A universal space for Cantor expansive dynamics

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jointly with

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Krakow, July 1st, 2012

Contents

Finite generated pseudogroups

A **pseudogroup** of transformations of X is a family of homeomorphisms between open sets of X closed w.r.t.

- inversion
- composition
- restriction to open sets
- gluing of homeomorphisms

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Pseudogroups on the Cantor set

Definition (Compact Generation**).** A pseudogroup Γ is compactly generated if X contains a relatively compact open set Y meeting all Γ-orbits and the reduced pseudogroup $\Gamma|_Y$ is generated by a finite set Λ_Y of elements of Γ such that each element $\lambda \in \Lambda_Y$ is the restriction of an element $\lambda' \in \Gamma$ with the closure of dom λ contained in dom λ'.

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Definition. A pseudogroup Γ is δ-expansive if for all $x \neq x' \in X$ with $d(x, x') < \delta$ there exists $\gamma \in \Gamma$ s.t. $d(\gamma(x), \gamma(x')) \geq \delta$.

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Gromov-Hausdorff space of trees

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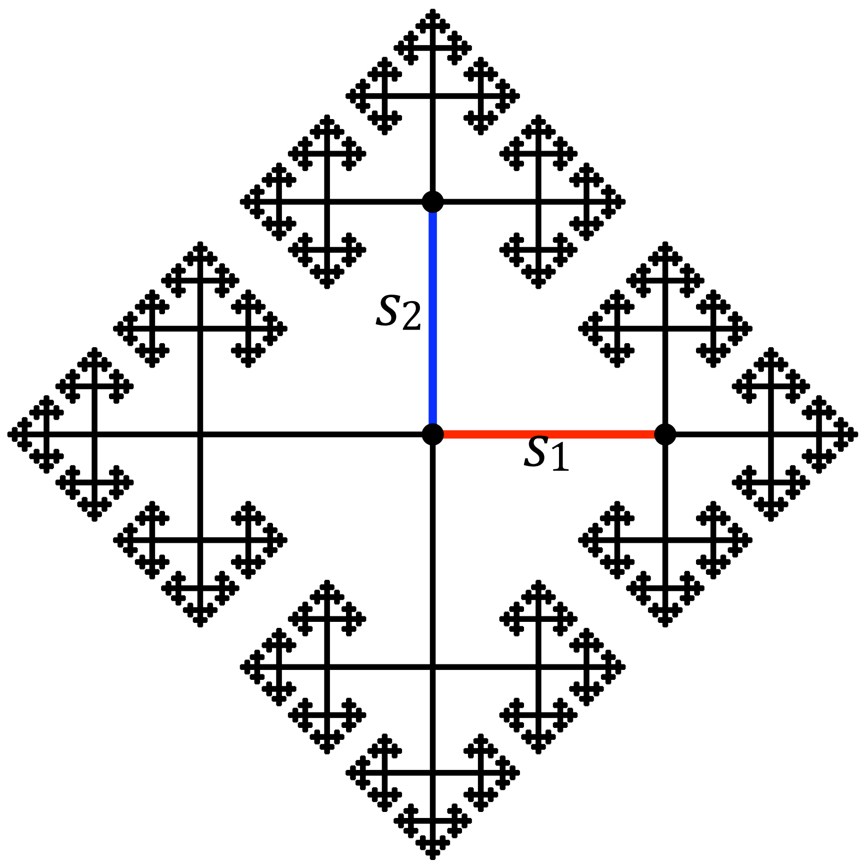

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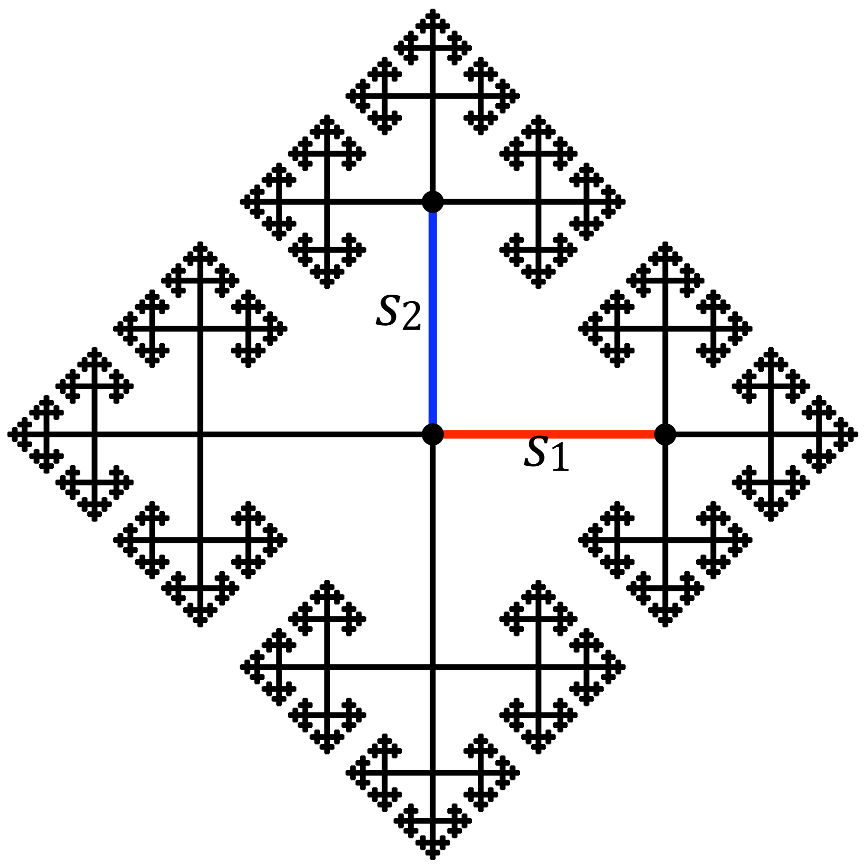
- $\mathcal{G}_n$ is the Cayley graph of $\mathbb{F}_n$
- $\mathfrak{X}_n$ is the space of rooted subtrees of $\mathcal{G}_n$

$$\mathfrak{X}_n = \{\, T \subset \mathcal{G}_n \mid T \text{ is a tree and } 1 \in T \,\}$$



