OPTIMAL SHAPE DESIGN OF A CAVITY FOR RADAR CROSS SECTION REDUCTION*

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Abstract. This paper focuses on the reduction of radar cross section (RCS) for a cavity embedded in an infinite ground plane. By introducing a transparent boundary condition, the unbounded scattering problem is reduced to a bounded domain problem. RCS reduction for the cavity is formulated as a shape optimization problem involving the Helmholtz equation. The existence of the minimizer is proved under an appropriate constraint. Descent directions of the objective function with respect to the boundary may be found via the domain derivative. It is used in a gradient-based optimization scheme to find the optimal shape of the cavity. Numerical examples show that the RCS is effectively reduced at different incident frequencies.

Key words. RCS reduction, shape optimization, Helmholtz equation, electromagnetic cavity

AMS subject classifications. 35Q60, 35J05, 35Q93, 78M50

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1. Introduction. Radar cross section (RCS) is an important measure for radar systems to detect a target. Intuitively, it measures how much energy reflected back compared to the incident wave. Reducing the RCS therefore is highly valuable in many industrial and military applications, for instance, stealth technology. Given the fact that RCS from a cavity can dominate the overall RCS for some objects, like the exhaust nozzles for a strike aircraft, it is important to study the RCS reduction for a cavity.

This paper considers the RCS reduction for a simplified model, which is a cavity embedded in an infinite ground plane, as shown in Figure 1. The cavity is assumed to be invariant in the $x_3$ direction. Hence, the scattering problem can be formulated as a two dimensional problem. With the incidence of a time-harmonic plane wave, the total field of the cavity is decomposed into three parts. Besides the incident field, there exist the reflected field and the scattered field. RCS measures the magnitude of the far field coefficient for the scattered field. When the observation angle is the same as the incident angle, it is called monostatic RCS or backscatter RCS. Since most radar systems are monostatic, we restrict our study to the reduction for backscatter RCS only in this paper.

A large number of studies have been done on both direct scattering and inverse scattering problems for such a cavity. Mathematical analysis for the scattering problem can be found in [13], [14], [15]. In particular, questions on the existence and uniqueness of the solution of the inverse scattering problem have been investigated in [8]. When the cavity is large and deep, the solution is highly oscillatory, which presents both a stability issue and computational challenge. The stability of the solu-

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Optimal Shape Design of a Cavity for RCS Reduction

The RCS reduction is studied in [10] for a two-dimensional rectangular cavity in the transverse magnetic (TM) polarization. In terms of numerical simulation, standard methods including finite difference [11] and finite element with boundary integral [16] have been developed. Mode matching methods have also been proposed in [7], [12]. Note that the large cavity problem is equivalent to a high wave number problem which is known to be computationally challenging. Therefore, high frequency techniques, such as bounding and shooting rays [21] and Gaussian beam asymptotics [20] were also applied to the large cavity scattering problem.

Our goal is to propose an optimization method to reduce the RCS by a proper shaping of the cavity. It is observed in [21] that using an appropriate longitudinal bending, the RCS of a cylindrical cavity was reduced significantly over a set of incident angles. In this paper, the RCS reduction is formulated as a minimization problem involving the Helmholtz equation. To reduce the open cavity problem into a bounded domain problem, an artificial boundary is introduced, which is essentially a Dirichlet to Neumann map. The minimization problem is solved by an iterative scheme. At each iteration, a forward scattering problem and an adjoint problem are solved. The solutions are then combined together to give a descent direction of the backscatter RCS. Since only the boundary data are needed, both of them are solved by the boundary integral method. The boundary of the cavity is parameterized by a summation of cosine functions, and the shape of the cavity is controlled by the amplitude of those functions. The relation between the perturbation of the amplitudes and the descent direction is obtained by solving a least squares problem. Numerical examples show the algorithm is effective in finding the optimal shape with reduced RCS at different frequencies.

The rest of the paper is organized as follows. Section 2 gives the formulation of the problem and define the objective function. In section 3, the existence of the minimizer for the optimization problem is established. Descent direction of the objective function is given in section 4 via the domain derivative. Section 5 gives the parametrization of the boundary and the optimization algorithm that leads to the optimal shape of the cavity. Numerical examples are provided in section 6 to illustrate the effectiveness of the algorithm. Section 7 concludes the whole discussion.

2. Model problem. Consider a cavity Ω embedded in an infinite ground plane \( \Gamma_c \), as shown in Figure 1. The opening of the cavity is denoted as \( \Gamma \). The cavity is assumed to be filled with the same material as the upper half space, which is vacuum. Thus the dielectric coefficient \( \varepsilon \) and magnetic permeability \( \mu \) are the same everywhere. We focus on the two dimensional problem by assuming that the cavity has no variance in the \( x_3 \) direction. There are two fundamental polarizations for the EM wave prop-

![Fig. 1. Geometry of the problem.](image-url)
agation: TM polarization and TE (transverse electric) polarization. For simplicity, we consider the TM polarization only, which is the case when the magnetic field is transverse to the invariant direction and the electrical field $E = (0, 0, u(x_1, x_2))$. The boundary of the cavity $\Omega$ is composed of a piecewise smooth curve. Denote $S_1$, $S_2$, and $S_3$ as the left, right, and bottom boundary, respectively and let $S = S_1 \cup S_2 \cup S_3$. The ground plane $\Gamma_c$ and the boundary $S$ are assumed to be perfect conductors, so the total field $u$ satisfies the following boundary condition:

$$u = 0 \text{ on } S \cup \Gamma_c. \quad (2.1)$$

Assume the incident electrical field is a time harmonic plane wave

$$u^i = e^{i(\alpha x_1 - \beta x_2)},$$

where $\alpha = k \cos \theta$, $\beta = k \sin \theta$, $k^2 = \omega^2 \varepsilon \mu$ is the wavenumber of the empty space with frequency $\omega$, and $\theta$ is the incident angle with respect to the positive $x_1$ direction. Following the formulation in [14], the total field $u$ after the illumination takes the form

$$u = u^s + u^i + u^r, \quad (2.2)$$

where $u^s$ is the scattered field and $u^r = -e^{i(\alpha x_1 + \beta x_2)}$ is the reflected field caused by the ground plane. Denote $R_+^2$ as the upper half-space. The equation for the total field is

$$\begin{cases} \Delta u + k^2 u = 0 \text{ in } \Omega \cup R_+^2, \\ u = 0 \text{ on } S \cup \Gamma_c, \\ u^s = \phi \text{ on } \Gamma_c, \end{cases} \quad (2.3)$$

together with the Sommerfeld radiation condition

$$\lim_{r \to \infty} \sqrt{r} \left( \frac{\partial u^s}{\partial r} - ik u^s \right) = 0, \quad (2.4)$$

where $r = \sqrt{x_1^2 + x_2^2}$.

