Mathematical analysis of surface plasmon resonance of a nano-gap in a plasmonic metal

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Abstract

We develop a mathematical theory for the excitation of surface plasmon resonance on an infinite thick metallic slab with a nano-gap defect. Using layer potential techniques, we establish the well-posedness of the involved scattering problem. We further derive asymptotic analysis to characterize the leading order term in the excited surface plasmon waves and derive sharp estimates for both the plasmonic part and non-plasmonic part. The explicit dependence of the surface plasmon resonance on the size of the nano-gap, and both the real and imaginary part of the metal dielectric constant is highlighted.

1 Introduction

Surface plasmonics is an emerging field of nanophotonic which studies the coupling of light and collective oscillations of free electron density on a metal-dielectric interface or localized metallic nanostructures. The resonant coupling results surface plasmon resonance, which enables localization of electromagnetic field at the sub-wavelength scale as well as enhancement of optical scattering and absorption. Many important applications in bio-sensing and design of novel optical devices [17, 21, 22, 26] have been proposed because of these extraordinary optical properties. There are two types of surface plasmon resonance: the first is called localized surface plasmon resonance which occurs on a metal nanostructure with finite size such as metallic nano-particles, and the second occurs on a infinite metal-dielectric interface which induces surface waves propagating along an infinite flat surface. The mathematical theory for the first type of plasmon resonance relies on the analysis of the spectral properties of the Neuman-Poincare operators, which has been studied extensively recently, see [2, 18, 10, 7, 9, 12, 13, 19, 20] and the references therein. See also [5, 6, 8] for the mathematical theory on the applications of localized surface plasmon in meta-surface and bio-sensing. Regarding to the second type of plasmon resonance, it is well known (through

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analytical solution) that they cannot be excited on an infinity flat metal-dielectric interface by an incident plane wave. To excite them, certain defect in the interface must be created. Few mathematical results exists for such a case. In the paper, we investigate a special case where the defect is formed by a nano-gap filled with perfect conducting materials. This conceptually simplified model allows us to derive asymptotic to the excitation of surface plasmon resonance which may provide valuable insight into the mechanism behind without too much technicalities while not losing essence of the physics. To our knowledge, the result in the paper provides the first rigorous mathematical theory. We expect that the approach developed here can be generalized to study surface plasmon resonance in the other types of defects in metal-dielectric interface.

![Figure 1: Geometry of the model. The domain of vacuum and metal is denoted by $\Omega_1$ and $\Omega_2$ respectively. The infinitely long and perfectly conducting slit $S_\delta$ perforated in the slab $\Omega_2$ has a width of $2\delta$. The remaining part of the metal consists of two disjoint semi-infinite domains $\Omega_2^-$ and $\Omega_2^+$. The scaling of the geometry is given by $\delta \ll \lambda$.](image)

To be more specific, we consider the model where the medium consists of two layers that are separated by the interface $\Gamma = \{(x_1, x_2) \mid x_2 = 0\}$. The top layer is vacuum that occupies the upper half plane $\Omega_1 := \{(x_1, x_2) \mid x_2 > 0\}$, and the bottom layer is metal that occupies the lower half plane $\Omega_2 := \{(x_1, x_2) \mid x_2 < 0\}$. The relative permittivity $\varepsilon$ on the $x_1x_2$ plane is given by

$$\varepsilon(x) = \begin{cases} 1 & x \in \Omega_1, \\ \varepsilon_m & x \in \Omega_2, \end{cases}$$

where $\varepsilon_m = \varepsilon'_m + i \varepsilon''_m$ is the relative permittivity for the metal. Note that $\varepsilon_m$ is a complex number depending on the frequency through the following Drude’s model [28]:

$$\varepsilon_m(\omega) = 1 - \frac{\omega_p^2}{\omega \left(\omega + i\gamma \right)}$$

where $\omega_p$ is the plasmon frequency of the metal and $\gamma$ is the damping coefficient. In this paper, we are interested in the frequency range where the real part $\varepsilon'_m$ is negative and it holds that $|\varepsilon'_m| \gg |\varepsilon''_m|$. This is true for most noble metals in certain optical frequency regime.
For metals like gold and silver, experimental data shows that $\varepsilon'_m < 0$ for frequencies in the range $200 - 700 \mu m$. Metals with such property are usually referred to as plasmonic metals.

To facilitate our asymptotic analysis, we assume that $|\varepsilon'_m| = O(\delta^\alpha)$ and $|\varepsilon''_m| = O(\delta^\beta)$, where $\alpha < 0$ and $\alpha < \beta$ [28]. In the main results obtained in this paper, we require that $-4 \leq \alpha < 0$ and $\beta < 2$. This will cover a broad frequency range that are of interest.

Assume that the lower half plane is perturbed by an infinitely long and perfectly conducting nano-gap $S_\delta^\infty := \{(x_1, x_2) \mid -\delta < x_1 < \delta, -\infty < x_2 < 0\}$, where its boundary consists of three segments $\Gamma_0^\delta$, $\Gamma_v^-$ and $\Gamma_v^+$ respectively. Then the remaining parts of $\Omega_2$ consist of two disjoint semi-infinite domains $\Omega_2^-$ and $\Omega_2^+$ as shown in Figure 1. Throughout the paper, it is assumed that the slit width is much smaller than the incident wavelength such that $\delta \ll \lambda$.

Let $\Gamma_\delta^-$ and $\Gamma_\delta^+$ be the left- and right- segment of the metal-vacuum interface respectively with the presence of the slit.

Let $u^i$ be a plane wave incident from the above and $u^s$ be the scattered field. The total field $u$ after the scattering consist of $u^i$ and $u^s$ in $\Omega_1$ and $u^s$ only in $\Omega_2^\pm$. It satisfies the following equations:

\[
\begin{cases}
\nabla \cdot \left( \frac{1}{\varepsilon(x)} \nabla u \right) + k^2 u = 0 & \text{in } \Omega_1 \cup \Omega_2^+ \cup \Omega_2^-,

[u] = 0, \quad \left[ \frac{\partial u}{\varepsilon \partial \nu} \right] = 0 & \text{on } \Gamma_\delta^- \cup \Gamma_\delta^+,

\partial u \partial \nu = 0 & \text{on } \Gamma_\delta^0 \cup \Gamma_v^- \cup \Gamma_v^+,
\end{cases}
\]

where $[\cdot]$ denotes the jump of the quantity when the limit is taken along the positive and negative unit normal direction $\nu$ respectively. Since the metallic structure is infinite in size, the usual Sommerfeld radiation condition does not hold for the scattered field and one should impose other radiation conditions which we shall address in this paper.

We are interested in the following issues:

1. The existence and uniqueness of the above equations (1.1). The case when $\varepsilon'_m > 0$ has been well-known in the literature, but no result is available for the case studied here.

2. The characterization of the surface plasmonic resonances and how it depends on the size of the nanogap $\delta$, and both the real part and imaginary part of the dielectric constant $\varepsilon_m$.

3. The stability or the estimate of the solution to the equations (1.1).

The study in this paper is also motivated by the recent attempt to the understanding of light interaction with subwavelength structures. This topic has drawn increasing interest in optics research ever since the report [16] by Ebbesen et al. on the so-called extraordinary optical transmission (EOT). See the review paper [17, 29] and the reference therein for investigation by various groups on various subwavelength structures, such as a single hole or slit, a periodic array of holes or slits for a variety of frequency regimes. In a series of studies [22]-[25], we established rigorous mathematical theories for the case of narrow slit structures perforated in a perfectly conducting metallic slab, both a single slit structure and a periodic
array of slit structures in various scaling regime were considered, and a complete mathematical theory for various interesting phenomenon such as the scattering enhancement, local field enhancement, existence of surface waves and total transmission were given. The results in this paper are expected to advance our study therein to the case of plasmonic metallic slabs where surface plasmon play a role in the involved physics. For related mathematical works, we would also like to refer the readers to [11, 15] for resonant scattering by closely related cavity structures in perfectly conductors, while resonant scattering by dielectric photonic crystal slabs can be found in [30, 31].

The rest of the paper is organized as follows. In Section 3, we investigate the integral operators thoroughly and present relevant estimates. The solution of the integral equation is then studied in Section 4 and 5 when the incident wave is even and odd with respect to the \(x_1\) variable.

### 2 Integral equation formulation

We use layer potentials to solve the scattering problem (1.1). See, for instance \([3, 4]\), for applications of the technique to various wave scattering problems. We first introduce two Green’s functions. Let \(G_1(x, y)\) be the Green’s function in the upper layer that satisfies

\[
\begin{aligned}
\Delta G_1(x, y) + k^2 G_1(x, y) &= \delta(x - y) & x, y &\in \Omega_1, \\
\frac{\partial G_1(x, y)}{\partial \nu_y} &= 0 & \text{on } \partial \Omega_1.
\end{aligned}
\]

Then

\[
G_1(x, y) = -\frac{i}{4} \left( H_0^{(1)}(k|x - y|) + H_0^{(1)}(k|x' - y|) \right),
\]

where \(H_0^{(1)}\) is the first kind Hankel function of order 0, and \(x' = (x_1, -x_2)\). From the Green’s second identity, one obtains an integral equation for the scattered field \(u^s\):

\[
u^s(x) = \int_{\Gamma} G_1(x, y) \frac{\partial u^s(y_1, 0^+)}{\partial \nu} ds_y, \quad x \in \Omega_1.
\]

From the continuity of the single-layer potential and the fact that \(\frac{\partial u}{\partial \nu} = 0\) on \(\Gamma_0^\delta\), it follows that the total field satisfies

\[
u(x) = \int_{\Gamma^-_\delta \cup \Gamma^+_\delta} G_1(x, y) \frac{\partial u}{\partial \nu} (y_1, 0^+) ds_y + 2u^i(y), \quad x \in \Gamma^-_\delta \cup \Gamma^+_\delta. \tag{2.1}
\]

Let \(G_{2,\pm}(x, y)\) be the Green’s function in the domain \(\Omega^\pm_2\) that satisfies

\[
\begin{aligned}
\Delta G_{2,\pm}(x, y) + k^2 \varepsilon_m G_{2,\pm}(x, y) &= \varepsilon_m \delta(x - y) & x, y &\in \Omega^\pm_2, \\
\frac{\partial G_{2,\pm}(x, y)}{\partial \nu_y} &= 0 & \text{on } \Gamma^\pm_\delta \cup \Gamma^\pm_v.
\end{aligned}
\]
It is easy to check that
\[ G_{2,\pm}(x, y) = G_{2}^{(0)}(x, y) + G_{2}^{(1)}(x, y), \]
where \( G_{2}^{(0)}(x, y) := \varepsilon_{m} H_{0}^{(1)}(k_{m}|x - y|) \) is the Green’s function in the homogeneous medium of metal. By the Green’s second identity, we obtain
\[ u(x) = -\int_{\Gamma_{\delta}^{\pm}} G_{2,\pm}(x, y) \frac{1}{\varepsilon_{m}} \frac{\partial u}{\partial \nu}(y, 0-) \, dy, \quad x \in \Omega_{2}^{\pm}. \]
Taking the limit leads to
\[ u(x) = -\int_{\Gamma_{\delta}^{\pm}} G_{2,\pm}(x, y) \frac{1}{\varepsilon_{m}} \frac{\partial u(y)}{\partial \nu}(y, 0-) \, dy, \quad x \in \Gamma_{\delta}^{\pm}. \]  
(2.2)

Let us define a function \( \varphi \in H^{-1/2}(\mathbb{R}) \) by letting
\[ \varphi(x_{1}) = \begin{cases} \partial_{y} u(x_{1}, 0) & x_{1} \in (-\infty, -\delta) \cup (\delta, \infty), \\ 0 & x_{1} \in (-\delta, \delta). \end{cases} \]
From the continuity conditions \( \partial_{y} u(x_{1}, 0+) = \frac{1}{\varepsilon_{m}} \partial_{y} u(x_{1}, 0-) \) along the interfaces \( \Gamma_{\delta}^{\pm} \), a combination of (2.1) and (2.2) leads to the system of integral equations
\[
\begin{cases}
\int_{-\delta}^{0} [G_{1}(x_{1}, 0; y_{1}, 0) + G_{2,\pm}(x_{1}, 0; y_{1}, 0)] \varphi(y_{1}) \, dy_{1} + \int_{\delta}^{\infty} G_{1}(x_{1}, 0; y_{1}, 0) \varphi(y_{1}) \, dy_{1} + 2u^{i} = 0, \\
\int_{-\infty}^{-\delta} G_{1}(x_{1}, 0; y_{1}, 0) \varphi(y_{1}) \, dy_{1} + \int_{\delta}^{\infty} [G_{1}(x_{1}, 0; y_{1}, 0) + G_{2,\pm}(x_{1}, 0; y_{1}, 0)] \varphi(y_{1}) \, dy_{1} + 2u^{i} = 0,
\end{cases}
\]  
(2.3)

where the first equation holds for \( x_{1} < -\delta \) and the second for \( x_{1} > \delta \).

We distinguish two cases when the incident wave \( u^{i} \) is even and odd respectively with respect to \( x_{1} \). To this end, let us define
\[ u^{i}_{e} = \frac{1}{2} (e^{i(k_{1}x_{1} + k_{2}x_{2})} + e^{i(-k_{1}x_{1} + k_{2}x_{2})}) = \cos(k_{1}x_{1}) \cdot e^{i k_{2} x_{2}}, \]
and
\[ u^{i}_{o} = \frac{1}{2i} (e^{i(k_{1}x_{1} + k_{2}x_{2})} - e^{i(-k_{1}x_{1} + k_{2}x_{2})}) = \sin(k_{1}x_{1}) \cdot e^{i k_{2} x_{2}}. \]

(i) \( u^{i} = u^{i}_{e} \) such that \( \varphi(x_{1}) = \varphi(-x_{1}) \). The system (2.3) reduces to the following integral equation on \( \Gamma_{\delta}^{+} \):
\[
\int_{\delta}^{\infty} [G_{1}(x_{1}, 0; y_{1}, 0) + G_{1}(x_{1}, 0; -y_{1}, 0) + G_{2,+}(x_{1}, 0; y_{1}, 0)] \varphi(y_{1}) \, dy_{1} + 2u^{i} = 0 \quad x_{1} > \delta.
\]  
(2.4)
(ii) \( u^i = u_o^i \) such that \( \varphi(x_1) = -\varphi(-x_1) \).

\[
\int_\delta^\infty \left[ G_1(x_1, 0; y_1, 0) - G_1(x_1, 0; -y_1, 0) + G_2(x_1, 0; y_1, 0) \right] \varphi(y_1) \, dy_1 + 2u^i = 0 \quad x_1 > \delta.
\]

(2.5)

Define the integral operators

\[
K^e_1 \varphi(x_1) = \int_\delta^\infty \left[ G_1(x_1, 0; y_1, 0) + G_1(x_1, 0; -y_1, 0) \right] \varphi(y_1) \, dy_1, \quad x_1 > \delta,
\]

(2.6)

\[
K^o_1 \varphi(x_1) = \int_\delta^\infty \left[ G_1(x_1, 0; y_1, 0) - G_1(x_1, 0; -y_1, 0) \right] \varphi(y_1) \, dy_1, \quad x_1 > \delta,
\]

(2.7)

\[
K_2 \varphi(x_1) = \int_\delta^\infty G_2(x_1, 0; y_1, 0) \varphi(y_1) \, dy_1, \quad x_1 > \delta.
\]

(2.8)

We express the integral equations (2.4) and (2.5) as

\[
(K^e_1 + K_2) \varphi = -2u^e \quad \text{and} \quad (K^o_1 + K_2) \varphi = -2u^o
\]

(2.9)

respectively.

3 Analysis of the integral operators

3.1 Preliminaries

Let \( s \in \mathbb{R} \), we denote by \( H^s(\mathbb{R}) \) the standard fractional Sobolev space with the norm

\[
\| u \|_{H^s(\mathbb{R})}^2 = \int_\mathbb{R} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 \, d\xi,
\]

where \( \hat{u} \) is the Fourier transform of \( u \) defined by

\[
\hat{u}(\xi) := \int_{-\infty}^{\infty} u(x_1) e^{-i\xi x_1} \, dx_1.
\]

Let \( I \) be an interval in \( \mathbb{R} \) and define

\[
H^s(I) := \{ u = U|_I \mid U \in H^s(\mathbb{R}) \}.
\]

Then \( H^s(I) \) is a Hilbert space with the norm

\[
\| u \|_{H^s(I)} = \inf \{ \| U \|_{H^s(\mathbb{R})} \mid U \in H^s(\mathbb{R}) \text{ and } U|_I = u \}.
\]

We also define

\[
\tilde{H}^s(I) := \{ u = U|_I \mid U \in H^s(\mathbb{R}) \text{ and } \text{supp} U \subset \bar{I} \}.
\]

One can show that (see [1]) the space \( \tilde{H}^s(I) \) is the dual of \( H^{-s}(I) \) and the norm for \( \tilde{H}^s(I) \) can be defined via the duality. As such \( \tilde{H}^s(I) \) is also a Hilbert space. We refer to [1] for
more details about the fractional Sobolev spaces. In what follows, we are mostly concerned with the case when \( s = \pm \frac{1}{2} \) and \( I = (\delta, \infty) \) for some \( \delta \geq 0 \). We denote

\[
\tilde{H}^{-\frac{1}{2}}(\delta, \infty) = \tilde{H}^{-\frac{1}{2}}(\Gamma_\delta^+), \quad H^{\frac{1}{2}}(\delta, \infty) = H^{\frac{1}{2}}(\Gamma_\delta^+).
\]

Remark For a given function \( \varphi \in \tilde{H}^{-\frac{1}{2}}(\Gamma_\delta^+) \), we may associate it with a function defined over the whole real line that vanishes on \( \mathbb{R} \setminus \Gamma_\delta^+ \). Without stating this explicitly and with the abuse of notation, we denote the new function as \( \varphi \) here and in several places throughout the paper, and it is obvious that \( \varphi \in H^{-\frac{1}{2}}(\mathbb{R}) \).