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Endow $\mathfrak{X}_n$ with the Gromov-Hausdorff metric

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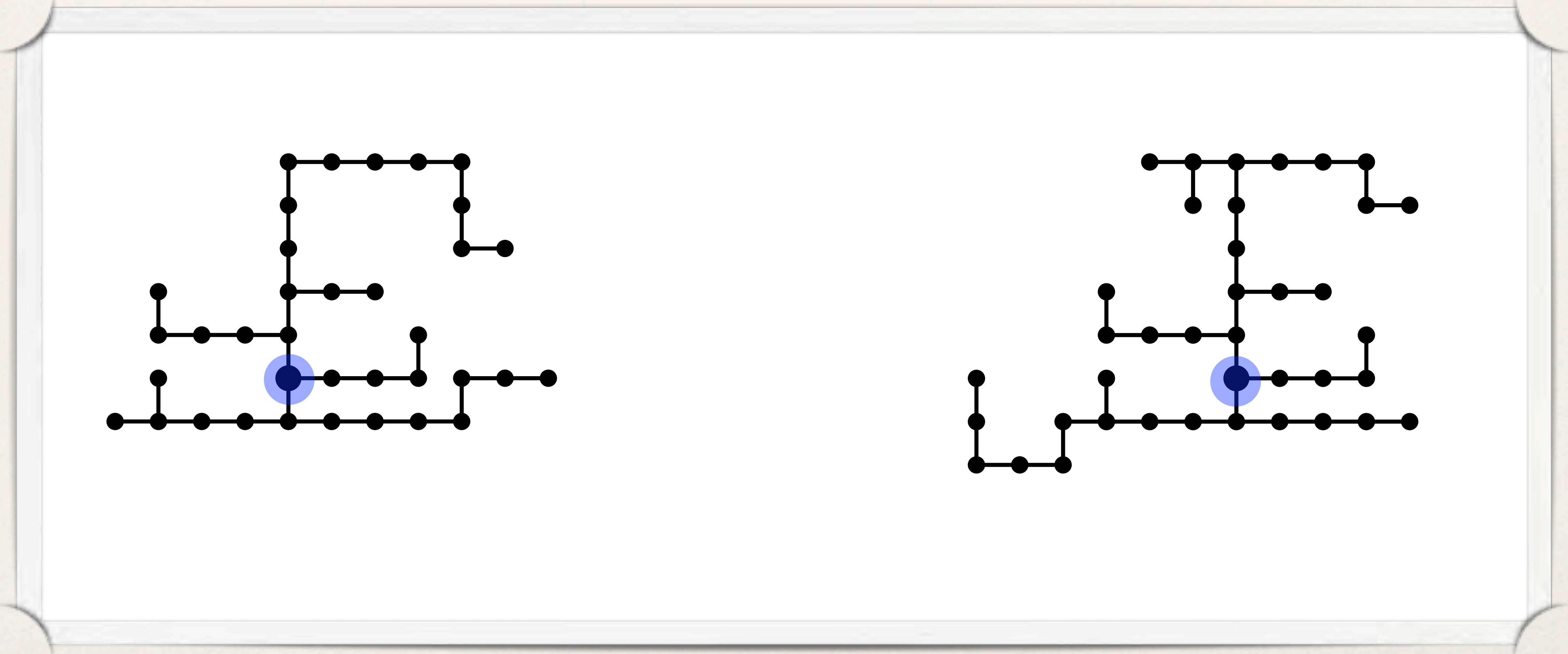

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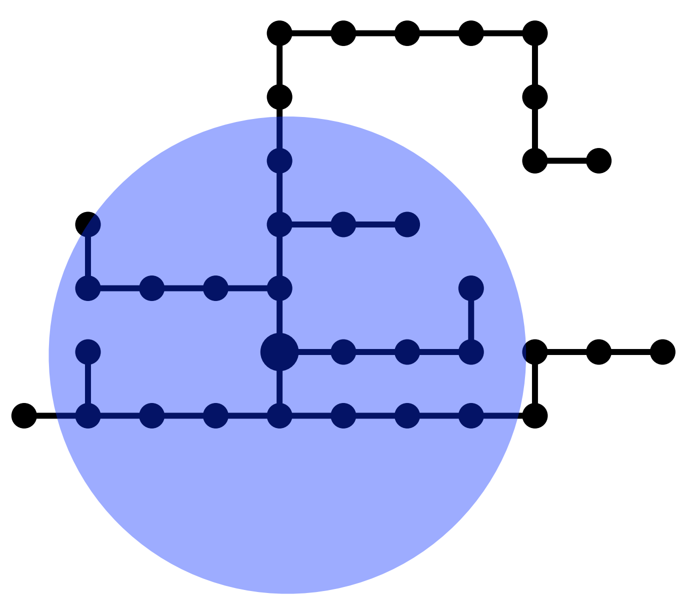