Since to analyze the problem (2.3) directly is difficult due to the unboundedness of the domain, we introduce a transparent boundary condition on $\Gamma$ to reduce the unbounded problem into a bounded one. The idea of a transparent boundary condition, which is essentially the Dirichlet to Neumann map, has been studied in many papers [8], [14], [15]. The boundary condition can be derived either by the Green’s function method [4] or by the Fourier transform [11]. Here we give the derivation based on the Fourier transform.

Since $u^i + u^r = 0$ on $\Gamma \cup \Gamma_c$, the scattered field satisfies the following equation in the upper half-space:

$$\begin{cases} \Delta u^s + k^2 u^s = 0 \text{ in } R_+^2, \\ u^s = 0 \text{ on } \Gamma_c, \\ u^s = \phi \text{ on } \Gamma, \end{cases} \quad (2.5)$$

along with the radiation condition (2.4), where $\phi$ is assumed to be known.

Take the Fourier transform of the Helmholtz equation in (2.5) with respect to $x_1$ and denote the Fourier transform of $u^s$ as $\hat{u}^s$,

$$\frac{\partial^2}{\partial x_2^2} \hat{u}^s + (k^2 - \xi^2) \hat{u}^s = 0 \quad \text{for } x_2 > 0. \quad (2.6)$$

It admits two linearly independent solutions. We choose the one that will satisfy the radiation condition and represent the “outgoing” wave,

$$\hat{u}^s = e^{i \sqrt{k^2 - \xi^2} x_2} \hat{u}^s(\xi, 0). \quad (2.7)$$
After the inverse Fourier transform, we obtain the identity
\begin{equation}
(2.8)
\tilde{u}^s = \frac{1}{2\pi} \int \frac{1}{R} e^{i\sqrt{k^2 - \xi^2} x_2} \tilde{u}^s(\xi, 0) e^{i\xi x_1} d\xi.
\end{equation}

Taking the normal derivative of both sides at \( x_2 = 0 \) yields the transparent boundary condition for the scattered field
\begin{equation}
(2.9)
\frac{\partial u^s}{\partial n} \big|_{x_2=0} = I(u^s),
\end{equation}
where the normal direction \( n \) is towards the upper half-space and the boundary operator is defined by
\begin{equation}
(2.10)
I(f) = \frac{i}{2\pi} \int \frac{1}{R} \sqrt{k^2 - \xi^2} \hat{f}(\xi, 0) e^{i\xi x_1} d\xi.
\end{equation}

Using the identity (2.2), we can derive the corresponding boundary condition for the total field \( u(x_1, x_2) \). Since on the ground plane it holds that \( u^i + u^r = 0 \), the same boundary operator \( I \) can be applied to the total field \( u \) at \( x_2 = 0 \). The resulting boundary condition for \( u \) satisfies
\begin{equation}
(2.11)
\frac{\partial u}{\partial n} = I(u) + g \quad \text{on} \quad \Gamma,
\end{equation}
where \( g = -2i\beta e^{i\alpha x_1} \) is the summation of the normal derivative of the incident and reflected fields. Now the equation for the total field \( u(x_1, x_2) \) becomes
\begin{equation}
(2.12)
\begin{cases}
\quad u + k^2 u = 0 \quad \text{in} \quad \Omega, \\
\quad u = 0 \quad \text{on} \quad S, \\
\quad \frac{\partial u}{\partial n} = I(u) + g \quad \text{on} \quad \Gamma.
\end{cases}
\end{equation}

For further discussions on the properties of the boundary operator \( I \), we define the space
\begin{equation}
H^{1/2}_{00} (\Gamma) = \{ u \in H^{1/2}(\Gamma) : \exists \tilde{u} \in H^{1/2}(R) \text{ such that } \tilde{u} = 0 \text{ on } \Gamma^c \text{ and } u = \tilde{u}|_{\Gamma} \}.
\end{equation}

In this setting, we call \( \tilde{u} \) the zero extension of \( u \) to \( H^{1/2}(R) \).

The scattering problem (2.12) has an equivalent weak formulation: Find \( u \in \tilde{H}^1_0 (\Omega) = \{ \phi \in H^1(\Omega), \phi = 0 \text{ on } S, \phi|_{\Gamma} \in H^{1/2}_{00} (\Gamma) \} \) such that
\begin{equation}
(2.13)
a(u, v) = \int_{\Gamma} g \overline{v} ds \quad \forall v \in \tilde{H}^1_0 (\Omega).
\end{equation}

Here the bilinear form is defined by
\begin{equation}
(2.14)
a(u, v) = \int_{\Omega} \nabla u \cdot \nabla \overline{v} - \int_{\Omega} k^2 u \overline{v} - \int_{\Gamma} I(\tilde{u}) \overline{v} ds.
\end{equation}

The proof of uniqueness and existence for the solution of (2.13) can be found in [14]. In this paper, we focus on the RCS reduction for the cavity. In the two dimensional case, the RCS is defined by:
\begin{equation}
(2.15)
\sigma(\varphi) = \lim_{r \to \infty} 2\pi r \frac{|u^s(r, \varphi)|^2}{|u^i|^2},
\end{equation}
where \( r = \sqrt{x_1^2 + x_2^2} \) and \( \varphi \) is the observation angle.

If the observation angle coincides with the incident angle, it is called the backscatter RCS or monostatic RCS. Since most radar systems are monostatic [6], we only focus on the reduction of the backscatter RCS in this paper. In order to evaluate the RCS, we need to find the far field pattern of the scattered field. Since we assume the
whole upper space is a vacuum, the far field pattern can be evaluated through the field at the opening of the cavity. In particular, for TM polarization, the backscatter RCS can be computed by:

\[
\sigma := 4 \kappa \frac{k}{k^2} \sin \theta \int_{\Gamma} \! \! \! e^{ikx_1 \cos \theta} \, dx \bigg|_{x_1 = 2}.
\]

(2.16)

We next introduce the optimal design problem. The idea is to change the shape of the cavity so that the backscatter RCS is minimized. It is easy to see that the design problem is trivial if no constraint is imposed, since the RCS is zero for any angle and any frequency if no cavity exists. Therefore, we impose the following constraints on the cavity:

- The opening \( \Gamma \) of \( \Omega \) is fixed;
- the area of \( \Omega \) is invariant;
- \( \Omega \) is bounded in a set \( D \).

