$ as $\tau \to 0$ for a wide class of reaction functions f (see Assumption 1).

The blow-up phenomena of the space-fractional reaction-diffusion equation remains partially understood. $(-\Delta)^{\alpha/2}$ is the fractional Laplace operator which is the non-local generalization of the classical Laplace operator. Hence, Equation (1) simplifies to the following standard reaction-diffusion equation for $\alpha = 2$ and $d = 1$

$$u_t - u_{xx} = f(u), \quad x \in \Omega \subset \mathbb{R}, t > 0 \tag{3}$$

whose theoretical properties as well as the discretization has been intensively studied previously. In particular [8] provides a complete discretization scheme for studying problems such as (3) with $f(u) = u^p$ which is the generalization of the work presented in [26] where a finite difference scheme was provided for the special case of equation (3) when $p = 2$. Moreover, it's known that under certain initial and boundary conditions solution blows-up in finite time (i.e solution becomes unbounded in finite time).

We aim to generalize convergence and error estimate results in [8] for the fractional Laplace operator and compute the numerical blow-up time for the fractional case. It's is important to have a numerical scheme that provides an accurate approximation of the blow-up time.

In this paper, we investigate the blow-up phenomena of FPDEs. In Section 2, some standard ODE and PDE blow-up results are presented. The fully discretized scheme of Equation (8) is provided where the Fractional Laplace Operator is dicretized based on the trapezoidal rule introduced in [14]. In Section 3, we perform convergence analysis and provide several error estimate results. In Section 4, the adapted numerical algorithm with auxiliary function H is applied to the blow-up problem given by (8). The blow-up times are estimated numerically for the proposed FPDE and also for the FPDE with exponential reaction term. Comparison of blow-up time for FPDE ($\alpha \to 2$) and PDE ($\alpha = 2$) is provided in Section 5 and consistent blow-up times are obtained. Some concluding remarks are given in Section 6.

2 The Numerical Blow-Up Time

The numerical criterion developed in [8] for estimating the blow up time is best understood in the context of the ODE initial value problem

$$\begin{aligned} u_t &= f(u), \qquad t > 0 \\ u(0) &= u_0 > 0, \end{aligned} \tag{4}$$

where f satisfies the following assumptions that guarantee the occurrence of finite time blow-up [?].

Assumption 1. Assume the reaction function $f : [0, \infty) \to \mathbb{R}$ is positive, increasing, and convex, i.e. $f(u), f'(u), f''(u) > 0$ for all $u > 0$. Moreover, assume that

$$T^*_{\text{ode}} = \int_{U^0}^{\infty} \frac{1}{f(s)} ds < \infty. \tag{5}$$

In the special case when Equation (4) is a Bernoulli equation, i.e. when $f(u) = u^p$ $(p > 1)$, the transformation $z = u^{-(p-1)}$ yields a linear equation which reduces the blow-up condition for $u(t)$ to that of finding a a zero for $z(t)$. Under a forward Euler discretization $U^{n+1} = U^n + \tau f(U^n)$ with time-step parameter τ, the blow up time T^*_{ode} can in turn be approximated by $\hat{T}_{\text{ode}}(\tau) = n_\tau \tau$, where $n_\tau \in \mathbb{N}$ is the time-step at which U^n crosses the threshold

$$\tau H(U^{n_\tau - 1}) < 1, \qquad \tau H(U^{n_\tau}) \geq 1, \tag{6}$$

with $H(u) = (p-1)u^{p-1}$. The positivity and convexity of $f(u)$ guarantee that $U^n \to \infty$ as $n \to \infty$ which, together with the fact that $H(u) \to \infty$ as $u \to \infty$, guarantees the existence of the integer n_τ satisfying (6). For more general forcing terms $f(u)$ satisfying Assumption 1, the relative largeness condition (6) can still be used to estimate the blow-up time, for any function $H(u)$ satisfying the following assumptions.

Assumption 2. Let $H : [0, \infty) \to [0, \infty)$ be any function such that $H(u) > 0$ for $u > 0$, $\lim_{u\to\infty} H(u) = \infty$, and

$$\lim_{\tau\to 0^+} \tau \ln \left(f \left(H^{-1} \left(\frac{1}{\tau} \right) \right) \right) = 0. \tag{7}$$

Under Assumptions 1 and 2, the numerical error in approximating the blow up time T^*_{ode} can be shown (see [8], Theorem 2.1) to satisfy

$$-\int_{H^{-1}(\frac{1}{\tau})}^{\infty} \frac{1}{f(s)} ds \leq \hat{T}_{\text{ode}}(\tau) - T^*_{\text{ode}} \leq -\int_{H^{-1}(\frac{1}{\tau})}^{\infty} \frac{1}{f(s)} ds + \tau \ln \left(f \left(H^{-1} \left(\frac{1}{\tau} \right) \right) \right),$$

and hence $\hat{T}_{\text{ode}}(\tau) - T^*_{\text{ode}} \to 0$ as $\tau \to 0$.

Next we consider modifications of Criterion (6) to estimate the blow up time for the fractional reaction-diffusion equation

$$\begin{aligned} \partial_t u + (-\Delta)^{\alpha/2} u &= f(u), \quad x \in \Omega, \ t > 0 \\ u(x, 0) = u_0(x) &\geq 0, \quad x \in \Omega \\ u(x, t) &= 0, \quad x \in \Omega^c, \ t > 0, \end{aligned} \tag{8}$$

defined over the spatial domain $\Omega \subset \mathbb{R}$ with exterior $\Omega^c = \mathbb{R} \setminus \Omega$. The operator $(-\Delta)^{\alpha/2}$ is the one-dimensional fractional Laplacian with $\alpha \in (0, 2)$, defined via the Cauchy principal value as

$$(-\Delta)^{\alpha/2} u(x,t) = c_{1,\alpha} \text{P.V.} \int_{\mathbb{R}} \frac{u(x,t) - u(y,t)}{|x-y|^{1+\alpha}} dy, \tag{9}$$

with normalizing constant

$$c_{1,\alpha} = \frac{2^{\alpha-1}\alpha\Gamma(\frac{\alpha+1}{2})}{\sqrt{\pi}\Gamma(1-\frac{\alpha}{2})},$$

where Γ denotes the Gamma function. Intuitively, the homogeneous Dirichlet conditions on the domain's exterior and the anomalous diffusion modeled by the fractional Laplacian inhibit the growth caused by the reaction term f. Nevertheless, finite-time blow-up is known to occur [] for any f satisfying Assumption 1 and large enough initial condition.

Let $U_i^n \approx u(x_i, t_n)$ be the finite difference approximation of the solution u of (8) on a spatial grid $\{x_i\}_{i=0}^K$ of width $h > 0$ and a temporal grid $t_n = \tau n$, $n \in \mathbb{N} \cup \{0\}$ with constant step size $\tau > 0$. Specifically, for a given discretized initial condition $U_i^0 = u_0(x_i)$, the grid function U_i^n is updated according to the forward Euler equation

$$\frac{U_i^{n+1} - U_i^n}{\tau} + (-\Delta)_h^{\alpha/2} U_i^n = f(U_i^n), \tag{10}$$

where $(-\Delta)_h^{\alpha/2}$, defined below, is an appropriate finite difference discretization of the operator $(-\Delta)^{\alpha/2}$. Further, let $H : [0, \infty) \to [0, \infty)$ be a monotone increasing function that satisfies Assumption 2. We then define our numerical approximation $\hat{T}_{\text{fpde}}(\tau)$ of the blow-up time $T^*_{\text{fpde}}(\tau) > 0$ to be the first time instant $\tau_n = n\tau$ at which

$$\tau_{n-1} H(\|U^{n-1}\|_\infty) < 1, \qquad \text{and} \qquad \tau_n H(\|U^n\|_\infty) \geq 1. \tag{11}$$

This criterion was proposed in [8] for the classical reaction diffusion equation and was shown there to lead to an approximate blow-up time that converges to the true blow-up time as $\tau, h \to 0$.