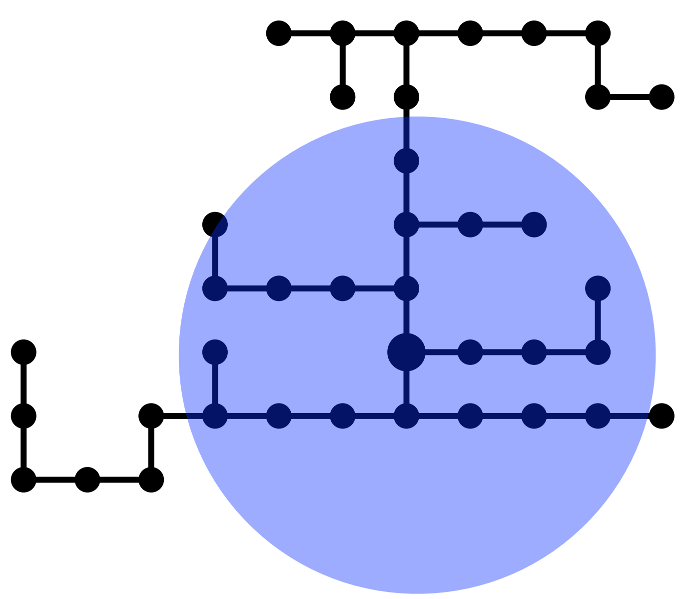

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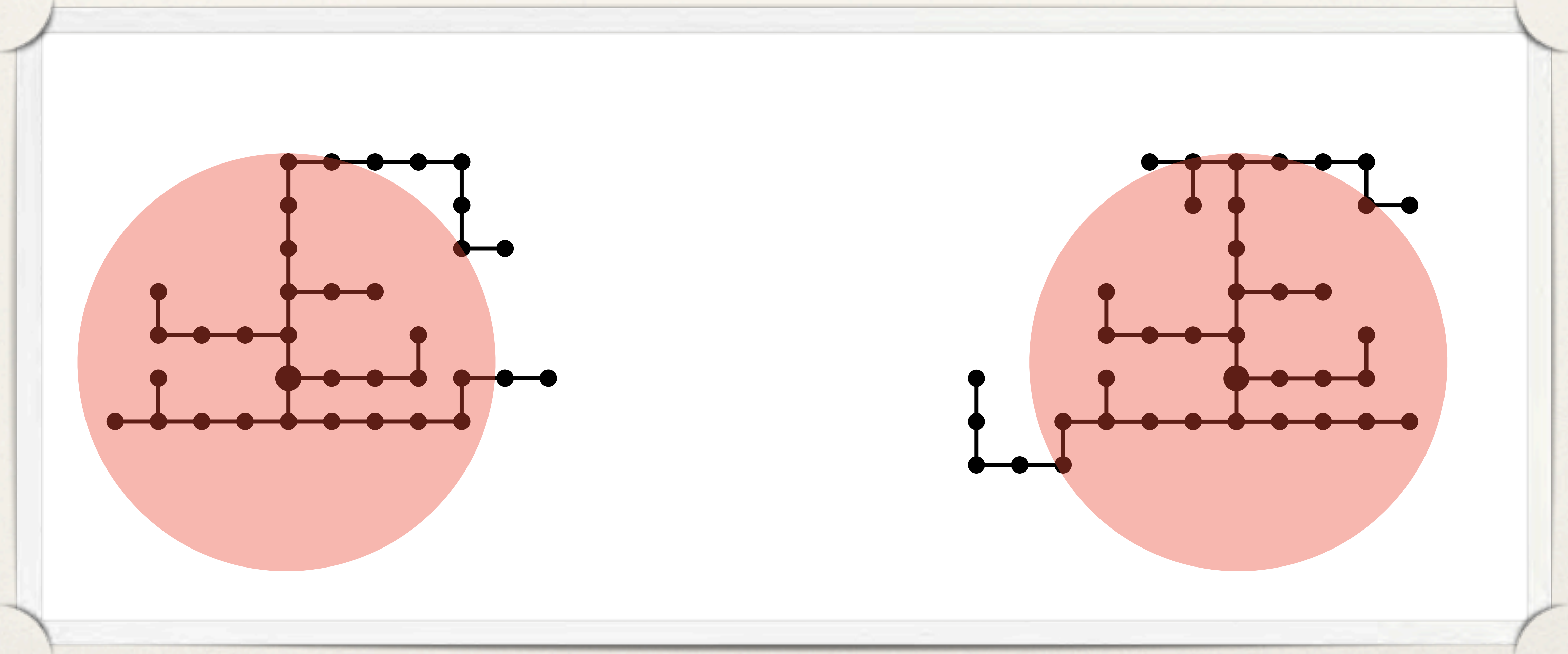

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$$d_{\mathrm{GH}}(T, T') = e^{-R(T, T')}$$

where

$$R(T, T') = \sup\{\, n \in \mathbb{N} \mid B_T(1, n) = B_{T'}(1, n) \,\}$$

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The partial action of $\mathbb{F}_2$

For each generator s_i of $\mathbb{F}_n$ there is a translation

- **τ_{s_i} is not globally defined**
- $\mathfrak{T}_n = \langle \tau_{s_1}, \tau_{s_2}, \ldots, \tau_{s_n} \rangle$ pseudogroup generated by them
- $\mathfrak{T}_n$ is expansive

