Geometric methods in the dynamics of dissipative PDEs

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Dissipative PDEs

'Definition' of Dissipative PDEs: - apparently all interesting (asymptotic) dynamics is 'finite dimensional'.

apparently= In numerical simulations increasing the dimension of Galerkin projection does not change the dynamics.

Examples:

$$u_t(t,x) = Lu + N(u, Du, \ldots, D^r u) + f(x),$$

L - smoothing operator, Laplacian or its power with correct sign. $r < s$, where s the order of L.

Includes: Navier-Stokes , Ginzburg-Landau, reaction diffusion, Kuramoto-Sivashinski PDEs

Our goal: tools for rigorous study of dissipative PDEs

Virtually all theorems and phenonomena from finite dimensional dynamics, which are stable with respect to multidimensional perturbations should hold also for dissipative PDEs.

Mutlidimensional perturbation: we add a 'contracting' dimension.

Ex. $f : \mathbf{R} \to \mathbf{R}$, $F : \mathbf{R} \times \mathbf{R}^s \to \mathbf{R} \times \mathbf{R}^s$, such that $\|(f(x), 0) - F(x, y)\|$ is small in a suitable sense

Needs both:

- Constructive proofs of stability with respect to multidimensional perturbations

- Effective algorithms - problem: this is an infinite dimension

Some related work

- "Classical" PDEs setting, use of Schauder or Banach fixed points in suitable function spaces : Nakao, Yamamoto, Plum, McKenna, Watanabe, Lessard and others. restricted to static problems, no dynamics

- functional analytic approach: Arioli and Koch - results on fixed points and bifurcations for Kuramoto-Sivashinski PDE , periodic orbit

- self-consistent bounds - good for dynamics of dissipative PDEs (and static problems too).

  – fixed points and heteroclinic connections for Cahn-Hillard (gradient system): Maier-Papper, Mischaikow, Wanner

- periodic orbits for Kuramoto-Sivashinski PDE in 1D - P. Z.
- others: Swift-Hohenberg eq. - steady states : Hiraoka, Ogawa, Mischaikow, Day
- bifurcations of steady states for KS eq. - P.Z.

Outline of this talk

1. Model problem - heteroclinic connection for Kuramoto-Sivashinsky Eq.

2. Rigorous integration of dissipative PDEs

3. Proving the existence of fixed points

4. Rigorous bounds for basin of attraction

5. Rigorous bounds for the unstable(stable) manifold

6. Some data from the proofs

7. Conclusions, future work

A Model Problem - Kuramoto-Sivashinsky PDE

Consider the Kuramoto-Sivashinsky (KS) ej.

$$u_t = -\nu u_{xxxx} - u_{xx} + 2uu_x, \qquad \nu > 0$$

where $(t, x) \in [0, \infty) \times \mathbf{R}$ subject to periodic and odd boundary conditions

$$\begin{aligned} u(t, 0) &= u(t, 2\pi) \\ u(t, -x) &= -u(t, x) \end{aligned}$$

For various values of ν a variety of dynamics,

fixed points,
periodic orbits,
heteroclinic orbits,
chaotic dynamics,

have been observed numerically.

Goal: A rigorous means of proving these numerical results.

A Model Problem - Kuramoto-Sivashinsky PDE, Fourier expansion

Fourier expansion is: $u(t,x) = \sum_{k=-\infty}^{\infty} b_k(t) e^{ikx}$

Substituting in KS and applying boundary conditions gives:

$$\dot{a}_k = k^2(1-\nu k^2)a_k - k\sum_{n=1}^{k-1} a_n a_{k-n} + 2k\sum_{n=1}^{\infty} a_n a_{n+k}$$

where $b_k = i a_k$ and $k = 1, 2, 3, \ldots$.