In particular, as shown in Figure 2, there exist two fixed regions \( K \) and \( D \) as the lower bound and upper bound of the cavity. In order to satisfy the constraints, we choose to design the cavity in the following way.

The left boundary \( S_1 \) of \( \Omega \) can be parameterized by a \( C^2 \) function with respect to \( x_2 \). The right boundary \( S_2 \) is kept parallel to \( S_1 \). To satisfy the area invariance, the depth of the cavity is assumed to be fixed, and the bottom of \( \Omega \) is simply a line segment with the end points connecting \( S_1 \) and \( S_2 \).

More specifically, \( S_1 \) is parameterized by \( f(x_2) \), where \( f(x_2) \) is a \( C^2 \) function chosen from a compact subset with respect to \( C^{1, \beta} \), \( 0 \leq \beta < 1 \). Denote the admissible set as \( \Lambda \):

\[
\Lambda = \{ f \in C^2([-h, 0]) \mid f(0) = 0, (f(x_2), x_2) \in D, (f(x_2), x_2) \notin K, \| f \|_{C^2} \leq C \},
\]

(2.17)

where \( h \) is the depth of the cavity, \( C \) is a fixed positive constant, and the norm \( \| \cdot \|_{C^2} \) is the usual maximal norm up to the second derivative, namely,

\[
\| f \|_{C^2} = \sum_{i=0}^{2} \max_{x \in [-h, 0]} |f^{(i)}(x)|.
\]

(2.18)

The compactness of the admissible set is used to ensure the convergence of the solution. We also denote the width of the opening \( \Gamma \) by \( l \).
Given the incident angle, our goal is to solve the following minimization problem:

\begin{equation}
\min_{S_1 \in \Lambda} \sigma(S_1).
\end{equation}

The existence of the minimizer for the objective functions follows from Theorem 2.1.

**Theorem 2.1.** There exists at least one minimizer for the problem (2.19) in \( \Lambda \).

The theorem will be proved in the following section. Note that we only give the existence of the minimizer. Whether the minimizer is unique or not is still open; nevertheless, numerical experiments show that there is generally more than one minimizer.

### 3. Existence of the minimizer.

The existence of the minimizer is based on the continuous dependence of the solution \( u \) to (2.13) on the boundary \( S_1 \). We first define the mapping

\begin{equation}
F : S_1 \rightarrow \frac{u}{\Gamma}.
\end{equation}

As the proof for the well-posedness of the solution \( u \), the property of the boundary operator \( I(\cdot) \) plays an important role in the continuity argument. We begin by recalling a lemma in [14].

**Lemma 3.1.** The boundary operator \( I : H_{00}^{1/2}(\Gamma) \rightarrow H^{-1/2}(\Gamma) \) is continuous, where \( H^{-1/2}(\Gamma) \) is denoted as the dual space of \( H_{00}^{1/2}(\Gamma) \).

The following theorem shows that the mapping \( F \) is continuous with respect to \( S_1 \). A similar result for the obstacle scattering problem can be found in [22].

**Theorem 3.2.** The mapping \( F : S_1 \rightarrow \frac{u}{\Gamma} \) is continuous from \( C^1 \) into \( H_{00}^{1/2}(\Gamma) \).

**Proof.** For \( u|_{\Gamma} \in H_{00}^{1/2}(\Gamma) \), a direct calculation shows:

\begin{equation}
-\Re\int_{\Gamma} I(\tilde{u})\overline{u} = \Im \int_{\mathbb{R}} \sqrt{k^2 - \xi^2} |\tilde{u}(\xi)|^2 d\xi \geq 0.
\end{equation}

Therefore, there exist positive constants \( C_1, C_2 \) such that

\begin{equation}
\Re\{a(u, u)\} = \int_{\Omega} \nabla u \cdot \nabla \overline{u} - \int_{\Omega} k^2 u \overline{u} - \Re\int_{\Gamma} I(\tilde{u})\overline{u} ds \geq C_1 \|
abla u\|_{L^2(\Omega)}^2 - C_2 \|u\|_{H_{00}^{1/2}(\Omega)}^2.
\end{equation}

Hence, by the Lax–Milgram lemma and the Fredholm alternative, the uniqueness of the scattering problem [14] implies the existence and the continuous dependence of the solution on the right-hand side. Define the operator \( B : \tilde{H}_{0}^{1}(\Omega) \rightarrow H^{-1}(\Omega) \), where \( H^{-1}(\Omega) \) is the dual space of \( \tilde{H}_{0}^{1}(\Omega) \), such that

\begin{equation}
\langle Bu, v \rangle = a(u, v).
\end{equation}

Here \( \langle \cdot, \cdot \rangle \) is the dual system between \( H^{-1}(\Omega) \) and \( \tilde{H}_{0}^{1}(\Omega) \).

From the discussion above, the operator \( B \) is invertible and holds

\begin{equation}
\|u|_{\Gamma}\|_{H_{00}^{1/2}(\Gamma)} \leq \|u\|_{\tilde{H}_{0}^{1}(\Omega)} \leq \|B^{-1}\||g\|_{H^{-1/2}(\Gamma)}.
\end{equation}

Define the transformation

\begin{equation}
(y_1(x), y_2(x)) = (x_1 - f(x_2), x_2) \quad \text{for} \quad (x_1, x_2) \in \Omega.
\end{equation}
where \( f(x_2) \) is the parametrization of \( S_1 \). It maps the cavity \( \Omega \) onto the rectangle \( R = [0, l] \times [-h, 0] \). Then the bilinear form can be rewritten as:

\[
(3.7) \quad a(u, v) = \int_R \left\{ \sum_{j,k=1}^2 b_{jk} \frac{\partial u}{\partial y_j} \frac{\partial \tau}{\partial y_k} - k^2 u \tau \right\} J - \int_{\Gamma} I(\tilde{u}) \tau ds,
\]

where \( J \) denotes the Jacobian of the transformation and the coefficients \( b_{jk} \) are given by

\[
(3.8) \quad b_{jk} = \sum_{i=1}^2 \frac{\partial y_j}{\partial x_i} \frac{\partial y_k}{\partial x_i}.
\]