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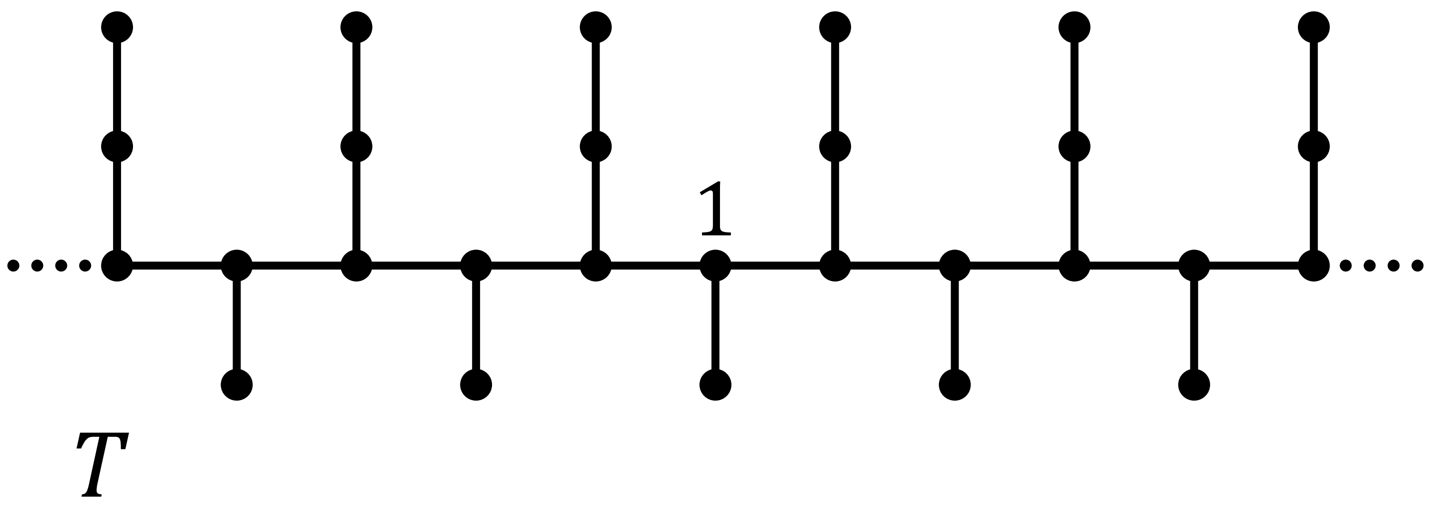

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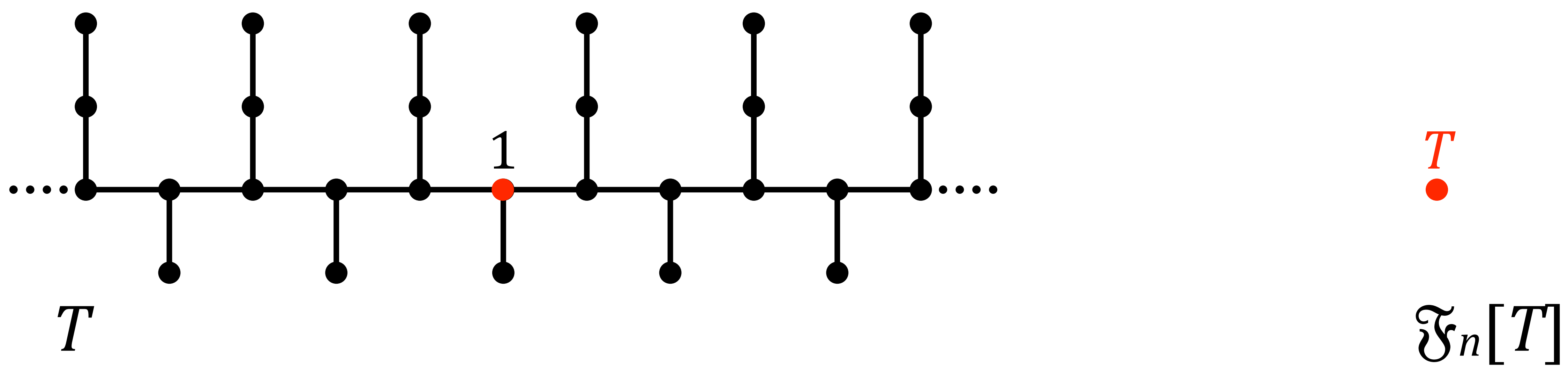

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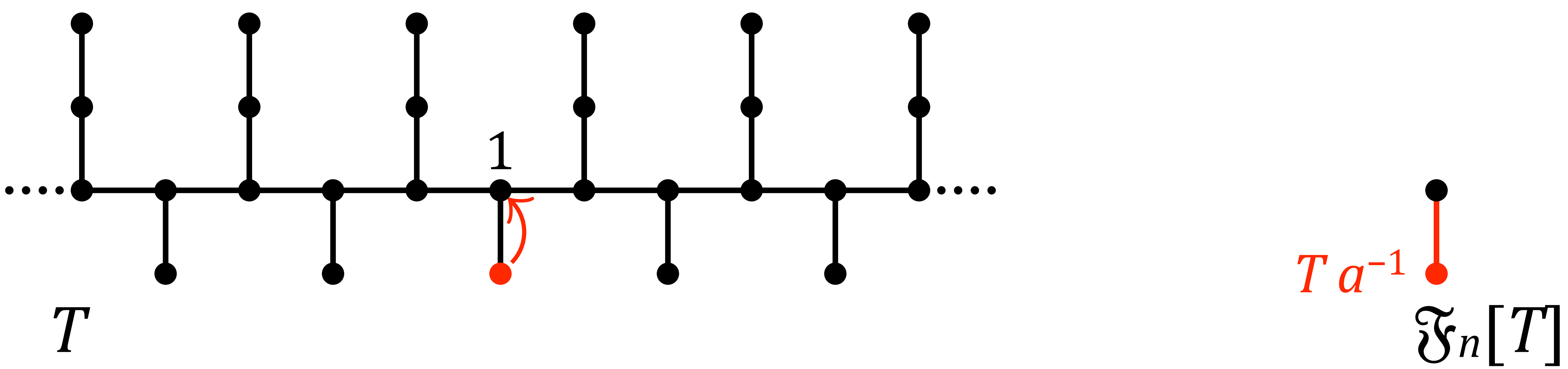