In this work we use the weighted trapezoidal rule introduced in [14] to approximate the fractional Laplacian on the bounded domain $\Omega = (-l, l)$, which, under a change of variables, can be written as

$$(-\Delta)^{\alpha/2} u(x) = -c_{1,\alpha} \text{P.V.} \int_0^\infty \frac{u(x+\xi) - 2u(x) + u(x-\xi)}{\xi^{1+\alpha}} d\xi. \tag{12}$$

To this end, we subdivide $(-l, l)$ into K sub-intervals of equal width $h = L/K$, where $L = 2l$. Let $\{U_i\}_{i=0}^K$ be a grid function defined at the grid points $x_i = -l + ih$ for $i = 0, \ldots, K$ and let $\xi_k = kh$ for $k = 0, ..., K$ denote the location of the offset variable ξ on the same grid. On every sub-interval $[\xi_{k-1}, \xi_k]$, we use a weighted trapezoidal rule to approximate the integral

$$\int_{\xi_{k-1}}^{\xi_k} \frac{u(x+\xi) - 2u(x) + u(x-\xi)}{\xi^{1+\alpha}} d\xi \approx \frac{1}{2}\left(\delta^{k-1} U_j + \delta^k U_j\right) \int_{\xi_{k-1}}^{\xi_k} \xi^{1-\alpha} d\xi,$$

where $\delta^k U_i = \frac{U_{i+k} - 2U_i + U_{i-k}}{\xi_k^2}$ denotes the central difference operator. By accounting for the homogeneous external conditions and the limiting behavior as $\xi \to 0^+$, the integral operator (12) can be discretized fully as

$$(-\Delta)_h^{\alpha/2} U_i = -\frac{c_{1,\alpha}}{2\nu}\left(2\delta^1 U_i \xi_1^\nu + \sum_{k=2}^K \left(\delta^{k-1} U_i + \delta^k U_i\right)(\xi_k^\nu - \xi_{k-1}^\nu) - \frac{U_i}{\xi_K^\alpha}\right), \tag{13}$$

where $\nu = 2 - \alpha$. The components of the discrete operator's matrix representation, also denoted here by $(-\Delta)_h^{\alpha/2}$, are given by

$$\left[(-\Delta)_h^{\alpha/2}\right]_{ij} = \frac{c_{1,\alpha}}{\nu h^\alpha} \begin{cases} \sum_{k=2}^{K-1} \frac{(k+1)^\nu - (k-1)^\nu}{k^2} + \frac{K^\nu - (K-1)^\nu}{K^2} + (2^\nu + 1) + \frac{2\nu}{\alpha K^\alpha}, & j = i, \\ -\frac{1}{2}(2^\nu + 1), & j = i \pm 1, \\ -\frac{(|j-i|+1)^\nu - (|j-i|-1)^\nu}{2|j-i|^2}, & \text{otherwise.} \end{cases} \tag{14}$$

This discretization has second order accuracy and generalizes the classical central difference method in the case when $\alpha \to 2^-$. It was extended in [15] to two and three dimensional domains, but we focus on the one-dimensional domain in this work.

3 Error Estimates and Convergence Analysis

In this section we prove stability and convergence of the finite difference scheme (10) before the blow-up time is reached, show that the finite difference solution is strictly increasing under conditions on f and u_0, and finally prove that the estimate (11) converges to the true blow-up time when τ and h tend to zero.

We first establish a suitable α-dependent Courant-Friedrichs-Lewy (CFL) condition on the spatial and temporal discretization parameters, τ and h, that ensures stability of the forward Euler scheme (10). To this end, we rewrite Equation (10) to obtain the explicit updating rule

$$U_j^{n+1} = LU_j^n + f(U_j^n), \tag{15}$$

where, throughout the paper, we define $L = (I - \tau(-\Delta)_h^{\alpha/2})$.

Lemma 1. *For any $K \in \mathbb{N}$, $\alpha \in (0, 2)$, and $\nu = 2 - \alpha$,*

$$\sum_{k=2}^{K} \frac{(k+1)^\nu - (k-1)^\nu}{k^2} + \frac{K^\nu - (K-1)^\nu}{K^2} + \frac{2\nu}{\alpha K^\alpha} \leq \frac{2\nu}{\alpha}. \tag{16}$$

Proof. We attain this bound by noting that the upper bound in (16) is equal to the integral $2\int_1^\infty x^{-2} d(x^\nu)$, and that the sum on its left represents an underestimating numerical quadrature rule. Specifically, for any $k \geq 2$,

$$\frac{(k+1)^\nu - (k-1)^\nu}{k^2} = \int_{k-1}^{k+1} k^{-2} d(x^\nu) \approx \int_{k-1}^{k+1} x^{-2} d(x^\nu)$$

represents a weighted midpoint rule, whose error is given by

$$\int_{k-1}^{k+1} (x^{-2} - k^{-2}) d(x^\nu) = \int_{k-1}^{k+1} \nu(x^{-2} - k^{-2}) x^{\nu-1} dx.$$

If $\nu - 1 < 0$, the mapping $x \mapsto x^{\nu-1}$ is decreasing. This, together with the convexity of $x \mapsto x^{-2}$, then imply

$$\begin{aligned} \int_{k-1}^{k+1} \nu(x^{-2} - k^{-2}) x^{\nu-1} dx &\geq \nu(k+1)^{\nu-1} \int_{k-1}^{k+1} (x^{-2} - k^{-2}) dx \\ &\geq \nu(k+1)^{\nu-1} \int_{k-1}^{k+1} (-2k(x-k)) dx = 0. \end{aligned} \tag{17}$$

If $\nu-1>0$, on the other hand, then it can readily be seen that the function $x \mapsto \nu x^{\nu-1}(x^{-2}-k^{-2})$ is decreasing, convex and has a zero at $x=k$. Similarly, the last term represents a weighted right-hand sum. If $\nu>1$, then since $x \mapsto x^{-2}$ is decreasing,

$$\int_{K-1}^{K}(x^{-2}-K^{-2})d(x^{\nu}) \geq 0,$$

whereas if $\nu<1$, the differential $d(x^{\nu})=\nu x^{\nu-1}dx$ is decreasing and hence

$$\begin{aligned}\int_{K-1}^{K}(x^{-2}-K^{-2})d(x^{\nu}) &= \int_{K-1}^{K}(x^{-2}-K^{-2})\nu x^{\nu-1}dx \\ &\geq \nu K^{\nu-1}\int_{K-1}^{K}(x^{-2}-K^{-2})dx \geq 0. \qquad (18)\end{aligned}$$

Estimates (17) and (18) now imply

$$\begin{aligned}&\sum_{k=2}^{K}\frac{(k+1)^{\nu}-(k-1)^{\nu}}{k^2}+\frac{K^{\nu}-(K-1)^{\nu}}{K^2}+\frac{2\nu}{K^{\alpha}} \\ \leq &\sum_{k=2}^{K}\int_{k-1}^{k+1}x^{-2}d(x^{\nu})+\int_{K-1}^{K}x^{-2}d(x^{\nu})+2\int_{K}^{\infty}x^{-2}d(x^{\nu}) \\ \leq &2\int_{1}^{\infty}x^{-2}d(x^{\nu})=\frac{2\nu}{\alpha}.\end{aligned}$$

□

Lemma 2. *If, for any $\eta \in (0,1)$, the discretization parameters h and τ satisfy the CFL condition*

$$\frac{\tau}{h^{\alpha}} \leq \left(\frac{c_{1,\alpha}}{\nu}\left[\frac{2\nu}{\alpha}+2^{\nu}+1\right]\right)^{-1}\eta, \qquad (19)$$

then the diagonal entries of the matrix $L=I-\tau(-\Delta)_h^{\alpha/2}$ satisfy $L_{ii} \geq 1-\eta$ for $i=1,\dots,K$. In this case, its induced maximum norm satisfies $\|L\|_{\infty} \leq 1$.

Proof. By virtue of the upper bound (16), we have

$$\tau\left[(-\Delta)_h^{\alpha/2}\right]_{ii} \leq \frac{\tau}{h^{\alpha}}\frac{c_{1,\alpha}}{\nu}\left[\frac{2\nu}{\alpha}+2^{\nu}+1\right].$$

To establish a lower bound on the diagonal term L_{ii}, it thus suffices to show that the expression

$$s(\alpha) := \frac{c_{1,\alpha}}{\nu}\left[\frac{2\nu}{\alpha}+2^{\nu}+1\right] = \frac{2^{\alpha}\Gamma\left(\frac{1+\alpha}{2}\right)}{\sqrt{\pi}\Gamma\left(1-\frac{\alpha}{2}\right)}+\frac{2^{\alpha-1}\alpha\Gamma\left(\frac{1+\alpha}{2}\right)}{\nu\sqrt{\pi}\Gamma\left(1-\frac{\alpha}{2}\right)}(2^{\nu}+1)$$

is positive and bounded above for all $\alpha \in [0,2]$. Clearly, $s(\alpha)$ is continuous and positive for $\alpha \in (0,2)$, since it is a composition of continuous, positive functions on this domain. Using the fact that $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$, we obtain

$$\lim_{\alpha\to 0^+} s(\alpha)=1.$$

Moreover, since $\nu\Gamma(1-\frac{\alpha}{2}) = 2\frac{\nu}{2}\Gamma(\frac{\nu}{2}) = 2\Gamma(1+\frac{\nu}{2})$, we have

$$\lim_{\alpha\to 2^-} \frac{2^{\alpha-1}\alpha\Gamma\left(\frac{1+\alpha}{2}\right)}{\nu\sqrt{\pi}\Gamma\left(1-\frac{\alpha}{2}\right)} = \lim_{\alpha\to 2^-} \frac{2^{\alpha-1}\alpha\Gamma\left(\frac{1+\alpha}{2}\right)}{\sqrt{\pi}2\Gamma\left(1+\frac{\nu}{2}\right))} = 1,$$

and since $\Gamma(\frac{\nu}{2}) \to \infty$ as $\alpha \to 2^-$ and hence $\nu \to 0^+$, we have

$$\lim_{\alpha\to 2^-} \frac{2^{\alpha}\Gamma\left(\frac{1+\alpha}{2}\right)}{\sqrt{\pi}\Gamma\left(1-\frac{\alpha}{2}\right)} = 0.$$

