Homeomorphism groups of commutator width one

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Introduction

**Definition**

A group is *uniformly perfect* if every element is written as a product of a bounded number of commutators. The least number of such bound is called the *commutator width*.

**Theorem**

Any element of \( \text{Homeo}(S^n)_0 \) can be written as one commutator, where \( \text{Homeo}(S^n)_0 \) is the identity component of the group of homeomorphisms of the \( n \)-dimensional sphere \( S^n \).

**Theorem**

Any element of \( \text{Homeo}(\mu^n) \) can be written as one commutator, where \( \text{Homeo}(\mu^n) \) is the group of homeomorphisms of the \( n \)-dimensional Menger compact space \( \mu^n \).
David Eisenbud, Ulrich Hirsch and Walter Neumann showed in


that the commutator width of $\text{Homeo}(S^1)_0$ is one.
In 1958, Anderson showed that in the group $\text{Homeo}_c(\mathbb{R}^n)$ of homeomorphisms of the $n$-dimensional Euclidean space $\mathbb{R}^n$ with compact support, any element can be written as one commutator.

Then it has been known that the commutator width of $\text{Homeo}(S^n)_0$ is at most 2.
It is worth recalling the construction by Anderson.

For given \( f \in \text{Homeo}_c(\mathbb{R}^n) \), we find a bounded ball \( U \) such that the support \( \text{supp}(f) \subset U \).

Then we can find an element \( g \in \text{Homeo}_c(\mathbb{R}^n) \) such that

1. \( g^n(U) \) (\( n \in \mathbb{Z} \)) are disjoint and
2. \( \lim_{n \to \infty} \text{diam}(g^n(U)) = 0 \).

Put \( F = \prod_{n=0}^{\infty} g^n f g^{-n} \), then we have \( gFg^{-1} = f^{-1}F \).

Thus \( f = FgF^{-1}g^{-1} \).
The construction by Anderson

\[ f \xrightarrow{g} g(U) \xrightarrow{g} g^2(U) \xrightarrow{g} g^3(U) \xrightarrow{g} g^4(U) \]
What is the meaning that the commutator width is one?

In the case of \( \text{Homeo}_c(\mathbb{R}^n) \), we see that for any element \( f \), there exist \( g \) such that \( g \) and \( fg \) are conjugate.

That is, \( g \) is dynamically so strong that \( fg \) and \( g \) have the same dynamics, and hence they are conjugate.
In the case of $\text{Homeo}_0(S^n)$ or $\text{Homeo}(\mu^n)$, we have the candidate which has the strong dynamics.

The candidate is the topologically hyperbolic homeomorphism.

**Definition**

A topologically hyperbolic homeomorphism is a homeomorphism $h$ with one source $s_+$ and one sink $s_-$ such that $\lim_{n \to +\infty} h^n(x) = s_-$ and $\lim_{n \to -\infty} h^n(x) = s_+$ for $x \notin \{s_-, s_+\}$.

Orientation preserving topologically hyperbolic homeomorphisms with nice fundamental domains outside the fixed points are conjugate.
Diffeo\((S^1)_0\)
Hence for a given homeomorphism $f$, we will construct a topologically hyperbolic homeomorphism $g$ which is so strong that $fg$ is topologically hyperbolic. See Figure.

Take four disjoint disks $f^{-1}(D^n_0)$, $D^n_0$, $D^n_1$, $f(D^n_1)$ and define $g$ such that $g(D^n_0) = S^n \setminus \text{int}(D^n_1)$ and $g(S^n \setminus \text{int}(D^n_0)) = D^n_1$, and observe how are the actions of $g$ and $fg$. 
$\text{Diffeo}(S^n)_0$
\[ \text{Diffeo}(S^n)_0 \]

\[
\begin{align*}
(gfg)(\Sigma) & \quad (fgf^{-1})(\Sigma) \\
(g^2)(\Sigma) & \quad (fg^2)(\Sigma) \\
(f^{-1}g^{-1}f^{-1})(\Sigma) & \quad (g^{-1}f)(\Sigma) \\
(f^{-1}g^{-1})(\Sigma) & \quad (g^{-1})(\Sigma) \\
(f^{-1}g^{-1}f)(\Sigma) & \quad (g^{-1}f^{-1})(\Sigma)
\end{align*}
\]
In the construction, we need the following deep theorems.