We now define the even and odd extension operators (\( \tilde{H}^{-\frac{1}{2}}(\Gamma_\delta^+) \to H^{-1/2}(\mathbb{R}) \)) by letting

\[
E\varphi(x_1) = \varphi(x_1) + \varphi(-x_1), \quad O\varphi(x_1) = \varphi(x_1) - \varphi(-x_1),
\]

It is clear that \( E\varphi \) and \( O\varphi \) is an even and odd function, respectively.

**Lemma 3.1** Let \( \varphi \in \tilde{H}^{-\frac{1}{2}}(\Gamma_\delta^+) \) where \( \delta > 0 \). Then \( E\varphi \) and \( O\varphi \) is an even and odd function respectively, and it holds that

\[
\tilde{E}\varphi(\xi) = \tilde{\varphi}(\xi) + \tilde{\varphi}(-\xi), \quad \tilde{O}\varphi(\xi) = \tilde{\varphi}(\xi) - \tilde{\varphi}(-\xi).
\]

Furthermore,

\[
\|\varphi\|_{\tilde{H}^{-\frac{1}{2}}(\Gamma_\delta^+)} \lesssim \|E\varphi\|_{H^{-1/2}(\mathbb{R})} \lesssim \|\varphi\|_{\tilde{H}^{-\frac{1}{2}}(\Gamma_\delta^+)},
\]

\[
\|\varphi\|_{\tilde{H}^{-\frac{1}{2}}(\Gamma_\delta^+)} \lesssim \|O\varphi\|_{H^{-1/2}(\mathbb{R})} \lesssim \|\varphi\|_{\tilde{H}^{-\frac{1}{2}}(\Gamma_\delta^+)}.\]

We postpone of the proof the lemma to the appendix.

### 3.2 Spectral representation of the integral operators and surface plasmonic polaritons

It is known that the Hankel functions adopt the spectral decomposition:

\[
-i \frac{H_0^{(1)}(k|x - y|)}{4} = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{2\rho(\xi)} e^{-\rho(\xi)(x_2 - y_2)} e^{i\xi(x_1 - y_1)} d\xi,
\]

\[
-i \frac{H_0^{(1)}(k_m|x - y|)}{4} = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{2\rho_m(\xi)} e^{-\rho_m(\xi)(x_2 - y_2)} e^{i\xi(x_1 - y_1)} d\xi,
\]

where

\[
\rho(\xi) = \sqrt{\xi^2 - k^2}, \quad \rho_m(\xi) = \sqrt{\xi^2 - k^2 \varepsilon_m}.
\]

With the above spectral decompositions, we may rewrite the operators \( K_1^\varepsilon, K_2^p \) and \( K_2 \) as follows:

\[
K_1^\varepsilon \varphi(x_1) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\xi x_1} + e^{-i\xi x_1}}{\rho(\xi)} \int_{\delta}^{\infty} \varphi(y_1) e^{-i\xi y_1} dy_1 d\xi
\]

\[
= -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\rho(\xi)} (\tilde{\varphi}(\xi) + \tilde{\varphi}(-\xi)) e^{i\xi x_1} d\xi
\]

\[
= -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\rho(\xi)} \tilde{E}\varphi(\xi) e^{i\xi x_1} d\xi; \quad (3.1)
\]
\[ K_1^\varphi(x_1) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\rho_0(\xi)} \int_\delta \varphi(y_1) e^{-iy_1} dy_1 d\xi \]
\[ = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\rho_0(\xi)} (\varphi(\xi) - \varphi(-\xi)) e^{i\xi x_1} d\xi \]
\[ = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\rho_0(\xi)} \widehat{\varphi}(\xi) e^{i\xi x_1} d\xi; \quad (3.2) \]

\[ K_2^\varphi(x_1) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\varepsilon_m(e^{i\xi x_1} + e^{i(2\delta-x_1)})}{\rho_m(\xi)} \int_\delta \varphi(y_1) e^{-iy_1} dy_1 d\xi \]
\[ = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\varepsilon_m}{\rho_m(\xi)} \left( \varphi(\xi) + e^{-i2\delta\xi} \varphi(-\xi) \right) e^{i\xi x_1} d\xi. \]
\[ = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\varepsilon_m}{\rho_m(\xi)} e^{-i\delta\xi} \widehat{\varphi}(\xi) e^{i\xi x_1} d\xi. \quad (3.3) \]

where \( \varphi_\delta \) is defined by \( \varphi_\delta(x_1) = \varphi(x_1 + \delta) \).

If \( \delta = 0 \), the symbol (multiplier) associated with the operator \( K_1^e + K_2 \) is given by \( \frac{\rho_0(\xi)\varepsilon_m + \rho_m(\xi)}{\rho_0(\xi)\rho_m(\xi)} \), which attains zeros at \( \xi_\pm(k) = \pm k\sqrt{\varepsilon_m/(\varepsilon_m + 1)} \). This implies that when inverting the operator \( K_1^e + K_2 \), the corresponding symbol will attain poles at \( \xi = \xi_\pm(k) \).

The poles are associated with the so-called surface plasmonic polariton, which gives rise to eigenmodes that are localized along the metal-vacuum interface [26]. Correspondingly, the spectral component of the current source function is amplified in the neighborhood the plasmonic frequency \( \xi_\pm \) when inverting the operator, and the surface plasmon is excited. It can be calculated that

\[ \xi_\pm = \xi_\pm^e + i\xi_\pm^e, \quad (3.4) \]

where \( \xi_\pm^e = k(1 + O(1/|\varepsilon_m|)) \) and \( \xi_\pm^e = O(k\varepsilon_m/|\varepsilon_m|^2) \). Namely, \( \xi_\pm \) lies in the vicinity of \( \pm k \).

To address the difficulties induced by the surface plasmonic poles for solving the integral equation, we decompose the operator \( K_1^e \) by treating the spectral components with and without the poles separately. To this end, let \( \Delta := \{ \xi | |\xi| \leq 2k \} \) and \( \chi \) be the corresponding characteristic function. We decompose the operator \( K_1^e \) as

\[ K_1^e = K_{1,0}^e + K_{1,1}^e, \]

where

\[ K_{1,0}^e \varphi(x_1) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1 - \chi\Delta(\xi)}{\rho_0(\xi)} \widehat{\varphi}(\xi) e^{i\xi x_1} d\xi, \quad (3.5) \]
\[ K_{1,1}^e \varphi(x_1) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\chi\Delta(\xi)}{\rho_0(\xi)} \widehat{\varphi}(\xi) e^{i\xi x_1} d\xi. \quad (3.6) \]

Similarly, the operator \( K_1^o \) is decomposed as

\[ K_1^o = K_{1,0}^o + K_{1,1}^o, \]
where
\[ K_{1,0}^\nu \varphi(x_1) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1 - \chi_{\delta}(\xi)}{\rho_0(\xi)} \hat{\varphi}(\xi) e^{i\xi x_1} \, d\xi, \quad (3.7) \]
\[ K_{1,1}^\nu \varphi(x_1) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\chi_{\delta}(\xi)}{\rho_0(\xi)} \hat{\varphi}(\xi) e^{i\xi x_1} \, d\xi. \quad (3.8) \]

It is clear that the operators \( K_{1,0}^\nu \) and \( K_{1,1}^\nu \) can be extended to bounded operators from \( \tilde{H}^{-1/2}(\Gamma_\delta^+) \) to \( H^{1/2}(\mathbb{R}) \), and hence are bounded from \( \tilde{H}^{-1/2}(\Gamma_\delta^+) \) to \( H^{1/2}(\Gamma_\delta^+) \).

We also decompose the operator \( K_2 \) as
\[ K_2 = K_{2,0} + K_{2,1}, \]
where \( K_{2,0} \) is the corresponding integral operator when \( \delta = 0 \). Namely,
\[ K_{2,0} \varphi(x_1) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\varepsilon_m}{\rho_m(\xi)} \hat{\varphi}(\xi) e^{i\xi x_1} \, d\xi, \quad (3.9) \]
\[ K_{2,1} \varphi(x_1) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\varepsilon_m}{\rho_m(\xi)} (e^{-i\delta \xi} - 1) \hat{\varphi}(-\xi) e^{i\xi x_1} \, d\xi. \quad (3.10) \]

Equivalently, this gives
\[ K_{2,0} \varphi(x_1) = 2 \int_\delta^\infty \left( G_2^{(0)}(x_1, 0; y_1, 0) + G_2^{(0)}(-x_1, 0; y_1, 0) \right) \varphi(y_1) \, dy_1, \quad (3.11) \]
\[ K_{2,1} \varphi(x_1) = \int_\delta^\infty \left( G_2^{(0)}(x_1, 0; y_1, 0) - G_2^{(0)}(-x_1, 0; y_1, 0) \right) \varphi(y_1) \, dy_1, \quad (3.12) \]
where \( G_2^{(0)}(x_1, x_2, y_1, y_2) = \varepsilon_m H_0^{(1)}(k_m|x - y|). \)

**Lemma 3.2** The operator \( K_2 \) is bounded from \( \tilde{H}^{-1/2}(\Gamma_\delta^+) \) to \( H^{1/2}(\Gamma_\delta^+) \). Furthermore, the inverse \( K_2^{-1} \) exists and there holds
\[ ||K_2^{-1}|| \lesssim \frac{1}{\sqrt{|\varepsilon_m|}}. \]

**Proof.** For \( \varphi \in \tilde{H}^{-1/2}(\Gamma_\delta^+) \), it is clear that \( K_2 \varphi \) can be extended naturally to \( H^{1/2}(\mathbb{R}) \). With abuse of notation, we also denote the extension as \( K_2 \varphi \). It is straightforward to check that \( K_2 \) bounded from \( \tilde{H}^{-1/2}(\Gamma_\delta^+) \) to \( H^{1/2}(\Gamma_\delta^+) \). We next show that \( K_2 \) is invertible. Indeed,
\[ \langle K_2 \varphi, \varphi \rangle_{L^2(\Gamma_\delta^+)} = \langle \tilde{K}_2 \varphi, \tilde{\varphi} \rangle_{L^2(\mathbb{R})} \]
\[ = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\varepsilon_m}{\rho_m(\xi)} e^{-i\delta \xi} \tilde{E}\varphi_{\delta}(\xi) \tilde{\varphi}(\xi) \, d\xi \]
\[ = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\varepsilon_m}{\rho_m(\xi)} \tilde{E}\varphi_{\delta}(\xi) \tilde{\varphi}(\xi) \, d\xi. \]
Note that $\hat{E}\varphi_\delta$ is an even function and $\hat{\varphi}_\delta - \hat{E}\varphi_\delta$ is an odd function, we have
\[
\int_{-\infty}^{\infty} \frac{1}{\rho_m(\xi)} \hat{E}\varphi_\delta(\xi) \left( \hat{\varphi}_\delta(\xi) - \hat{E}\varphi_\delta(\xi) \right) \, d\xi = 0.
\]
Consequently,
\[
\langle K_2 \varphi, \varphi \rangle = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\varepsilon_m}{\rho_m(\xi)} |\hat{E}\varphi_\delta(\xi)|^2 \, d\xi.
\]
Therefore,
\[
|\langle K_2 \varphi, \varphi \rangle| \geq \Re \left\{ -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\varepsilon_m}{\rho_m(\xi)} |\hat{E}\varphi_\delta(\xi)|^2 \, d\xi \right\} 
\geq C \int_{-\infty}^{\infty} \frac{|\varepsilon_m|}{\sqrt{\xi^2 - k^2 \varepsilon_m}} \left| \hat{E}\varphi_\delta(\xi) \right|^2 \, d\xi, \tag{3.13}
\]
for some universal constant $C$. Here we have used the fact that $|\varepsilon'_m| \gg |\varepsilon''_m|$ in the last inequality. Note that $\varepsilon'_m < 0$, hence
\[
\int_{\{|\xi| > k^2 |\varepsilon'_m|\}} \frac{|\varepsilon'_m|}{\sqrt{\xi^2 - k^2 \varepsilon'_m}} \left| \hat{E}\varphi_\delta(\xi) \right|^2 \, d\xi \geq \frac{|\varepsilon'_m|}{\sqrt{2}} \int_{\{|\xi| > k^2 |\varepsilon'_m|\}} \frac{1}{\sqrt{\xi^2 + 1}} \left| \hat{E}\varphi_\delta(\xi) \right|^2 \, d\xi.
\]
\[
\int_{\{|\xi| < k^2 |\varepsilon'_m|\}} \frac{|\varepsilon'_m|}{\sqrt{\xi^2 - k^2 \varepsilon'_m}} \left| \hat{E}\varphi_\delta(\xi) \right|^2 \, d\xi \geq \frac{\sqrt{|\varepsilon'_m|}}{\sqrt{2k}} \int_{\{|\xi| < k^2 |\varepsilon'_m|\}} \frac{1}{\sqrt{\xi^2 + 1}} \left| \hat{E}\varphi_\delta(\xi) \right|^2 \, d\xi.
\]
Substituting into (3.13) yields
\[
|\langle K_2 \varphi, \varphi \rangle| \geq C \sqrt{|\varepsilon'_m|} \cdot \|E\varphi_\delta\|_{H^{-1/2}(\mathbb{R})} \geq C \sqrt{|\varepsilon'_m|} \cdot ||\varphi_\delta||_{H^{-1/2}(\mathbb{R}^+)} = C \sqrt{|\varepsilon'_m|} \cdot ||\varphi||_{H^{-1/2}(\Gamma^+_\delta)}^2
\]
Therefore, we obtain
\[
||K_2 \varphi||_{H^{1/2}(\Gamma^+_\delta)} \geq C \sqrt{|\varepsilon'_m|} \cdot ||\varphi||_{H^{-1/2}(\Gamma^+_\delta)}.
\tag{3.14}
\]
It follows that $K_2$ is injective and the range $\text{Ran}(K_2)$ is closed.

A parallel calculation as above shows that the adjoint operator $K_2 \varphi^*$ has similar property. Especially, $K_2 \varphi^*$ is injective. Thus we have
\[
\text{Ran}(K_2) = (\text{Ker}((K_2)^*)^\perp = \{0\}^\perp = H^{1/2}(\Gamma^+_\delta).
\tag{3.15}
\]
From (3.14) and (3.15), we conclude that $K_2$ is invertible and
\[
||K_2^{-1}|| \leq \frac{C}{\sqrt{|\varepsilon'_m|}}.
\]
□
4 Solution of the integral equation for the even case

4.1 An overview of the methodology

We investigate the solution of the integral equation (2.9) for the even case in this section. The odd case is presented in Section 5. Let us introduce the integral operator

\[ D := K_1^e + K_2, \]

and the integral equation reads

\[ D \varphi = -2u_i^e. \] (4.1)

Let \( \tilde{D} : H^{-1/2}(\mathbb{R}^+) \to H^{1/2}(\mathbb{R}^+) \) be sum of \( K_1^e \) and \( K_2 \) when the slit width \( \delta = 0 \). Its spectral representation is given by

\[ \tilde{D} \varphi = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \frac{1}{\rho_0(\xi)} + \frac{\varepsilon_m}{\rho_m(\xi)} \right) \hat{E} \varphi(\xi)e^{i\xi x_1} d\xi, \quad x_1 > 0. \]

It follows by a direct calculation that the solution of the integral equation \( \tilde{D} \varphi_{00} = -2u_i^e \) is given by

\[ \varphi_{00} = R \cdot \cos(k_1 x_1) \cdot \chi_{(0,+\infty)}, \] (4.2)

where the coefficient \( R = -\frac{2\rho_0(k_1)\rho_m(k_1)}{\rho_m(k_1) + \varepsilon_m\rho_0(k_1)} \).

We view the operator \( D \) at the presence of slit as a perturbation of \( \tilde{D} \). As such let us decompose the solution of (2.9) as \( \varphi = \varphi_0 + \varphi_1 \), where

\[ \varphi_0 = \varphi_{00} \cdot \chi_{(\delta,\infty)}, \]

and \( \varphi_1 \) satisfies

\[ D \varphi_1 = \tilde{D} \varphi_{00} - D \varphi_0; \] (4.3)

Therefore, with a suitable decomposition of \( \tilde{D} \varphi_{00} - D \varphi_0 \) to be accomplished in Section 4.4, we will need to investigate the solution of the following integral equation in order to obtain \( \varphi_1 \):

\[ D \varphi = f, \] (4.4)

where \( f \) lies in some finite energy space to be specified in Section 4.3. From the spectral representation of integral operators in Section 3.2, the symbol of \( D \) contains plasmonic poles in the frequency band \( \Delta = \{ \xi \mid |\xi| \leq 2k \} \). Consequently, the spectral component of the source function \( f \) would be amplified in the neighborhood the plasmonic frequency \( \xi'_\pm \) when inverting the operator \( D \), and the surface plasmonic polariton occurs.

Following the decomposition of the integral operators (3.7) - (3.9), we decompose the operator \( D \) as \( D = D_0 + K_{1,1}^e \), where the operator \( D_0 : H^{-1/2}(\Gamma_{\delta}^+) \to H^{1/2}(\Gamma_{\delta}^+) \) is given by

\[ D_0 := K_{1,0}^e + K_2. \] (4.5)

We can view \( D_0 \) as a pre-conditional for the operator \( D \). It is clear that the symbol of \( D_0^{-1} \) does not contain plasmonic poles. In fact, it can be shown that \( D_0 \) is invertible.
Proposition 4.1 The operator $D_0 : \tilde{H}^{-1/2}(\Gamma^\perp_\delta) \to H^{1/2}(\Gamma^\perp_\delta)$ is invertible and there holds

$$||D_0^{-1}|| \lesssim \frac{1}{\sqrt{\varepsilon_m}}.$$ 

**Proof.** From Lemma 3.2, $K_2^{-1}$ is invertible and we may rewrite $D_0$ as

$$D_0 = K_2 \cdot \left[ I + (K_2)^{-1}K_{1,0}^\varepsilon \right].$$

Since $||K_{1,0}^\varepsilon|| \lesssim 1$ and $||K_2^{-1}|| \lesssim \frac{1}{\sqrt{\varepsilon_m}}$, it follows that $I + K_2^{-1}K_{1,0}^\varepsilon$ is invertible. Therefore,

$$||D_0^{-1}|| \leq \left\| \left[ I + K_2^{-1}K_{1,0}^\varepsilon \right]^{-1} \right\| \cdot ||K_2^{-1}|| \lesssim \frac{1}{\sqrt{\varepsilon_m}}.$$

□

To analyze the operator $D$, we need to introduce two function spaces:

$$V_1 = \{ \varphi \in \tilde{H}^{-1/2}(\Gamma^\perp_\delta) : \int_\rho_0(\xi) |\hat{E}\varphi(\xi)|^2 d\xi < \infty \},$$

$$V_2 = \{ \varphi = U|_{\Gamma^\perp_\delta} : \int_\rho_0(\xi) |\hat{U}(\xi)|^2 d\xi < \infty \}. $$

One can show that $V_1$ is a Hilbert space with the norm

$$||\varphi||_{V_1}^2 = \int_\rho_0(\xi) |\hat{E}\varphi(\xi)|^2 d\xi.$$ 

Moreover, one can show that $V_2$ is the dual space of $V_1$.