Combining these two limits yields

$$\lim_{\alpha\to 2^-} s(\alpha) = 2.$$

If τ and h are chosen to satisfy Inequality (3.1), then

$$L_{ii} = 1 - \tau\left[(-\Delta)_h^{\alpha/2}\right]_{ii} \geq 1 - \frac{\tau}{h^\alpha}s(\alpha) \geq 1-\eta. \tag{20}$$

To bound $\|L\|_\infty$, we note that the CFL condition (3.1) guarantees $L_{ii} > 0$. Moreover, for $j = 1, ..., K-1$ and $j \neq i$, the off-diagonal terms

$$L_{ij} = -\tau\left[(-\Delta)_h^{\alpha/2}\right]_{ij} = \begin{cases} \frac{\tau}{h^\alpha}\frac{c_{1,\alpha}}{2\nu}(2^\nu+1), & \text{if } j = i\pm 1 \\ \frac{\tau}{h^\alpha}\frac{c_{1,\alpha}}{2\nu}\left[\frac{(|j-i|+1)^\nu-(|j-i|-1)^\nu}{|j-i|^2}\right], & \text{if } |j-i|>1 \end{cases} \tag{21}$$

are also nonnegative. Hence, the matrix norm is given by the maximum row sum, i.e.

$$\|L\|_\infty = \max_i \sum_{j=1}^{K-1} L_{ij} = 1 + \sum_{j=1}^{K-1} -\tau\left[(-\Delta)_h^{\alpha/2}\right]_{ij}.$$

We will show that $\sum_{j=1}^{K-1} -\tau\left[(-\Delta)_h^{\alpha/2}\right]_{ij} \leq 0$ for any $i = 1, ..., K-1$, from which the result follows. Indeed, letting $k = |i-j|$, we note that

$$\begin{aligned}\sum_{\substack{j=1\\|j-i|>1}}^{K-1} -\tau\left[(-\Delta)_h^{\alpha/2}\right]_{ij} &= \frac{\tau}{h^\alpha}\frac{c_{1,\alpha}}{2\nu}\sum_{\substack{j=1\\|j-i|>1}}^{K-1}\left[\frac{(|j-i|+1)^\nu-(|j-i|-1)^\nu}{|j-i|^2}\right] \\ &\leq \frac{\tau}{h^\alpha}\frac{c_{1,\alpha}}{\nu}\sum_{k=2}^{K-1}\left[\frac{(k+1)^\nu-(k-1)^\nu}{k^2}\right].\end{aligned}$$

Also,

$$-\tau\left[(-\Delta)_h^{\alpha/2}\right]_{i,i-1} - \tau\left[(-\Delta)_h^{\alpha/2}\right]_{i,i+1} = \frac{\tau}{h^\alpha}\frac{c_{1,\alpha}}{\nu}(2^\nu+1).$$

On the other hand,

$$\begin{aligned}-\tau\left[(-\Delta)_h^{\alpha/2}\right]_{ii} = \frac{\tau}{h^\alpha}\frac{c_{1,\alpha}}{\nu}\Bigg[&-\sum_{k=2}^{K-1}\frac{(k+1)^\nu-(k-1)^\nu}{k^2} - \frac{K^\nu-(K-1)^\nu}{K^2} \\ &-(2^\nu+1) - \frac{2\nu}{\alpha K^\alpha}\Bigg].\end{aligned}$$

Therefore,

$$\sum_{j=1}^{K-1} -\tau\left[(-\Delta)_h^{\alpha/2}\right]_{ij} \leq \frac{\tau}{h^\alpha}\frac{c_{1,\alpha}}{\nu}\left[-\frac{K^\nu-(K-1)^\nu}{K^2} - \frac{2\nu}{\alpha K^\alpha}\right] \leq 0.$$

□

Remark 1. Note that, since $s(\alpha) \to 2$ as $\alpha \to 2^-$, Inequality (3.1) reduces to the CFL condition

$$\frac{\tau}{h^2} \leq \frac{1}{2}\eta, \qquad \text{for } \eta \in (0,1),$$

which is used to ensure stability for the classical central difference scheme.

It follows from Theorem 3.2 in [14] and simple Taylor expansion arguments that the local truncation error $\zeta(\tau, h)$ of the scheme in (10), given by

$$\zeta(\tau,h) := \frac{u(x_j,t_{n+1}) - u(x_j,t_n)}{\tau} + (-\Delta)_h^{\alpha/2} u(x_j,t_n) - f(u(x_j,t_n)), \tag{22}$$

satisfies $\zeta(\tau, h) = O(\tau + h^2)$ as $\tau, h \to 0^+$, provided the solution $u \in C^2([0,T], C^{3,\alpha}(\Omega))$ for any time $T < T^*$ before the blow-up. The global convergence before onset of the blow-up now follows by standard arguments, as stated in the following theorem.

Theorem 1. *Let the error $E^n \in \mathbb{R}^{K-1}$ be given by $E_j^n = u(x_j,t_n) - U_j^n$, for $j = 1, \dots K-1$ and $n = 1, ..., N$, and for any time $t_N < T^*$, let*

$$C = \max_{\substack{0<t<t_N \\ x\in\Omega}} |f'(u(x,t))|.$$

Then, for any $\tau > 0$ and $h > 0$ satisfying the CFL condition (3.1), we have

$$\|E^N\|_\infty \leq \frac{e^{Ct_N}}{C}\left(C\|E^0\|_\infty + \zeta(\tau,h)\right) = O(\tau + h^2). \tag{23}$$

Proof. Applying the updating formula (15) to the error and invoking Equation (22) gives

$$E_j^N = LE_j^{N-1} + \tau\left[f(u(x_j,t_{N-1})) - f(U_j^{N-1})\right] + \tau\zeta(\tau,h),$$

We proceed to bound $\|E^N\|_\infty$ recursively, making use of geometric series and the bound $\|L\|_\infty \leq 1$ under Condition (3.1), to obtain

$$\begin{aligned}
\|E^N\|_\infty &\leq (1+\tau C)\|E^{N-1}\|_\infty + \tau\zeta(\tau,h) \\
&\leq (1+\tau C)^N\|E^0\|_\infty + \tau\zeta(\tau,h)\sum_{n=0}^{N-1}(1+\tau C)^n \\
&= (1+\tau C)^N\|E^0\|_\infty + \left(\frac{(1+\tau C)^N - 1}{C}\right)\zeta(\tau,h) \\
&\leq \frac{(1+\tau C)^N}{C}\left(C\|E^0\|_\infty + \zeta(\tau,h)\right).
\end{aligned}$$

The result now follows from the inequality $(1+\tau C)^N \leq e^{N\cdot\tau C} = e^{Ct_N}$. ∎

3.1 Existence and Error of Numerical Blow-Up Time

In this section we show that for a sufficiently large initial condition, the finite difference approximation U_j^n diverges as $n \to \infty$ so that the numerical blow-up estimate given in (11) is well-defined. Specifically, we make the following assumption.

Assumption 3. Assume that the *non-trivial* and *non-negative* initial condition U^0 satisfies

$$-(-\Delta)_h^{\alpha/2} U_i^0 + f(U_i^0) \geq a f(U_i^0), \; i = 0, \dots, K, \tag{24}$$

for some $a \in (0,1)$, where $U_i^0 = u_0(x_i)$ for $i = 0, \dots K$.

Lemma 3. *For the function f satisfying Assumptions 1 above and any grid function U_i on Ω that is constant on Ω^c, we have*

$$(-\Delta)_h^{\alpha/2} f(U_i) \leq f'(U_i)(-\Delta)_h^{\alpha/2} U_i, \qquad i = 1, ..., K-1. \tag{25}$$

Proof. Applying a first order Taylor approximation to the central difference and using the convexity of f, we obtain

$$\begin{aligned}
\delta_2^k f(U_i) &= \frac{f(U_{i+k}) - 2f(U_i) + f(U_{i-k})}{\xi_k^2} \\
&= f'(U_i)\delta_2^k U_i + \frac{f''(\zeta_1)}{2\xi_k^2}(U_{i+k} - U_i)^2 + \frac{f''(\zeta_2)}{2\xi_k^2}(U_{i-k} - U_i)^2 \\
&\geq f'(U_i)\delta_2^k U_i,
\end{aligned}$$

where ζ_1 is some point between U_i and U_{i+k} and ζ_2 lies between U_i and U_{i-k}. Similarly,

$$f(U_i) - f(U_\infty) \geq f'(U_i)(U_i - U_\infty).$$

The result now follows from the definition of the discrete fractional Laplacian given in terms of central differences by Equation (13). □

The following lemma shows that under the above assumptions, the finite difference solution is increasing with time and hence there exists an approximate blow up time.

Lemma 4. *Let U_i^n solve Equation (10) and suppose Assumptions 1 and 3 hold, then*

$$\frac{U_i^{n+1} - U_i^n}{\tau} = -(-\Delta)_h^{\alpha/2} U_i^n + f(U_i^n) \geq a f(U_i^n), \tag{26}$$

for any $i = 1, ..., K-1$, and $n \geq 0$.

