AFFABILITY OF LAMINATIONS DEFINED BY REPETITIVE PLANAR TILINGS

Pablo González Sequeiros
Joint work with F. Alcalde and Á. Lozano
Objective

Theorem

The continuous hull of any repetitive and aperiodic planar tiling is affable
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The continuous hull of any repetitive and aperiodic planar tiling is affable

Summary

1. Tilings
2. Laminations defined by tilings
3. Affable equivalence relations
4. Affability Theorem for planar tilings
5. Robinson Inflation
Theorem

*The continuous hull of any aperiodic and repetitive planar tiling is affable*
Definition

A *planar tiling* $\mathcal{T}$ is a partition of $\mathbb{R}^2$ into *tiles*, polygons touching face-to-face obtained by translation from a finite set of *prototiles*. 
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Definition

A tiling $\mathcal{T}$ is \textit{aperiodic} if it has no translation symmetries:

$$\mathcal{T} \neq \mathcal{T} + v, \forall v \in \mathbb{R}^2$$

A set of prototiles $\mathcal{P}$ is said to be \textit{aperiodic} if any tiling obtained from $\mathcal{P}$ is aperiodic.
Definition

A tiling $\mathcal{T}$ is **repetitive** if for any radius $r > 0$, there exists $R = R(r) > 0$ such that any ball $B(x, R)$ contains a translated copy $M + v$ of any patch $M$ with diameter $\delta(M) < r$. 

\[ R \]
2. Laminations defined by tilings
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Let \( \mathbb{T}(\mathcal{P}) \) be the set of tilings \( \mathcal{T} \) constructed from a finite set of prototiles \( \mathcal{P} \)
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Gromov-Hausdorff topology

\[
U^R_{\varepsilon, \varepsilon'}(\mathcal{T}) = \{ \mathcal{T}' \in \mathbb{T}(\mathcal{P}) \mid \exists v, v' \in \mathbb{R}^2 : \|v\| < \varepsilon, \|v'\| < \varepsilon', R(\mathcal{T} + v, \mathcal{T}' + v') > R \}
\]

\[
R(\mathcal{T}, \mathcal{T}') = \sup\{ R > 0 : B_{\mathcal{T}}(0, R) = B_{\mathcal{T}'}(0, R) \}
\]

Proposition

If \( \mathcal{P} \) is **finite**, then \( \mathbb{T}(\mathcal{P}) \) is a **compact** metrizable space laminated by the orbits of the natural action of \( \mathbb{R}^2 \).

The **leaf through** \( \mathcal{T} \in \mathbb{T}(\mathcal{P}) \) is its orbit \( \mathcal{L}_\mathcal{T} = \{ \mathcal{T} + v \mid v \in \mathbb{R}^2 \} \)
Fixing a base point in any prototile of $\mathcal{P}$, we obtain a **Delone set** $\mathcal{D}_T$ in any tiling $T \in \mathcal{T}(\mathcal{P})$.
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$r$-discrete and $R$-dense

The subspace $T = \{ T \in \mathbb{T}(\mathcal{P}) : 0 \in D_T \}$ is a totally disconnected and compact \textit{total transversal} of $\mathbb{T}(\mathcal{P})$. 
2. Laminations defined by tilings

Foliated atlas / Flow box decomposition

\[ \varphi : (v, \mathcal{T}') \in B(0, r) \times U_r(\mathcal{T}) \mapsto \mathcal{T}' + v \in \mathbb{T}(\mathcal{P}) \]
2. Laminations defined by tilings

Foliated atlas / Flow box decomposition

\[ \varphi : (v, \mathcal{T}') \in M \times B_M(\mathcal{T}) \quad \mapsto \quad \mathcal{T}' + v \in \mathcal{T}(\mathcal{P}) \]
Minimal subsets

The \textit{continuous hull of } $\mathcal{T}$ is the closure of its orbit $\mathcal{L}_T = \{\mathcal{T} + v \mid v \in \mathbb{R}^2\}$

Proposition

A tiling $\mathcal{T} \in \mathbb{T}(\mathcal{P})$ is \textit{repetitive} if and only if its continuous hull $\mathbb{X} = \overline{\mathcal{L}_T}$ is \textit{minimal}.