Linearization: $\dot{a}_k = k^2(1-\nu k^2)a_k$

- k-th mode is unstable for $k < \frac{1}{\sqrt{\nu}}$
- k-th mode is stable for $k > \frac{1}{\sqrt{\nu}}$
- the modes with $k >> \frac{1}{\sqrt{\nu}}$ should be irrelevant for the dynamics

A Model Problem - Kuramoto-Sivashinsky PDE, known results

Known results:

- the existence of global attractor, the functions from attractor are analytic - Fourier series converge at geometric rate (Foias, Temam)

- the existence of finite dimensional inertial manifold (Foias, Nicolaenko, Sell, Temam, Rossa, Jolly) (not of much use in rigorous numerics)

No analytical results on dynamics more complicated than fixed points bifurcating from zero solution

Our rigorous results for Kuramoto-Sivashinsky PDE

- the existence of multiple periodic orbits for various parameter values $\nu \approx 0.1215$, 0.1212, 0.125, 0.032, 0.02991, both stable and unstable orbits

- the existence of multiple fixed points for various values o f ν and their bifurcations

- the existence of attractive fixed points for various values of ν

- today: the existence of heteroclinic connection between zero and unimodal fixed point for $\nu = 0.75$

How to establish the existence of heteroclinic connection in the flow?

Draw picture on the blackboard, discuss how to do it for ODEs

The proof consists of the following stages:

1) the proof of the existence of two fixed points, "the source" and "the target"

2) rigorous estimates for the attracting region around the target point

3) rigorous estimates for one dimensional unstable manifold of the source point

4) rigorous integration of PDE - the propagation of the unstable manifold of the source until it enters the basin of attraction of the target point.

Challenge of infinite dimension

Our PDE $\dot{x} = f(x)$, x - a sequence of Fourier coefficients

Problems:

- f is not continuous and only densely defined

- how to represent in finite form elements of our phasespace? What phasespace?

Our approach: We restrict our attention to the sets of the form

$$W \oplus \Pi_{k>m}^{\infty}[a_k^-, a_k^+], \quad a_k^{\pm} = \pm C/k^s$$

$W \subset \mathbf{R}^m$ - compact set, s - large enough

Why $W \oplus \Pi_{k>m}^{\infty}[-C/k^s, C/k^s]$

Let $T = \Pi_{k>m}^{\infty}[a_k^-, a_k^+]$, where $a_k^{\pm} = \pm C/k^s$

$$W \oplus \Pi_{k>m}^{\infty}[a_k^-, a_k^+] = \{(a_k)_{k\in\mathbf{N}} \mid (a_1, \ldots, a_m) \in W, \; a_k \in [a_k^-, a_k^+], \text{for } k > m\}$$

- any continuous periodic odd function of class C^s is contained in some $W \oplus T$

- $W \oplus T$ is compact in topology of component-wise convergence, l_2 etc

- on $W \oplus T$ our vector field becomes very smooth, we can use Taylor formula, everything what is needed converges

- on $W \oplus T$ our PDE defines local semiflow, the flows for Galerkin projections on $W \oplus T$ have uniformly bounded Lipschitz constants on a compact time intervals and converge uniformly to the semiflow for full system (**logarithmic norms**)

Most important: While $W \oplus T$ is not invariant under the flow of our PDE, it may have an important dynamical property - an isolation for the tail.

ISOLATION for $n > m$

For $a \in W \oplus T$ and $k > m$ holds

$$a_k = a_k^+ \quad \Rightarrow \quad \dot{a}_k < 0$$
$$a_k = a_k^- \quad \Rightarrow \quad \dot{a}_k > 0$$

Gives some kind of invariance for tail (high modes): **draw picture , explain**

This is an infinite set of inequalites, but it turns out to be relatively easy to satisfy!!!