Proof. To prove Inequality (26), we show that the quantity

$$V_i^n := -(-\Delta)_h^{\alpha/2} U_i^n + (1-a) f(U_i^n)$$

is non-negative for $i = 1, \dots, K-1$ and $n \geq 0$. Note that $V_i^0 \geq 0$ for $i = 1, \dots K-1$ by Assumption 3. Suppose by way of induction that $V_i^n \geq 0$ for $i = 1, \dots, K-1$ and some $n \geq 0$. Then, by definition,

$$\frac{V_i^{n+1} - V_i^n}{\tau} = -(-\Delta)_h^{\alpha/2} \left(\frac{U_i^{n+1} - U_i^n}{\tau} \right) + (1-a) \left(\frac{f(U_i^{n+1}) - f(U_i^n)}{\tau} \right). \tag{27}$$

Using the finite difference update given in Equation (10) and the result from Lemma 3, the first term in Expression (27) can be bounded below by

$$\begin{aligned}
-(-\Delta)_h^{\alpha/2} \left(\frac{U_i^{n+1} - U_i^n}{\tau} \right) &= -(-\Delta)_h^{\alpha/2} (-(-\Delta)_h^{\alpha/2} U_i^n + f(U_i^n)) \\
&= -(-\Delta)_h^{\alpha/2} V_i^n - a(-\Delta)^{\alpha/2} f(U_i^n) \\
&\geq -(-\Delta)_h^{\alpha/2} V_i^n - a f'(U_i^n)(-\Delta)_h^{\alpha/2} U_i^n.
\end{aligned}$$

Moreover, using the Taylor expansion of f together with its convexity and Equation (10), the second term in (27) can be bounded below by

$$\begin{aligned}(1-a)\left(\frac{f(U_i^{n+1})-f(U_i^n)}{\tau}\right) &= (1-a)f'(U_i^n)\left(\frac{U_i^{n+1}-U_i^n}{\tau}\right)+\frac{f''(\zeta_i^n)}{2\tau}\left(U_i^{n+1}-U_i^n\right)^2\\ &\geq (1-a)f'(U_i^n)\left(-(-\Delta)_h^{\alpha/2}U_i^n+f(U_i^n)\right),\end{aligned}$$

where ζ_i^n is some point between U_i^n and U_i^{n+1}. Combining these two estimates in Equation (27) gives

$$\begin{aligned}\frac{V_i^{n+1}-V_i^n}{\tau} &\geq (-\Delta)_h^{\alpha/2}V_i^n + af'(U_i^n)(-\Delta)_h^{\alpha/2}U_i^n+(1-a)f'(U_i^n)\left((-\Delta)_h^{\alpha/2}U_i^n+f(U_i^n)\right)\\ &= (-\Delta)_h^{\alpha/2}V_i^n+f'(U_i^n)V_i^n \geq (-\Delta)_h^{\alpha/2}V_i^n,\end{aligned}$$

where the final inequality is due to the positive slope of f and the induction hypothesis $V_i^n \geq 0$. To prove $V_i^{n+1}\geq 0$ for $i=1,\dots,K-1$, it thus suffices to show that $(I-\tau(-\Delta)_h^{\alpha/2})V_i^n \geq 0$. This in turn follows from the fact is only possible if *all* entries of $I-\tau(-\Delta)_h^{\alpha/2}$ are non-negative, which is guaranteed under the CFL condition (3.1), specifically by Inequalities (20) and (21). □

Corollary 1. *Under Assumptions 1 and 3, as well as the CFL condition (3.1), the solution U_i^n of Equation (10) satisfies $U_i^n \geq 0$ for $i=0,...,K$, $n\geq 0$. Moreover, $\|U^n\|_\infty := \max_j |U_j^n|$ satisfies*

$$\frac{\|U^n\|_\infty - \|U^{n-1}\|_\infty}{\tau} \geq af(\|U^{n-1}\|_\infty), \qquad \text{for } n\geq 1, \tag{28}$$

and $\|U^n\|_\infty \to \infty$ as $n\to\infty$.

Proof. By Assumption 3, we have $U_i^0 \geq 0$ for $i=0,...,K$. Now Assumption 1, guaranteeing that $f(u)\geq 0$ for $u\geq 0$, together with Inequality (26) imply

$$U_i^1 \geq U_i^0 + a\tau f(U_i^0) \geq U_i^0 \geq 0, \qquad \text{for } i=0,...,K.$$

Invoking this argument recursively yields $U_i^{n+1}\geq U_i^n \geq 0$ for $i=0,...,K$ and $n\geq 0$. Further, let $k=\text{argmax}_j U_j^{n-1}$, so that $\|U^{n-1}\|_\infty = U_k^{n-1}$. By Inequality (26),

$$\|U^n\|_\infty \geq U_k^n \geq U_k^{n-1}+a\tau f(U_k^{n-1}) = \|U^{n-1}\|_\infty + a\tau f(\|U^{n-1}\|_\infty),$$

from which Inequality (28) follows. Similarly, letting $l=\text{argmax}_j U_j^0$ so that $\|U^0\|_\infty = U_l^0 > 0$ and using Assumption 1, specifically the fact that $f'(u)>0$ for $u>0$, we again make use of Inequality (26) to obtain

$$\begin{aligned}\|U^n\|_\infty \geq U_l^n &\geq U_l^{n-1}+\tau af(U_l^{n-1})\\ &\geq U_l^{n-2}+2a\tau f(U_l^{n-2}) \geq \ldots \geq U_l^0 + na\tau f(U_l^0) \to \infty, \quad \text{as } n\to\infty.\end{aligned}$$

□

The above corollary implies that, for a fixed discretization level τ satisfying the CFL condition, the numerical stopping time exists, i.e. for any H with $H(s)>0$, $H'(s)>0$ for $s>0$ and $\lim_{s\to\infty} H(s)=\infty$ there is n_τ such that

$$\tau H(||U^{n_\tau - 1}||_\infty) < 1, \quad \tau H(||U^{n_\tau}||_\infty) \geq 1.$$

It now remains to show that the numerical blow-up time approaches the actual blow-up time as the discretization is refined.

Recall that the L^p norm is defined by

$$\|U^n\|_p = \begin{cases} \left(\sum_{j=1}^{N-1} h|U_j^n|^p\right)^{\frac{1}{p}}, & \text{if } 1 \leq p < \infty \\ \max_{j=1,\dots N-1} |U_j^n|, & \text{if } p = \infty \end{cases}$$

Definition 1. Let T be the blow-up time of the solution u of equation (8). Define $T_\infty = \tau n_\tau$ to be the numerical blow-up time.

Theorem 2. *If H satisfies*

$$\tau \ln f\left(H^{-1}\left(\frac{1}{\tau}\right)\right) \to 0 \quad as \quad \tau \to 0,$$

then $T_\infty \to T$ as $\tau \to 0$.

Proof. The theorem is essentially proved in the following papers of Nakagawa [26] and Cho [8, 7]. The outline is as follows: we prove the convergence by showing that

$$t_{\text{inf}} = \liminf_{\tau \to 0} T_\infty \geq T$$

$$t_{\text{sup}} = \limsup_{\tau \to 0} T_\infty \leq T.$$

From Lemma 4 and Corollary 1 it follows

$$||U^{n_{\tau_i}}|| \to \infty \quad \text{as } i \to \infty. \tag{29}$$

Next, assume $t_{inf} < T$. We pick τ_i and h_i such that the CFL condition

$$\frac{\tau}{h^\alpha} \leq \left(\frac{c_{1,\alpha}}{\nu}\left[\frac{2\nu}{\alpha} + 2^\nu + 1\right]\right)^{-1} \eta,$$

is satisfied and

$$T_\infty(\tau_i, h_i) < \frac{t_{\text{inf}} + T}{2}$$

The solution $u(t_n, x_j)$ remains bounded on the interval $[0, (t_{inf} + T)/2]$ which together with (29) contradicts to Theorem 1. A similar argument helps to establish the inequality for t_{sup}. □

4 Numerical Experiments

In this section, we apply the discretization described in Section 2 for the fractional Laplace operator $-(-\Delta)^{\alpha/2}$ to detect the numerical blow-up and get error estimates and convergence results.