Leaves without holonomy

Lemma

The \textit{orbit} of any \textit{aperiodic} tiling is \textit{homeomorphic to } $\mathbb{R}^2$.
2. Laminations defined by tilings

Theorem

The continuous hull $X$ of any repetitive and aperiodic planar tiling $\mathcal{T}$ is a minimal transversally Cantor lamination without holonomy.

The lamination $\mathcal{F} = \{L_{\mathcal{T}}\}_{\mathcal{T} \in X}$ defines on the total transversal $X$ an étale equivalence relation (EER)

$$\mathcal{R} = \{ (\mathcal{T}, \mathcal{T} - \nu) \in X \times X \mid \nu \in D_{\mathcal{T}} \}$$

which represents its transverse dynamics.
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Definition

Two EER’s $\mathcal{R}$ and $\mathcal{R}'$ on $X$ and $X'$ are orbit equivalent if there exists a homeomorphism $\varphi : X \to X'$ such that $\varphi(\mathcal{R}[x]) = \mathcal{R}'[\varphi(x)]$, $\forall x \in X$. 

AFFABILITY OF LAMINATIONS DEFINED BY REPETITIVE PLANAR TILING

Pablo González Sequeiros
Theorem

*The continuous hull of any aperiodic and repetitive planar tiling is affable*
3. Affable equivalence relations
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Let \( \mathcal{R} \) be an EER on a totally disconnected space \( X \)

**Definition** [Giordano-Putnam-Skau, Renault]

\( \mathcal{R} \) is **compact (CEER)** if \( \mathcal{R} \setminus \Delta \) is compact subset of \( X \times X \)

\[ \Rightarrow \exists N \geq 1 : \# \mathcal{R}[x] \leq N, \ \forall x \in X \]
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\]

\( \mathcal{R} \) is **approximately finite (AF)** if there exists an increasing sequence of CEERs \( \mathcal{R}_n \) such that \( \mathcal{R} = \lim_{n \to \infty} \mathcal{R}_n \)
3. Affable equivalence relations

Let $\mathcal{R}$ be an EER on a totally disconnected space $X$

Definition [Giordano-Putnam-Skau, Renault]

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$\mathcal{R}$ is **approximately finite** (AF) if there exists an increasing sequence of CEERs $\mathcal{R}_n$ such that $\mathcal{R} = \lim_{n \to \infty} \mathcal{R}_n$

$\mathcal{R}$ is said to be **affable** if it is orbit equivalent to an AF equivalence relation
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A transversally Cantor lamination $(M, \mathcal{F})$ without holonomy is **affable** if the equivalence relation $\mathcal{R}$ induced on any total transversal $T$ is affable
3. Affable equivalence relations

\[ X = \text{Space of infinite paths from } V_0 \]

\[(e_i) R_{\text{tail}} (e'_i) \iff \exists N : e_i = e'_i, \forall i \geq N \]

\[ R_{\text{tail}} = \lim_{n \to \infty} R_n \]

\[(e_i) R_n (e'_i) \iff e_i = e'_i, \forall i \geq n \]
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$(e_i) R_n (e'_i) \iff e_i = e'_i, \forall i \geq n$

Theorem [Giordano-Putnam-Skau, Renault]

Any AF equivalence relation is isomorphic to the tail equivalence relation on the infinite path space of a certain Bratteli diagram.
3. Affable equivalence relations

\[ X = \text{Space of infinite paths from } V_0 \]

\[ (e_i) R_{tail} (e_i') \iff \exists N : e_i = e_i', \ \forall i \geq N \]

\[ R_{tail} = \lim_{n \to \infty} R_n \]

\[ (e_i) R_n (e_i') \iff e_i = e_i', \ \forall i \geq n \]