Why it is a easy to find a good tail = self-consistent bounds

$$u_t = Lu + N(u, Du, \ldots, D^r u)$$

$x \in \mathbf{T}^n$ (periodic boundary conditions),
L - linear, diagonal, N - polynomial

Fourier expansion $u(t) = \sum_{k \in \mathbf{Z}^n} a_k(t) e^{ik \cdot x}$

Lemma. Let $s > s_0$. If $|a_k| \leq C/|k|^s$, $|a_0| \leq C$, then there exists $D = D(C, s)$

$$|N_k| \leq \frac{D}{|k|^{s-r}}, \qquad |N_0| \leq D$$

This is in fact a statement about regularity. a is of "class C^s" then $N(a)$ is of "class C^{s-r}"

Isolation. Assume $L(a)_k = -|k|^p a_k$, $p > r$.

Assume $|a_k| \leq \frac{C}{|k|^s}$, $|a_{k_0}| = \frac{C}{|k_0|^s}$, then

$$\frac{d|a_{k_0}|}{dt} \leq -|k_0|^p |a_{k_0}| + |N_{k_0}(a)| \leq$$

$$-C|k_0|^{p-s} + D|k_0|^{r-s}$$

$$\frac{d|a_{k_0}|}{dt} < 0, \qquad |k_0| > M$$

Rigorous integration of dissipative PDEs - the general idea

$$u_t = Lu + N(u, Du, \ldots, D^r u) + f(x), \qquad (1)$$

$u \in \mathbf{R}^n$, $x \in \mathbf{T}^d$, L is a linear, N - a polynomial (or analytic), f smooth enough.

L is diagonal in the Fourier basis $\{e^{kx}\}_{k \in \mathbf{Z}^d}$

$$Le^{ikx} = \lambda_k e^{ikx}, \qquad (2)$$

$$\begin{aligned} \lambda_k &= -v(|k|)|k|^p && (3) \\ 0 &< v_0 \leq v(|k|) \leq v_1, && \text{for } |k| > K_- \quad (4) \\ p &> r. && (5) \end{aligned}$$

1 We replace PDE by an infinite ladder of ODEs for Fourier coefficients of $u(t,x)$.

$$\frac{du_k}{dt} = \lambda_k u_k + N_k(u), \qquad \text{for all } k \in \mathbf{Z}^d. \quad (6)$$

2 we split 'the phase space' for (6) into two parts: the finite dimensional part, X, containing the Fourier modes most relevant for the dynamics of (1) and the tail in $X^\perp$. Now problem (6) is replaced by two problems (7) and (8).

3 The first part consist of a finite dimensional differential inclusion for $p \in X$, given by

$$\frac{dp}{dt} \in P(Lp + N(p+T)), \qquad p \in X \qquad (7)$$

P is a projection onto X. The second part is concerned with the evolution of T

$$\lambda_k u_{k,j} + N^-_{k,j} < \frac{du_{k,j}}{dt} < \lambda_k u_{k,j} + N^+_{k,j}, \qquad \text{"}k \text{ not in } X\text{"} \qquad (8)$$

where $N^\pm_{k,j}$ are suitably chosen constants.

Obviously, to infer from (7) and (8) any information on the behavior of solutions of the full system (6) one needs some **consistency conditions** and **fast decay of Fourier coefficients**.

Tails $T = \Pi_{k>m}[-\frac{C}{k^s}, \frac{C}{k^s}]$ do the job through **the isolation property**.

Our algorithm gives uniform and compact bounds for all Galerkin projections of PDE. The solution of PDE is obtained through passing to the limit with the dimension of Galerkin projection.

The method of self-consistent bounds

H - Hilbert space,
$e_1, e_2, \ldots$ - an orthogonal basis in H
The corresponding projections are

$$\begin{aligned} p_m = P_m a &:= (a_1, a_2, \ldots, a_m) \\ q_m = Q_m a &:= (a_{m+1}, a_{m+2}, \ldots) \end{aligned}$$

The problem:

$$\dot{a} = F(a) \tag{9}$$

F is not continuous, with dense domain in H.