Example 1. In the first example we demonstrate the convergence of our method as $h, \tau \to 0$, while satisfying (3.1). To this end, consider Equation (8) on the spatial domain $\Omega = (-1, 1)$ with reaction term $f(u) = u^2$ and initial condition

$$u(x, 0) = \begin{cases} 5(1 - x^2)^5, & x \in (-1, 1) \\ 0, & \text{otherwise} \end{cases}.$$

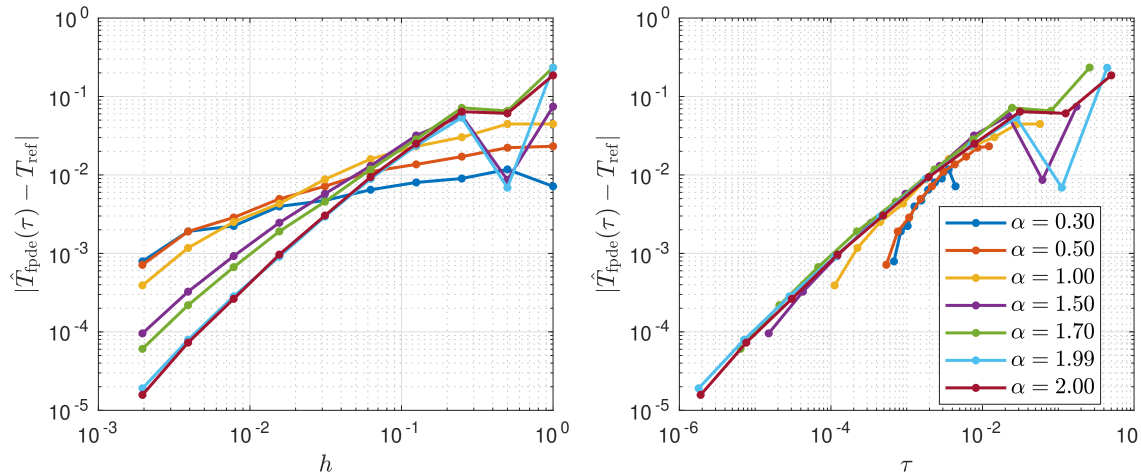
We use the auxiliary scaling function $H(s) = s$, which readily satisfies Assumption 2. Specifically

$$\lim_{\tau \to 0^+} \tau \ln \left(f \left(H^{-1} \left(\frac{1}{\tau} \right) \right) \right) = \lim_{\tau \to 0^+} \tau \ln(\tau^{-2}) = 0.$$

To investigate blowup estimate's convergence rate, we compute the numerical blowup time $\hat{T}_{\text{fpde}}$ at different spatial discretization levels $h = 2^{-l}$ where $l = 0, ..., 10$ with corresponding time discretization parameters τ chosen in accordance with the CFL condition (3.1). Specifically, we let

$$\tau = 0.9 \left(\frac{c_{1,\alpha}}{\nu} \left[\frac{2\nu}{\alpha} + 2^{\nu} + 1 \right] \right)^{-1} h^{\alpha}, .$$

In the absence of an explicit formula for the exact blowup time, we use a numerical solution based on $h = 2^{-10}$ as a reference. The results for different values of fractional exponents $\alpha \in (0, 2]$ are shown in Figure 1. As expected, the accuracy of the numerical estimate $\hat{T}_{\text{fpde}}$ improves as both τ and $h \to 0$. Moreover, the estimated rate of convergence are consistent with the log-linear decay of the error term.



(a) Log-log plot of the error vs h.

(b) Log-log plot of the error vs τ.

Figure 1: Convergence of the numerical blowup time as $h, \tau \to 0^+$.

Figure 1 cannot be used directly to compare the convergence of the method for different values of α. For every value of h, the time step τ was chosen based solely on stability requirements. Figure 2 shows the relation between h and τ for these α-values. It illustrates how, for a given meshwidth h, the CFL condition (3.1) requires a far smaller time step τ when α is near 2, and allows for a larger τ when α is small. Thus, even though Figure 1a shows that, for a given value of h, the error decreases as α increases, it may be due to smaller values of τ for large α's. Similarly, a better accuracy at smaller α-values for a given τ, as shown in Figure 1b, may be the result of a much finer spatial mesh.

Figure 3 shows the blowup time $\hat{T}_{\text{fpde}}$ for the problem above, estimated at the finest refinement levels, for different values of $\alpha \in (0, 2]$. Since diffusion counteracts the growth caused by the reaction term f, it is reasonable to expect a delay in the onset of blowup as the fractional exponent α, and hence the speed of diffusion, is increased.

Example 2. In this example, we investigate the effect of the auxiliary scaling function $H(s)$ on the accuracy of the blow-up estimate $\hat{T}$. In particular, we compute estimates of $\hat{T}$ using both power

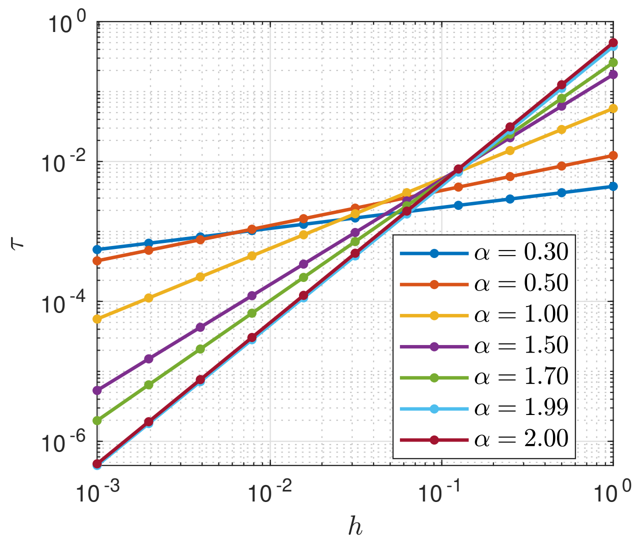


Figure 2: The largest temporal step size τ guaranteeing stability for different mesh-width values h.

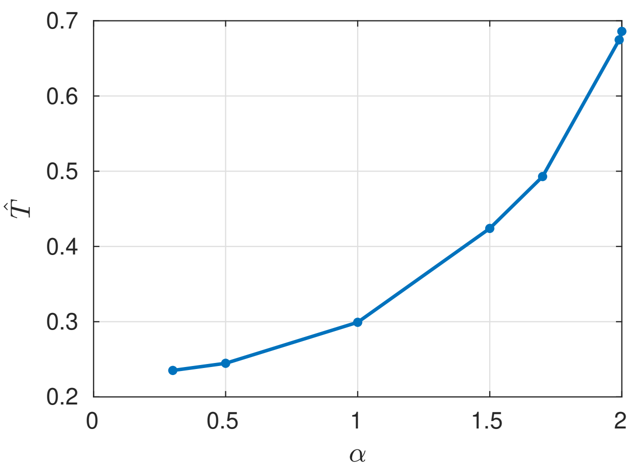


Figure 3: The estimated of blowup time $\hat{T}$ for the system in Example 1 for different values of α.

functions and exponential functions and compare their accuracy. Moreover, we try to establish whether there is a relation between the optimal form of H and the error term $t\ln(f(H^{-1}(1/\tau)))$.

Consider the equation $\partial_t u + (-\Delta)^{\alpha/2}u = u^2, x \in \Omega = (-1,1), t > 0$ with the following non-negative initial condition

and the exterior Dirichlet boundary condition

$$u(x,t) = 0 \quad x \in \mathbb{R} \setminus \Omega, t > 0.$$

We should choose an increasing function $H(s)$ such that $\lim\limits_{s\to\infty} H(s) = \infty$ and

$$\tau \ln f\Big(H^{-1}\Big(\frac{1}{\tau}\Big)\Big) \to 0 \quad \text{as} \quad \tau \to 0,$$

for $\tau > 0$. And we stop the numerical computation when $\tau H(\|u^n\|_p) > 1$ at the step n. There is no well ordered way to choose the function H. However, following [13] we will begin with $H(s) = s$ as a starting choice. The discrete initial and boundary conditions are

$$U_j^0 = u_0(x_j), \quad j = 1, ..., J-1$$

and

$$U_0^n = U_J^n = 0 \quad n \geq 0.$$

where h is the spacial grid size and τ is the temporal mesh size.

Next, we need to compute the blow-up time for the fractional PDE. The estimated numerical blow-up time is $T_\infty = 0.209880$.

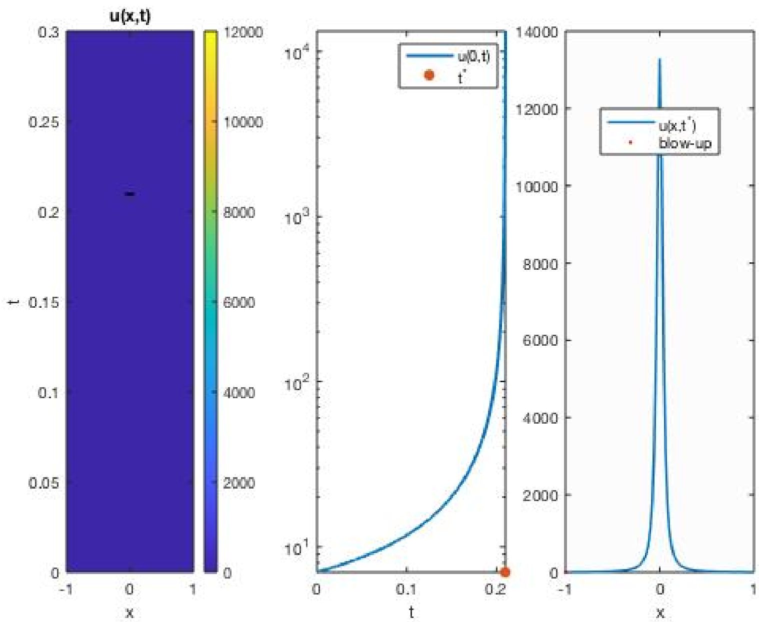


Figure 4: $\partial_t u + (-\Delta)^{\alpha/2} u = u^2, x \in (-1, 1)$, with $\alpha = 1.5$, $T_\infty = 0.209880$

As mentioned before different choices of function H and different norms will lead to different numerical blow-up times. Our choice of H is $H(s) = s$. The table below shows blow-up times for different choices of H.