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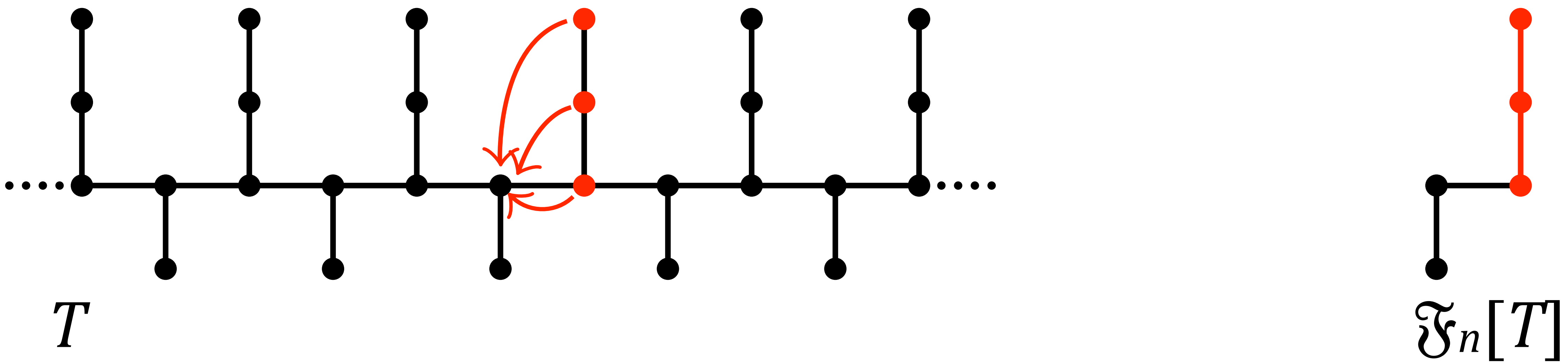

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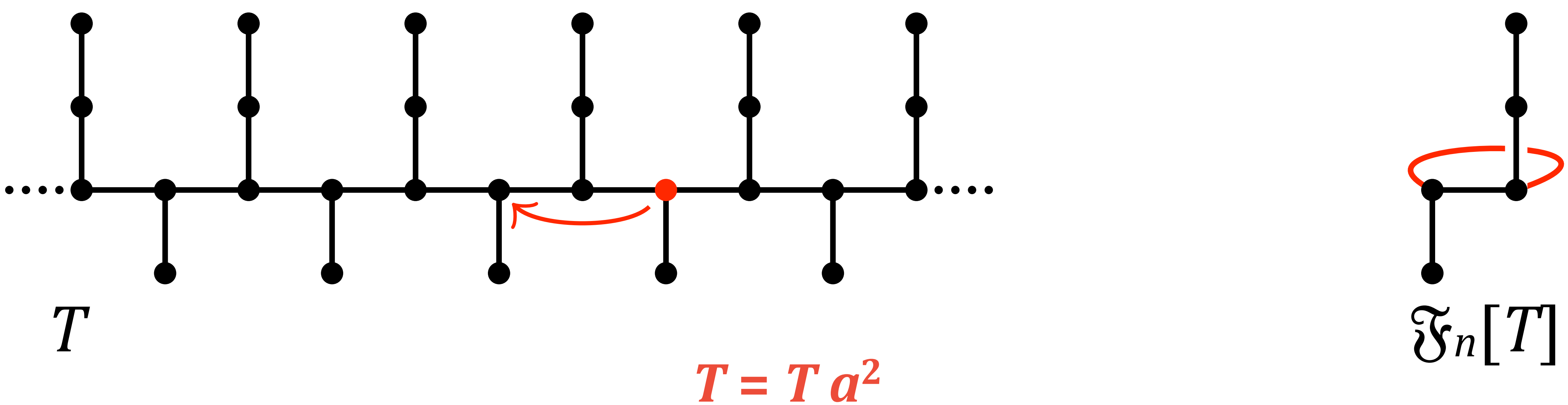

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Assign a tree to each point of X describing its orbit.