Since \( f(x_2) \in C^1 \), the coefficients in (3.8) are well defined. Denote the \( C^1 \) norm of \( f(x_2) \) as \( ||f||_{C^1} \). Then \( b_{jk} \) depends continuously on \( ||f(x_2)||_{C^1} \) and the Jacobian \( J \) is identically one, hence the bilinear form \( a \) also depends continuously on the \( C^1 \) norm of \( f(x_2) \). Let \( B_f \) denote the operator \( B \) that depends on \( f \). Using the perturbation argument on the Neumann series, a small perturbation \( \delta f \) to \( f \) results in

\[
(3.9) \quad ||B_f^{-1} - B_{f+\delta f}^{-1}|| \leq C_1(f)||\delta f||_{C^1},
\]

where \( C_1 \) is a constant depending on \( f \). Therefore, from (3.5), there exists another constant \( C_2 \) such that

\[
(3.10) \quad ||u_f||_\Gamma - ||u_{f+\delta f}||_\Gamma \leq C_2(f)||\delta f||_{C^1}.
\]

The proof is now complete. \( \square \)

Now we are ready to give the proof of Theorem 2.1 for the existence of the minimizer.

\textbf{Proof.} Since the admissible set \( \Lambda \) is a compact set with respect to \( C^{1,\beta}, 0 \leq \beta < 1 \), the existence result immediately follows from Theorem 3.2 and the fact that \( \sigma \) depends continuously on \( u_1 \). \( \square \)

We remark that the proof is given for the specific admissible set. For a more general admissible set, such as nonsmooth curves for \( S_1 \), the existence of the minimizer will be more involved. We next proceed to give the derivation of the domain derivative, which leads to the descent directions for the objective function.

\section{Domain derivative.} Domain derivatives have been studied extensively for various problems, including shape optimization problems [2, 17, 19], inverse scattering problems [1], and the image of local surface displacement [9]. The general idea is based on the transformation of coordinates. The perturbed domain is mapped back to the original domain, so that the difference at the boundary is transformed into the difference at the coefficients in the variational equation. Rigorous calculations on the coefficients lead to the equation for the domain derivative.

Since the boundary of the cavity is not globally smooth, a domain derivative is not guaranteed to exist for a general perturbation. We restrict our discussion to a smooth perturbation of the cavity. More specifically, let \( V(x) \) be a vector field defined on \( S_1 \) such that \( V(x) = (s(x_2), 0) \in C^2_0(S_1; R^2) \), where \( C^2_0(S_1; R^2) \) denotes the twice differentiable vector fields on \( S_1 \) with \( s(0) = 0 \). For such a fixed vector field \( V(x) \), define

\[
(4.1) \quad S_1^d = \{ x + \delta V(x)| x \in S_1, V(x) \in C^2_0(S; R^2) \}
\]
as the perturbation of $S_1$ with respect to $V(x)$. Since $S_2$ is parallel to $S_1$, $V(x)$ is also defined on $S_2$ by a horizontal shift. To define the domain derivative, extend the definition of $V(x)$ to every point in $\Omega$ in a way that for $(x_1, x_2) \in \Omega$, $V(x) = (s(x_2), 0)$. Denote $\Omega_\delta$ as the perturbed cavity with respect to $\Omega$ along the direction $V$. Let $u^\delta \in H_0^1(\Omega_\delta)$ be the corresponding solution for the total field in $\Omega_\delta$. Let $T$ denote the map from $\Omega$ to $\Omega_\delta$, and $\tilde{u}^\delta = u^\delta \circ T \in H_0^1(\Omega)$. The domain derivative of the forward mapping $F$ with respect to the direction $V(x)$ is defined as

\[
F'(S_1) := \lim_{\delta \to 0} \frac{\tilde{u}^\delta|_\Gamma - u|_\Gamma}{\delta}.
\]

The limit of (4.2) exists in the sense of $H_{00}^{1/2}(\Gamma)$, as shown in the following theorem.

**Theorem 4.1.** Let $u$ be the solution of (2.3). If $S_1$ is $C^2$ and $V(x) \in C_0^2(S_1; R^2)$, then the domain derivative $F'(S_1)$ exists in $H_{00}^{1/2}(\Gamma)$. Moreover, $F'(S_1) = u'|_{\Gamma}$, where $u' \in H^1(\Omega)$ solves

\[
\begin{aligned}
\Delta u' + k^2 u' &= 0 & \text{in } \Omega, \\
u' &= -(V \cdot n) \frac{\partial^\delta}{\partial n} & \text{on } S_1 \cup S_2, \\
u' &= 0 & \text{on } S_3, \\
\frac{\partial u'}{\partial n} &= I(\tilde{u}') & \text{on } \Gamma.
\end{aligned}
\]

**Proof.** Assume $V(x_1, x_2) = (s(x_2), 0)$. For a small number $\delta$, let $\Omega_\delta$ denote the perturbed cavity with respect to $\Omega$ along the direction $V$. The total field $u^\delta$ satisfies the following equation:

\[
\begin{aligned}
u^\delta + k^2 u^\delta &= 0 & \text{in } \Omega_\delta, \\
u^\delta &= 0 & \text{on } S_i, \\
\frac{\partial u^\delta}{\partial n} &= I(\tilde{u}^\delta) + g & \text{on } \Gamma.
\end{aligned}
\]

with the weak form

\[
\int_{\Omega_\delta} \nabla u^\delta \cdot \nabla v - \int_{\Omega_\delta} k^2 u^\delta v - \int_{\Gamma} I(\tilde{u}^\delta)vds = \int_{\Gamma} gvd\Gamma, \forall v \in H_0^1(\Omega_\delta).
\]