Theorem 4.2 The operator $D : V_1 \to V_2$ is bounded and is invertible. Moreover,

$$||D^{-1}|| \lesssim \frac{\varepsilon'_m}{\varepsilon_m}.$$ 

**Proof.** First, it is clear that $D$ is bounded. We need only to show that $D$ is invertible and the inverse is also bounded. Let $\varphi$ be a solution to $D\varphi = f$, we first show that

$$||\varphi||_{V_1} \lesssim \frac{\varepsilon'_m}{\varepsilon_m} ||f||_{V_2}.$$ 

Indeed,

$$\langle D\varphi, \varphi \rangle = \langle K_1^\varepsilon \varphi, \varphi \rangle + \langle K_2 \varphi, \varphi \rangle$$

$$= -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\rho_0(\xi)} \hat{E}\varphi(\xi) \hat{\phi}(\xi) d\xi - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\varepsilon_m}{\rho_m(\xi)} \hat{E}\varphi(\xi) \hat{\phi}(\xi) d\xi$$

$$= -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\rho_0(\xi)} |\hat{E}\varphi(\xi)|^2 d\xi - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\varepsilon_m}{\rho_m(\xi)} |\hat{E}\varphi(\xi)|^2 d\xi.$$
Using $|\varepsilon''_m| \gg \varepsilon'_m$, one can show that
\[
\text{Im} \frac{\varepsilon_m}{\rho_m(\xi)} \geq \frac{1}{3} \varepsilon''_m, \quad \text{Re} \frac{\varepsilon_m}{\rho_m(\xi)} \leq 2 \frac{|\varepsilon'_m|}{|\rho_m(\xi)|}.
\]

As a result,
\[
|\text{Im} \langle D\varphi, \varphi \rangle| \geq \frac{1}{2\pi |\rho_0(\xi)|} |\hat{E}\varphi(\xi)|^2 d\xi + \frac{1}{6\pi} \int_{-\infty}^{\infty} \frac{\varepsilon''_m}{\rho_m(\xi)} |\hat{E}\varphi_\delta(\xi)|^2 d\xi.
\]

On the other hand,\[
|\langle D\varphi, \varphi \rangle| \leq |\langle f, \varphi \rangle| \leq \|f\|_{V_2} \cdot \|\varphi\|_{V_1}.
\]

We obtain
\[
\int_{-\infty}^{\infty} \frac{\varepsilon''_m}{|\rho_m(\xi)|} |\hat{E}\varphi_\delta(\xi)|^2 d\xi \lesssim \|f\|_{V_2} \cdot \|\varphi\|_{V_1}.
\]

Therefore,
\[
\int_{|\xi| \geq k} \frac{1}{|\rho_0(\xi)|} |\hat{E}\varphi(\xi)|^2 d\xi \leq |\text{Re} \langle D\varphi, \varphi \rangle| + \left| \text{Re} \int_{-\infty}^{\infty} \frac{\varepsilon_m}{\rho_m(\xi)} |\hat{E}\varphi_\delta(\xi)|^2 d\xi \right|
\]
\[
\lesssim \|f\|_{V_2} \cdot \|\varphi\|_{V_1} + \frac{|\varepsilon'_m|}{\varepsilon''_m} \left| \text{Im} \int_{-\infty}^{\infty} \frac{\varepsilon_m}{\rho_m(\xi)} |\hat{E}\varphi_\delta(\xi)|^2 d\xi \right|
\]
\[
\lesssim \|f\|_{V_2} \cdot \|\varphi\|_{V_1} + \frac{|\varepsilon'_m|}{\varepsilon''_m} \|f\|_{V_2} \cdot \|\varphi\|_{V_1}
\]
\[
\lesssim \frac{|\varepsilon'_m|}{\varepsilon''_m} \|f\|_{V_2} \cdot \|\varphi\|_{V_1}.
\]

It follows that
\[
\|\varphi\|^2_{V_1} = \int \frac{1}{|\rho_0(\xi)|} |\hat{E}\varphi(\xi)|^2 d\xi \lesssim \frac{|\varepsilon'_m|}{\varepsilon''_m} \|f\|_{V_2} \cdot \|\varphi\|_{V_1},
\]
whence
\[
\|\varphi\|_{V_1} \lesssim \frac{|\varepsilon'_m|}{\varepsilon''_m} \|f\|_{V_2}.
\]

We conclude that the $D$ is injective, moreover, the range of $D$ is closed in $V_2$. We next show that the range of $D$ is dense in $V_2$. For this, we consider the adjoint of $D$, denoted by $D^*$, which is defined by the following identity:
\[
\langle D\varphi, \psi \rangle = \langle \varphi, D^* \psi \rangle
\]
where $\psi \in V_1$. A direct computation shows that
\[
D^* \psi(x_1) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\rho_0(\xi)} \hat{E}\psi(\xi) e^{i\xi x_1} d\xi - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\varepsilon_m}{\rho_m(\xi)} e^{i\delta x_1} \hat{E}\psi(\xi) e^{i\xi x_1} d\xi, \quad x_1 > \delta.
\]

Therefore, a similar argument as for the operator $D$ shows that $D^*$ is injective. Consequently, the range of $D$ is dense in $V_2$. This combines with the fact that the range of $D$ is also closed in $V_2$ yields that $D$ is onto the space $V_2$. Recall that $D$ is also injective. The open mapping theorem gives that $D$ is invertible and the inverse is also bounded. Moreover, the inverse satisfies that desired estimate.

□
Remark 1 The above theorem established the existence and uniqueness of the solution to the integral equation (4.1) when the source term is in the space $V_2$. It also provides a rough estimate for the solution without giving much insight on the physics. In what follows, we shall derive a sharp estimate. Moreover, we shall characterize the excited surface plasmon resonance.

Observe that the operator equation (4.4) can be rewritten as

$$\varphi + D_0^{-1}K_{1,1}^e \varphi = D_0^{-1}f.$$ (4.6)

We first aim to express the term $D_0^{-1}K_{1,1}^e \varphi$ in the above equation in terms of $\varphi$ by solving the equation $D_0 \phi = K_{1,1}^e \varphi$, which excludes plasmon resonances. Then we study the enhancement effect induced from the surface plasmon resonances by solving the whole equation (4.6). These two steps are addressed in Section 4.2 and 4.3 respectively. Finally, we summarize the solution of (4.1) in Section 4.4.

4.2 Solution of $D_0 \phi = K_{1,1}^e \psi$

In this section, we solve the equation

$$D_0 \phi = K_{1,1}^e \psi,$$ (4.7)

where it holds that $\| \sqrt{\frac{E_0}{\rho_0}} \|_{L^2(\Delta)} < \infty$. It is clear that $K_{1,1}^e \psi \in V_2$.

Using the spectral decomposition of the operators (3.1) and (3.3), let us decompose the operator $D_0$ as $D_0 = D_{0,0} + D_{0,1}$, where

$$D_{0,0} := (K_{1,0}^e + K_{2,0}) \quad \text{and} \quad D_{0,1} := K_{2,1}.$$ (4.8)

We also introduce the operator $\tilde{D}_{0,0} : H^{-1/2}(\mathbb{R}^+) \rightarrow H^{1/2}(\mathbb{R}^+)$ as follows:

$$\tilde{D}_{0,0} \varphi(x_1) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \frac{\varepsilon_m}{\rho_m(\xi)} + \frac{1 - \chi_\Delta}{\rho_0(\xi)} \right) \hat{E} \varphi(\xi) e^{i\xi x_1} d\xi, \quad x_1 > 0.$$ (4.9)

Define the function $\hat{\phi}_{00,\psi}(x_1)$ over the whole real line such that its Fourier transform is given by

$$\hat{\phi}_{00,\psi}(\xi) = \frac{\chi_\Delta \cdot \hat{\psi}(\xi)}{\rho_0(\xi) \left( \frac{\varepsilon_m}{\rho_m(\xi)} + \frac{1 - \chi_\Delta}{\rho_0(\xi)} \right)} = \frac{\rho_m(\xi)}{\rho_0(\xi) \varepsilon_m} \cdot \hat{E} \hat{\psi}(\xi) \cdot \chi_\Delta.$$ (4.10)

Lemma 4.3 The following estimate holds:

$$\| \hat{\phi}_{00,\psi} \|_{L^1(\Delta)} \lesssim \frac{1}{\sqrt{\varepsilon_m}} \frac{\| \hat{E} \hat{\psi} \|_{L^2(\Delta)}}{\sqrt{\rho_0}}.$$ (4.11)

Moreover, $\phi_{00,\psi}$ is a smooth and even function with

$$\| \phi_{00,\psi} \|_{C^3(\mathbb{R})} \lesssim \frac{1}{\sqrt{\varepsilon_m}} \frac{\| \hat{E} \hat{\psi} \|_{L^2(\Delta)}}{\sqrt{\rho_0}}.$$ (4.12)
Proof. It is clear that
\[ \left| \tilde{\phi}_{00,\psi}(\xi) \right| \lesssim \frac{1}{\sqrt{|\varepsilon_m'|}} \frac{1}{\rho_0(\xi)} \cdot \tilde{E}\psi(\xi). \]
Therefore,
\[ \left\| \tilde{\phi}_{00,\psi} \right\|_{L^1(\Delta)} \lesssim \frac{1}{\sqrt{|\varepsilon_m'|}} \left\| \tilde{\psi} \cdot \chi_\Delta \right\|_{L^2(\Delta)} \cdot \frac{\chi_\Delta}{\sqrt{|\rho_0|}} \left\| \phi_{00,\psi} \right\|_{L^2(\Delta)} \lesssim \frac{1}{\sqrt{|\varepsilon_m'|}} \cdot \left\| \frac{\tilde{E}\psi}{\sqrt{|\rho_0|}} \right\|_{L^2(\Delta)}. \]
The second estimate follows immediately. \( \square \)

Let \( \phi_{00,+} = \phi_{00,\psi} \cdot \chi_{(0,\infty)} \). By observing that
\[ \phi_{00,\psi}(x_1) = \phi_{00,+}(x_1) + \phi_{00,+}(-x_1) = E\phi_{00,+}(x_1) \quad \text{and} \quad \tilde{\phi}_{00,\psi}(\xi) = \tilde{E}\phi_{00,+}(\xi), \]
we have
\[ \left( \frac{\varepsilon_m}{\rho_m(\xi)} + \frac{1 - \chi_\Delta}{\rho_0(\xi)} \right) \tilde{E}\phi_{00,+}(\xi) = \frac{\chi_\Delta}{\rho_0(\xi)} \tilde{E}\psi(\xi). \]
Consequently, it holds that
\[ \tilde{D}_{0,0} \phi_{00,+} = K^{e}_{1,1} \psi, \] (4.11)
where we have extended \( K^{e}_{1,1} \psi \) naturally to \( \mathbb{R}^+ \).

We decompose the solution \( \phi \) of the operator equation (4.7) as \( \phi_0 + \phi_1 \), where
\[ \phi_0 = \phi_{00,\psi} \cdot \chi_{[\delta,\infty)}. \]
Using (4.11), it is seen that \( \phi_1 \) satisfies
\[ D_0 \phi_1 = q, \] (4.12)
where
\[ q(x_1) := \tilde{D}_{0,0} \phi_{00,+}(x_1) - [D_{0,0} \phi_0(x_1) + D_{0,1} \phi_0(x_1)] \quad \text{for} \quad x_1 > \delta. \] (4.13)

For brevity of notation, here and henceforth, we let
\[ g(x_1 - y_1) := G^{(0)}_2(x_1, 0; y_1, 0) = \varepsilon_m H^{(1)}_0(k_m|x_1 - y_1|). \] (4.14)

**Lemma 4.4** Let \( q \) be defined in (4.13), then \( q(x_1) = q_1(x_1) + q_2(x_1) \), where
\[ q_1(x_1) = -2 \int_0^\infty g(x_1 + y_1 - \delta)(\phi_{00,\psi}(y_1 + \delta) - \phi_{00,\psi}(y_1 - \delta)) dy_1, \quad x_1 > \delta, \] (4.15)
\[ q_2(x_1) = -\frac{1}{2\pi} \int_{-\infty}^\infty \frac{1 - \chi_\Delta}{\rho_0(\xi)} (\phi_{00,\psi} \cdot \chi_{(-\delta,\delta)})^\wedge(\xi) e^{i\xi x_1} d\xi, \quad x_1 > \delta. \] (4.16)

Moreover, the following asymptotic expansions hold for \( x_1 > \delta \):
\[ q_1(x_1) = \phi_{00,\psi}''(0) \cdot q_{1,0}(x_1) \cdot \varepsilon_m \delta^3 + O(\varepsilon_m \delta^4) \cdot \left\| \phi_{00,\psi} \right\|_{C^3(\mathbb{R})}, \] (4.17)
\[ q_2(x_1) = \phi_{00,\psi}(0) \cdot q_{2,0}(x_1) + O(\delta^2 \sqrt{\ln(1/\delta)} \cdot \left\| \phi_{00,\psi} \right\|_{C^3(\mathbb{R})}), \] (4.18)
where
\[ \left\| q_{1,0} \right\|_{H^{3/2}(\mathbb{R}_x^+)} \lesssim 1 \quad \text{and} \quad \left\| q_{2,0} \right\|_{H^{3/2}(\mathbb{R}_x^+)} \lesssim \delta \sqrt{\ln(1/\delta)}. \]
**Proof.** From the definition (3.12),

\[ D_{0,1} \phi_0(x_1) = \int_{\delta}^{\infty} \left( G_{2,+}(x_1, 0; y_1, 0) - 2G_{2}^{(0)}(x_1, 0; y_1, 0) - 2G_{2}^{(0)}(-x_1, 0; y_1, 0) \right) \phi_0(y_1) \, dy_1. \]

Therefore,

\[ D_{0,1} \phi_0(x_1) = 2 \int_{\delta}^{\infty} (g(x_1 + y_1 - 2\delta) - g(x_1 + y_1)) \, \phi_0(y_1) \, dy_1 \]

\[ = 2 \int_{-\delta}^{\infty} g(x_1 + y_1) \, \phi_0(y_1 + 2\delta) \, dy_1 - 2 \int_{\delta}^{\infty} g(x_1 + y_1) \, \phi_0(y_1) \, dy_1 \]

\[ = 2 \int_{-\delta}^{\infty} g(x_1 + y_1) \, (\phi_{00,\psi}(y_1 + 2\delta) - \phi_{00,\psi}(y_1)) \, dy_1 + 2 \int_{-\delta}^{\delta} g(x_1 + y_1) \, \phi_{00,\psi}(y_1) \, dy_1 \]

\[ = 2 \int_{0}^{\infty} g(x_1 + y_1 - \delta) \, (\phi_{00,\psi}(y_1 + \delta) - \phi_{00,\psi}(y_1 - \delta)) \, dy_1 + 2 \int_{-\delta}^{\delta} g(x_1 + y_1) \, \phi_{00,\psi}(y_1) \, dy_1. \]

(4.19)

On the other hand, from (4.8) and (4.9), it follows that for \( x_1 > \delta \),

\[ \tilde{D}_{0,0} \phi_{00,\psi}(x_1) - D_{0,0} \phi_0(x_1) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \frac{\varepsilon_m}{\rho_m(\xi)} + \frac{1 - \chi(\xi)}{\rho_0(\xi)} \right) (\phi_{00,\psi} \cdot \chi_{(-\delta,\delta)}) (\xi) e^{i\xi x_1} \, d\xi. \]

(4.20)

Note that

\[ -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\varepsilon_m}{\rho_m(\xi)} (\phi_{00,\psi} \cdot \chi_{(-\delta,\delta)}) (\xi) e^{i\xi x_1} \, d\xi = 2 \int_{-\delta}^{\delta} g(x_1 + y_1) \, \phi_{00,\psi}(y_1) \, dy_1, \]

(4.21)

then (4.15) and (4.16) follows by combining (4.19) - (4.21).

We now derive the asymptotics for \( q_1 \) and \( q_2 \). Note that

\[ \phi_{00,\psi}(y_1 + \delta) - \phi_{00,\psi}(y_1 - \delta) = 2\delta \phi_{00,\psi}''(0) y_1 + R_1(y_1) y_1^3 + R_2(y_1) y_1^2 \delta + R_3(y_1) y_1 \delta^2, \]

where \( R_1, R_2, R_3 \) are smooth functions such that

\[ \| R_j \|_{C^1(\mathbb{R})} \lesssim \| \phi_{00,\psi} \|_{C^3(\mathbb{R})}, \quad j = 1, 2, 3. \]

Correspondingly, we decompose \( q_1 \) as

\[ q_1 =: q_{1,0} + q_{1,1} + q_{1,2} + q_{1,3}, \]

where \( q_{1,j} \) is the integral the above density.

