Lefschetz sequences
and chaotic dynamics

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Lefschetz number of $h$

\[ \text{Lefschetz number of } h = \sum_{n \in \mathbb{N}} (-1)^n \text{trace}(h_n). \]

Euler characteristic of $E$:

\[ \chi(E) = \text{Lefschetz number of } \text{id}_E = \sum_{n \in \mathbb{N}} (-1)^n \dim E_n. \]
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Dual Lefschetz sequence of $h : E \to E$:

$$L^* k = L((h^* - \text{id}_E) k) = \sum_{m=0}^{k} (-1)^{k-m} L_m.$$ 

Question: Why is a dual Lefschetz sequence interesting from a dynamical point of view? 

Motivation: Topological methods for detecting chaotic dynamics.
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$f : \mathbb{R} \times M \rightarrow TM$ is a smooth time dependent vector field on manifold $M$, $T$-periodic with respect to time.
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**Definition**

The Poincaré map of the equation

\[(\ast) \quad \dot{x} = f(t, x)\]

is the function

$$P : x_0 \mapsto P(x_0) := \varphi_{(0,T)}(x_0).$$

$\varphi_{(0,T)}(x_0)$ is the value of the solution of the problem

$$x' = f(t, x), \quad x(0) = x_0$$

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at time $T$.

**Remark**

$$\{ k\text{-periodic points of } P \} = \{ \text{Initial points of } kT\text{-periodic solutions of } (*) \}$$
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if

1. \( W \) is a block for the vector-field \([1 \ f]\),
2. there is a compact set \( W^- \) contained in \( W \) such that
   \[
   W^- = W^- \cup (\{T\} \times W_T),
   \]
3. there exists a homeomorphism \( h : [0, T] \times W_0 \to W \) such that \( \pi_1 \circ h = \pi_1 \) and \( h([0, T] \times W_0^-) = W^- \),
4. \( W_0 = W_T, W_0^- = W_T^- \) are compact ENR's.
$W$ is a segment over $[0, T]$

$m: W_0 \ni x \mapsto \pi_2 h(T, \pi_2 h^{-1}(0, x)) \in W_0$
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$$\mu_{\mathcal{W}} := H(m): H(\mathcal{W}_0, \mathcal{W}_0^{--}) \to H(\mathcal{W}_0, \mathcal{W}_0^{--})$$
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$H$ a homology functor over $\mathbb{Q}$

$$\mu_W := H(m): H(W_0, W_0 \rightarrow) \rightarrow H(W_0, W_0 \rightarrow)$$

$$L(\mu_W) := \sum_{n=0}^{\infty} (-1)^n \text{trace } H_n(m)$$
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**Definition**
The equation 
\[ \dot{x} = f(t, x) \]
is \( \Sigma_2 \)-chaotic if there exists a compact set \( I \subset \mathbb{M} \), invariant with respect to the Poincaré map \( P \) and a function \( g : I \rightarrow \Sigma_2 \) such that

1. \( g \) is continuous and surjective,
2. \( \sigma \circ g = g \circ P \),
3. for every \( k \)-periodic sequence \( s \in \Sigma_2 \) the set \( g^{-1}(s) \) contains at least one \( k \)-periodic point of \( P \).
Let \( W \) and \( U \) be segments over \([0, T]\) for the equation

\[(\ast) \quad \dot{x} = f(t, x),\]

where \( f \) is smooth and \( T \)-periodic with respect to \( t \) and

1. \((W_0, W_0^{-}) = (U_0, U_0^{-})\),
2. \( U \subset W \),
3. \( \mu_U = \text{id}_{H(W_0, W_0^{-})} \).
$I \subset \mathcal{W}_0$ is the set of all points in $\mathcal{W}_0$ whose full trajectories are contained in the bigger segment $\mathcal{W}$. 

Define $g: I \rightarrow \Sigma^2$ by

$$g(x)_n = \begin{cases} 1, & \text{if } P_n(x_0) \text{ leaves } U \text{ in time less than } T, \\ 0, & \text{otherwise}. \end{cases}$$

One can prove that $g: I \rightarrow \Sigma^2$ is continuous and $\sigma \circ g = g \circ P$. 
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$$L_k = L(\mu^{k}_W), \quad k \geq 0$$

where $\mu^{k}_W = \underbrace{\mu_W \circ \ldots \circ \mu_W}_{k}$, $\mu^0_W = \text{id}_{H(W_0, W_0^-)}$. 

Theorem (Srzednicki, KW) $L^* k \neq 0$ implies that $g^{-1}(c^n_k) \cap \text{Fix}(P^n)$ is non-empty.

Corollary If $L_1 \neq L_0$ then $g : I \to \Sigma_2$ is surjective. Moreover, if $p > |L_1 - L_0|$ is prime, then $L^{*} p \neq 0$. 