Define the mapping $g$,

\[
g(y_1, y_2) = (x_1 - \delta s(x_2), x_2),
\]

which maps $\Omega_\delta$ back to $\Omega$. Let $\tilde{u}^\delta = u^\delta \circ q^{-1}$ and $\tilde{v} = v \circ q^{-1}$. The weak form in (4.5) can be rewritten as:

\[
\int_{\Omega} \left( \sum_{j,k=1}^{2} b_{jk} \frac{\partial \tilde{u}^\delta}{\partial y_j} \frac{\partial \tilde{\pi}}{\partial y_k} - k^2 \tilde{u}^\delta \tilde{\pi} \right)J - \int_{\Gamma} I(\tilde{u}^\delta)\tilde{v} = \int_{\Gamma} g\tilde{v}ds,
\]

where $b_{jk}$ has the same definition as (3.8). Here, the $J$ is the Jacobian for the transformation $q$, which is one. Define the new bilinear form

\[
a^\delta(\tilde{u}, v) = \int_{\Omega} \sum_{j,k=1}^{2} b_{jk} \frac{\partial \tilde{u}}{\partial y_j} \frac{\partial \tilde{\pi}}{\partial y_k} - k^2 \tilde{u} \tilde{\pi} - \int_{\Gamma} I(\tilde{u})\tilde{v},
\]

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where \( \tilde{u}, v \in \tilde{H}^1_0(\Omega) \). In particular, \( \tilde{u}^\delta \) in (4.7) is the unique solution of

\[
\tag{4.9}
\forall v \in \tilde{H}^1_0(\Omega), a^\delta(\tilde{u}^\delta, v) = \int_B g v ds.
\]

Assume \( u \) is the solution of (2.13); it holds that

\[
\tag{4.10}
a(\tilde{u}^\delta - u, v) = a(\tilde{u}^\delta, v) - a^\delta(\tilde{u}^\delta, v) = \int_\Omega \nabla \tilde{u}^\delta \cdot \nabla v - \int_\Omega \sum_{j,k=1}^2 b_{jk} \frac{\partial \tilde{u}^\delta}{\partial y_j} \frac{\partial v}{\partial y_k}.
\]

Extend the definition of \( V(x) \) to \( \Omega \) by the following:

\[
\tag{4.11}
V(x) = (s(x_2), 0) \quad \text{for} \quad (x_1, x_2) \in \Omega.
\]

It is easy to see that the matrix

\[
\tag{4.12}
(b_{jk})_{2 \times 2} = I - \delta(\nabla V + (\nabla V)^T) + O(\delta^2),
\]

where \( I \) is the 2 \( \times \) 2 identity matrix. In order to find the limit of \( \frac{\delta u^\delta}{\delta} - u \), we begin by proving that:

\[
\tag{4.13}
\| \tilde{u}^\delta - u \|_{\tilde{H}^1_0(\Omega)} \leq C|\delta|\| u \|_{\tilde{H}^1_0(\Omega)},
\]

where \( C \) is independent of \( \delta \) and \( u \). From (4.10) and (4.12), it holds

\[
\tag{4.14}
a(\tilde{u}^\delta - u, v) = \delta \int_\Omega (\nabla \tilde{u}^\delta)^T (\nabla V + (\nabla V)^T) \nabla v + O(\delta^2).
\]

As we mentioned in Theorem 3.2, the operator \( B \) associated with the quadratic form \( a \) has a bounded inverse from \( H^{-1}(\Omega) \) to \( \tilde{H}^1_0(\Omega) \). We therefore try to estimate the \( H^{-1}(\Omega) \) norm of the first term on the right-hand side of (4.14), i.e.,

\[
\tag{4.15}
\sup_{v \in \tilde{H}^1_0(\Omega)=1} \left( \int_\Omega (\nabla \tilde{u}^\delta)^T (\nabla V + (\nabla V)^T) \nabla v \right).
\]

By Hölder’s inequality, for any \( v \in \tilde{H}^1_0(\Omega) \), it satisfies

\[
\delta \int_\Omega (\nabla \tilde{u}^\delta)^T (\nabla V + (\nabla V)^T) \nabla v \\
\leq 2|\delta| \left( \max_{\Omega} |\nabla V| \right) \int_\Omega |(\nabla \tilde{u}^\delta)^T \nabla v| dx \\
\leq C|\delta| \cdot \| \tilde{u}^\delta \|_{\tilde{H}^1_0(\Omega)} \| v \|_{\tilde{H}^1_0(\Omega)}.
\]

With (4.16), the boundedness of \( \tilde{u}^\delta - u \) follows from the continuous dependence, which implies

\[
\tag{4.17}
\| \tilde{u}^\delta - u \|_{\tilde{H}^1_0(\Omega)} \leq C|\delta| \| \tilde{u}^\delta \|_{\tilde{H}^1_0(\Omega)}.
\]

For sufficiently small \( \delta \), (4.17) implies (4.13). Let \( u_0 \) solve the following problem

\[
\tag{4.18}
a(u_0, v) = \int_\Omega \nabla u_0^T (\nabla V + (\nabla V)^T) \nabla v.
\]
Due to the zero boundary condition for $u$, the divergence theorem and the fact that $\nabla \cdot V = 0$, it holds

$$\begin{align*}
\nabla u^T (\nabla V + (\nabla V)^T) \nabla \sigma &= \nabla (V \cdot \nabla \sigma) \cdot \nabla u + \nabla (V \cdot \nabla u) \cdot \nabla \sigma - \nabla \cdot [(\nabla u \cdot \nabla \sigma) V].
\end{align*}$$

Using Green’s formula and the fact that $V = 0$ on $\Gamma$, we have

$$a(u_0, v) = \int_{\Omega} - (V \cdot \nabla \sigma) \Delta u + \nabla (V \cdot \nabla u) \cdot \nabla \sigma + \int_{\partial \Omega} (V \cdot \nabla \sigma) \frac{\partial u}{\partial n} - (\nabla u \cdot \nabla \sigma) V \cdot nds.$$

Since $u = v = 0$ on $S$, it follows

$$\begin{align*}
(V \cdot \nabla \sigma) \frac{\partial u}{\partial n} - (\nabla u \cdot \nabla \sigma) V \cdot n = 0 \text{ on } S.
\end{align*}$$

Based on the equation for $u$, $a(u_0, v)$ can be rewritten as

$$a(u_0, v) = \int_{\Omega} k^2 u (V \cdot \nabla \sigma) + \nabla (V \cdot \nabla u) \cdot \nabla \sigma.$$

The divergence theorem and $\nabla \cdot V = 0$ yield that

$$a(u_0, v) = \int_{\Omega} k^2 u (V \cdot \nabla \sigma) + \nabla (V \cdot \nabla u) \cdot \nabla \sigma + k^2(\nabla \cdot V) u \nabla \sigma$$

$$= \int_{\Omega} \nabla (V \cdot \nabla u) \cdot \nabla \sigma - k^2(V \cdot \nabla u) \nabla \sigma.$$