Setting \( x'_1 = (x - \delta)/\delta, \ y'_1 = y_1/\delta \), and \( k' = k_m \delta \), then it follows that \( k' = O(1) \) and

\[ \tilde{q}_{1,0}(x'_1) := q_{1,0}(\delta x'_1 + \delta) = 2\delta^3 \phi_{00,\psi}''(0) \cdot \varepsilon_m \cdot \int_{0}^{\infty} H_0^{(1)}(ik' x'_1 + y'_1) \, y'_1 \, dy'_1, \]

\[ \tilde{q}_{1,1}(x'_1) := q_{1,1}(\delta x'_1 + \delta) = \delta^4 \cdot \varepsilon_m \cdot \int_{0}^{\infty} H_0^{(1)}(ik' x'_1 + y'_1) R_1(\delta y'_1) (y'_1)^3 \, dy'_1, \]

\[ \tilde{q}_{1,2}(x'_1) := q_{1,2}(\delta x'_1 + \delta) = \delta^4 \cdot \varepsilon_m \cdot \int_{0}^{\infty} H_0^{(1)}(ik' x'_1 + y'_1) R_2(\delta y'_1) (y'_1)^2 \, dy'_1, \]

\[ \tilde{q}_{1,3}(x'_1) := q_{1,3}(\delta x'_1 + \delta) = \delta^4 \cdot \varepsilon_m \cdot \int_{0}^{\infty} H_0^{(1)}(ik' x'_1 + y'_1) R_3(\delta y'_1) y'_1 \, dy'_1. \]
Since $H_0^{(1)}(ik'|y'_1|)$ decays exponentially, we can show that
\[
\| \int_0^\infty H_0^{(1)}(ik'|x'_1 + y'_1|) y'_1 d y'_1 \|_{H^{1/2}(0,\infty)} = O(1),
\]
\[
\| \int_0^\infty H_0^{(1)}(ik'|x'_1 + y'_1|) R_1(\delta y'_1) (y'_1)^3 d y'_1 \|_{H^{1/2}(0,\infty)} \lesssim \| R_1 \|_{C^3(\mathbb{R})},
\]
\[
\| \int_0^\infty H_0^{(1)}(ik'|x'_1 + y'_1|) R_2(\delta y'_1) (y'_1)^2 d y'_1 \|_{H^{1/2}(0,\infty)} \lesssim \| R_2 \|_{C^3(\mathbb{R})},
\]
\[
\| \int_0^\infty H_0^{(1)}(ik'|x'_1 + y'_1|) R_3(\delta y'_1) (y'_1)^2 d y'_1 \|_{H^{1/2}(0,\infty)} \lesssim \| R_3 \|_{C^3(\mathbb{R})}.
\]

By the translation and scaling invariance of $\| \cdot \|_{H^{1/2}}$ norm, we deduce that the integral $q_1 \in H^{1/2}(\Gamma_\delta^+)$. Furthermore,
\[
q_1(x_1) = \phi''_{00,\psi}(0) \varepsilon_m \delta^3 \cdot q_{1,0}(x_1) + O(\varepsilon_m \delta^4) \cdot \| \phi_{00,\psi} \|_{C^3(\mathbb{R})}, \quad \text{in } H^{1/2}(\Gamma_\delta^+) \tag{4.22}
\]
where $\| q_{1,0} \|_{H^{1/2}(\Gamma_\delta^+)} = O(1)$.

We extend $q_2$ naturally to the whole real line and still denotes it as $q_2$. Applying the Taylor expansion, we see that
\[
q_2(x_1) = \phi_{00,\psi}(0) \cdot (1 + O(\delta^2)) \cdot q_{2,0}(x_1), \tag{4.23}
\]
where
\[
q_{2,0} := \frac{1}{2\pi} \int_{-\infty}^\infty \frac{1}{\rho_0(\xi)} \chi_{\Delta}(\xi) e^{i\xi x_1} d \xi.
\]

It follows that
\[
\| q_{2,0} \|_{H^{1/2}(\mathbb{R})}^2 \lesssim \int_{-\infty}^\infty \frac{1}{\sqrt{1 + |\xi|^2}} \frac{\sin^2(\delta \xi)}{\xi^2} d \xi
\]
\[
= \delta^2 \cdot \int_{-\infty}^\infty \frac{1}{\sqrt{\delta^2 + \xi^2}} \frac{\sin^2 \xi}{\xi^2} d \xi
\]
\[
\leq C \delta^2 \ln(1/\delta). \tag{4.24}
\]

The proof is complete by combining (4.23) and (4.24).}

From the above discussions, we can obtain the expansion of the solution for the operator equation (4.7). In particular, by virtue of the equation (4.12), Proposition 4.1, Lemma 4.3 - 4.4, we arrive at the following conclusion.

**Theorem 4.5** Let $\phi$ be the solution of the quation $D_0 \phi = K_{1,1}^e \psi$. Let $\phi_{00,\psi}$ be defined by
\[
\hat{\phi}_{00,\psi}(\xi) = \frac{\rho_m(\xi)}{\rho_0(\xi) \varepsilon_m} \cdot \hat{E}_{\psi}(\xi) \cdot \chi_\Delta(\xi).
\]

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Then $\phi = \phi_0 + \phi_1$, where

$$
\begin{align*}
\phi_0 &= \phi_{00,\psi} \cdot \chi(\delta, \infty), \\
\phi_1 &= \delta^{3+\alpha} \cdot \phi'''_{00,\psi}(0) \cdot D_0^{-1} q_{1,0} + \phi_{00,\psi}(0) \cdot D_0^{-1} q_{2,0} + D_0^{-1} q_h.
\end{align*}
$$

In addition,

$$
\begin{align*}
\|\phi_{00,\psi}\|_{C^3(\mathbb{R})} &\lesssim \delta^{-\frac{3}{2}} \cdot \|\hat{E}_\psi\|_{L^2(\Delta)} \\
\|q_{1,0}\|_{H^{1/2}(\Gamma_+^1)} &\lesssim 1, \quad \|q_{2,0}\|_{H^{1/2}(\Gamma_+^1)} \lesssim \delta \sqrt{\ln(1/\delta)}, \\
\|q_h\|_{H^{1/2}(\Gamma_+^1)} &\lesssim \left(\delta^{4+\alpha} + \delta^3 \sqrt{\ln(1/\delta)}\right) \|\phi_{00,\psi}\|_{C^3(\mathbb{R})} \lesssim \left(\delta^{4+\frac{3}{2}} + \delta^{3-\alpha/2} \sqrt{\ln(1/\delta)}\right) \|\hat{E}_\psi\|_{L^2(\Delta)}, \\
\|\phi_1\|_{H^{1/2}(\Gamma_+^1)} &\lesssim \left(\delta^{3+\frac{3}{2}} + \delta^{1-\alpha/2} \sqrt{\ln(1/\delta)}\right) \|\hat{E}_\psi\|_{L^2(\Delta)}.
\end{align*}
$$

### 4.3 Solution of $D\varphi = f$ and excitation of surface plasmon

Following (4.6), we rewrite the operator equation $D\varphi = f$ as

$$
\varphi + D_0^{-1} K_{1,1}^e \varphi = D_0^{-1} f. \tag{4.25}
$$

By Theorem 4.2, we see that $\varphi \in V_1$ and

$$
\|\hat{E}_\varphi\|_{L^2(\Delta)} \lesssim \|\varphi\|_{V_1} \lesssim \frac{\varepsilon_m'}{\varepsilon_m''} \|f\|_{V_2} = O(\delta^{\alpha-\beta}) \|f\|_{V_2}. \tag{4.26}
$$

From Theorem 4.5, it is seen that $D_0^{-1} K_{1,1}^e \varphi = \phi_{00} \chi(\delta, \infty) + \phi_1$, where $\phi_{00}$ is defined by

$$
\hat{\phi}_{00}(\xi) = \frac{\rho_m(\xi)}{\rho_0(\xi) \varepsilon_m} \cdot \hat{E}_\varphi(\xi) \cdot \chi(\xi),
$$

and

$$
\phi_1 = \delta^{3+\alpha} \cdot \phi'''_{00,\psi}(0) \cdot D_0^{-1} q_{1,0} + \phi_{00,\psi}(0) \cdot D_0^{-1} q_{2,0} + D_0^{-1} q_h.
$$

By virtue of Lemma 4.3 and (4.26), we have the following estimate.

**Lemma 4.6** The following estimate holds

$$
\|\hat{\phi}_{00}\|_{L^1(\mathbb{R})} \lesssim \frac{\varepsilon_m'}{\varepsilon_m''} \|f\|_{V_2}
$$

Moreover, $\phi_{00}$ is a smooth and even function with

$$
\|\phi_{00}\|_{C^3(\mathbb{R})} \lesssim \frac{\varepsilon_m'}{\varepsilon_m''} \|f\|_{V_2}.
$$
Substituting the expansion for $D_0^{-1}K_{1,1}^* \varphi$ into (4.25), we obtain

$$\varphi + \phi_0 + \delta^{3+\alpha} \cdot \phi_{00,\psi}(0) \cdot D_0^{-1}q_{1,0} + \phi_{00,\psi}(0) \cdot D_0^{-1}q_{2,0} + D_0^{-1}q_h = D_0^{-1}f. \quad (4.27)$$

Extending evenly over the whole real line yields

$$E\varphi + \phi_0(1 - \chi(-\delta, \delta)) + \delta^{3+\alpha} \cdot \phi''_{00}(0) \cdot ED_0^{-1}q_{1,0} + \phi_0(0) \cdot ED_0^{-1}q_{2,0} + ED_0^{-1}q_h = E(D_0^{-1}f).$$

This leads to the following equation in the Fourier domain:

$$\hat{E}\varphi(\xi) + \hat{\phi}_0(\xi) + \hat{Q}(\xi) = (ED_0^{-1}f)^*(\xi), \quad (4.28)$$

where

$$\hat{Q}(\xi) := -(\phi_0(0) + O(\delta^2)) \cdot \hat{\chi}(-\delta, \delta)(\xi) = \phi_0(0) \cdot \hat{q}_{2,1}(\xi) + \hat{q}_{2,2}(\xi),$$

where $\hat{q}_{2,1}(\xi) = \frac{\sin \delta \xi}{\xi}$ and $q_{2,2}$ satisfy the estimate

$$\|\hat{q}_{2,1}(\xi)\|_{L^2(\Delta)} \lesssim \delta, \quad \|\frac{1}{\sqrt{1 + |\xi|}} \hat{q}_{2,1}(\xi)\|_{L^2(\mathbb{R})} \lesssim \delta \sqrt{\ln \frac{1}{\delta}},$$

$$\|\hat{q}_{2,2}\|_{L^2(\Delta)} \lesssim \|\phi_0\|_{C^3(\mathbb{R})} \cdot \delta^3, \quad \|\frac{1}{\sqrt{1 + |\xi|}} \hat{q}_{2,2}\|_{L^2(\mathbb{R})} \lesssim \|\phi_0\|_{C^3(\mathbb{R})} \cdot \delta^3 \sqrt{\ln \frac{1}{\delta}}.$$ 

Correspondingly, we express $\hat{Q}(\xi)$ as

$$\hat{Q}(\xi) = \delta^{3+\alpha} \cdot \phi''_{00}(0) \cdot (ED_0^{-1}q_{1,0})^*(\xi) + \phi_0(0) \cdot [(ED_0^{-1}q_{2,0})^*(\xi) + \hat{q}_{2,1}(\xi)] + (ED_0^{-1}q_h + q_{2,2})^*(\xi).$$

**Lemma 4.7** The following estimate holds:

$$\left\|\hat{Q}(\xi)\right\|_{L^2(\Delta)} \lesssim (|\phi_0(0)| + |\phi''_{00}(0)|) \cdot (\delta^{3+\alpha/2} + \delta^{1-\alpha/2}) + (\delta^{4+\alpha/2} + \delta^{3-\alpha/2}) \cdot \|\phi_0\|_{C^3(\mathbb{R})},$$

$$\left\|\frac{1}{\sqrt{1 + |\xi|}} \hat{Q}(\xi)\right\|_{L^2(\mathbb{R})} \lesssim (|\phi_0(0)| + |\phi''_{00}(0)|) \cdot (\delta^{3+\alpha/2} + \delta^{1-\alpha/2} \sqrt{\ln \frac{1}{\delta}}) + (\delta^{4+\alpha/2} + \delta^{3-\alpha/2} \sqrt{\ln \frac{1}{\delta}}) \cdot \|\phi_0\|_{C^3(\mathbb{R})}.$$ 

In light of the formula (4.10), the following equation holds for $\xi \in \Delta$:

$$\hat{\phi}_0(\xi) \left(\frac{\rho_0 \xi_m}{\rho_m} + 1\right) + \hat{Q}(\xi) = (ED_0^{-1}f)^*(\xi).$$
Hence we can express the Fourier transform of $\widetilde{\phi}_0$ as

$$\widetilde{\phi}_0(\xi) = \frac{\rho_m(\xi)}{\rho_0(\xi)\varepsilon_m + \rho_m(\xi)} \left[ (ED_0^{-1} f)(\xi) - \hat{Q}(\xi) \right], \quad \xi \in \Delta. \quad (4.29)$$

On other hand, note that

$$\phi_0(0) = \int_{\Delta} \widetilde{\phi}_0(\xi) d\xi, \quad \phi''_0(0) = -\int_{\Delta} \xi^2 \widetilde{\phi}_0(\xi) d\xi.$$

Substituting (4.29) into the above two formulas yields a linear system for $\phi_0(0)$ and $\phi''_0(0)$:

$$\Pi \begin{bmatrix} \phi_0(0) \\ \phi''_0(0) \end{bmatrix} = b, \quad (4.30)$$

where

$$\Pi = \begin{bmatrix} 1 + A_1(ED_0^{-1} q_{2,0}) + A_1(q_{2,1}) & \delta^{3+\alpha} A_1(ED_0^{-1} q_{1,0}) \\ -A_2(ED_0^{-1} q_{2,0}) - A_2(q_{2,1}) & 1 - \delta^{3+\alpha} A_2(ED_0^{-1} q_{1,0}) \end{bmatrix},$$

$$b = \begin{bmatrix} A_1(ED_0^{-1} f) - A_1(ED_0^{-1} q_h) - A_1(q_{2,2}) \\ -A_2(ED_0^{-1} f) + A_2(ED_0^{-1} q_h) + A_2(q_{2,2}) \end{bmatrix}.$$

In addition, the functional $A_1$ and $A_2$ are defined as

$$A_1(\varphi) := \int_{\Delta} \frac{\rho_m(\xi)}{\rho_0(\xi)\varepsilon_m + \rho_m(\xi)} \varphi(\xi) d\xi,$$

$$A_2(\varphi) := \int_{\Delta} \frac{\xi^2 \cdot \rho_m(\xi)}{\rho_0(\xi)\varepsilon_m + \rho_m(\xi)} \varphi(\xi) d\xi.$$

We first solve the above linear system to obtain $\phi_0(0)$ and $\phi''_0(0)$, which will lead to the estimation for $\hat{Q}(\xi)$ in (4.29). To this end, we study the entries in the matrix $\Pi$ and the vector $b$. This is given in what follows.

**Lemma 4.8** The following inequality holds:

$$\left\| \frac{\rho_m}{\rho_0\varepsilon_m + \rho_m} \right\|_{L^2(\Delta)} \lesssim \left( 1 + \frac{1}{\sqrt{\varepsilon'_m}} \right)^{1/2}, \quad (4.31)$$

$$\left\| \frac{\rho_m}{\sqrt{|\rho_0| (\rho_0\varepsilon_m + \rho_m)}} \right\|_{L^2} \lesssim |\varepsilon'_m|^{1/2} \left( 1 + \frac{1}{\varepsilon''_m} \right). \quad (4.32)$$

In addition, if $\varphi \in L^2(\Delta)$, then

$$|A_j(\varphi)| \leq C \left( 1 + \frac{1}{\sqrt{\varepsilon''_m}} \right) \|\varphi\|_{L^2(\Delta)}, \quad j = 1, 2.$$

**Proof.** See Appendix B.
Lemma 4.9  The following expansions hold for $\Pi$ and $b$:

$$
\Pi = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (1 + \delta^{-\beta/2}) \begin{bmatrix} O(\delta) & O(\delta^{3+\alpha/2}) \\ O(\delta) & O(\delta^{3+\alpha/2}) \end{bmatrix},
$$

$$
b = \begin{bmatrix} A_1(ED_0^{-1}f) \\ -A_2(ED_0^{-1}f) \end{bmatrix} + O(1 + \delta^{-\beta/2}) \left( \delta^{4+\alpha/2} + \delta^3 \sqrt{\ln(1/\delta)} \right) \|\tilde{\phi}_{00}\|_{L^1(\mathbb{R})}.
$$

Moreover, assume that $-4 \leq \alpha < 0$, $\beta < 2$, then $\phi_{00}(0)$ and $\phi''_{00}(0)$ have the following estimates:

$$
\phi_{00}(0) = A_1(ED_0^{-1}f) \cdot (1 + o(1)) + O(1 + \delta^{-\beta/2}) \delta^3 \sqrt{\ln(1/\delta)} \|\tilde{\phi}_{00}\|_{L^1(\mathbb{R})},
$$

$$
\phi''_{00}(0) = A_2(ED_0^{-1}f) \cdot (1 + o(1)) + O(1 + \delta^{-\beta/2}) \delta^3 \sqrt{\ln(1/\delta)} \|\tilde{\phi}_{00}\|_{L^1(\mathbb{R})}.
$$

Proof. From the estimation in Theorem 4.5 and Lemma 4.8, we have

$$
|A_1(ED_0^{-1}q_h)| \lesssim \frac{\rho_m(\xi)}{\rho_0(\xi) \varepsilon_m + \rho_m(\xi)} \cdot \|(D_0^{-1}q_h)^{\wedge}(\xi)\|_{L^2(\Delta)}
$$

$$
\lesssim (1 + \delta^{-\beta/2}) \delta^{-\alpha/2} \|q_h\|_{H^{1/2}(\Gamma_0^+)}
$$

$$
\lesssim (1 + \delta^{-\beta/2}) \left( \delta^{4+\alpha} + \delta^3 \sqrt{\ln(1/\delta)} \right) \|\phi_{00}\|_{C^3(\mathbb{R})},
$$

$$
|A_1(q_{2,2})| \lesssim \frac{\rho_m(\xi)}{\rho_0(\xi) \varepsilon_m + \rho_m(\xi)} \cdot \|q_{2,2}\|_{L^2(\Delta)}
$$

$$
\lesssim (1 + \delta^{-\beta/2}) \delta^3 \sqrt{\ln(1/\delta)} \|\phi_{00}\|_{C^3(\mathbb{R})}.
$$

Similarly,

$$
|A_2(ED_0^{-1}q_h)| \lesssim (1 + \delta^{-\beta/2}) \left( \delta^{4+\frac{3}{2}+\alpha} + \delta^3 \sqrt{\ln(1/\delta)} \right) \|\phi_{00}\|_{C^3(\mathbb{R})},
$$

$$
|A_1(q_{1,0})| \lesssim (1 + \delta^{-\beta/2}) \delta^{-\alpha/2},
$$

$$
|A_2(q_{1,0})| \lesssim (1 + \delta^{-\beta/2}) \delta^{-\alpha/2},
$$

$$
|A_1(q_{2,0})| \lesssim (1 + \delta^{-\beta/2}) \delta^{1-\alpha/2} \sqrt{\ln(1/\delta)},
$$

$$
|A_2(q_{2,0})| \lesssim (1 + \delta^{-\beta/2}) \delta^{1-\alpha/2} \sqrt{\ln(1/\delta)},
$$

$$
|A_1(q_{2,1})| \lesssim (1 + \delta^{-\beta/2}) \delta,
$$

$$
|A_2(q_{2,1})| \lesssim (1 + \delta^{-\beta/2}) \delta,
$$

$$
|A_1(q_{2,2})| \lesssim (1 + \delta^{-\beta/2}) \delta^3 \sqrt{\ln(1/\delta)} \|\phi_{00}\|_{C^3(\mathbb{R})},
$$

$$
|A_2(q_{2,2})| \lesssim (1 + \delta^{-\beta/2}) \delta^3 \sqrt{\ln(1/\delta)} \|\phi_{00}\|_{C^3(\mathbb{R})}.
$$

Finally, using the estimate

$$
\|\phi_{00}\|_{C^3(\mathbb{R})} \lesssim \|\tilde{\phi}_{00}\|_{L^1(\mathbb{R})}
$$

we obtain the desired estimate for $\phi_{00}(0)$ and $\phi''_{00}(0)$. \hfill \Box

Now, we are ready to discuss the solution of the operator equation $D\varphi = f$. We distinguish two types of source function $f$:
(i) \( f \in H^{1/2}(\Gamma^+_\delta) \).