Theorem [Giordano-Putnam-Skau]

Any minimal affable equivalence relation on a compact space is orbit equivalent to a minimal \( \mathbb{Z} \)-action (Bratteli-Vershik system).
4. Affability Theorem
Theorem

The continuous hull of any aperiodic and repetitive planar tiling is affable
Sketch of the proof

1. Applying the inflation process to obtain an affable equivalence subrelation $R_\infty$ of $R$

2. Defining the boundary of $R_\infty$ and studying its properties

3. Applying the Absorption Theorem
Inflation [Bellisard-Benedetti-Gambaudo]

For any flow box decomposition $B$ of $X$ there exists another one $B'$ inflated of $B$:
4. Affability Theorem

Let \( \{B^{(n)}\}_{n \in \mathbb{N}} \) a sequence of flow box decompositions obtained by inflation

Compact subrelations \( R_n \)

For any \( n \in \mathbb{N} \) and \( \mathcal{T} \in X \), we define:

\[
R_n[\mathcal{T}] = \mathbb{P}_n \cap X
\]
4. Affability Theorem

Let \( \{B^{(n)}\}_{n \in \mathbb{N}} \) a sequence of flow box decompositions obtained by inflation

**Compact subrelations** \( \mathcal{R}_n \)

For any \( n \in \mathbb{N} \) and \( \mathcal{T} \in X \), we define:

\[
\mathcal{R}_n = \bigcup_{\mathcal{T} \in X^{(n)}} \mathcal{R}_n[\mathcal{T}]
\]

\[
\mathcal{R}_n[\mathcal{T}] = \mathbb{P}_n \cap X
\]
Let $\{B^{(n)}\}_{n \in \mathbb{N}}$ a sequence of flow box decompositions obtained by inflation

**Compact subrelations $\mathcal{R}_n$**

For any $n \in \mathbb{N}$ and $\mathcal{T} \in X$, we define:

\[
\mathcal{R}_n = \bigcup_{\mathcal{T} \in X^{(n)}} \mathcal{R}_n[\mathcal{T}] = \bigcup_{B_n \in B^{(n)}} B_n
\]

\[
B_n = B_n \cap X, \ B_n \in B^{(n)}
\]
4. Affability Theorem

Let \( \{B^{(n)}\}_{n \in \mathbb{N}} \) a sequence of flow box decompositions obtained by inflation

**Compact subrelations** \( R_n \)

For any \( n \in \mathbb{N} \) and \( T \in X \), we define:

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R_n = \bigcup_{T \in X^{(n)}} R_n[T] = \bigcup_{B_n \in B^{(n)}} B_n
\]

\[
B_n = B_n \cap X, \quad B_n \in B^{(n)}
\]

\( R_n \) is a CEER

**Proposition**

The direct limit \( R_\infty = \lim_{n \to \infty} R_n \) is a **minimal open AF equivalence subrelation** of \( R \).
4. Affability Theorem

Sketch of the proof

1. Applying the inflation process to obtain an affable equivalence subrelation $\mathcal{R}_\infty$ of $\mathcal{R}$

2. Defining the boundary of $\mathcal{R}_\infty$ and studying its properties

3. Applying the Absorption Theorem
4. Affability Theorem

Boundary of $\mathcal{R}_\infty$

For any $n \in \mathbb{N}$ and $\mathcal{T} \in X$, we define:

$$\partial \mathcal{R}_n[\mathcal{T}] = \partial \mathcal{P}_n \cap X$$
Boundary of $\mathcal{R}_\infty$

For any $n \in \mathbb{N}$ and $\mathcal{T} \in X$, we define:

$$
\partial \mathcal{R}_n = \bigcup_{\mathcal{T} \in X^{(n)}} \partial \mathcal{R}_n[\mathcal{T}] = \bigcup_{B_n \in \mathcal{B}^{(n)}} \partial_v B_n
$$

$$
\partial_c \mathcal{R}_n = \bigcup_{\mathcal{P}_n \in \mathcal{P}^{(n)}} \partial \mathcal{P}_n = \bigcup_{B_n \in \mathcal{B}^{(n)}} \partial_v B_n
$$

$$
\partial \mathcal{R}_n[\mathcal{T}] = \partial \mathcal{P}_n \cap X
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Boundary of $\mathcal{R}_\infty$