$F_k \circ P_n$ is a C^1-function for $n, k \in \mathbf{N}$

Later $F(a) = L(a) + N(a)$, L - linear, N- non-linear
$e_1, e_2, \ldots$ - eigenvectors of L - very helpful

The method:

Def. Fix m, M ($m \leq M$). *A compact set* $W \subset P_m(H)$ *and a sequence of pairs* $\{a_k^{\pm} \in \mathbf{R} \mid a_k^- < a_k^+,\ k \in \mathbf{Z}^+\}$ *are self-consistent a-priori bounds for* F *if:*

C1 *For* $k > M$, $a_k^- < 0 < a_k^+$.

C2 *Let* $\hat{a}_k := \max |a_k^{\pm}|$ *and set* $\hat{u} = \sum_{k=0}^{\infty} \hat{a}_k e_k$. *Then,* $\hat{u} \in H$, $(\{\hat{a}_k\} \in l_2)$

C3 *The function* $u \mapsto F(u)$ *is continuous on*

$$W \oplus \prod_{k=m+1}^{\infty} [a_k^-, a_k^+] \subset H.$$

Moreover, if we define
$\hat{f}_k = \max_{u \in W \oplus \prod_{k=m+1}^{\infty} [a_k^-, a_k^+]} |F_k(u)|$ *and set*
$\hat{f} = \sum \hat{f}_k e_k$, *then* $\hat{f} \in H$. $(\{\hat{f}_k\} \in l_2)$

Notation: $T = \prod_{k=m+1}^{\infty} [a_k^-, a_k^+]$ - Tail

C4. ISOLATION for $n > m$

For $a \in W \oplus T$ and $k > m$ holds

$$a_k = a_k^+ \qquad \Rightarrow \quad \dot{a}_k < 0$$
$$a_k = a_k^- \qquad \Rightarrow \quad \dot{a}_k > 0$$

C1,C2,C3 - give convergence

C4 - gives a priori bounds

C1,C2,C3,C4 - easy to satisfy. It is enough to take $|a_k| \leq \frac{C}{|k|^s}$ for s large enough

Finite dimensional part

Basic Differential Inclusion:

$$\dot{p} \in P_m F(p) + \Gamma_m, \quad p \in \mathbf{R}^m, \qquad (10)$$

where $\Gamma_m = \{P_m F(p+q) - P_m F(p) \mid p \in W, q \in T\}$

We say a multivalued map $p_I : [0,h] \to H$ is *upper attainable set (uas) map* for (10) if the following is true

- any C^1 function satisfying (10) and defined on the maximum interval of existence is defined on $[0,h]$

- if a C^1-function $p : [0,h] \to X_m$ satisfies (10), then $p(t) \in p_I(t)$ for $t \in [0,h]$

Theorem: Assume $W \oplus T$ are self-consistent bounds for F. If $p_I : [0, t_1] \to X_m = P_m(H)$ is uas map for (10), such that $p_I([0, t_1]) \subset W$.

Then for any $q_0 \in T$, the problem $u' = F(u)$ (and all its Galerkin projections $u' = P_n F(u)$, $n > M$) has a solution $u(t) = (p(t), q(t))$ for $t \in [0, t_1]$, such that

$$p(t) \in p_I(t), \qquad q(t) \in T, \qquad \text{for } t \in [0, t_1]$$

Basin of attraction. Logarithmic norms

Logarithmic norm: $Q \in R^{n \times n}$

$$\mu(Q) = \lim_{h>0, h\to 0} \frac{\|I + hQ\| - 1}{h}$$

can be negative !!!

- for Euclidean norm

  $\mu(Q) =$ the largest eigenvalue of $1/2(Q+Q^T)$.

- for max norm $\|x\| = \max_k |x_k|$

$$\mu(Q) = \max_k (q_{kk} + \sum_{i, i\neq k} |q_{ki}|)$$

- for norm $\|x\| = \sum_k |x_k|$

$$\mu(Q) = \max_i (q_{ii} + \sum_{k, k \neq i} |q_{ki}|)$$

Convergence of Galerkin projections.