Table 1: T_∞ and T_1 numerical blow-up times for $\partial_t u + (-\Delta)^{\alpha/2} u = u^2, x \in (-1, 1), \alpha = 1.5$.

$H(s) = s^m$	T_∞	T_1
m=1	0.209880	0.210120
m=1.1	0.209640	0.210120
m=4/3	0.208560	0.209880
m=3/2	0.207120	0.209400
m=2	0.197520	0.204000

This table suggests a pattern which we illustrate graphically below. The convergence is from below for large H while the convergence is from above for small H. See Figure 5.

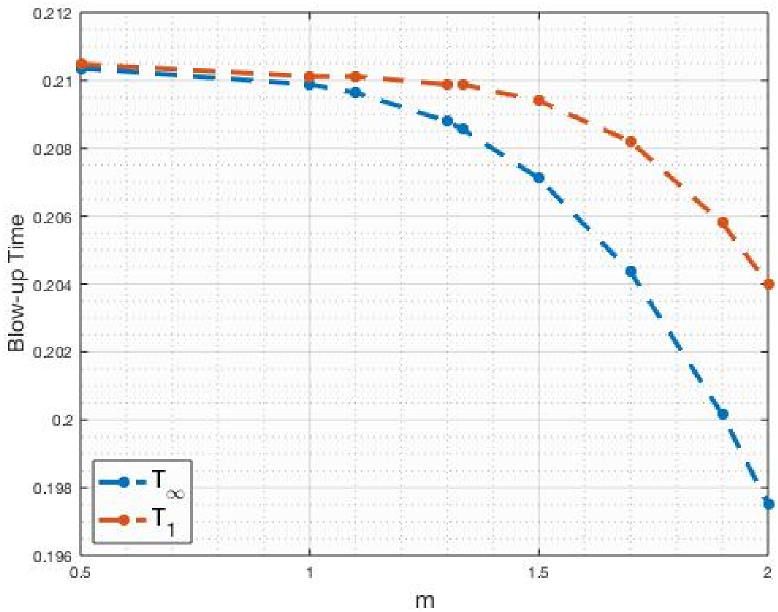


Figure 5: T_∞ and T_1 blow-up times for different choices of H.

However, we aim to find an optimal H numerically. The reference blow-up time corresponding to $nt = 2^{19}$ is $T_\infty = 0.209056$ and is $T_1 = 0.209058$. Next we estimate the error to find what H will minimize the error. The error is defined as

$$\text{error} = |\text{ reference blow-up time - blow-up time}|.$$

We find that for T_∞ the error is minimized when $m = 1.3$ and for T_1 when $m = 1.5$. We illustrate that graphically. See Figure 6.

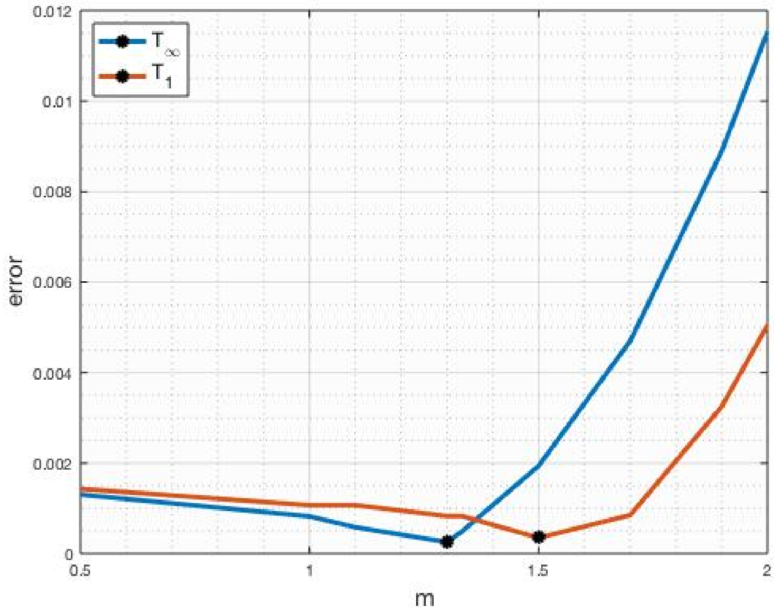


Figure 6: The error for different choices of H.

It's very difficult to observe a systematic way for choosing the function H. Next, we set the splitting parameter $\gamma = 2$ and we compute the blow-up times for different choices of α. The table below suggests a pattern.

Table 2: T_∞ and T_1 for $\partial_t u + (-\Delta)^{\alpha/2} u = u^2, x \in (-1,1), \gamma = 2, H(s) = s$.

α	T_∞	T_1
0.5	0.161040	0.161400
0.8	0.168480	0.168840
1	0.176280	0.176640
1.3	0.193920	0.194160
1.5	0.209880	0.210120
1.7	0.228840	0.229200
1.9	0.251040	0.251280
1.999	0.263760	0.264000

We observe that the blow-up happens at later time when α gets larger. We also notice that the blow-up times T_∞ and T_1 are different as it was expected but they are very close. The plot below is the illustration of Table 2.

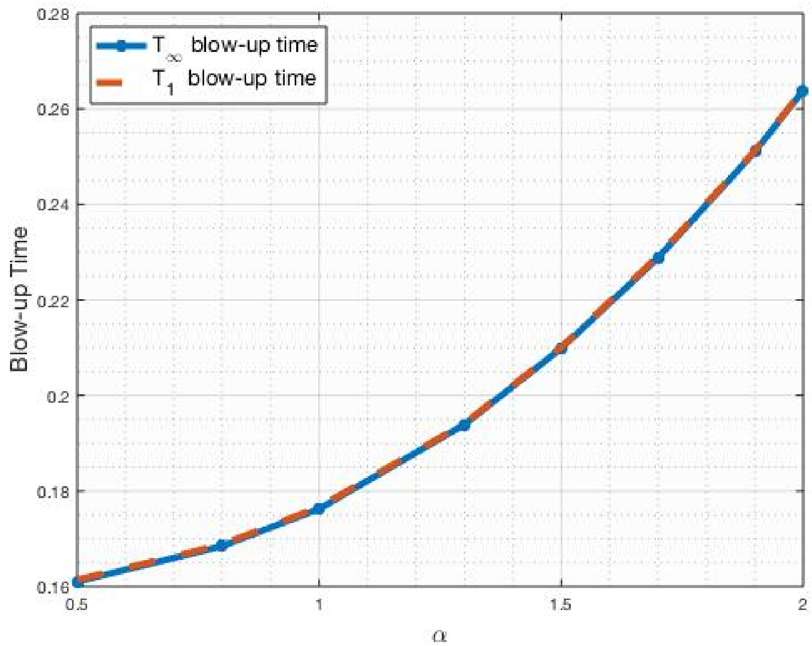


Figure 7: T_∞ and T_1 blow-up times using $\alpha \in [0.3, 1.999]$.

Next, we numerically estimate the T_1 blow-up time. When $\alpha = 1.5$ and the $H(s) = s$ the blow-up time is $T_1 = 0.210120$.

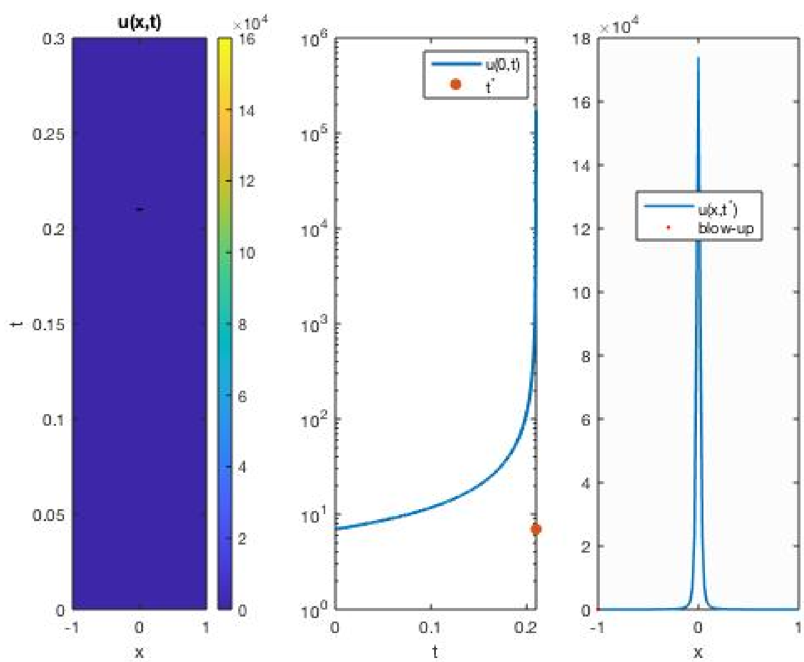


Figure 8: $\partial_t u + (-\Delta)^{\alpha/2} u = u^2, x \in (-1, 1)$, with $\alpha = 1.5$, $T_1 = 0.210120$.