(ii) \( f = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\chi_\Lambda(\xi)}{\rho_0(\xi)} \hat{\psi}(\xi) e^{i\xi x_1} \, d\xi, \) where \( \hat{\psi}(\xi) \) is even and it holds that \( \| \frac{\hat{\psi}}{\sqrt{\rho_0}} \|_{L^2(\Delta)} < \infty \).

The estimates for the energy of the solution in the frequency band \( \Delta \) and \( \mathbb{R}\setminus\Delta \) are given in Theorems 4.10 and 4.12 respectively for the above two cases.

Theorem 4.10 Assume that \( -4 \leq \alpha < 0 \) and \( \beta < 2 \). If \( f \in H^{1/2}(\Gamma^+_\delta) \), the following holds for the solution of \( D\varphi = f \):

\[
\left\| \frac{\chi_\Lambda(\xi)}{\rho_0(\xi)} \hat{E}\varphi(\xi) \right\|_{L^1(\mathbb{R})} \lesssim (1 + \delta^{-\beta/2}) \| f \|_{H^{1/2}(\Gamma^+_\delta)}
\]

and

\[
\left\| \frac{1 - \chi_\Lambda(\xi)}{\sqrt{1 + |\xi|}} \hat{E}\varphi(\xi) \right\|_{L^2(\mathbb{R})} \lesssim \delta^{-\alpha/2} \| f \|_{H^{1/2}(\Gamma^+_\delta)}.
\]

Proof. First, using Lemma 4.8 and Proposition 4.1, we obtain

\[
|A_1(ED_0^{-1}f)| \lesssim \delta^{-\alpha/2} (1 + \delta^{-\beta/2}) \| f \|_{H^{1/2}(\Gamma^+_\delta)}, \quad |A_2(ED_0^{-1}f)| \lesssim \delta^{-\alpha/2} (1 + \delta^{-\beta/2}) \| f \|_{H^{1/2}(\Gamma^+_\delta)}.
\]

On the other hand, from Lemma 4.6, we have \( \| \hat{\phi}_{00} \|_{L^1(\mathbb{R})} \lesssim \delta^{\alpha/2-\beta} \| f \|_{L^2(\mathbb{R})} \lesssim \delta^{\alpha/2-\beta} \| f \|_{H^{1/2}(\Gamma^+_\delta)} \).

Therefore, using Lemma 4.9, it follows that

\[
|\phi_{00}(0)| \lesssim (1 + \delta^{-\beta/2}) \delta^{-\alpha/2} \left( 1 + \delta^{3+\alpha-\beta} \sqrt{\ln(1/\delta)} \right) \| f \|_{H^{1/2}(\Gamma^+_\delta)}
\]

\[
\lesssim (1 + \delta^{-\beta/2}) \delta^{-\alpha/2} \| f \|_{H^{1/2}(\Gamma^+_\delta)},
\]

\[
|\phi_{00}''(0)| \lesssim (1 + \delta^{-\beta/2}) \delta^{-\alpha/2} \| f \|_{H^{1/2}(\Gamma^+_\delta)}.
\]

By Lemma 4.7, we have

\[
\left\| \hat{Q} \right\|_{L^2(\Delta)} \lesssim (1 + \delta^{-\beta/2}) \delta^{-\alpha/2} \left( \delta^{3+\alpha/2} + \delta^{1-\alpha/2} \right) \| f \|_{H^{1/2}(\Gamma^+_\delta)} + \left( \delta^{4+\alpha/2} + \delta^{3-\alpha/2} \right) \| \hat{\phi}_{00} \|_{L^1(\mathbb{R})}.
\]

Now from the formula (4.29), the Cauchy-Schwartz inequality leads to updated estimate for \( \phi_{00} \):

\[
\left\| \frac{\hat{\phi}_{00}(\xi)}{\rho_0(\xi)E_m(\xi)} \right\|_{L^2(\Delta)} \lesssim \left( \| ED_0^{-1}f \|_{L^2(\Delta)} + \| \hat{Q} \|_{L^2(\Delta)} \right) \lesssim \left( 1 + \delta^{-\beta/2} \right) \left( \| ED_0^{-1}f \|_{L^2(\Delta)} + \| \hat{Q} \|_{L^2(\Delta)} \right)
\]

\[
\lesssim \left( 1 + \delta^{-\beta/2} \right) \left[ \delta^{-\alpha/2} + (1 + \delta^{-\beta/2}) \delta^{-\alpha/2} \left( \delta^{3+\alpha/2} + \delta^{1-\alpha/2} \right) \right] \| f \|_{H^{1/2}(\Gamma^+_\delta)}
\]

\[
+ \left( 1 + \delta^{-\beta/2} \right) \left( \delta^{4+\alpha/2} + \delta^{3-\alpha/2} \right) \| \hat{\phi}_{00}(\xi) \|_{L^1(\mathbb{R})}.
\]
Using the condition that $\alpha \geq -2$ and $\beta < 2$, we have
\[
\|\hat{\phi}_{00}(\xi)\|_{L^1(\mathbb{R})} \lesssim \delta^{-\alpha/2} \left(1 + \delta^{-\beta/2}\right) \|f\|_{H^{1/2}(\Gamma^+_d)}.
\]
This also implies the improved estimates for $\phi_{00}$ and $Q$:
\[
\|\phi_{00}\|_{C^3(\mathbb{R})} \lesssim \delta^{-\alpha/2} \left(1 + \delta^{-\beta/2}\right) \|f\|_{H^{1/2}(\Gamma^+_d)},
\]
and
\[
\left\|\frac{\hat{Q}(\xi)}{\sqrt{1 + |\xi|}}\right\|_{L^2(\mathbb{R})} \lesssim \delta^{-\alpha/2} \left(1 + \delta^{-\beta/2}\right) \left(\delta^{3+\alpha/2} + \delta^{1-\alpha/2}\sqrt{\ln \frac{1}{\delta}}\right) \|f\|_{H^{1/2}(\Gamma^+_d)}.
\]

Now, by (4.10),
\[
\hat{E}\varphi(\xi) = \frac{\rho_0(\xi)\varepsilon_m}{\rho_m(\xi)} \cdot \hat{\phi}_{00}(\xi) \quad \text{for} \ \xi \in \Delta.
\]
Using the estimate for $\hat{\phi}_{00}(\xi)$, we obtain
\[
\left\|\frac{\chi_{\Delta}(\xi)\hat{E}\varphi(\xi)}{\rho_0(\xi)}\right\|_{L^1(\mathbb{R})} \lesssim \sqrt{\varepsilon_m} \left\|\hat{\phi}_{00}(\xi)\right\|_{L^1(\mathbb{R})} \lesssim \left(1 + \delta^{-\beta/2}\right) \|f\|_{H^{1/2}(\Gamma^+_d)}.
\]
Finally, note that the support of $\hat{\phi}_{00}(\xi)$ belongs to $\Delta$, the formula (4.28) leads to
\[
\hat{E}\varphi(\xi) = (ED_0^{-1}f)^\wedge(\xi) - \hat{Q}(\xi) \quad \text{for} \ \xi \notin \Delta.
\]
We obtain
\[
\left\|\frac{1 - \chi_{\Delta}(\xi)}{\sqrt{1 + |\xi|}}\hat{E}\varphi(\xi)\right\|_{L^2(\mathbb{R})} \lesssim \|D_0^{-1}\| \left\|f\right\|_{H^{1/2}(\Gamma^+_d)} + \left\|\frac{1}{\sqrt{1 + |\xi|}}\hat{Q}(\xi)\right\|_{L^2(\mathbb{R})} \lesssim \delta^{-\alpha/2} \|f\|_{H^{1/2}(\Gamma^+_d)}.
\]

**Remark 2** In the above theorem and subsequent theorems as well, we have estimated the excited surface plasmon wave by their $L^1$-norm in the Fourier space. This estimate controls the $L^\infty$-norm of the excited surface plasmon wave. On the other hand, the standard $L^2$-norm estimate or energy estimate for the excited surface plasmon can also be derived from our analysis. That bound is significantly greater than that for the $L^1$-norm.

For a source function $f$ that takes the form in (ii), the solution of $D_0\phi = f$ can be expressed in the following lemma.

**Lemma 4.11** Let $f \in V_2$ be in the form of (ii), then $D_0^{-1}f$ has the following expansion
\[
\hat{E}D_0^{-1}f(\xi) = \frac{\rho_m(\xi)}{\varepsilon_m} \cdot \frac{\hat{\psi}}{\rho_0} \cdot \chi_{\Delta}(\xi) + \hat{R}(\xi).
\]
where the lower order term \( R \) satisfies the estimate
\[
\left\| \frac{\hat{R}(\xi)}{\sqrt{1 + |\xi|}} \right\|_{L^2} \lesssim \left( \delta^{3 + \frac{1}{2} \alpha} + \delta^{1 - \alpha/2} \sqrt{\ln(1/\delta)} \right) \cdot \left\| \frac{\hat{\psi}}{\sqrt{|\rho_0|}} \right\|_{L^2(\Delta)}.
\]
Moreover,
\[
\left\| \frac{\hat{E}D^{-1}_0 f \cdot \sqrt{|\rho_0|}}{\sqrt{1 + |\xi|}} \right\|_{L^2(\Delta)} \lesssim \delta^{-\alpha/2} \left\| \frac{\hat{\psi}}{\sqrt{|\rho_0|}} \right\|_{L^2(\Delta)},
\]
\[
\left\| \frac{\hat{E}D^{-1}_0 f \cdot (1 - \chi_\Delta)}{\sqrt{1 + |\xi|}} \right\|_{L^2} \lesssim \left( \delta^{3 + \frac{1}{2} \alpha} + \delta \sqrt{\ln(1/\delta)} \right) \left\| \frac{\hat{\psi}}{\sqrt{|\rho_0|}} \right\|_{L^2(\Delta)},
\]
\[
|A_j(\hat{E}D^{-1}_0 f)| \lesssim (1 + \delta^{-\beta}) \left\| \frac{\hat{\psi}}{\sqrt{|\rho_0|}} \right\|_{L^2(\Delta)}, \quad j = 1, 2.
\]

**Proof.** From a parallel argument as in Theorem 4.5, we obtain
\[
D^{-1}_0 f = \phi_{0,\psi} \cdot \chi_{(\delta,\infty)} + \phi_{1,f}.
\]
Using the Fourier transform of \( \phi_{0,\psi} \), it follows that
\[
\hat{E}D^{-1}_0 f(\xi) = \hat{\phi}_{0,\psi} + \hat{\phi}_{0,\psi} \chi_{(-\delta,\delta)}(\xi) + \hat{E}\phi_{1,f}
\]
\[
= \frac{\rho_m(\xi)}{\varepsilon_m} \cdot \frac{\hat{\psi}}{\rho_0} \cdot \chi_\Delta(\xi) + (\phi_{0,\psi}(0) + O(\delta^2)) \cdot \hat{\chi}_{(-\delta,\delta)}(\xi) + \hat{E}\phi_{1,f}(\xi)
\]
\[
= \frac{\rho_m(\xi)}{\varepsilon_m} \cdot \frac{\hat{\psi}}{\rho_0} \cdot \chi_\Delta(\xi) + \hat{R}(\xi).
\]
By Theorem 4.5, we have
\[
\left\| \frac{\hat{R}(\xi)}{\sqrt{1 + |\xi|}} \right\|_{L^2} \lesssim \left( \delta^{3 + \frac{1}{2} \alpha} + \delta^{1 - \alpha/2} \sqrt{\ln(1/\delta)} \right) \cdot \left\| \frac{\hat{\psi}}{\sqrt{|\rho_0|}} \right\|_{L^2(\Delta)}.
\]
Therefore,
\[
\left\| \frac{\hat{E}D^{-1}_0 f \cdot \sqrt{|\rho_0|}}{\sqrt{1 + |\xi|}} \right\|_{L^2(\Delta)} \lesssim \delta^{-\alpha/2} \left\| \frac{\hat{\psi}}{\sqrt{|\rho_0|}} \right\|_{L^2(\Delta)},
\]
\[
\left\| \frac{\hat{E}D^{-1}_0 f \cdot (1 - \chi_\Delta)}{\sqrt{1 + |\xi|}} \right\|_{L^2} \lesssim \left( \delta^{3 + \frac{1}{2} \alpha} + \delta^{1 - \alpha/2} \sqrt{\ln(1/\delta)} \right) \left\| \frac{\hat{\psi}}{\sqrt{|\rho_0|}} \right\|_{L^2(\Delta)}.
\]
Hence the estimate for \( ED^{-1}_0 f \) holds. Finally, by applying Lemma 4.8, we arrive at
\[
|A_1(\hat{E}D^{-1}_0 f)| \lesssim (1 + \delta^{-\beta/2} + \delta^{1 - \alpha} \sqrt{\ln(1/\delta)}) \left\| \frac{\hat{\psi}}{\sqrt{|\rho_0|}} \right\|_{L^2(\Delta)} \lesssim (1 + \delta^{-\beta/2}) \left\| \frac{\hat{\psi}}{\sqrt{|\rho_0|}} \right\|_{L^2(\Delta)}.
\]
Theorem 4.12 Assume that $-4 \leq \alpha < 0$ and $\beta < 2$. If $f$ is given as in (ii), the following holds for the solution of $D\varphi = f$:

$$
\hat{E}\varphi(\xi)\chi_\Delta(\xi) = \frac{\rho_m(\xi)\hat{\psi}}{\rho_0(\xi)\varepsilon_m + \rho_m(\xi)} + \frac{\rho_0(\xi)\varepsilon_m}{\rho_0(\xi)\varepsilon_m + \rho_m(\xi)} \cdot [\hat{R}(\xi) - \hat{Q}(\xi)]
$$

where the lower order terms $R$ and $Q$ have the estimate

$$
\|\hat{R}(\xi)\|_{L^2(\Delta)} + \|\hat{Q}(\xi)\|_{L^2(\Delta)} \lesssim \left(\delta^{3+\frac{1}{2}\alpha} + \delta^{1-\alpha/2}\sqrt{\ln(1/\delta)}\right) \cdot \|\frac{\hat{\psi}}{\sqrt{\rho_0}}\|_{L^2(\Delta)}.
$$

Moreover,

$$
\left\|\frac{X_{\Delta}(\xi)}{\rho_0(\xi)} E\varphi(\xi)\right\|_{L^1(\mathbb{R})} \lesssim (1 + \delta^{-\beta/2}) \delta^{\alpha/2} \|\frac{\hat{\psi}}{\sqrt{\rho_0}}\|_{L^2(\Delta)}
$$

and

$$
\left\|\frac{1 - X_{\Delta}(\xi)}{\sqrt{1 + |\xi|}} E\varphi(\xi)\right\|_{L^2(\mathbb{R})} \lesssim (1 + \delta^{-\beta/2}) \delta^{1-\alpha/2} \sqrt{\ln(1/\delta)} \|\frac{\hat{\psi}}{\sqrt{\rho_0}}\|_{L^2(\Delta)}.
$$

Proof. The proof is similar to that of Theorem 4.10. First, by Lemma 4.9 and Lemma 4.11, we can show that

$$
|\phi_{00}(0)| \lesssim (1 + \delta^{-\beta/2}) \|\frac{\hat{\psi}}{\sqrt{\rho_0}}\|_{L^2(\Delta)} \quad \text{and} \quad |\phi_{00}''(0)| \lesssim (1 + \delta^{-\beta/2}) \|\frac{\hat{\psi}}{\sqrt{\rho_0}}\|_{L^2(\Delta)}.
$$