Logarithmic norms - Fundamental lemma

Lemma: Let $\phi(t,x)$ be a flow induced by

$$x' = f(x).$$

Assume that Z is a convex set,

$$y([0,T]), \varphi([0,T],x_0) \in Z$$
$$\mu\left(\frac{\partial f}{\partial x}(\eta)\right) \leq l, \qquad \text{for } \eta \in Z$$
$$\left\|\frac{dy}{dt}(t) - f(y(t))\right\| \leq \delta.$$

Then for $0 \leq t \leq T$ we have

$$\|\varphi(t,x_0) - y(t)\| \leq e^{lt}\|y(0) - x_0\| + \delta\frac{e^{lt}-1}{l}, \qquad \text{if } l \neq 0.$$

For $l = 0$ we have

$$\|\varphi(t,x_0) - y(t)\| \leq \rho + \delta t.$$

In particular: e^{lT} is a Lipschitz constant for $\phi(t,\cdot)$ in Z (if Z is forward invariant).

Convergence of Galerkin projections.

$$x' = F(x) = Lx + N(x) \tag{11}$$

$e_1, e_2, \ldots$- eigenvectors for L, $Le_k = \lambda_k e_k$, $\lambda_k \to -\infty$

W,$\{a_k^{\pm}\}$ - self-consistent bounds,
$T = \Pi_{k=m+1}^{\infty}[a_k^-, a_k^+]$
W - convex
$P_n(W \oplus T)$ is a trapping region (an isolating block with $W^- = \emptyset$) for n-dim Galerkin projections of (11), $n > m$

Condition *D*: there exists $l \in \mathbf{R}$ such that for all $k = 1, 2, \ldots$, $a \in W \oplus T$

$$\sum_{i=1}^{\infty} \left| \frac{\partial N_k}{\partial x_i} \right| (a) + \lambda_k \leq l$$

Idea: the logarithmic norms for all Galerkin projections are uniformly bounded.

Convergence of Galerkin projections:

Theorem:

1. **Uniform convergence and existence** For $x_0 \in W \oplus T$, let $x_n : [0, \infty] \to P_n(W \oplus T)$ be a solution of $x' = P_n(F(x))$, $x(0) = P_n x_0$.

   Then x_n converges uniformly on compact intervals to a function $x^* : [0, \infty] \to W \oplus T$, which is a solution of (11) and $x^*(0) = x_0$. The convergence of x_n on compact time intervals is uniform with respect to $x_0 \in W \oplus T$.

2. **Lipschitz constant**. Let $x : [0, \infty] \to W \oplus T$ and $y : [0, \infty] \to W \oplus T$ be solutions of (11), then

$$|y(t) - x(t)| \leq e^{lt} |x(0) - y(0)|$$

Convergence of Galerkin projections - comments

- We got a semiflow on $W \oplus T$

- A computable expression for a Lipschitz constant of the induced semiflow
  Application: If $W \oplus T$ - a trapping region isolating a fixed point and $l < 0$, then we have an attracting fixed point - gives the verified basin of attraction

- we have a formula for the error of Galerkin projection

Cone condition - bounds on the unstable manifold

$$x' = F(x) = Lx + N(x) \qquad (12)$$

$N_1 \times N_2 \subset \mathbf{R}^m$,$\{a_k^{\pm}\}$ - self-consistent bounds,
$T = \Pi_{k=m+1}^{\infty}[a_k^-, a_k^+]$
$N_1 \subset \mathbf{R}^u$ is u-dimensional cube
$N_2 \subset \mathbf{R}^{m-u}$ is $m-u$-dimensional cube
$P_n((N_1 \times N_2) \oplus T)$ is an isolating block for a fixed point "with u-unstable directions" for n-dim Galerkin projections of (12), $n > m$

$V = (N_1 \times N_2) \oplus T$

Coordinates on V: (x, y), $x \in \mathbf{R}^u$, $y \in \mathbf{R}^{m-u} \oplus T$.