For any $n \in \mathbb{N}$ and $\mathcal{T} \in X$, we define:

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$$\partial_c \mathcal{R}_n = \bigcup_{\mathcal{P}_n \in \mathcal{P}^{(n)}} \partial \mathcal{P}_n = \bigcup_{\mathcal{B}_n \in \mathcal{B}^{(n)}} \partial_v \mathcal{B}_n$$

$$\partial \mathcal{R}_{n+1} \subset \partial \mathcal{R}_n$$
Boundary of $\mathcal{R}_\infty$

For any $n \in \mathbb{N}$ and $\mathcal{T} \in X$, we define:

$$\partial \mathcal{R}_n = \bigcup_{\mathcal{T} \in X^{(n)}} \partial \mathcal{R}_n[\mathcal{T}] = \bigcup_{\mathcal{B}_n \in \mathcal{B}^{(n)}} \partial_v \mathcal{B}_n$$

$$\partial_c \mathcal{R}_n = \bigcup_{\mathcal{P}_n \in \mathcal{P}^{(n)}} \partial \mathcal{P}_n = \bigcup_{\mathcal{B}_n \in \mathcal{B}^{(n)}} \partial_v \mathcal{B}_n$$

$$\partial \mathcal{R}_\infty = \bigcap_{n \in \mathbb{N}} \partial \mathcal{R}_n$$

$$\partial_c \mathcal{R}_\infty = \bigcap_{n \in \mathbb{N}} \partial_c \mathcal{R}_n$$

$\partial \mathcal{R}_{n+1} \subset \partial \mathcal{R}_n$
4. Affability Theorem

Boundary of $\mathcal{R}_\infty$

$$\partial \mathcal{R}_\infty = \bigcap_{n \in \mathbb{N}} \partial \mathcal{R}_n$$

$$\partial_c \mathcal{R}_\infty = \bigcap_{n \in \mathbb{N}} \partial_c \mathcal{R}_n$$
4. Affability Theorem

Sketch of the proof

1. Applying the inflation process to obtain an affable equivalence subrelation \( R_\infty \) of \( R \)

2. Defining the boundary of \( R_\infty \) and studying its properties

3. Applying the Absorption Theorem
Absorption

Theorem [Giordano-Matui-Putnam-Skau]

- Let $\mathcal{R}$ be a **minimal AF** equivalence relation on the Cantor set $X$
- Let $\mathcal{K}$ be a CEER on a **closed** subset $Y \subset X$ such that

\[ \ldots \]

Then $\mathcal{R} \vee \mathcal{K}$ is **affable**
Absorption

Theorem [Giordano-Matui-Putnam-Skau]

- Let $\mathcal{R}$ be a minimal AF equivalence relation on the Cantor set $X$
- Let $\mathcal{K}$ be a CEER on a closed subset $Y \subset X$ such that

\[ Y \text{ is } \mathcal{R}\text{-thin} \iff \mu(Y) = 0 \text{ for any } \mathcal{R}\text{-invariant measure} \]

Then $\mathcal{R} \vee \mathcal{K}$ is affable
Absorption

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  \begin{align*}
  &Y \text{ is } \mathcal{R}\text{-thin} \iff \mu(Y) = 0 \text{ for any } \mathcal{R}\text{-invariant measure} \\
  &\mathcal{R}|_Y \text{ remains étale and } \mathcal{K} \text{ is transverse to } \mathcal{R}|_Y
  \end{align*}

Then $\mathcal{R} \vee \mathcal{K}$ is affable

\[ \mathcal{R}|_Y \cap \mathcal{K} = \Delta_Y \]
Absorption

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Absorption

Theorem [Giordano-Matui-Putnam-Skau]