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**Theorem** (Srzednicki, KW)

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**Theorem (Srzednicki, KW)**

$L_k^* \neq 0$ implies that $g^{-1}(c_k^n) \cap \text{Fix}(P^n)$ is non-empty.

**Corollary**

If $L_1 \neq L_0$ then $g : I \to \Sigma_2$ is surjective. Moreover, if $p > |L_1 - L_0|$ is prime, then $L_p^* \neq 0$. 
If $L_k$ is constant, then $L^*_k = 0$ for $k \geq 1$. 
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**Example**

For 2-periodic $L_k$ we have

$$L_k^* = \begin{cases} L_0, & \text{if } k = 0; \\ (-2)^{k-1}(L_1 - L_0), & k \geq 1, \end{cases}$$
Theorem (Pieniążyk, KW)

Assume that $L_n$ is $m$-periodic and

$$L_1 = \ldots = L_{m-1}, \quad 0 \neq L_m = L_0 \neq L_1.$$ 

If $m$ is even then $L_k^* \neq 0$ for $k \geq 0$. 

If $m$ is odd then $L_k^* = 0$ iff $k$ is odd multiplicity of $m$. 
**Theorem** (Pieniążek,KW)

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**Theorem** (Pieniążek, Srzednicki,KW)

For $n \geq 2$ and $\phi > 0$ small enough the equation

$$\dot{z} = (1 + e^{i\phi t}|z|^2)z^n$$

is $\Sigma_2$-chaotic for odd $n$. If $n$ is even, then $g$ is surjective and $F(c_m^k) \neq \emptyset$ whenever $k$ is not an odd multiplicity of $m = n + 1$. 
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Möbius function $\mu : \mathbb{N} \to \mathbb{Z}$ is given by

$$\mu(n) = \begin{cases} 
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(-1)^k & \text{if } n = p_1 \ldots p_k, \ p_i \text{ different primes,} \\
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A sequence $a_n$ of integers satisfies the Dold’s relations if

$$\forall n \in \mathbb{N} \quad \sum_{d|n} \mu(n/d) a_d \equiv 0 \mod n.$$
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In particular,

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a_p \equiv a_1 \mod p,
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for prime $p$. 
Assume that $p$ is odd prime and $L_k$ is $p$-periodic and non-constant.
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Then

$$L_1 = \ldots = L_{p-1}, \quad L_p = L_0 \neq L_1.$$ 

It follows that $L_k^* = 0$ if and only if $k$ is odd multiplicity of $p$. 
Lemma
Assume that $a_n$ is $m$-periodic sequence of integers such that for each prime $p$

$$a_p \cong a_1 \mod p. \quad (1)$$

If $(r, m) = 1$, then $a_r = a_1$
Lemma
Assume that $a_n$ is $m$-periodic sequence of integers such that for each prime $p$

$$a_p \cong a_1 \mod p.$$ (1)

If $(r, m) = 1$, then $a_r = a_1$

Proof: Since $(r, m) = 1$, so by Dirichlet’s theorem there are $n_k \to \infty$ such that

$$p_k = r + n_k m$$

is a prime number for each $k \in \mathbb{N}$. In particular, there is $k$ such that

$$p_k > |a_r - a_1|.$$ 

Then, by

$$a_r = a_{p_k} \cong a_1 \mod p_k$$

we get that $p_k | a_r - a_1$, hence $a_r = a_1$. 
Theorem (Marzantowicz, KW)

(a) Sequences $L_n$ and $L^*_n$ satisfy Dold’s relations,
(b) If $L_n$ is bounded but not the constant sequence (so it is periodic), then $L^*_n$ is unbounded.
Theorem (Marzantowicz, KW)

(a) Sequences $L_n$ and $L^*_n$ satisfy Dold’s relations,

(b) If $L_n$ is bounded but not the constant sequence (so it is periodic), then $L^*_n$ is unbounded. Moreover there exists $\rho > 1$ such that

$$\lim_{k \to \infty} \frac{L^*_{2lk}}{\rho^{2lk}} = a \neq 0.$$
For \( n \geq |L_1 - L_0| \) we put

\[ S(n) = \{ p \in \mathbb{N} : |L_1 - L_0| < p < n, \ p \nmid n, \ p \text{ prime} \}. \]

and

\[ \nu(n) = \sum_{p \in S(n)} \binom{n}{p}. \]
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**Theorem (Gierzkiewicz, KW)**

If $n \geq |L_1 - L_0|$, then $P$ has at least $\nu(n)$ periodic points with minimal period $n$. 
**Theorem**

Let $k \geq 1$ be fixed. If the sequence $\{L^*_m\}_{m \geq 1}$ is unbounded, then for each $n$-periodic sequence $c^n_k$ the set $g^{-1}(c^n_k)$ contains infinitely many periodic points of $P$.

**Corollary**

Assume that $L_n$ is a bounded but not the constant sequence. Then, $g^{-1}(1^\infty)$ contains infinitely many periodic points of $P$. 