Quadratic form: $Q((x, y)) = \sum_i x_i^2 - \sum_i y_i^2$

Cone condition - bounds on the unstable manifold

Theorem Let V, Q be as above. Assume that

$$w^t \left([dF(V)]^T Q + Q[dF(V)] \right) w > 0 \qquad (13)$$

for all $w \neq 0$ and such that $w = x - y$, where $x, y \in V$.

Then there exists a unique fixed point $z \in V$, such that the local unstable manifold of z

$$W_V^u(z) = \{p \mid \varphi((-\infty, 0], p) \in V, \lim_{t \to -\infty} \varphi(t, p) = z\}$$

is a graph of a Lipschitz function:

$$W_V^u(z) = \{(x, y(x)), x \in N_1\}$$

, where $y : N_1 \to N_2 \oplus T$. Moreover,

$$Q(p_1 - p_2) > 0, \qquad \text{for } p_i \in W_V^u(z),\ p_1 \neq p_2$$

Meaning of cone condition (13):

Cone condition(13) implies that:

$$\frac{d}{dt}Q(\varphi(t,p_1)-\varphi(t,p_2)) = (p_1-p_2)^t \left(D^tQ + QD\right)(p_1-p$$

where $D = \int_0^1 dF(p_2 + t(p_1 - p_2))dt$.

$L(p_1, p_2) = Q(p_1 - p_2)$ is a two-point Lapunov function. It is increasing along the orbits.

$L(z_0, \cdot)$ is our usual Lapunov function.

Verification of cone condition (13):

Cone condition (13) is implied by the following one:

for some $\epsilon > 0$ and all i holds

$$2 \inf_{x \in V} \left| \frac{\partial F_i}{\partial x_i}(x) \right| - \sum_{j, j \neq i} \sup_{x \in V} \left| Q_{jj} \frac{\partial F_j}{\partial x_i}(x) + Q_{ii} \frac{\partial F_i}{\partial x_j}(x) \right| \geq \epsilon. \tag{14}$$

which implies that the i-th diagonal element of $dF^t(V)Q + QdF(V)$ is larger then half of the sum of all elements in the i-th row and i-th column.

Likely to hold when the diagonal in dF dominates.

For KS-equation: $\frac{\partial F_i}{\partial x_i} \approx -i^4$, $\frac{\partial F_i}{\partial x_j} \approx \frac{iD}{|i-j|^s}$

Data from the proof of the connection

Source: the zero point

- $m = 8$, $M = 16$

- the half-size of the neighborhood $\delta = 0.075$ in the unstable direction

- next directions: $7e-4$, $1.2e-5$, $6e-8$, ...
  Tail - $1.28151e + 007/k^{16}$

- cone condition: $\epsilon = 0.0329203$

Target: $\approx (0.712361, -0.123239, 0.0101786)$

- $m = 3$, $M = 10$
- $\delta = 0.06$ - the half-size of the neighborhood - in the dominant directions
- tail: $9077.32/k^{10}$
- $l = -0.022517$ - logarithmic norm

The integration:

- $m = 8$, $M = 16$
- $order = 4$, $h = 5e - 4$
- a time needed to get from the source to the target $t = 12.4$, around $6.2 \cdot 10^4$ time steps
- size of the set at the end of computations: $4.50517e - 005$, tail: $3.926578e - 004/k^{14}$

Conclusions

- computer assisted proofs in the dynamics of dissipative PDEs are possible

- rigorous numerics for time evolution of dissipative PDEs is possible

- the global existence, uniqueness and regularity theorems are not required, interesting solutions are constructed

- could be applied to (I hope): Ginzburg-Landau, Navier-Stokes in 2D and 3D

Future work

• prove chaos (symbolic dynamics) for KS $\nu \approx 0.029$ or $\nu \approx 0.1212$

• Construct an *rigorous* C^1*-algorithm* for dissipative PDE (Arioli and Koch have such algorithm based on different approach).

This will make possible to **rigorously** apply a lot of dynamical system theory to dissipative PDEs.

• Finite elements ?