The above identity implies $u_0$ satisfies

$$\begin{align*}
\begin{cases}
    u_0 + k^2 u_0 = (\Delta + k^2)(V \cdot \nabla u) & \text{in } \Omega, \\
    u_0 = 0 & \text{on } S, \\
    \frac{\partial u_0}{\partial n} = I(\bar{u}_0) & \text{on } \Gamma.
\end{cases}
\end{align*}$$

Let $u' = u_0 - V \cdot \nabla u$. Then $u'$ satisfies the equation:

$$\Delta u' + k^2 u' = 0 \text{ in } \Omega.$$

Due to the zero boundary condition for $u$ on $S_1 \cup S_2 \cup S_3$, the tangential derivative for $u$ is zero on the boundary. Hence, on $S_1 \cup S_2$, it holds $u' = -(V \cdot n) \partial_n u$. On $S_3$, we have $u' = 0$ since $V \cdot n = 0$. The boundary condition for $u'$ on $\Gamma$ can be derived similarly by using the fact that $V = 0$ and $V \cdot n = 0$ on $\Gamma$. We conclude that $u'$ satisfies (4.3).
In particular, since \( V = 0 \) on \( \Gamma \), we have

\[
(4.24) \quad u'|_{\Gamma} = u_0|_{\Gamma} = \lim_{\delta \to 0} \frac{\tilde{u}^\delta - u}{\delta}|_{\Gamma} = \mathcal{F}'(S).
\]

According to (4.20), the limit exists in \( H^{1/2}_{00}(\Gamma) \). The proof is complete. \( \square \)

Remark. The domain derivative can be obtained for more general vector fields (i.e., nonzero component in the vertical direction) following the same idea.

As the following theorem shows, the domain derivative offers a way to find the descent direction of the objective function (2.19), which is crucial for an optimization algorithm.

**Theorem 4.2.** For \( S_1 \in C^2 \), let \( u \) be the solution of the forward scattering problem (2.12) and \( v \) be the solution of the following boundary value problem:

\[
(4.25) \quad \begin{cases} 
\Delta v + k^2 v = 0 & \text{in } \Omega, \\
v = 0 & \text{on } S, \\
\frac{\partial v}{\partial n} = I^*(\tilde{v}) + 2k \sin^2 \theta e^{-ikx' \cos \theta} \int_\Gamma u e^{ikx' \cos \theta} ds' & \text{on } \Gamma,
\end{cases}
\]

where \( I^* \) is the adjoint operator of \( I \). If \( V(x) \in C^2_0(S_1; \mathbb{R}^2) \) satisfies that

\[
(4.26) \quad \int_{S_1} (V(x) \cdot n) R e \left( \frac{\partial u}{\partial n} \frac{\partial v}{\partial n} \right) ds + \int_{S_2} (V(x) \cdot n) R e \left( \frac{\partial u}{\partial n} \frac{\partial v}{\partial n} \right) ds < 0,
\]

then \( \frac{d\sigma(S_1)}{d\delta}|_{\delta=0} = \lim_{\delta \to 0} \frac{\sigma(S_1') - \sigma(S_1)}{\delta} < 0 \). Here \( V(x) \) is also defined on \( S_2 \) by a horizontal shift.

**Proof.** Assume \( u' \) solves the equation (4.3). Apply Green’s theorem

\[
(4.27) \quad \int_\Omega \nabla u' + k^2 u' = \int_\Omega \nabla u \frac{\partial u'}{\partial n} ds - \int_{\partial \Omega} \frac{\partial \nabla u'}{\partial n} u' ds + \int_\Omega \Delta u' + k^2 u'.
\]

Since both \( u' \) and \( \nabla u \) satisfy the Helmholtz equation, the above equation implies

\[
(4.28) \quad \int_{\partial \Omega} \frac{\partial u'}{\partial n} ds = \int_{\partial \Omega} \frac{\partial \nabla u'}{\partial n} u' ds.
\]

Recalling the boundary condition for \( u' \) and \( \nabla u \), we have

\[
(4.29) \quad \int_{\partial \Omega} \frac{\partial u'}{\partial n} ds = \int_\Gamma \nabla I(\tilde{u}') ds
\]

and

\[
\int_{\partial \Omega} \frac{\partial \nabla u'}{\partial n} ds = \int_\Gamma I'(\tilde{v}) u' ds + \int_\Gamma 2k \sin^2 \theta e^{ikx' \cos \theta} u' \int_\Gamma u e^{ikx' \cos \theta} ds' ds' \\
- \int_{S_1 + S_2} (V(x) \cdot n) \frac{\partial u}{\partial n} \frac{\partial \nabla u'}{\partial n} ds.
\]

The property of the adjoint operator implies

\[
(4.30) \quad \int_\Gamma \nabla I(\tilde{u}') ds = \int_\Gamma I'(\tilde{v}) u' ds.
\]
Combining (4.28) and (4.30), we obtain
\[
(4.31) \quad \int_{\Gamma} 2k \sin^2 \theta e^{ikx \cos \theta} u f'(x) \frac{d\sigma}{ds} = \int_{S^1 + S^2} (V(x) \cdot n) \frac{\partial \varphi}{\partial n} \, ds.
\]

The conclusion follows immediately from the fact that the real part of the left-hand side is exactly \( \frac{d\varphi(S)}{ds} \bigg|_{\delta=0} \).

For brevity, let us denote
\[
\varphi_1(x) = \text{Re} \left( \frac{\partial u}{\partial n} \right) \bigg|_{S_1},
\]
\[
\varphi_2(x) = \text{Re} \left( \frac{\partial u}{\partial n} \right) \bigg|_{S_2},
\]
where \( x = (f(x_2), x_2) \). Since \( S_1 \) and \( S_2 \) are parallel to each other, according to the theorem, the descent direction can be chosen to satisfy:
\[
(5.1) \quad V(x) \cdot n = \varphi_2(x) - \varphi_1(x),
\]
where \( n = \left( \frac{-1}{\sqrt{f'^2(x_2) + 1}} \right) \) is the unit out normal for \( S_1 \). Then the RCS may be reduced after a small perturbation to the cavity along the direction \( V(x) \). However, such a \( V(x) \) may not be in \( C^2(S_1; R^2) \). In order to keep the perturbed boundary in the admissible set, our idea is to project \( V(x) \) into a given vector space by solving a least squares problem, and then move the boundary along the projected direction and expect the RCS to decrease along that direction. Details will be given in the following section.