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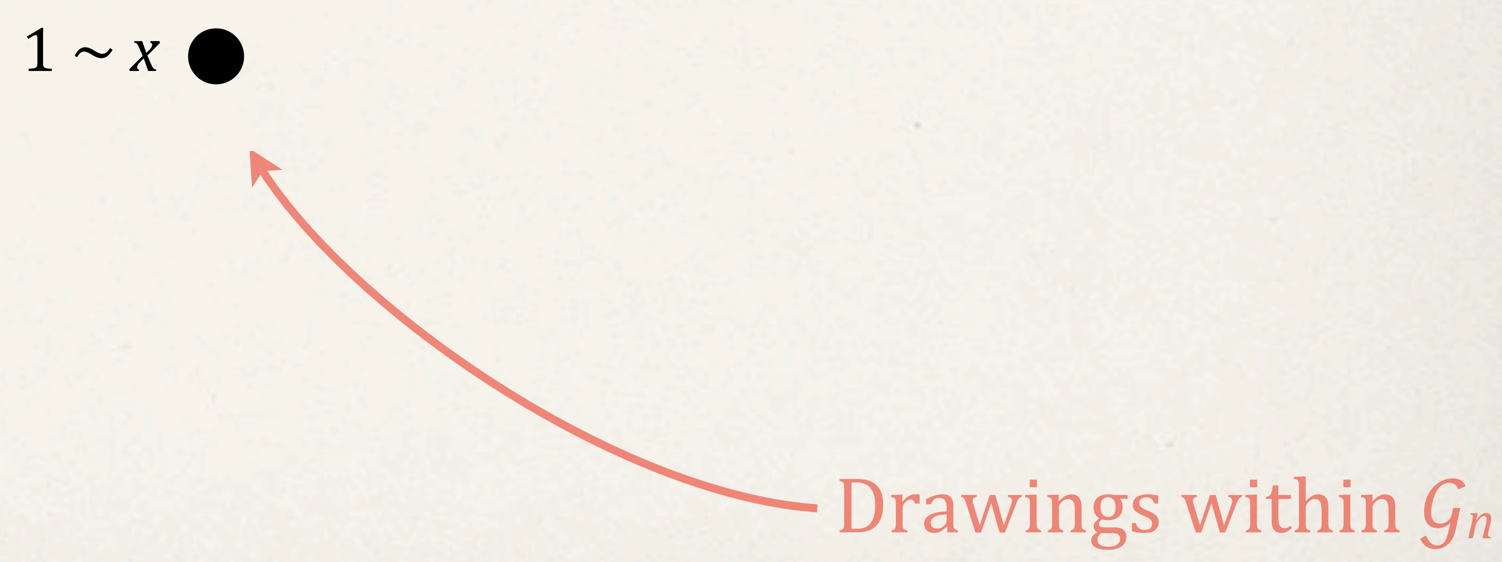

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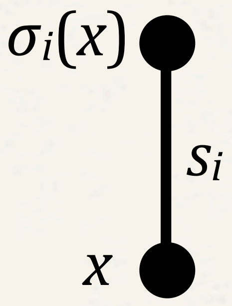

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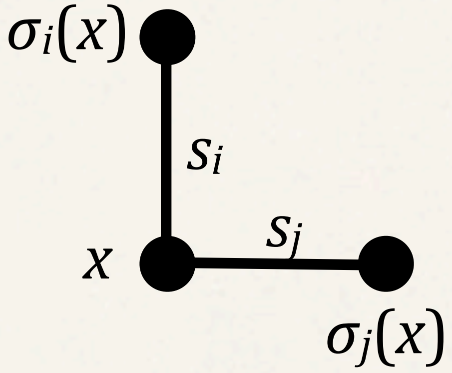

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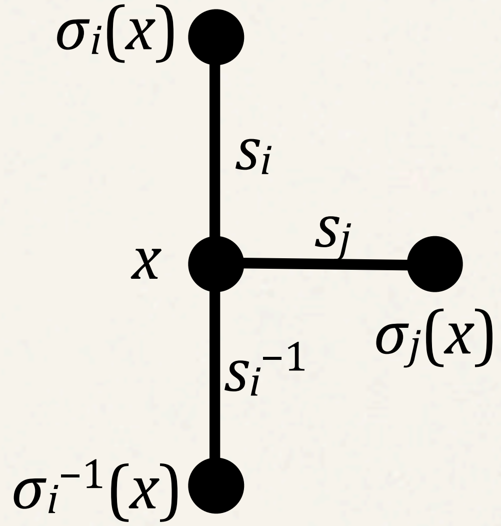

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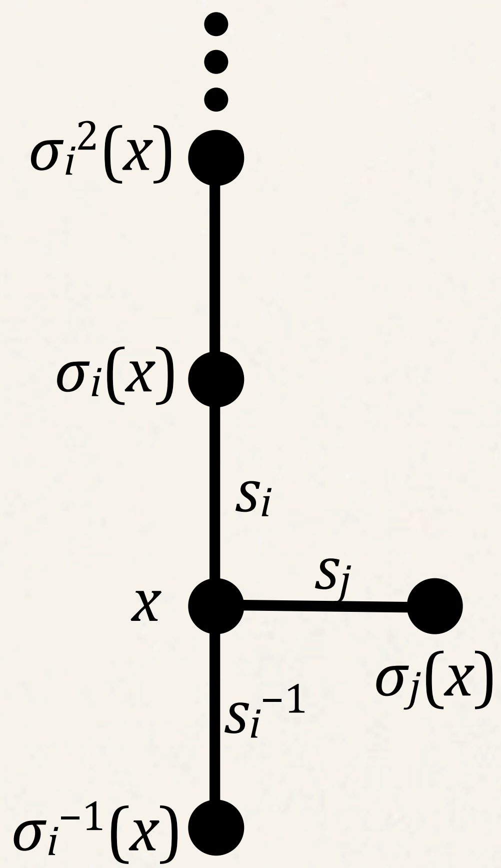