Example 3. Consider the following fractional equation

$$\partial_t u + (-\Delta)^{\alpha/2} u = e^u, x \in \Omega = (-1,1), t > 0$$

with the exponential reaction term and with the following non-negative initial condition

$$u(0,x) = u_0 = \begin{cases} 3(1-x^2)^{1+m} & x \in (-1,1) \\ 0 & else \end{cases}$$

and the exterior Dirichlet boundary condition.

$$u(x,t) = 0 \quad x \in \mathbb{R} \setminus \Omega, t > 0.$$

We apply the same discretization described in Sections 2 and choose a non-negative increasing function $H(s) = e^s - 1$ to help to detect the numerical blow-up time. The estimated blow-up time is $T_\infty = 0.083040$ when $\alpha = 1.5$. We observe the blow-up happens at an earlier time compared to $f(u) = u^2$. Also, notice that the initial condition is not taken as large as it is for $f(u) = u^2$.

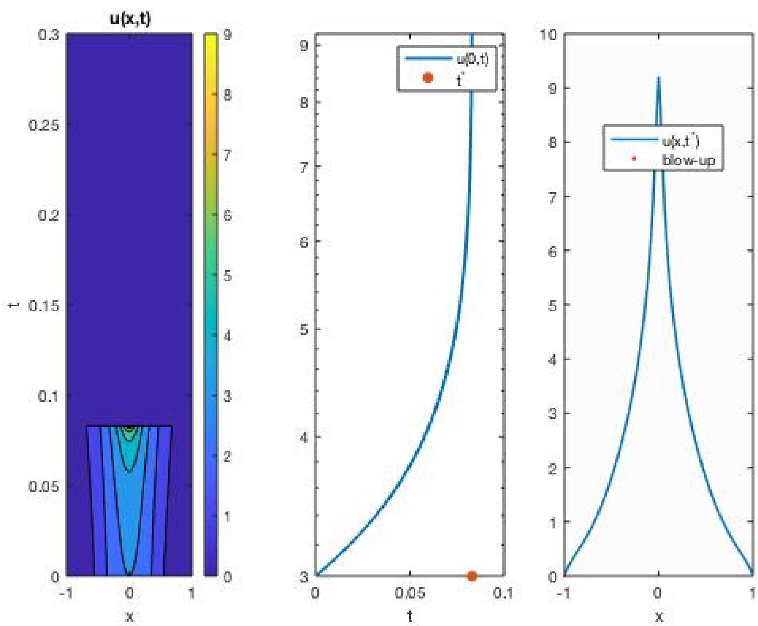


Figure 9: $\partial_t u + (-\Delta)^{\alpha/2} u = e^u, x \in (-1,1)$, $\alpha = 1.5, H(s) = e^s - 1$, $T_\infty = 0.083040$.

Next, we compute the numerical blow-up time for different α-s. We can recognize a pattern similar to $f(u) = u^2$. We notice that the T_∞ and T_1 numerical blow-up times are different as it was expected, and the difference is within the following range $(0.00036, 0.00048)$.

Table 3: T_∞ and T_1 for $\partial_t u + (-\Delta)^{\alpha/2} u = e^u, x \in (-1,1), \gamma = 2, H(s) = e^s - 1$.

α	T_∞	T_1
0.5	0.057360	0.057720
0.8	0.060360	0.060720
1	0.063720	0.064080
1.3	0.072480	0.072960
1.5	0.083040	0.083520
1.7	0.099960	0.100320
1.9	0.125640	0.126000
1.999	0.141960	0.142440

We observe that the blow-up happens at a later time as α gets larger.

4.1 Convergence Results

In this section, we would like to show mesh refinement. We fix the mesh ratio $\lambda = 0.3$ to ensure the stability and start to refine the time mesh by powers of 2 and compute the T_∞ and T_1 numerical blow-up times. Since $\lambda = 0.3$ and $\alpha = 1.5$ then by solving for h we get

$$h = \left(\frac{\tau}{0.3}\right)^{2/3}.$$

The tables below represent the estimated error and the rate for T_∞ and T_1 with $\alpha = 1.5$ and $H(s) = s$.

Table 4: T_∞, error and rate for $\partial_t u + (-\Delta)^{\alpha/2} u = u^2, x \in (-1,1), \gamma = 2$.

nt	time mesh τ	h	T_∞	error e	rate
2^4	0.01875000	0.15749013	0.093750	0.115306	1.16867269
2^5	0.00937500	0.09921257	0.065625	0.143431	0.84047137
2^6	0.00468750	0.06250000	0.056250	0.152806	0.67755673
2^7	0.00234375	0.03937253	0.100781	0.108275	0.68726313
2^8	0.00117188	0.02480314	0.214453	0.005397	1.41255499
2^9	0.00058594	0.01562500	0.212109	0.003053	1.39259278
2^{10}	0.00029297	0.00984313	0.210645	0.001589	1.39464977
2^{11}	0.00014648	0.00620079	0.210059	0.001003	1.35838127
2^{12}	0.00007324	0.00390625	0.209546	0.000490	1.37436633
2^{13}	0.00003662	0.00246078	0.209326	0.000270	1.36785611
2^{14}	0.00001831	0.00155020	0.209198	0.000142	1.36948087
2^{15}	0.00000916	0.00097656	0.209134	0.000078	1.36461664
2^{16}	0.00000458	0.00061520	0.209093	0.000037	1.38019830
2^{17}	0.00000229	0.00038755	0.209072	0.000016	1.40572664
2^{18}	0.00000114	0.00024414	0.209062	0.000006	1.44555051

Note that the error is defined defined as

$$e = |T_\infty - ref|$$

where ref is the reference blow-up time $T_\infty = 0.209056$ corresponding to $nt = 2^{19}$. The rate is

$$r = \frac{\log e}{\log h}$$

as

$$e = ch^r$$

where c is a constant.

Also, notice that the smaller time step results in a more refined blow-up time and a smaller error, i.e. from $nt = 2^4$ to $nt = 2^{18}$ we have an error refinement from 0.1153060 to 0.0000060 making 0.1153000 difference. And the space mesh h is getting smaller by a factor of 1.6. We can perform similar analysis for the T_1 blow-up time.

Table 5: T_1, error and rate for $\partial_t u + (-\Delta)^{\alpha/2} u = u^2, x \in (-1,1), \gamma = 2$.

nt	time mesh τ	h	T_1	error e	rate
2^4	0.01875000	0.15749013	0.037500	0.171558	0.95371188
2^5	0.00937500	0.09921257	0.075000	0.134058	0.86972123
2^6	0.00468750	0.06250000	0.056250	0.152808	0.67755200
2^7	0.00234375	0.03937253	0.103125	0.105933	0.69402343
2^8	0.00117188	0.02480314	0.216797	0.007739	1.31505696
2^9	0.00058594	0.01562500	0.213281	0.004223	1.31458601
2^{10}	0.00029297	0.00984313	0.211523	0.002465	1.29962950
2^{11}	0.00014648	0.00620079	0.210498	0.001440	1.28723393
2^{12}	0.00007324	0.00390625	0.209839	0.000781	1.29029873
2^{13}	0.00003662	0.00246078	0.209473	0.000415	1.29630012
2^{14}	0.00001831	0.00155020	0.209271	0.000213	1.30680632
2^{15}	0.00000916	0.00097656	0.209171	0.000113	1.31113896
2^{16}	0.00000458	0.00061520	0.209111	0.000053	1.33159201
2^{17}	0.00000229	0.00038755	0.209081	0.000023	1.35953000
2^{18}	0.00000114	0.00024414	0.209066	0.000008	1.41096405

Note that for this case the reference blow-up time is $T_1 = 0.209058$ corresponding to $nt = 2^{19}$. Hence, error refinement from 2^4 to 2^{18} is 0.171550.

The plot below represent the numerical blow-up times corresponding to different times steps, in particular to $nt = 2^8, 2^9, 2^{10}, 2^{11}$... and etc. We can see from the plot the estimated blow-up times converge to their reference blow-up times, $T_\infty = 0.209056$ and $T_1 = 0.209058$.

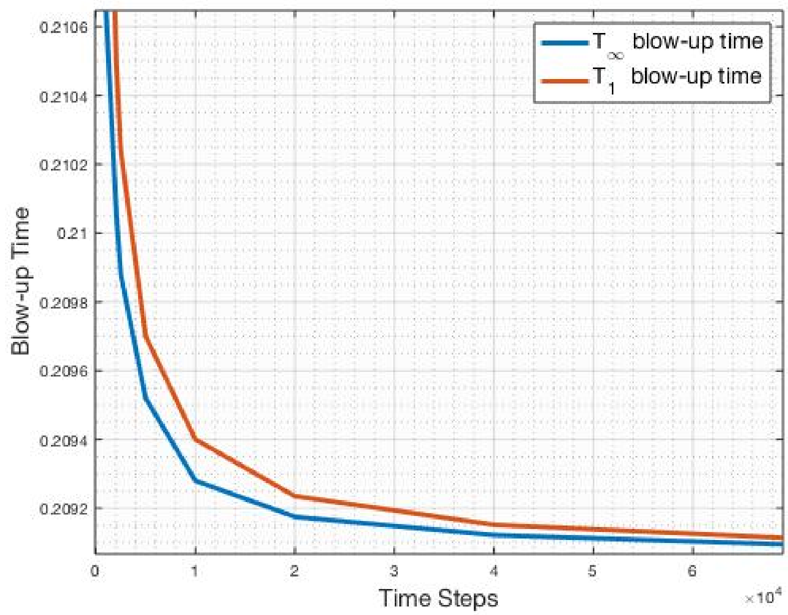


Figure 10: T_∞ and T_1 blow-up times convergence with respect to different times steps for a fixed λ. $\partial_t u + (-\Delta)^{\alpha/2} u = u^2, x \in (-1, 1)$, with $\alpha = 1.5, H(s) = s$.