**Generalized Schoenflies Theorem (Brown)**

Let $\Sigma$ be a locally flat $(n - 1)$-dimensional sphere in the $n$-dimensional sphere $S^n$. Then the closures of the complementary domains of $\Sigma$ are homeomorphic to the $n$-dimensional disk $D^n$.

**Annulus conjecture (Kirby, Quinn)**

Let $\Sigma_0$ and $\Sigma_1$ be disjoint locally flat $(n - 1)$-dimensional spheres in the $n$-dimensional sphere $S^n$. Then the closure of the region between them is homeomorphic to $S^{n-1} \times [0, 1]$. 
For the proof

Lemma

For any compact set $K$ in the interior $\text{int}(D^n)$ of the standard disk $D^k$ and any positive real number $\varepsilon$, there is a homeomorphism $\varphi_{K,\varepsilon} : D^n \to D^n$ which is the identity on $\partial D^n$ such that $\text{diam}(\varphi_{K,\varepsilon}(K)) \leq \varepsilon$. 
For the \( n \)-dimensional Menger compact space \( \mu^n \), we use the following propositions in the place of the above theorems and lemma. They are shown by using Bestvina’s Z-set unknotting theorem.
Proposition

Let $A$ be a closed set in the compact $n$-dimensional Menger space $\mu^n$ such that

(1) $A$ is homeomorphic to the compact $(n - 1)$-dimensional Menger space $\mu^{n-1}$,

(2) $\mu^n \setminus A = U_1 \cup U_2$, $U_1 \neq \emptyset$, $U_2 \neq \emptyset$ and $U_1 \cap U_2 = \emptyset$, and

(3) $U_1 \cup A$ and $U_2 \cup A$ are $n$-dimensional Menger manifolds and $A$ is a $Z$-set in $U_1 \cup A$ and in $U_2 \cup A$.

Then $U_1 \cup A$ and $U_2 \cup A$ are homeomorphic to $\mu^n$. 
Prpopsition

Let $A_1$ and $A_2$ be a closed set in the compact $n$-dimensional Menger space $\mu^n$ such that

1. $A_1$ and $A_2$ are homeomorphic to the disjoint union of two compact $(n - 1)$-dimensional Menger space $\mu^{n-1}$,

2. $\mu^n \setminus A_i = U_{i1} \cup U_{i2} \cup U_{i3}$ (disjoint union of nonempty open sets; $i = 1, 2$) and

3. $\overline{U_{i1}}, \overline{U_{i2}}$ and $\overline{U_{i3}}$ are $n$-dimensional Menger manifolds and $A_i \subset \overline{U_{i2}}$ is a Z-set as well as $A_i \cap \overline{U_{i1}} \subset \overline{U_{i1}}$ and $A_i \cap \overline{U_{i3}} \subset \overline{U_{i3}}$.

Then any homeomorphism $A_1 \rightarrow A_2$ extends to a homeomorphism $h : \mu^n \rightarrow \mu^n$ such that $h(U_{1j}) = U_{2j}$ after changing the indices $i_1$ and $i_3$ if necessary.
Lemma

Under the assumption of the former Proposition for any compact set $K$ in $U_1$ and any positive real number $\varepsilon$, there is a homeomorphism $\varphi_{K,\varepsilon}$ of $\mu^n$ such that $\varphi_{K,\varepsilon}|(U_2 \cup A) = \text{id}_{U_2 \cup A}$, $\text{diam}(\varphi_{K,\varepsilon}(K)) \leq \varepsilon$. 