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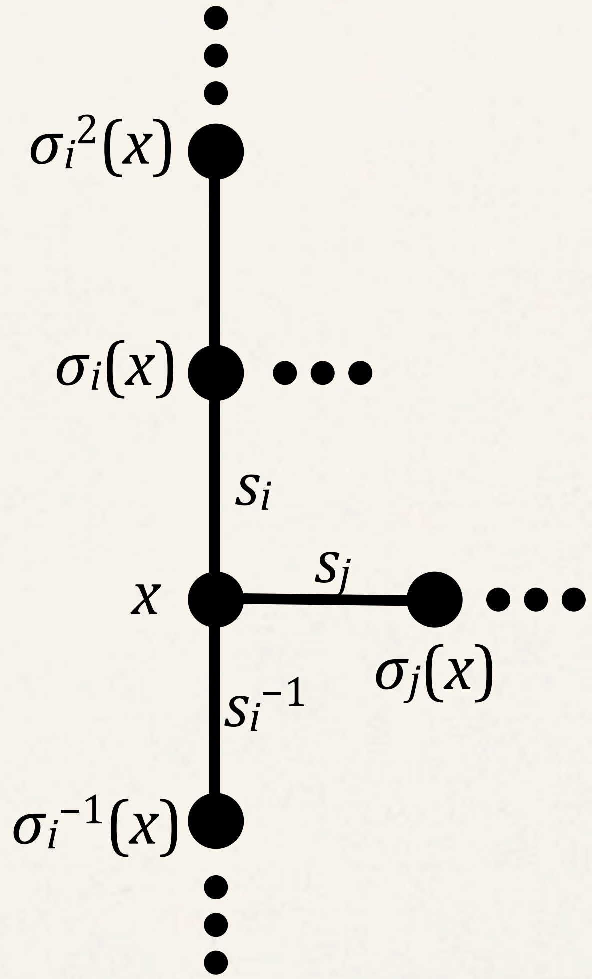

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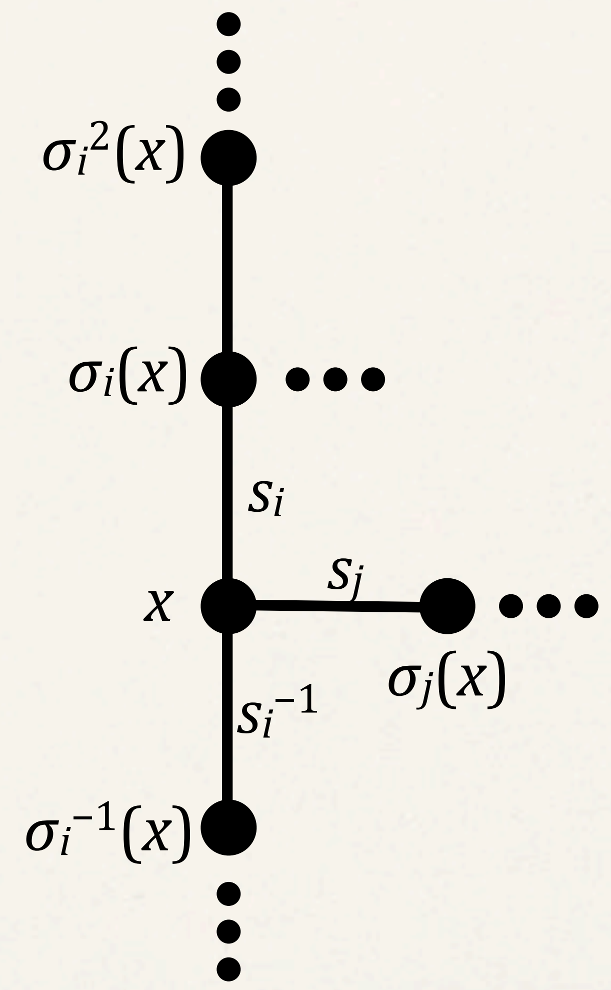

$$\Psi(x) = T_x$$

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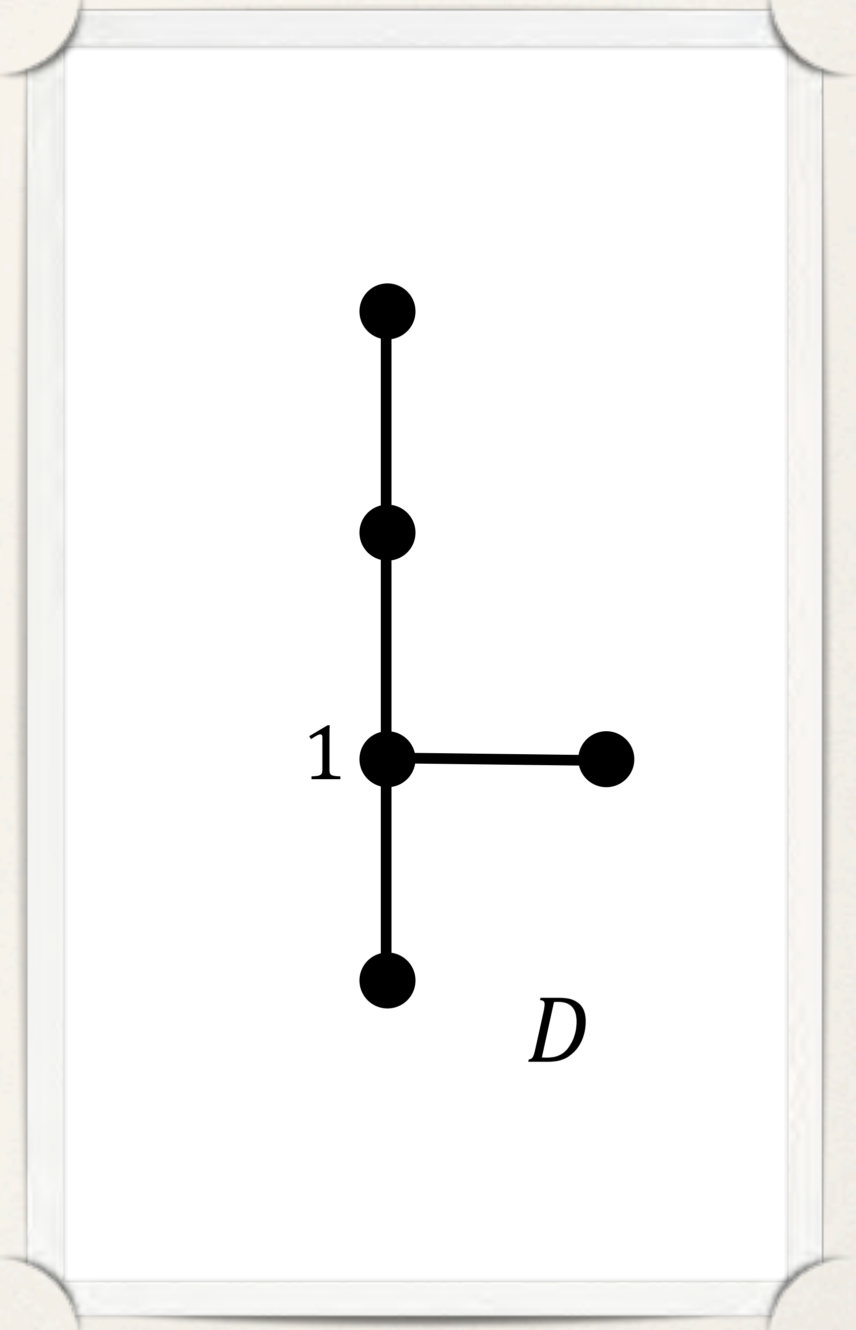