- Let $\mathcal{R}$ be a minimal AF equivalence relation on the Cantor set $X$
- Let $\mathcal{K}$ be a CEER on a closed subset $Y \subset X$ such that
  \[
  \text{Y is } \mathcal{R}\text{-thin } \iff \mu(Y) = 0 \text{ for any } \mathcal{R}\text{-invariant measure}
  \]
  $\mathcal{R}|_Y$ remains étale and $\mathcal{K}$ is transverse to $\mathcal{R}|_Y$

Then $\mathcal{R} \vee \mathcal{K}$ is affable

$\mathcal{R}|_Y \cap \mathcal{K} = \Delta_Y$

$\begin{align*}
x & \sim \mathcal{K} \quad y \quad \mathcal{R}
\end{align*}$

$\begin{align*}
x' & \sim \mathcal{K} \quad y'
\end{align*}$
Absorption

Theorem [Giordano-Matui-Putnam-Skau]

- $\mathcal{R}_\infty$ minimal AF equivalence relation on $X$
- $\mathcal{K}$ a CEER on $Y = \partial \mathcal{R}_\infty$

$\mu(\partial \mathcal{R}_\infty) = 0$ for any $\mathcal{R}_\infty$-invariant measure

Any $\mathcal{R}$-equivalence class separates into a uniformly bounded number of $\mathcal{R}_\infty$-equivalence classes

Then $\mathcal{R} = \mathcal{R}_\infty \vee \mathcal{K}$ affable
5. Robinson inflation
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Step 1: Reduction to square tilings

Theorem [Sadun-Williams]

The continuous hull of a any planar tiling is orbit equivalent to the continuous hull of a tiling whose tiles are marked squares

We can assume that $\mathcal{P}$ is a finite set of marked squares and all leaves in the continuous hull $X$ are endowed with the max-distance
Step 2: Partial decomposition

Let $\mathcal{B}_1$ be a finite family of compact flow boxes

$$\mathcal{B}_{1,i} \cong \mathcal{P}_{1,i} \times C_{1,i}$$

where the plaques $\mathcal{P}_{1,i}$ are square patterns of side $N_1$ and the axis

$$C_1 = \bigcup_{i=1}^{k_1} C_{1,i}$$

is $N_1$-dense in $X$.  

5. Robinson inflation
Step 3: Inflated decomposition

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Step 3: Inflated decomposition

$B'_1$, flow box decomposition with plaques $P'$
5. Robinson inflation

Step 3: Inflated decomposition

$B'_1$, flow box decomposition with plaques $P'$

$B''_1$, inflated flow box decomposition with plaques $P''$

$\{B''_n\}_{n \in \mathbb{N}}$, sequence of flow box decompositions obtained by Robinson inflation
5. Robinson inflation

Step 4: Properties of the boundary

Lemma

The isoperimetric ratio of the plaques tends to zero uniformly:

$$\lim_{n \to \infty} \frac{L(\partial P''_n)}{A(P''_n)} = 0$$

Proposition

The boundary $\partial R_\infty$ is $R_\infty$-thin.
5. Robinson inflation

Step 4: Properties of the boundary

Proposition

Any $\mathcal{R}$-equivalence class separates into at most into $4 \mathcal{R}_\infty$-equivalence classes.
Step 5: Absorption

Applying inductively the Absorption Theorem, we can exhaust $\mathcal{R}$ by a countable family of affable increasing EERs:

$$\mathcal{R}_\infty \lor \mathcal{K}_2 \lor \mathcal{K}_{3,1} \lor \cdots \lor \mathcal{K}_{3,m} \lor \mathcal{K}_{4,0} \lor \cdots \lor \mathcal{K}_{4,l}$$

$\mathcal{R}$ is affable
Thank you for your attention

Note: Figures of Robinson tilings have been made from the applet P. Ofella (with C. Woll): *Robinson Tiling*, from *The Wolfram Demonstrations Project*. http://demonstrations.wolfram.com/RobinsonTiling/