We notice that the error gets close to 0 when the time mesh τ gets smaller. The plots below confirm that as long as the space mesh h ($\log(h)$) gets smaller the error ($\log(e)$) gets closer to zero.

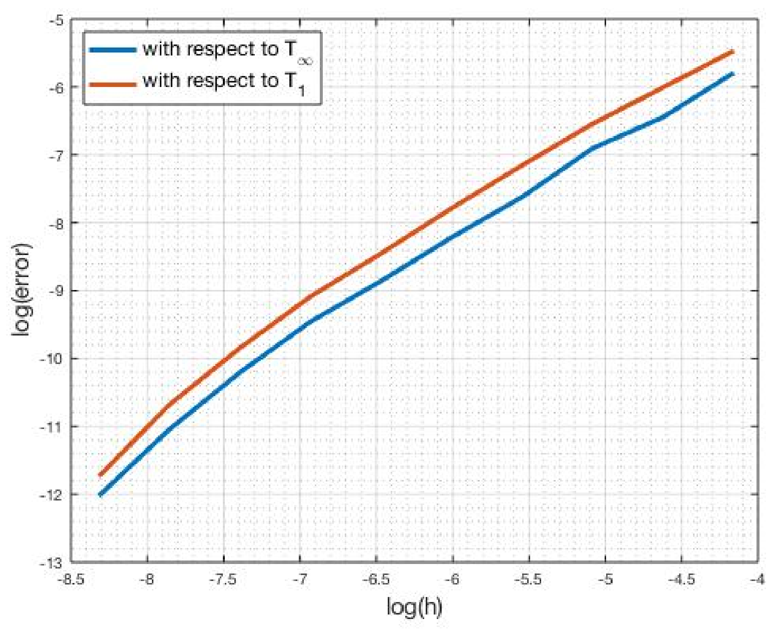


Figure 11: Mesh in space ($\log h$) with respect to errors ($\log e$) for T_∞ and T_1.

5 FPDE ($\alpha \to 2$) and PDE ($\alpha = 2$) Comparison

We would also like make a comparison between a PDE and a FPDE when classical and fractional Laplace operators are considered, respectively. The goal is to inspect how close the computed blow-up times are when central difference scheme is used for PDE, i.e. $\alpha = 2$, and weighted trapezoidal is used for the FPDE with $\alpha = 1.999$. In order to make the comparison possible same initial condition and the same choice of $H(s) = s$ is considered

$$u(0,x) = u_0 = \begin{cases} 7(1-x^2)^{1+m} & x \in (-1,1) \\ 0 & else \end{cases}$$

and the exterior Dirichlet boundary condition

$$u(x,t) = 0 \quad x \in \mathbb{R} \setminus \Omega, t > 0.$$

First we consider equation (3) with 0 boundary conditions and $f(u) = u^2$. Spacial grid size is chosen such that $\lambda = \dfrac{\tau}{h^2} = 0.3$ is less than 0.5. The numerical blow-up time for the classical heat equation with appropriate boundary and large enough initial condition is $T_\infty = 0.264000$.

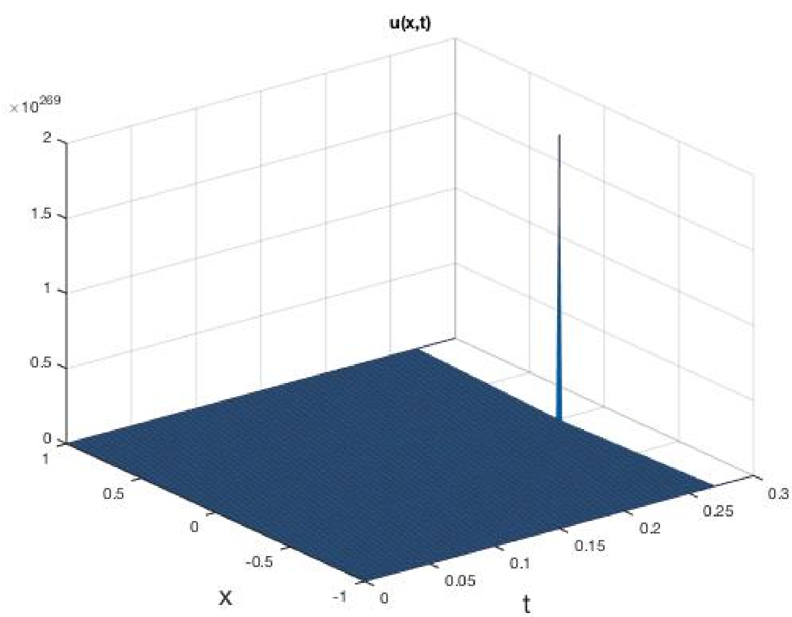


Figure 12: $u_t - u_{xx} = u^2, x \in \Omega = (-1,1), t > 0$ with $\tau = \dfrac{0.3}{50^2}$.

Next, consider the fractional problem again with 0 boundary conditions. Illustrated in Figure 12. Note that the $\lambda = 0.2988$ in this case. The numerical blow-up time for the fractional PDE is $T_\infty = 0.263760$.

We observe that the numerical blow-up times detected by the numerical algorithms for the classical and fractional reaction-diffusion equation are close when the same boundary and initial conditions are considered. Hence, we can claim that the numerical algorithm implemented to discretize the fractional heat equation provide an accurate blow-up time.

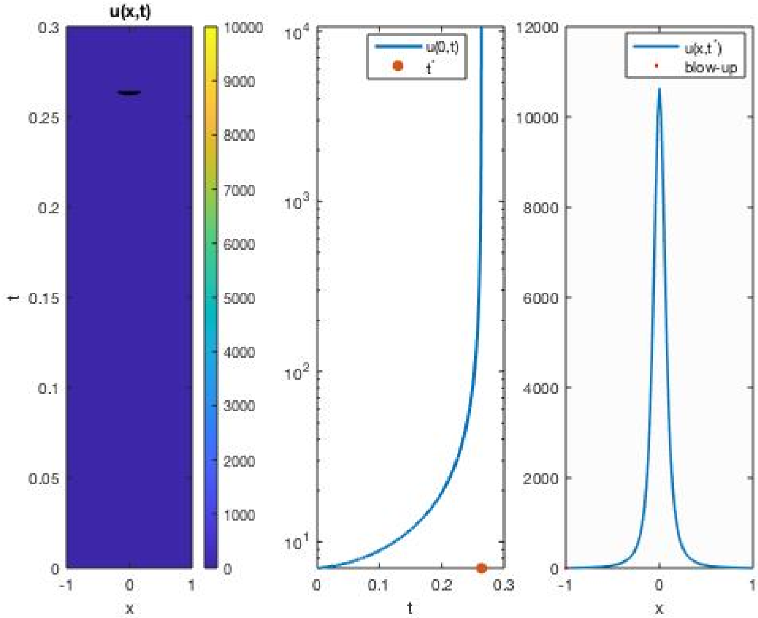


Figure 13: $\partial_t u + (-\Delta)^{\alpha/2} u = u^2, x \in (-1, 1)$, $\alpha = 1.999$, $\gamma = 2$, $T_\infty = 0.263760$.

6 Conclusion

In this work, we extended a numerical method for detecting the blowup time introduced by Cho in [8], to space-fractional reaction diffusion equations with homogeneous exterior conditions. We used a weighted trapezoidal method [15] to discretize the fractional Laplacian and propagated the numerical solution in time by a forward Euler time-stepping scheme. In our convergence analysis, we derived CFL conditions under which the scheme is stable and showed that the estimated blow-up time converges to the true blowup time under appropriate assumptions on the auxiliary scaling function H. The form of H can influence the accuracy of the approximated blowup time, and we investigated its choice based on the form of the error term given in (7). In our numerical experiments, we demonstrated the convergence of the blowup time estimates, investigated their dependence on the fractional exponent α and reaction term f, and showed that they converge to those of the classical reaction-diffusion system (3) as $\alpha \to 2^-$. The main advantage of this blowup detection method over that proposed by [27], is that it does not require the time-step to be decreased as the solution approaches a blowup; the choice of the fixed time-step is based solely on accuracy and stability considerations. Nevertheless, for small spatial mesh sizes, the stability requirement can place severe constraints on the temporal step size, leading to an increase in computational cost, especially since the discretized fractional Laplacian is not sparse. One avenue of future work would be to extend the method to include implicit schemes. Since the spatial location of the solution's blowup is often localized, another interesting direction would be to adapt the moving mesh methods developed in [6] to discretizations of the fractional Laplacian. And use them to approximate the spatial

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