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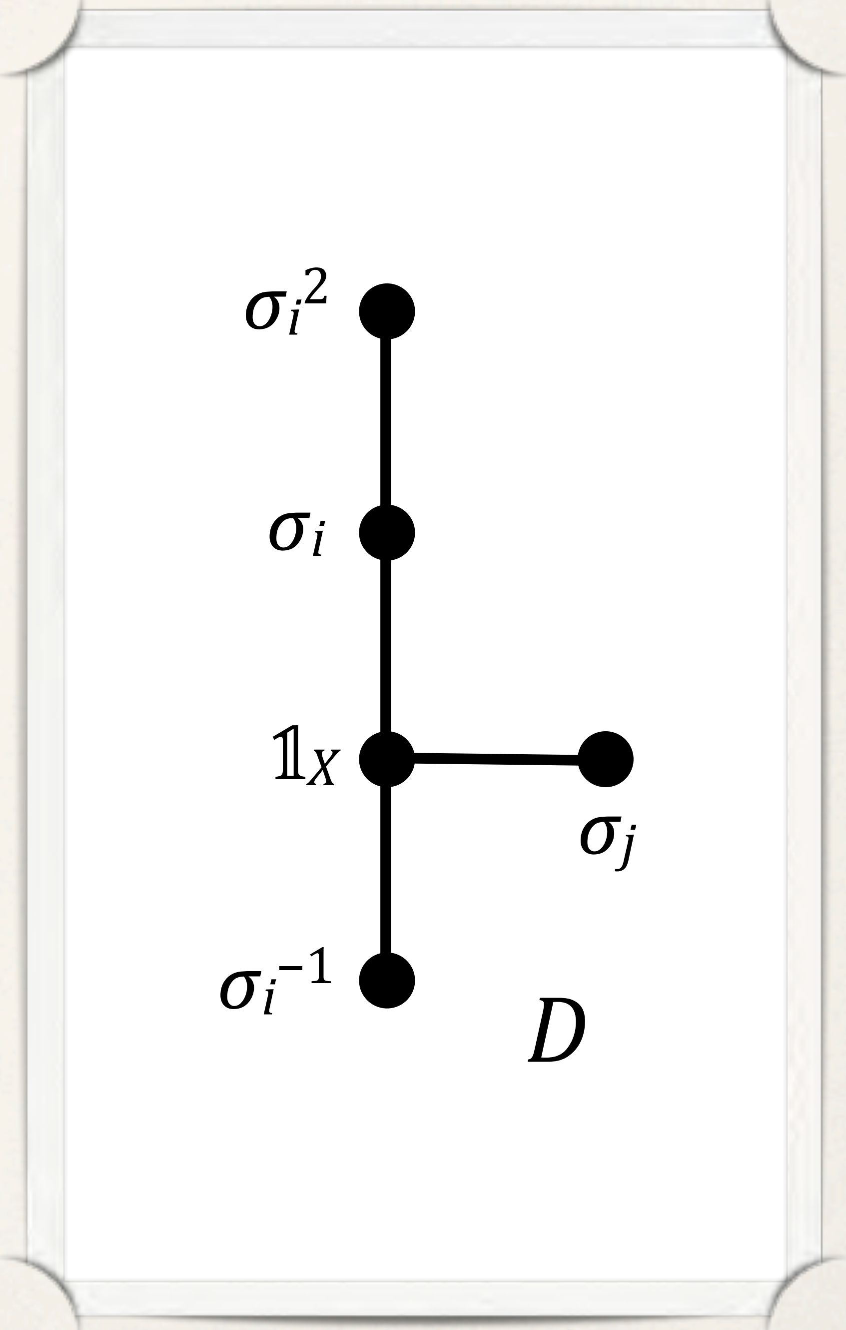
$$\Psi^{-1}(O) = \bigcap_{w \in D} \text{dom}\ w = \text{a clopen set}$$



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Thanks!