5. Optimization algorithm. Without loss of generality, assume the depth of the cavity \( h = \pi \). We choose to represent the boundary by the finite Fourier series
\[
(5.2) \quad f(x_2) = \sum_{j=1}^{M} \beta_j (\cos j x_2 - 1), \quad x_2 \in [-\pi, 0],
\]
where \( \beta \) belongs to the admissible set \( U_M \)
\[
U_M = \{ \beta \in R^M : \rho_{1j} \leq \beta_j \leq \rho_{2j}, \quad j = 1, 2, \ldots, M \}.
\]

Here we drop the sine modes since cosine modes are already sufficient to be a base for a function \( f \in C^2[-\pi, 0] \) when \( M \to \infty \) if we add a technical assumption that \( f'(0) = f'(-\pi) = 0 \). In that case, \( f \) can be extended to a \( 2\pi \) periodic \( C^2 \) function by an even extension over \([-\pi, \pi]\), and all the sine modes will vanish in the Fourier series. In (5.2), the additional term “-1” for each cosine function is used to guarantee \( f(0) = 0 \).

This kind of parametrization can be seen in [21] for the design of a cylindrical cavity using longitudinal bending with \( M = 1 \). It is also a common parametrization for the boundary of the obstacle in the setting of inverse scattering problems [22]. Compared to directly controlling the value on \( f \) with uniform discretization, controlling the coefficients can avoid high oscillations for the output boundary. It is easy to see that, by imposing small changes for large \( j \) in the admissible set \( U_M \), we can obtain a smooth curve with low oscillation, which is more useful from the practical point of view.
Assume we perturb the coefficients $\beta$ by adding $\delta \beta = \{\delta \beta_1, \delta \beta_2, \ldots, \delta \beta_M\}$. By linearity, the corresponding change to $f(x_2)$ is:

$$\delta f = \sum_{j=1}^{M} \delta \beta_j (\cos j x_2 - 1).$$

(5.3)

From the discussion above, $\delta f$ can be chosen based on (4.32) in order to get the RCS reduced. We therefore solve the least squares problem

$$\min_{\delta \beta \in \mathbb{R}^M} \int_{-h}^{h} \left( \frac{\delta f(x_2)}{\sqrt{f^2(x_2) + 1}} - (\varphi_2(f(x_2), x_2) - \varphi_1(f(x_2), x_2)) \right)^2 \, dx_2.$$ 

(5.4)

The equation can be solved by any linear solver after finite discretization. If $\beta^{new}$ is the perturbed coefficients based on $\delta \beta$, it is necessary for $\beta^{new}$ to satisfy the admissible condition. This can be accomplished by projecting $\beta^{new}$ back into the admissible set $U_M$. Let $\beta = \beta^{old} + \alpha \delta \beta$ for $\alpha \in (0, 1]$. Then $\beta^{new} = P(\beta)$, which is defined as the following:

$$\beta^{new}_j = \begin{cases} 
\rho_{1j} & \text{if } \beta_j < \rho_{1j}, \\
\rho_{2j} & \text{if } \beta_j > \rho_{2j}, \\
\beta_j & \text{otherwise}
\end{cases} \quad \text{for } j = 1, 2, \ldots, M.
$$

Here we do not try to find the optimal step length $\alpha$ due to the computational cost. Instead, we use the bisection search on the interval $[0, 1]$ to find a step length that can reduce the RCS, and then we continue to the next iteration. This algorithm follows the general idea in [5] for the optimal design of periodic antireflective structures.

We now state the minimization algorithm:

- Choose an initial value $\beta^0 \in U_M$, set $m = 0$.
- for $m = 1, 2, \ldots$, until stop
  - Solve the forward problem (2.13), adjoint problem (4.25), and evaluate $\varphi_1(x)$ and $\varphi_2(x)$. Solve the least squares problem (5.4) to obtain $\delta \beta$.
  - if $\|\delta \beta\| < \epsilon$, stop;
  - else let $\alpha = 1$, while $\alpha > \epsilon$ do
    - $\beta^{m} = P(\beta^{m-1} + \alpha \delta \beta)$;
    - if $\sigma(S_1(\beta^{m})) < \sigma(S_1(\beta^{m-1}))$, stop while;
    - $\alpha \leftarrow \alpha/2$;
  - end.
- end.

To solve the scattering problem for a cavity, since only the boundary data are needed during the optimization, we thus choose the boundary integral method. In particular, for the forward scattering problem, we solve the equation

$$\frac{1}{2} u(x) = - \int_S G \partial_n u|_s + \int_\Gamma \partial_n Gu|_\Gamma - \int_\Gamma GI(u|_\Gamma) - \int_\Gamma Gg \quad \forall x \in \Gamma,$$

(5.5)

$$\int_S G \partial_n u|_s = \int_\Gamma \partial_n Gu|_\Gamma - \int_\Gamma GI(u|_\Gamma) - \int_\Gamma Gg \quad \forall x \in S,$$

where $G(\rho, \rho') = -\frac{i}{4} H_0^1(|k_0| \rho - |\rho'|)$. For the adjoint problem, we simply replace $g$ by

$$2k \sin^2 \theta e^{ikx \cos \theta} \int_\Gamma u e^{ikx' \cos \theta} \, ds'.
$$

Special care must be taken in dealing with the singularity of $G$ and the corner point on the boundary $S$. Effective numerical methods for (5.5) can be seen in [3].

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6. Numerical experiments. In this section, we present three numerical examples to illustrate the effectiveness of the algorithm and show that the backscatter RCS can be effectively reduced by appropriate shaping. Throughout all the examples, we assume the admissible set $U_M$ for the coefficient is

$$\{\beta \in \mathbb{R}^{20} : -1/j^{1.5} \leq \beta_j \leq 1/j^{1.5}, \quad j = 1, 2, \ldots, 20\}.$$  

As we mentioned before, imposing smaller change for larger $j$ (high frequency component) ensures a less oscillatory shape. However, for specific engineering problems, those design variables should be chosen accordingly. Our optimization starts from a serpentine shape as shown in Figure 3(a), with the height $h = 0.3m$ and width of the opening $l = 1m$. The boundary is obtained by simply assuming $\beta_1$ equals 0.5 and all the other $\beta_j$ equal zero. We compare the optimized RCS with the corresponding cavity with rectangular shape, as shown in Figure 3(b).