On the other hand, note that

$$
\hat{\phi}_{00}(\xi) = \frac{\rho_m(\xi)}{\rho_0(\xi)\varepsilon_m + \rho_m(\xi)} \cdot [(ED\hat{D}^{-1} f)^\wedge(\xi) - Q(\xi)] \quad \text{for} \quad \xi \in \Delta. \quad (4.33)
$$

With the help of Lemma 4.11, we have

$$
\left\|\hat{\phi}_{00}(\xi)\right\|_{L^1(\mathbb{R})} \lesssim (1 + \delta^{-\beta/2}) \delta^{\alpha/2} \cdot \|\frac{E\hat{D}^{-1} f \cdot \sqrt{\rho_0}}{L^2(\Delta)}\| + \|\hat{Q} \cdot \sqrt{\rho_0}\|_{L^2(\Delta)}
$$

$$
\lesssim (1 + \delta^{-\beta/2}) \delta^{\alpha/2} \cdot \delta^{-\alpha/2} \|\frac{\hat{\psi}}{\sqrt{\rho_0}}\|_{L^2(\Delta)}
$$

$$
+ (1 + \delta^{-\beta/2}) \delta^{\alpha/2} \cdot \delta^{1-\alpha/2} \delta^{3+\alpha} \|\frac{\hat{\psi}}{\sqrt{\rho_0}}\|_{L^2(\Delta)}
$$

$$
+ (1 + \delta^{-\beta/2}) \delta^{\alpha/2} \left(\delta^{4+\frac{3}{2}} + \delta^{3}\sqrt{\ln(1/\delta)} \left\|\hat{\phi}_{00}(\xi)\right\|_{L^1(\mathbb{R})}\right)
$$

As a result,

$$
\left\|\hat{\phi}_{00}(\xi)\right\|_{L^1(\mathbb{R})} \lesssim (1 + \delta^{-\beta/2}) \|\frac{\hat{\psi}}{\sqrt{\rho_0}}\|_{L^2(\Delta)},
$$

25
which further implies the improved estimate for $Q$:
\[
\left\| \hat{Q}(\xi) \right\|_{L^2(\Delta)} \lesssim (1 + \delta^{-\beta/2}) \cdot \left( \delta^{3+\alpha/2} + \delta^{1-\alpha/2} \right) \left\| \hat{\psi} \right\|_{L^2(\Delta)},
\]
\[
\left\| \frac{1}{\sqrt{1 + |\xi|}} \hat{Q}(\xi) \right\|_{L^2(\mathbb{R})} \lesssim (1 + \delta^{-\beta/2}) \cdot \left( \delta^{3+\alpha/2} + \delta^{1-\alpha/2} \sqrt{\ln \frac{1}{\delta}} \right) \left\| \hat{\psi} \right\|_{L^2(\Delta)}.
\]

Next, recall that
\[
\hat{E}_{\psi}(\xi_\Delta) = \frac{\rho_0(\xi) \varepsilon_m + \rho_m(\xi)}{\rho_0(\xi) \varepsilon_m + \rho_m(\xi)} \cdot \hat{\psi}_{00}.
\]

By (4.33) and Lemma 4.11, we have for $\xi \in \Delta$,
\[
\hat{E}_{\psi}(\xi) = \frac{\rho_0(\xi) \varepsilon_m + \rho_m(\xi)}{\rho_0(\xi) \varepsilon_m + \rho_m(\xi)} \cdot \left[ (ED_0^{-1} f)^\wedge(\xi) - \hat{Q}(\xi) \right]
\]
\[
= \frac{\rho_m(\xi) \hat{\psi}}{\rho_0(\xi) \varepsilon_m + \rho_m(\xi)} + \frac{\rho_0(\xi) \varepsilon_m}{\rho_0(\xi) \varepsilon_m + \rho_m(\xi)} \cdot \left[ \hat{R}(\xi) - \hat{Q}(\xi) \right]
\]
with $R, Q$ satisfying the desired estimate. This also implies that
\[
\left\| \frac{\chi_\Delta(\xi)}{\rho_0(\xi)} \hat{E}_{\psi}(\xi) \right\|_{L^1(\mathbb{R})} \lesssim \sqrt{\varepsilon_m} \left\| \hat{\psi}_{00} \right\|_{L^1(\Delta)} \lesssim (1 + \delta^{-\beta/2}) \delta^{1/2} \left\| \frac{\hat{\psi}}{\sqrt{|\rho_0|}} \right\|_{L^2(\Delta)}.
\]

Finally, from (4.28),
\[
(1 - \chi_\Delta)\hat{E}_{\psi}(\xi) = (ED_0^{-1} f)^\wedge(\xi) - \hat{Q}(\xi).
\]

By Lemma 4.11 and Lemma 4.7, we have
\[
\left\| \frac{1 - \chi_\Delta(\xi)}{\sqrt{1 + |\xi|}} \hat{E}_{\psi}(\xi) \right\|_{L^2(\mathbb{R})} \lesssim \left( \delta^{3+\frac{1}{2}\alpha} + \delta \sqrt{\ln(1/\delta)} \right) \left\| \frac{\hat{\psi}}{\sqrt{|\rho_0|}} \right\|_{L^2(\Delta)}
\]
\[
\quad + (1 + \delta^{-\beta/2}) \left( \delta^{3+\alpha/2} + \delta^{1-\alpha/2} \sqrt{\ln(1/\delta)} \right) \left\| \frac{\hat{\psi}}{\sqrt{|\rho_0|}} \right\|_{L^2(\Delta)}
\]
\[
\lesssim (1 + \delta^{-\beta/2}) \delta^{1-\alpha/2} \sqrt{\ln(1/\delta)} \left\| \frac{\hat{\psi}}{\sqrt{|\rho_0|}} \right\|_{L^2(\Delta)}.
\]

\[\square\]

**Remark 3** In Theorem 4.12, the leading order term for the excited surface plasmon is characterized by the term
\[
\frac{\rho_m(\xi) \hat{\psi}}{\rho_0(\xi) \varepsilon_m + \rho_m(\xi)},
\]
which shows that the excitation is mainly due to the spectral component near the plasmonic frequency $\xi_\pm$ in the source term.
Remark 4 By Theorem 4.10 and 4.12, we see that the amplitude of the excited surface plasmon depends mainly on $\alpha$ and $\beta$, or the real part $\varepsilon'_m$ and imaginary part $\varepsilon''_m$ of the metal relative permittivity. To have strong surface plasmon excitation, one need to have small $\varepsilon''_m$ and $|\varepsilon'_m|$.

Remark 5 The approach in this paper relies on the assumption that $\alpha < 0$ which corresponding to the case when the skin depth of the metal is much smaller than one. It does not apply to the case when $\alpha = 0$ or $|\varepsilon'_m| = O(1)$. We expect some other interesting phenomenon to occur in that case. The case $\beta > 2$ corresponding to extremely small $\varepsilon''_m$ can lead to strong surface plasmon excitation. In practice, $\varepsilon''_m$ cannot be too small since it is bounded by the damping coefficient in the Drude model. It is a interesting mathematical question to investigate the limiting case when $\varepsilon''_m$ tend to zero. This will require more delicate analysis. We leave the case when $\alpha = 0$ and $\beta > 2$ as an open problem for future investigation.

4.4 Solution of the operator equation (4.1)

From the discussions in Section 4.1, the solution of the operator equation (4.1) can be decomposed as $\varphi = \varphi_0 + \varphi_1$, where $\varphi_0 = \varphi_{00} \cdot \chi(\delta,\infty)$ and $\varphi_{00}$ is given by (4.2). In addition, $\varphi_1$ satisfies the operator equation

$$D \varphi_1 = p, \quad \text{where} \quad p := \tilde{D} \varphi_{00} - D \varphi_0. \quad (4.34)$$

Lemma 4.13 Let $p$ be defined in (4.34), then $p = p_1 + p_2 + p_3$, and the following asymptotic expansions hold for $x_1 > \delta$:

$$p_1(x_1) = \varphi_{00}'(0) \cdot p_{1,0}(x_1) \cdot \varepsilon_n \delta^3 + O(\delta^{4+\alpha/2}) \quad \text{in } H^{1/2}(\Gamma^+_\delta),$$

$$p_2(x_1) = \varphi_{00}(0) \cdot p_{2,0}(x_1) + O\left(\delta^{3-\alpha/2} \sqrt{\ln(1/\delta)}\right) \quad \text{in } H^{1/2}(\Gamma^+_\delta),$$

$$p_3(x_1) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\chi_\Delta(\xi)}{\rho_0(\xi)} \left(\varphi_{00} \cdot \chi(-\delta,\delta)^\wedge(\xi) e^{i\xi x_1} d\xi.\right.$$

where

$$\|p_{1,0}\|_{H^{1/2}(\Gamma^+_\delta)} \lesssim 1 \quad \text{and} \quad \|p_{2,0}\|_{H^{1/2}(\Gamma^+_\delta)} \lesssim \delta \sqrt{\ln(1/\delta)}.$$

Moreover,

$$\|p_1\|_{H^{1/2}(\Gamma^+_\delta)} \lesssim \delta^{3+\alpha/2}, \quad \|p_2\|_{H^{1/2}(\Gamma^+_\delta)} \lesssim \delta^{1-\alpha/2} \sqrt{\ln(1/\delta)}.$$

Proof. We first note from the explicit expression of $\varphi_{00}$ that,

$$\|\varphi_{00}\|_{C^3(\mathbb{R})} \lesssim \frac{1}{\sqrt{|\varepsilon'_m|}} = \delta^{-\alpha/2}.$$

Recall that $D = K^e_1 + K_2 = K^e_1 + K_{2,0} + K_{2,1}$, thus

$$p = \tilde{D} \varphi_{00} - D \varphi_0 = \left[\tilde{D} \varphi_{00} - (K^e_1 + K_{2,0}) \varphi_0\right] - K_{2,1} \varphi_0.$$
More explicitly,
\[ K_{2,1} \varphi_0 = \int_{\delta}^{\infty} \left( G_{2,1} (x, 0; y_1, 0) - G_{2}^{(0)} (x_1, 0; y_1, 0) - G_{2}^{(0)} (-x_1, 0; y_1, 0) \right) \varphi_0 (y_1) \, dy_1, \]
\[ \tilde{D} \varphi_{00} - (K_{1} + K_{2,0}) \varphi_0 = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \frac{\varepsilon_m}{\rho_m (\xi)} + \frac{1}{\rho_0 (\xi)} \right) \cdot (\varphi_{00} \cdot \chi_{(\delta, \delta)}) (\xi) e^{ix_1} \, d\xi. \]

By a parallel calculation in Lemma 4.4, we can decompose \( p \) as \( p = p_1 + p_2 + p_3 \), where
\[ p_1 (x_1) = -2 \int_0^\infty g (x_1 + y_1 - \delta) (\varphi_{00} (y_1 + \delta) - \varphi_{00} (y_1 - \delta)) \, dy_1, \quad x_1 > \delta, \]
\[ p_2 (x_1) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1 - \chi_{\Delta} (\xi)}{\rho_0 (\xi)} (\varphi_{00} \cdot \chi_{(-\delta, \delta)}) (\xi) e^{ix_1} \, d\xi, \quad x_1 > \delta, \]
\[ p_3 (x_1) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\chi_{\Delta} (\xi)}{\rho_0 (\xi)} (\varphi_{00} \cdot \chi_{(-\delta, \delta)}) (\xi) e^{ix_1} \, d\xi \quad x_1 > \delta. \]

The same proof as in Lemma 4.4 for \( p_1 \) and \( p_2 \) leads to the assertion. \( \square \)

**Theorem 4.14** Let \( \varphi_{00} \) be defined in \((4.2)\). Assume that \(-4 \leq \alpha < 0 \) and \( \beta < 2 \). The solution of \((4.1)\) admits the decomposition \( \varphi = \varphi_0 + \varphi_1 \), where
\[ \varphi_0 = \varphi_{00} \cdot \chi_{(\delta, \infty)} \quad \text{and} \quad \varphi_1 = D^{-1} p_1 + D^{-1} p_2 + D^{-1} p_3. \]

In addition,
\[ \left\| \frac{\chi_{\Delta} (\xi)}{\rho_0 (\xi)} \widehat{E \varphi_1 (\xi)} \right\|_{L^1(\mathbb{R})} \lesssim \delta \cdot (1 + \delta^{-\beta/2}) \quad \text{and} \quad \left\| \frac{1 - \chi_{\Delta} (\xi)}{\sqrt{1 + |\xi|}} \widehat{E \varphi_1 (\xi)} \right\|_{L^2(\mathbb{R})} \lesssim \delta^{1-\alpha} \sqrt{\ln(1/\delta)}. \]

**Proof.** Based on the decomposition of the source function \( p \) in the above lemma 4.13, we write the solution of the equation \( \varphi_0 \) as \( \varphi_1 = \varphi_1^{(1)} + \varphi_1^{(2)} \), where
\[ D\varphi_1^{(1)} = p_1 + p_2 \quad \text{and} \quad D\varphi_1^{(2)} = p_3. \]

We apply Theorem 4.10 and Lemma 4.13 for the equation \( D\varphi_1^{(1)} = p_1 + p_2 \) to obtain the following estimates:
\[ \left\| \frac{\chi_{\Delta} (\xi)}{\rho_0 (\xi)} \widehat{E \varphi_1^{(1)} (\xi)} \right\|_{L^1(\mathbb{R})} \lesssim (1 + \delta^{-\beta/2}) \left( \delta^{3+\alpha/2} + \delta^{1-\alpha/2} \sqrt{\ln(1/\delta)} \right), \]
\[ \left\| \frac{1 - \chi_{\Delta} (\xi)}{\sqrt{1 + |\xi|}} \widehat{E \varphi_1^{(1)} (\xi)} \right\|_{L^2(\mathbb{R})} \lesssim \delta^{-\alpha/2} \left( \delta^3 + \delta^{1-\alpha} \sqrt{\ln(1/\delta)} \right). \]

On the other hand, applying Theorem 4.12 to the equation \( D\varphi_1^{(2)} = p_3 \) together with the estimate
\[ \left\| \varphi_{00} \cdot \chi_{(-\delta, \delta)} \right\|_{L^2(\Delta)} \lesssim \delta^{1-\alpha/2} \]
leads to
\[
\left\| \frac{\chi_\Delta(\xi)}{\rho_0(\xi)} E\varphi_1^{(2)}(\xi) \right\|_{L^1(\mathbb{R})} \lesssim (1 + \delta^{-\beta/2}) \delta, \\
\left\| \frac{1 - \chi_\Delta(\xi)}{\sqrt{1 + |\xi|}} E\varphi_1^{(2)}(\xi) \right\|_{L^2(\mathbb{R})} \lesssim (1 + \delta^{-\beta/2}) \delta^2 \sqrt{\ln(1/\delta)}.
\]

The proof is complete by combining the above estimates. \(\square\)

**Remark 6** The above theorem shows that the excited of surface plasmon by an incident even plane wave is weak in the amplitude unless \(\beta \geq 2\), in which case the imaginary part \(\varepsilon''\) of the metal relative permittivity is extremely small.

## 5 Solution of the integral equation for the odd case

Let us define \(D := K_1^o + K_2\) and write the integral equation for the odd case as
\[D \varphi = -2u_o^i.\] (5.1)

We would like to apply an analogous perturbation argument as in Section 4 to obtain the solution \(\varphi\). To this end, let \(\tilde{D} : H^{-1/2}(\mathbb{R}^+) \to H^{1/2}(\mathbb{R}^+)\) be given by
\[\tilde{D}\varphi := -\frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \frac{1}{\rho_0(\xi)} + \frac{\varepsilon_m}{\rho_m(\xi)} \right) \hat{\varphi}(\xi)e^{i\xi x_1} d\xi, \quad x_1 > 0.
\]

It can be calculated that the solution of \(\tilde{D}\varphi_0 = -2u_o^i\) takes the form of
\[\varphi_{00} = R \cdot \sin(k_1 x_1) \cdot \chi(0, \infty),\] (5.2)

where the coefficient \(R = -\frac{2\rho_0(k_1)\rho_m(k_1)}{\rho_m(k_1) + \varepsilon_m\rho_0(k_1)}\). Now if one decomposes the solution of (5.1) as \(\varphi = \varphi_0 + \varphi_1\), where \(\varphi_0 = \varphi_{00} \cdot \chi(0, \infty)\), then it is clear that \(\varphi_1\) satisfies the equation
\[D\varphi_1 = \tilde{D}\varphi_{00} - D\varphi_0.\] (5.3)

In order to distinguish the frequency component near and away from the surface plasmon frequency when solving the operator equation (5.3), we introduce the operator \(D_0 : H^{-1/2}(\Gamma^+_\delta) \to H^{1/2}(\Gamma^+_\delta)\) that excludes the surface plasmonic resonances by letting
\[D_0 \varphi := (K_{1,0}^o + K_2)\varphi = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \frac{1 - \chi_\Delta(\xi)}{\rho_0(\xi)} \hat{\varphi}(\xi) + \frac{\varepsilon_m}{\rho_m(\xi)} e^{-i\delta \xi} \hat{E}\varphi(\xi) \right) e^{i\xi x_1} d\xi.
\]

Following the argument in Proposition 4.1, it holds that \(D_0\) is invertible. As such (5.3) can be rewritten as
\[\varphi_1 + D_0^{-1}K_{1,1}^o \varphi_1 = D_0^{-1} f, \quad \text{where } f = \tilde{D}\varphi_{00} - D\varphi_0.\] (5.4)

We discuss the solution of the above operator equation in the rest of this section. The derivation shares similarity with the one for the even case, and we skip some of technical calculations for conciseness.
5.1 Solution of \( D_0 \phi = K_{1,1}^0 \psi \)

For a function \( \psi \) that satisfies \( \| \frac{\partial \phi}{\partial |\mu|} \|_{L^2(\Delta)} < \infty \), consider solving the operator equation

\[
D_0 \phi = K_{1,1}^0 \psi. \tag{5.5}
\]

We decompose the operator \( D_0 \) as \( D_0 = D_{0,0} + D_{0,1} \), where

\[
D_{0,0} := (K_{1,0}^0 + K_{2,0}) \quad \text{and} \quad D_{0,1} := K_{2,1}. \tag{5.6}
\]

Define the operator \( \tilde{D}_{0,0} : H^{-1/2}(\mathbb{R}^+) \to H^{1/2}(\mathbb{R}^+) \) as

\[
\tilde{D}_{0,0} \psi(x_1) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \frac{1 - \chi_\Delta(\xi)}{\rho_0(\xi)} + \frac{\varepsilon_m}{\rho_m(\xi)} \right) \hat{\psi}(\xi) e^{i\xi x_1} d\xi, \quad x_1 > 0. \tag{5.7}
\]

Let \( \phi_{00,\psi}(x_1) \) be a function over the whole real line such that its Fourier transform is given by

\[
\hat{\phi}_{00,\psi}(\xi) = \frac{\chi_\Delta(\xi) \cdot \hat{\psi}(\xi)}{\rho_0(\xi)} = \frac{\rho_m(\xi)}{\rho_0(\xi) \varepsilon_m} \cdot \hat{\psi}(\xi) \cdot \chi_\Delta(\xi). \tag{5.8}
\]

Then \( \phi_{00,\psi} \) is a smooth and odd function, and it holds that

\[
\| \phi_{00,\psi} \|_{L^1(\Delta)} \lesssim \frac{1}{\sqrt{\varepsilon_m}} \frac{\| \hat{\psi} \|_{L^2(\Delta)}}{\sqrt{\rho_0}} \quad \text{and} \quad \| \phi_{00,\psi} \|_{C^2(\mathbb{R})} \lesssim \frac{1}{\sqrt{\varepsilon_m}} \frac{\| \hat{\psi} \|_{L^2(\Delta)}}{\sqrt{\rho_0}}. \tag{5.9}
\]

Let \( \phi_{00,\psi} = \phi_{00,\psi}|_{(0,\infty)} \), then it follows that

\[
\tilde{D}_{0,0} \phi_{00,\psi} = K_{1,1}^0 \psi, \tag{5.10}
\]

where we have extended \( K_{1,1}^0 \) naturally to \( \mathbb{R}^+ \). If one decomposes the solution \( \phi \) of the operator equation \( (5.5) \) as \( \phi = \phi_0 + \phi_1 \), where \( \phi_0 = \phi_{00,\psi} \cdot \chi(\delta,\infty) \), then \( \phi_1 \) satisfies the equation

\[
D_0 \phi_1 = q, \quad \text{where} \quad q := \tilde{D}_{0,0} \phi_{00,\psi} - D_0 \phi_0. \tag{5.11}
\]

**Lemma 5.1** Let \( q \) be defined in \( (5.11) \), then \( q(x_1) = q_1(x_1) + q_2(x_1) \), where

\[
q_1(x_1) = -2 \int_0^\infty g(x_1 + y_1 - \delta) (\phi_{00,\psi}(y_1 + \delta) - \phi_{00,\psi}(y_1 - \delta)) dy_1 \tag{5.12}
\]

\[
= -4 \int_0^\infty g(x_1 + y_1) \phi_{00,\psi}(y_1) dy_1 + 4 \int_0^\delta g(x_1 - y_1) \phi_{00,\psi}(y_1) dy_1, \quad x_1 > \delta,
\]

\[
q_2(x_1) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1 - \chi_\Delta(\xi)}{\rho_0(\xi)} (\phi_{00,\psi} \cdot \chi(-\delta,\delta) \cdot \hat{\psi}(\xi) e^{i\xi x_1} d\xi, \quad x_1 > \delta. \tag{5.13}
\]

In addition, the following asymptotic expansions hold for \( x_1 > \delta \):

\[
q_1(x_1) = \phi'_{00,\psi}(0) \cdot q_{1,0}(x_1) + O(\varepsilon_m \delta^3), \tag{5.14}
\]

\[
q_2(x_1) = \phi'_{00,\psi}(0) \cdot q_{2,0}(x_1) + O(\delta^3 \sqrt{\ln(1/\delta)}), \tag{5.15}
\]

where

\[
\|q_{1,0}\|_{H^{1/2}((1/\delta)^\perp)} \lesssim 1 \quad \text{and} \quad \|q_{2,0}\|_{H^{1/2}((1/\delta)^\perp)} \lesssim \delta^2 \sqrt{\ln(1/\delta)}. \]
Proof. Define the operator $\hat{D}_{0,0} : H^{-1/2}(\mathbb{R}^+) \to H^{1/2}(\mathbb{R}^+)$ as

$$\hat{D}_{0,0} \psi(x_1) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \frac{1 - \chi_1(\xi)}{\rho_0(\xi)} \hat{O} \psi(\xi) + \frac{\varepsilon_m}{\rho_m(\xi)} \hat{E} \psi(\xi) \right) e^{i \xi x_1} d\xi, \quad x_1 > 0. \quad (5.16)$$

We write $q$ as

$$q = (\hat{D}_{0,0} \phi_{00,+} - \hat{D}_{0,0} \phi_{00,+}) + (\hat{D}_{0,0} \phi_{00,+} - D_0 \phi_0) =: J_1 + J_2.$$

For $x_1 > \delta$, 

$$J_1(x_1) = \hat{D}_{0,0} \phi_{00,+} - \hat{D}_{0,0} \phi_{00,+} = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\varepsilon_m}{\rho_m(\xi)} \left( \hat{O} \phi_{00,+}(\xi) - \hat{E} \phi_{00,+}(\xi) \right) e^{i \xi x_1} d\xi = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\varepsilon_m}{\rho_m(\xi)} (\phi_{00,\psi} \cdot \chi_{(-\infty,0)})(\xi) e^{i \xi x_1} d\xi = \frac{4}{\pi} \int_{-\infty}^{\infty} g(x_1 - y_1) \phi_{00,\psi}(y_1) dy_1$$

$$= -4 \int_{0}^{\infty} g(x_1 + y_1) \phi_{00,\psi}(y_1) dy_1 = -4 \int_{0}^{\infty} g(x_1 + y_1) \phi_{00,\psi}(y_1) dy_1. \quad (5.17)$$

Note that $D_0 = D_{0,0} + D_{0,1}$, we have

$$J_2 = \hat{D}_{0,0} \phi_{00,+} - D_{0,0} \phi_0 - D_{0,1} \phi_0. \quad (5.18)$$

For $x_1 > \delta$,

$$\hat{D}_{0,0} \phi_{00,+} - D_{0,0} \phi_0 = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\varepsilon_m}{\rho_m(\xi)} \left( E(\phi_{00,\psi} \chi_{(0,\delta)}) \right)^{\wedge}(\xi) e^{i \xi x_1} d\xi = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1 - \chi_1(\xi)}{\rho_0(\xi)} (\phi_{00,\psi} \cdot \chi_{(-\delta,0)})(\xi) e^{i \xi x_1} d\xi = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\varepsilon_m}{\rho_m(\xi)} \left( E(\phi_{00,\psi} \chi_{(0,\delta)}) \right)^{\wedge}(\xi) e^{i \xi x_1} d\xi.$$

Further calculation yields

$$-\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\varepsilon_m}{\rho_m(\xi)} \left( E(\phi_{00,\psi} \chi_{(0,\delta)}) \right)^{\wedge}(\xi) e^{i \xi x_1} d\xi$$

$$= -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\varepsilon_m}{\rho_m(\xi)} (\phi_{00,\psi} \cdot \chi_{(-\delta,0)})(\xi) e^{i \xi x_1} d\xi + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\varepsilon_m}{\rho_m(\xi)} (\phi_{00,\psi} \cdot \chi_{(-\delta,0)})(\xi) e^{i \xi x_1} d\xi$$

$$= 2 \int_{-\delta}^{\delta} g(x_1 - y_1) \phi_{00,\psi}(y_1) dy_1 - 4 \int_{-\delta}^{0} g(x_1 - y_1) \phi_{00,\psi}(y_1) dy_1.$$

Thus

$$\hat{D}_{0,0} \phi_{00,+} - D_{0,0} \phi_0 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1 - \chi_1(\xi)}{\rho_0(\xi)} (\phi_{00,\psi} \cdot \chi_{(-\delta,0)})(\xi) e^{i \xi x_1} d\xi + 2 \int_{-\delta}^{\delta} g(x_1 - y_1) \phi_{00,\psi}(y_1) dy_1 - 4 \int_{-\delta}^{0} g(x_1 - y_1) \phi_{00,\psi}(y_1) dy_1. \quad (5.19)$$
On the other hand, from Lemma 4.4, it is known that

\[
D_{0,1} \phi_0 = 2 \int_0^\infty g(x_1 + y_1 - \delta) (\phi_{00,\psi}(y_1 + \delta) - \phi_{00,\psi}(y_1 - \delta)) dy_1 + 2 \int_{-\delta}^\delta g(x_1 + y_1) \phi_{00,\psi}(y_1) dy_1
= 2 \int_0^\infty g(x_1 + y_1 - \delta) (\phi_{00,\psi}(y_1 + \delta) - \phi_{00,\psi}(y_1 - \delta)) dy_1 - 2 \int_{-\delta}^\delta g(x_1 - y_1) \phi_{00,\psi}(y_1) dy_1.
\]

(5.20)

Then (5.12) and (5.13) follow by combining (5.17) - (5.20). The asymptotics of \( q_1 \) and \( q_2 \) can be obtained in an analogous way as in Lemma 4.4, by noting that \( \phi_{00,\psi} \) is now an odd function. \( \square \)

**Theorem 5.2** Let \( \phi \) be the solution of the equation \( D_0 \phi = K_{1,1}^e \psi \). Let \( \phi_{00,\psi} \) be defined by

\[
\hat{\phi}_{00,\psi}(\xi) = \frac{\rho_0(\xi)}{\rho_0(\xi) \in_m} \cdot \hat{E}(\xi) \cdot \chi(\xi).
\]

Then \( \phi = \phi_0 + \phi_1 \), where

\[
\phi_0 = \phi_{00,\psi} \cdot \chi(\delta, \infty),
\]

\[
\phi_1 = \phi'_{00,\psi}(0) \cdot \left( \delta^{2+\alpha} \cdot D_0^{-1} q_{1,0} + \delta^2 \sqrt{\ln(1/\delta)} \cdot D_0^{-1} q_{2,0} \right) + D_0^{-1} q_h.
\]

In addition,

\[
\| \phi_{00,\psi} \|_{C^2(\mathbb{R})} \lesssim \delta^{-\frac{3}{2}} \cdot \frac{\| \hat{E} \|_{L^2(\Delta)}}{\sqrt{|\rho_0|}},
\]

\[
\| q_{1,0} \|_{H^{1/2}(\Gamma^+_d)} \lesssim 1, \quad \| q_{2,0} \|_{H^{1/2}(\Gamma^+_d)} \lesssim 1, \quad \| q_0 \|_{H^{1/2}(\Gamma^+_d)} \lesssim 1,
\]

\[
\| q_h \|_{H^{1/2}(\Gamma^+_d)} \lesssim \delta^{3+\alpha} \| \phi_{00,\psi} \|_{C^2(\mathbb{R})} \lesssim \delta^{3+\frac{3}{2}} \| \hat{E} \|_{L^2(\Delta)}},
\]

\[
\| \phi_1 \|_{H^{1/2}(\Gamma^+_d)} \lesssim \delta^{2+\frac{3}{2}} \frac{\| \hat{E} \|_{L^2(\Delta)}}{\sqrt{|\rho_0|}}.
\]

**5.2 Solution of \( D \varphi = f \) and excitation of surface plasmon**

Let us introduce the function space

\[
V_1 = \{ \varphi \in H^{-1/2}(\Gamma^+_d) : \int \frac{1}{|\rho_0(\xi)|} |\hat{\varphi}(\xi)|^2 d\xi < \infty \}.
\]

First, by a parallel proof as in Theorem 4.2, it can be shown that \( D \) is invertible, and the following holds for the solution of \( D \varphi = f \):

\[
\| \frac{\hat{\varphi}}{\sqrt{|\rho_0|}} \|_{L^2(\Delta)} \lesssim \| \varphi \|_{V_1} \lesssim \frac{|\varepsilon_m|}{\varepsilon_m} \| f \|_{V_2} = O(\delta^{\alpha-\beta}) \| f \|_{V_2}. \]

(5.21)
Following (5.4), the operator equation $D\varphi = f$ is recast as
\[ \varphi + D_0^{-1}K_{1,1}^\varepsilon\varphi = D_0^{-1}f. \]  
(5.22)

Using Theorem 5.2, it follows that $D_0^{-1}K_{1,1}^\varepsilon\varphi = \phi_{00}\chi_{(\delta,\infty)} + \phi_1$, where $\phi_{00}$ is defined by
\[ \hat{\phi}_{00}(\xi) = \frac{\rho_m(\xi)}{\rho_0(\xi)\varepsilon_m} \cdot \hat{\varphi}(\xi) \cdot \chi_\Delta(\xi), \]
and
\[ \phi_1 = \delta^{2+\alpha} \cdot \phi_{00}'(0) \cdot D_0^{-1}q_0 + D_0^{-1}q_h. \]

Moreover, in light of the estimates (5.9) and (5.21), we have
\[ |\hat{\phi}_{00}(\xi)| \lesssim \frac{|\varepsilon_m'|^2}{\varepsilon_m^2} \|f\|_2, \quad \|\phi_{00}\|_{C^2(\mathbb{R})} \lesssim \frac{|\varepsilon_m'|^2}{\varepsilon_m^2} \|f\|_2. \]  
(5.23)

Substituting into (5.22) and extending both sides of the above equation as odd functions over the whole real line, we obtain
\[ O\varphi + \phi_{00}(1 - \chi_{(\delta,\delta)}) + \delta^{2+\alpha} \cdot \phi_{00}'(0) \cdot OD_0^{-1}q_0 + OD_0^{-1}q_h = O(D_0^{-1}f). \]

This is equivalent to the following equation in the Fourier domain:
\[ \hat{O}\varphi(\xi) + \hat{\phi}_{00}(\xi) + \hat{Q}(\xi) = (OD_0^{-1}f)^\wedge(\xi), \]  
(5.24)

where
\[ \hat{Q}(\xi) := -(\phi_{00}\chi_{(\delta,\delta)})^\wedge(\xi) + \delta^{2+\alpha} \cdot \phi_{00}'(0) \cdot (OD_0^{-1}q_0)^\wedge(\xi) + (OD_0^{-1}q_h)^\wedge(\xi). \]

Note that the Taylor expansion gives
\[ (\phi_{00}\chi_{(\delta,\delta)})^\wedge(\xi) = \phi_{00}'(0) \cdot \hat{q}_1(\xi) + \hat{q}_2(\xi), \]

where $\hat{q}_1(\xi) = \delta \sin(\delta\xi)/\xi$, and it holds that
\[ \|\hat{q}_1(\xi)\|_{L^2(\Delta)} \lesssim \delta^2, \quad \|\hat{q}_1(\xi)\|_{L^2(\mathbb{R})} \lesssim \delta^2 \sqrt{\ln \frac{1}{\delta}}, \]
\[ \|\hat{q}_2(\xi)\|_{L^2(\Delta)} \lesssim \|\phi_{00}\|_{C^2(\mathbb{R})} \cdot \delta^3, \quad \|\hat{q}_2(\xi)\|_{L^2(\mathbb{R})} \lesssim \|\phi_{00}\|_{C^2(\mathbb{R})} \cdot \delta^3 \sqrt{\ln \frac{1}{\delta}}. \]

Hence $\hat{Q}(\xi)$ may be expressed as
\[ \hat{Q}(\xi) = \phi_{00}'(0) \cdot \left[ (\delta^{2+\alpha}(OD_0^{-1}q_0)^\wedge(\xi) + \hat{q}_1(\xi)) + (OD_0^{-1}q_h + q_2)^\wedge(\xi) \right]. \]  
(5.25)

Using the formula (5.8), we obtain the Fourier transform of $\tilde{\phi}_0$ as follows:
\[ \hat{\phi}_{00}(\xi) = \frac{\rho_m(\xi)}{\rho_0(\xi)\varepsilon_m + \rho_m(\xi)} \cdot \left[ (OD_0^{-1}f)^\wedge(\xi) - \hat{Q}(\xi) \right], \quad \xi \in \Delta. \]  
(5.26)
This leads to the following equation for $\phi'_0(0)$:

$$\Pi \cdot \phi'_0(0) = b,$$

where

$$\Pi = \left[ 1 + \delta^{2+\alpha} A(OD_0^{-1}q_0) + A(q_1) \right],$$

$$b = A(OD_0^{-1}f) - A(OD_0^{-1}q_h) - A(q_2),$$

and the functional $A$ is defined by

$$A(\varphi) := \int_\Delta \frac{i\xi \cdot \rho_m(\xi)}{\rho_0(\xi) + \rho_m(\xi)} \hat{\varphi}(\xi) d\xi.$$

Now a parallel proof of Lemma 4.9 gives the expansion of $\phi'_0(0)$ in the following Lemma.

**Lemma 5.3** Assume that $-4 < \alpha < 0$ and $\alpha - \beta > -4$, then $\Pi$ and $b$ adopt the expansions

$$\Pi = 1 + \left( 1 + \delta^{-\beta/2} \right) \cdot O \left( \delta^{2+\alpha/2} \right)$$

and

$$b = A(OD_0^{-1}f) + \left( 1 + \delta^{-\beta/2} \right) \cdot O \left( \delta^{3+\alpha/2} \right) \|\hat{\varphi}_0\|_{L^1(\Delta)}.$$

Moreover, the following holds for $\phi'_0(0)$:

$$\phi'_0(0) = A(ED_0^{-1}f) \cdot (1 + o(1)) + \left( 1 + \delta^{-\beta/2} \right) \cdot O \left( \delta^{3+\alpha/2} \right) \|\hat{\varphi}_0\|_{L^1(\Delta)}.$$

**Remark 7** The assumption that $-4 < \alpha < 0$ and $\alpha - \beta > -4$ is used throughout the subsequent analysis. We note that this assumption allows the particularly interesting case where $\alpha \geq -2$ and $\beta < 2$. We also note that this assumption is stricter than the one for even case. We leave it as an open question for the other cases which are not covered in this assumption.