The forward scattering problem and the adjoint problem are solved by the boundary integral method given in (5.5). The opening of the cavity is discretized uniformly by 512 subintervals as well as the bottom. For the curved boundary $S_1$ and $S_2$, since both of them are parameterized by a function on $[-h, 0]$, we simply discretize $[-h, 0]$ uniformly by 512 subintervals and assume $S_1$ and $S_2$ are piecewise constant on each subinterval. Note that higher order elements will result in a more accurate design, and our approximation is only given for illustration. The nonlocal boundary integral in (2.10) is transformed into a hypersingular integral by the Fourier transform:

$$I(u) = -\frac{k_0}{4\pi} \int_{\Gamma} \frac{1}{|x-x'|} H_1^{(1)}(k_0|x-x'|)u(x', 0)dx',$$

where $\int dx$ denotes the Hadamard principal value (or finite part) integral. We refer readers to [11] for details. We numerically evaluate the hypersingular integral by a Newton–Cotes formula proposed by Sun and Wu in [23]. Although these two integrals are equivalent mathematically, the hypersingular integral form does not need to truncate into finite terms compared to the Fourier-type integral.

In practice, the unit of RCS is in dB and the value is expressed as the logarithm of $\sigma$:

$$RCS = 10\log_{10} \sigma \text{ dB}.$$  

For the first example, we focus on reducing the backscatter RCS at the three wave numbers $k = 2\pi$, $k = 8\pi$, and $k = 32\pi$. We consider the minimization of the backscatter RCS at the normal incidence. It corresponds to a low frequency wave ($\omega = 300$ MHz) for $k = 2\pi$. After 500 iterations, we obtain the optimized shape shown in Figure 4(a). Fortunately, for such a low frequency case, the backscatter
RCS is constantly decreasing during the iteration. The reduction finally becomes extremely small and we take it to be convergent. The comparison between the rectangular shape and the optimal shape for the backscatter RCS is shown in Figure 4(b). From the graph we can see the an average of 15 dB reduction is achieved on all the incident angles, although the cavity is only designed to reduce the backscatter RCS at normal incidence. This speciality can only be found in the low frequency case, where there is almost no oscillation in the backscatter RCS across all the incident angles.

For $k = 8\pi$, the optimal shape is given in Figure 5. The RCS has been decreased by more than 15 dB at the normal incidence. As shown in Figure 4, the range of incident angles with RCS reduction is from $90^\circ$ to $105^\circ$, which is smaller than the interval for $k = 2\pi$.

For $k = 32\pi$, the incident frequency is considerably high (4.8 GHZ); the optimal shape is given in Figure 6. Although the RCS has been decreased at the normal inci-
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Fig. 6. (a) Optimal shape for $k = 32\pi$ at normal incidence; (b) RCS comparison for $k = 32\pi$, “- - -” for rectangular shape, “—–” for optimal shape.

Fig. 7. (a) Optimal shape for $k = 32\pi$ at the incidence angle $\theta = 110^\circ$; (b) RCS comparison for $k = 32\pi$, “- - -” for rectangular shape, “—–” for optimal shape.

dence as expected, the range of incident angles with RCS reduction is much smaller compared to $k = 2\pi$ and $k = 8\pi$. This is understandable since roughly, high frequency scattering can be viewed as “billiard-ball-bouncing scattering,” where the scattered field can be approximated by the ray-tracing technique [21]. In such a case, reduction of RCS at one incident angle always accompanies an increase at the other angle. Nevertheless, it is still very useful since the angles near the normal are considered as the “threat sector” during the detection of an object [6].

For the second example, we try to reduce the backscatter RCS with the incidence angle $\theta = 120^\circ$ for $k = 32\pi$. The result is shown in Figure 7. We see the RCS is decreased successfully around $\theta = 120^\circ$, but increased around $\theta = 105^\circ$. This confirms the conclusion that RCS reduction through shaping cannot be expected for all angles for high frequency incidence [6].

From the examples above, we show the algorithm works efficiently in finding the local minimum for the objective function. However, as the wavenumber increases, the
Fig. 8. (a) Optimal shape for $k = \frac{3\pi}{2}$ at normal incidence; (b) RCS comparison for $k = \frac{3\pi}{2}$, “- - -” for rectangular shape, “—–” for optimal shape; (c) RCS comparison for $k = \frac{3\pi}{2}$ between $[90^\circ, 120^\circ]$, “- - -” for rectangular shape, “—–” for optimal shape in Example 1, “ — ” for optimal shape in Example 3.

posibility that the solution being trapped in a local minimum also increases. For $k = \frac{3\pi}{2}$, we see the difference between the initial shape and the optimized shape from Figure 6(a) is insignificant, which implies the value of those local minima highly depends on the initial guess $\beta_0$. Since the local minimum we obtain may not be the most “optimal” one, we therefore add a random perturbation to the result after the solution is trapped in a local minimum and restart the iteration again. If the solution gets no better than the original one, we go back to the original solution, otherwise we choose the perturbed result. This process itself can be iterated several times until no improvement appears anymore. The new algorithm can be easily modified from the existing algorithm.

In the last example, we reconsider reducing the backscatter RCS with the wavenumber $k = \frac{3\pi}{2}$ at normal incidence. After 100 times perturbation, the optimal shape as shown in Figure 8(a) has a much greater difference from the initial shape compared to
the result in the first example. In particular, the RCS reduction gets improved near the normal incidence as shown in Figures 8(b) and 8(c).

7. Conclusion. In this paper, we consider the optimal design of an open cavity for RCS reduction. The problem is formulated as a minimization problem with the Helmholtz equation as the constraint. In addition, we make several assumptions on the shape of the cavity in order to make the problem nontrivial. The Helmholtz equation satisfied by the total field is reduced to a bounded domain problem by introducing a transparent boundary condition. The existence of the minimizer has been proven based on the continuous dependence of the boundary. The descent direction with respect to the boundary is also derived based on the domain derivative. A simple gradient-based optimization method is introduced with the integration of the descent direction. We also test our algorithm by several numerical examples. The result shows our approach is fairly effective in the RCS reduction.

Future works include extending the method to TE polarization, where the electronic field is transverse to the invariant direction. Another interesting direction is that the RCS reduction can also be obtained by using a nonsmooth curve for the boundary. More challenging work would be to investigate the optimal design of the three dimensional EM cavity problem.

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