With the above preparation, we are ready to present the solution of the operator equation $D\varphi = f$. Again we distinguish two types of source function $f$ when $f \in H^{1/2}(\Gamma^+_\delta)$ and $f = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\chi(\xi)}{\rho_0(\xi)} \hat{\varphi}(\xi) e^{i\xi x_1} d\xi$. The estimates for the energy of the solution in the frequency band $\Delta$ and $\mathbb{R} \setminus \Delta$ are given in Theorems 5.4 and 5.5 respectively. This can be estalished by estimating $Q(\xi)$ using the formula (5.25) and Lemma 5.3, which then leads to the estimation for $\hat{\varphi}_0$ and the solution $\varphi$. The proof is parallel to Theorems 4.10 and 4.12 and we omit here for conciseness.

**Theorem 5.4** Assume that $-4 < \alpha < 0$ and $\alpha - \beta > -4$. If $f \in H^{1/2}(\Gamma^+_\delta)$, the following holds for the solution of $D\varphi = f$:

$$\left\| \frac{\chi(\xi)}{\rho_0(\xi)} \hat{\varphi}(\xi) \right\|_{L^1(\mathbb{R})} \lesssim \left( 1 + \delta^{-\beta/2} \right) \|f\|_{H^{1/2}(\Gamma^+_\delta)},$$

$$\left\| \frac{1 - \chi(\xi)}{\sqrt{1 + |\xi|}} \hat{\varphi}(\xi) \right\|_{L^2(\mathbb{R})} \lesssim \delta^{-\alpha/2} \|f\|_{H^{1/2}(\Gamma^+_\delta)}.$$
Theorem 5.5 Assume that $-4 < \alpha < 0$ and $\alpha - \beta > -4$. If $f = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\chi_\Delta(\xi)}{\rho_0(\xi)} \hat{\psi}(\xi) e^{i\xi x_1} d\xi$, where $\hat{\psi}(\xi)$ is odd and $\left\| \frac{\hat{\psi}}{\sqrt{\rho_0}} \right\|_{L^2(\Delta)} < \infty$, then the following holds for the solution of $D\varphi = f$:

$$
\left\| \frac{\chi_\Delta(\xi)}{\rho_0(\xi)} \hat{\varphi}(\xi) \right\|_{L^1(\mathbb{R})} \lesssim (1 + \delta^{-\beta/2})^2 \delta^{\alpha/2} \left\| \frac{\hat{\psi}}{\sqrt{\rho_0}} \right\|_{L^2(\Delta)}
$$

and

$$
\left\| \frac{1 - \chi_\Delta(\xi)}{\sqrt{1 + |\xi|}} \hat{\varphi}(\xi) \right\|_{L^2(\mathbb{R})} \lesssim (1 + \delta^{-\beta/2}) \delta^{1 - \alpha/2} \left\| \frac{\hat{\psi}}{\sqrt{\rho_0}} \right\|_{L^2(\Delta)}.
$$

5.3 Solution of the operator equation (5.1)

We decompose the solution of the operator equation (5.1) as $\varphi = \varphi_0 + \varphi_1$, where $\varphi_0 = \varphi_{00} \cdot \chi(\delta, \infty)$ and $\varphi_{00}$ is given by (5.2). In addition, $\varphi_1$ satisfies the operator equation

$$
D\varphi_1 = p, \quad \text{where} \quad p := \hat{D}\varphi_0 - D\varphi_0.
$$

From a parallel calculation as in Lemmas 5.1, we obtain the following lemma.

Lemma 5.6 Let $p$ be defined in (5.27), then $p = p_1 + p_2 + p_3$, and the following asymptotic expansions hold for $x_1 > \delta$:

$$
p_1(x_1) = \varphi_{00}'(0) \cdot p_{1,0}(x_1) \cdot e^{-\epsilon_m \delta^2} + O(\delta^{3 + \alpha/2}) \quad \text{in} \ H^{1/2}(\Gamma_+^\delta),
$$

$$
p_2(x_1) = \varphi_{00}'(0) \cdot p_{2,0}(x_1) + O \left( \delta^{3 - \alpha/2} \sqrt{\ln(1/\delta)} \right) \quad \text{in} \ H^{1/2}(\Gamma_+^\delta),
$$

$$
p_3(x_1) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\chi_\Delta(\xi)}{\rho_0(\xi)} (\varphi_{00} \cdot \chi(-\delta, \delta) \cdot \xi^2) e^{i\xi x_1} d\xi.
$$

where

$$
\left\| p_{1,0} \right\|_{H^{1/2}(\Gamma_+^\delta)} \lesssim 1 \quad \text{and} \quad \left\| p_{2,0} \right\|_{H^{1/2}(\Gamma_+^\delta)} \lesssim \delta^2 \sqrt{\ln(1/\delta)}.
$$

Moreover,

$$
\left\| p_1 \right\|_{H^{1/2}(\Gamma_+^\delta)} \lesssim \delta^{2 + \alpha/2} \quad \text{and} \quad \left\| p_2 \right\|_{H^{1/2}(\Gamma_+^\delta)} \lesssim \delta^{2 - \alpha/2} \sqrt{\ln(1/\delta)}.
$$

Theorem 5.7 Let $\varphi_{00}$ be defined in (5.2). Assume that $-4 < \alpha < 0$ and $\alpha - \beta > -4$. The solution of (5.1) admits the decomposition $\varphi = \varphi_0 + \varphi_1$, where

$$
\varphi_0 = \varphi_{00} \cdot \chi(\delta, \infty) \quad \text{and} \quad \varphi_1 = D^{-1} p_1 + D^{-1} p_2 + D^{-1} p_3.
$$

In addition,

$$
\left\| \frac{\chi_\Delta(\xi)}{\rho_0(\xi)} \tilde{E} \varphi_1(\xi) \right\|_{L^1(\mathbb{R})} \lesssim \delta \cdot (1 + \delta^{-\beta/2}) \quad \text{and} \quad \left\| \frac{1 - \chi_\Delta(\xi)}{\sqrt{1 + |\xi|}} \tilde{E} \varphi_1(\xi) \right\|_{L^2(\mathbb{R})} \lesssim \delta^2 + \delta^{2 - \alpha/2}.
$$
Proof. Based on the decomposition of the source function \( p \) in the above lemma, we decompose the solution of the equation (5.27) as \( \varphi_1 = \varphi_1^{(1)} + \varphi_1^{(2)} \), where

\[
D\varphi_1^{(1)} = p_1 + p_2 \quad \text{and} \quad D\varphi_1^{(2)} = p_3.
\]

Now if one applies Theorem 5.4 for the equation \( D\varphi_1^{(1)} = p_1 + p_2 \), it follows that from the estimate in Lemma 5.6 that

\[
\|\chi \Delta (\xi) \rho_0(\xi) \frac{\tilde{O}\varphi_1^{(1)}(\xi)}{\rho_0(\xi)}\|_{L^1(\mathbb{R})} \lesssim (1 + \delta^{-\beta/2}) \|p_1 + p_2\|_{H^{1/2}(\mathbb{R}^+)} \lesssim (1 + \delta^{-\beta/2}) \delta^{2+\alpha/2},
\]

\[
\|\frac{1 - \chi \Delta (\xi)}{\sqrt{1 + |\xi|}} \tilde{O}\varphi_1^{(1)}(\xi)\|_{L^2(\mathbb{R})} \lesssim \delta^{-\alpha/2} \|p_1 + p_2\|_{H^{1/2}(\mathbb{R}^+)} \lesssim \delta^2.
\]

On the other hand, applying Theorem 5.5 to the equation \( D\varphi_1^{(2)} = p_3 \), we obtain

\[
\|\chi \Delta (\xi) \rho_0(\xi) \frac{\tilde{O}\varphi_1^{(2)}(\xi)}{\rho_0(\xi)}\|_{L^1(\mathbb{R})} \lesssim (1 + \delta^{-\beta/2}) \delta^{\alpha/2} \|\frac{\varphi_0 \cdot \chi(-\delta,\delta)}{\rho_0}\|_{L^2(\Delta)} \lesssim \delta^{1-\beta/2},
\]

\[
\|\frac{1 - \chi \Delta (\xi)}{\sqrt{1 + |\xi|}} \tilde{O}\varphi_1^{(2)}(\xi)\|_{L^2(\mathbb{R})} \lesssim (1 + \delta^{-\beta/2}) \delta^{1-\alpha/2} \|\frac{\varphi_0 \cdot \chi(-\delta,\delta)}{\rho_0}\|_{L^2(\Delta)} \lesssim (1 + \delta^{-\beta/2}) \delta^{2-\alpha}.
\]

The proof is complete by combining the above estimations. □

Appendix A Proof of Lemma 3.1

We first prove some auxiliary results.

Lemma A.1 For any \( L^2 \) even function \( f : \mathbb{R} \to \mathbb{R} \), we have

\[
\|\langle x \rangle^{-\frac{1}{2}} \mathcal{H}(\langle y \rangle^{\frac{1}{2}} f)\|_2 \lesssim \|f\|_2,
\]

where \( \mathcal{H} \) is the usual Hilbert transform on \( \mathbb{R} \) and \( \langle x \rangle = (1 + x^2)^{\frac{1}{2}} \).

Proof. First observe that the regime \( |y| \lesssim 1 \) is easily handled. Therefore we may assume that \( f \) is supported in \( |y| \gtrsim 1 \). It then suffices for us to prove the inequality (for \( f \) even and supported in \( |y| \gtrsim 1 \)):

\[
\|\langle x \rangle^{-\frac{1}{2}} \mathcal{H}(\langle y \rangle^{\frac{1}{2}} f)\|_2 \lesssim \|f\|_2.
\]

By using parity, this is equivalent to showing:

\[
\|x^{-\frac{1}{2}} \text{PV} \int_{y \in (0,\infty)} \frac{x}{x^2 - y^2} y^{\frac{1}{2}} f(y) dy\|_{L^2(0,\infty)} \lesssim \|f\|_{L^2(0,\infty)}.
\]

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Now introduce change of variable $x = e^t$, $y = e^s$, $y^{1/2}f(y) = \tilde{f}(s)$, $x^{1/2}f(x) = \tilde{f}(t)$. Note that $s, t \in \mathbb{R}$. Then we just need to show

$$\|PV \int K(t-s)\tilde{f}(s)ds\|_{L^2(\mathbb{R})} \lesssim \|\tilde{f}\|_2,$$

where the kernel $K$ is given by

$$K(z) = \frac{1}{e^z - e^{-z}}.$$

It is easy to check that $K$ is a standard Calderon-Zygmund operator (in particular $K$ is an odd function and the Hörmander gradient condition $|K'(z)| \lesssim |z|^{-2}$ on $\mathbb{R} \setminus \{0\}$ is obviously true.) The desired result then easily follows.

**Lemma A.2** Suppose $f \in H^{-\frac{1}{2}}(\mathbb{R})$ and $f$ is supported on $(0, \infty)$. Then

$$\|f\|_{H^{-\frac{1}{2}}(\mathbb{R})} \lesssim \|Ef\|_{H^{-\frac{1}{2}}(\mathbb{R})}.$$

**Proof.** Denote $g = Ef$, then obviously $f = g \cdot \chi_{(0,\infty)}$, i.e. $f$ is simply the restriction of $g$ to the half line. Denote $g = \langle \nabla \rangle^{\frac{1}{2}}h$ with $h \in L^2$, then the desired inequality is equivalent to

$$\|\langle \nabla \rangle^{\frac{1}{2}}\chi_{(0,\infty)}\langle \nabla \rangle^{\frac{1}{2}}h\|_2 \lesssim \|h\|_2.$$

Observe that the Fourier transform of $\chi_{(0,\infty)}$ is simply the sum of a delta distribution and the Hilbert transform. The contribution of the delta-part is harmless. Now denote $F = \hat{h}$. Then we only need to show for even function $F : \mathbb{R} \to \mathbb{R}$:

$$\|\langle x \rangle^{\frac{1}{2}}\mathcal{H}(\langle y \rangle^{\frac{1}{2}}F)\|_2 \lesssim \|F\|_2.$$

The result then follows from Lemma A.1.

**Proof of Lemma 3.1:** it is straightforward to check that

$$\hat{E}\varphi(\xi) = \hat{\varphi}(\xi) + \hat{\varphi}(-\xi), \quad \hat{O}\varphi(\xi) = \hat{\varphi}(\xi) - \hat{\varphi}(-\xi).$$

and

$$\|E\varphi\|_{H^{-1/2}(\mathbb{R})} \lesssim \|\varphi\|_{\dot{H}^{-1/2}(\mathbb{R}^+)}; \quad \|O\varphi\|_{H^{-1/2}(\mathbb{R})} \lesssim \|\varphi\|_{\dot{H}^{-1/2}(\mathbb{R}^+)}.$$

On the other hand, Lemma A.2 implies that

$$\|\varphi\|_{\dot{H}^{-1/2}(\mathbb{R}^+)} \lesssim \|E\varphi\|_{H^{-1/2}(\mathbb{R})}.$$

A similar argument as in Lemma A.1 and A.2 yields that

$$\|\varphi\|_{\dot{H}^{-1/2}(\mathbb{R}^+)} \lesssim \|O\varphi\|_{H^{-1/2}(\mathbb{R})}.$$
Appendix B  Proof of Lemma 4.8

Proof. We first prove (4.31), the proof for (4.32) follows from a similar argument. Let
\[ r(\xi) := \frac{\rho_0(\xi)\varepsilon_m}{\rho_m(\xi)} \quad \text{and} \quad w(\xi) := \frac{\rho_0(\xi)\varepsilon_m + \rho_m(\xi)}{\rho_m(\xi)} = 1 + r(\xi). \]
Without loss of generality, we assume that \( k = 1 \) and write \( \varepsilon_m = -a + bi \) where \( a, b > 0 \) and \( a > b \).

For \( \xi \in (0, 1) \), note that \( \rho_0 = i \sqrt{k^2 - \xi^2} \) and it can be shown that \( |\text{Re} \ r| \sim O(b/a) \). Thus
\[ |w| = |1 + r| \geq 1 - O(b/a) > 1/2. \]
We obtain
\[ \int_0^1 \frac{1}{|w(\xi)|^2} d\xi \lesssim 1. \quad (B.1) \]
Next, we aim to show
\[ \int_1^2 \frac{1}{|w(\xi)|^2} d\xi \lesssim \frac{1}{|b|}. \quad (B.2) \]
To this end, note that
\[ \frac{1}{\rho_m} = \frac{1}{\sqrt{\xi^2 + a - bi}} = \frac{1}{\sqrt{\xi^2 + a}} \left( 1 + \frac{b}{2(\xi^2 + a)} i + \frac{3b^2}{4a^2} + O\left(\frac{b^3}{a^3}\right) \right). \]
Let \( t = t(\xi) = \frac{\rho_0}{\rho_m} = \frac{\xi^2 - 1}{\xi^2 + a} \). Then \( 1 \leq \xi \leq 2 \) is equivalent to \( 0 \leq t \leq \frac{\sqrt{3}a}{a} \). Moreover, we have
\[ \text{Re} \ r(\xi) = t \left( -a - \frac{b}{\xi^2 + a} + O\left(\frac{b^2}{a^2}\right) \right) = -t \left( a + O\left(\frac{b^2}{a}\right) \right), \]
\[ \text{Im} \ r(\xi) = t \left( b - \frac{ab}{\xi^2 + a} + O\left(\frac{b^3}{a^2}\right) \right). \]
Therefore, \( |\text{Im} \ r(\xi)| \geq \frac{2eb}{3} \). It follows that
\[ |1 + r(\xi)|^2 \geq 1 + \text{Re} r(\xi)^2 + |\text{Im} r(\xi)|^2 \geq (1 - ta)^2 + \frac{1}{4} t^2 b^2. \]
On the other hand, since \( t = t(\xi) = \frac{\xi^2 - 1}{\xi^2 + a^2} \), we have
\[ \xi = \sqrt{\frac{at^2 + 1}{1-t^2}}. \]
A direct calculation shows that \( |\xi'(t)| \lesssim at \). Therefore
\[ \int_1^2 \frac{1}{|w(\xi)|^2} d\xi = \int_1^2 \frac{1}{(1 + |r(\xi)|)^2} d\xi \leq \int_0^{\sqrt{\frac{3}{a}}} \frac{at \, dt}{(1-at)^2 + \frac{1}{4} t^2 b^2}. \]
We now consider three regions: $I_1 = \{0 \leq t \leq \frac{1}{a+b^2}\}$, where $|1 - ta| \leq \frac{a^2}{b^2}$; $I_2 = \{\frac{1}{a+b^2} \leq t \leq \frac{a^2}{b^2}\}$, where $|1 - ta| \geq \frac{a^2}{b^2}$; and $I_3 = \{\frac{a^2}{b^2} \leq t \leq \sqrt{\frac{3}{a}}\}$, where $|1 - ta| \leq \frac{a^2}{b^2}$. We have

\[
\int_{I_1} \frac{at dt}{(1-at)^2 + \frac{1}{4}t^2b^2} \leq \int_{I_1} \frac{at dt}{(1-at)^2} = \frac{1}{a} \int_{0}^{\frac{a}{a+b^2}} \frac{tdt}{(1-t)^2} \lesssim \frac{1}{b},
\]

\[
\int_{I_2} \frac{at dt}{(1-at)^2 + \frac{1}{4}t^2b^2} \leq \int_{I_2} \frac{at dt}{t^2b^2} = \int_{I_2} \frac{at dt}{t^2b^2} \lesssim \frac{1}{b},
\]

\[
\int_{I_3} \frac{at dt}{(1-at)^2 + \frac{1}{4}t^2b^2} \leq \int_{I_3} \frac{at dt}{(1-at)^2} = \frac{1}{a} \int_{\frac{a}{a+b^2}}^{\sqrt{3a}} \frac{tdt}{(1-t)^2} \lesssim \frac{1}{b}.
\]

It follows that

\[
\int_{1}^{2} \frac{1}{|w(\xi)|^2} d\xi \lesssim \frac{1}{b},
\]

which is the desired estimate.

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References


