Nonlinear Algebra Meets Tensors in Quantum Information

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Goals of this Presentation

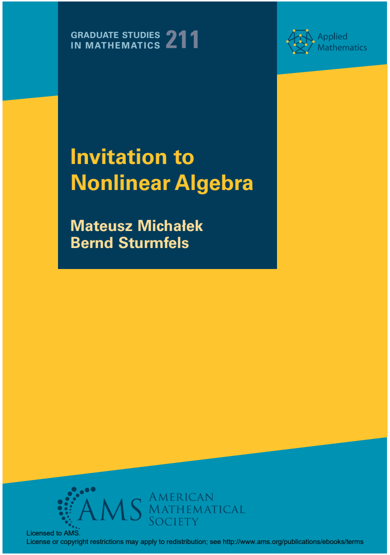
Study the mathematical concept of quantum entanglement:

- Develop methods for classifying entanglement up to symmetry (SVD and its higher order variants)
- Find the states with the most entanglement (Invariant Theory, Polynomial Optimization, Homotopy Continuation)

What is Nonlinear Algebra?

"A focus on computations and applications, and the theoretical underpinnings that this requires, results in a body of inquiry that is complementary to the existing curriculum."

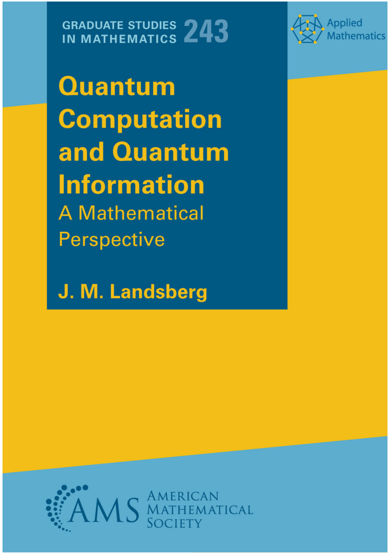
The art of solving polynomial equations, "fueled by recent theoretical advances, development of efficient software, and an increased awareness of these tools."



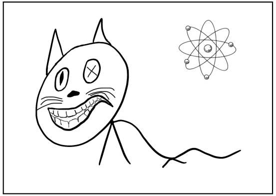
What is Quantum Information?

A beautiful theory rooted in Shannon's foundations on Information Theory, connected to tensors and representation theory.

The promised *quantum advantage* comes from the ability to store information utilizing the properties of quantum entanglement, and the different paradigm of computing made possible by the ability to manipulate qubits.



Quantum States and Qubits



Classical bits: 0 & 1, the smallest pieces of information.

Qubit: a unit vector in a Hilbert space $\mathcal{H} = \mathbb{C}\{|0\rangle, |1\rangle\}$.
a superposition of (orthonormal) basic states $|0\rangle, |1\rangle$.

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \text{ with } |\alpha|^2 + |\beta|^2 = 1.$$

Notation: $|a\rangle = \langle a|^\dagger$, outer product: $|a\rangle\langle b|$, inner product: $\langle a|b\rangle$.

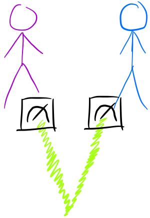
Basic projection operators:

$P_0 := |0\rangle\langle 0|$, so $P_0|\psi\rangle = \alpha|0\rangle$, and $P_1 = |1\rangle\langle 1|$, so $P_1|\psi\rangle = \beta|1\rangle$.

Measurements and expectation: e.g. probability that $|\psi\rangle$ is in state $|0\rangle$:

$$\langle\psi| P_0 |\psi\rangle = (\overline{\alpha}\langle 0| + \overline{\beta}\langle 1|)|0\rangle\langle 0|(\alpha|0\rangle + \beta|1\rangle) = |\alpha|^2$$

Entanglement for Multiple Particle Quantum States



Multi-particle state measurement basis: $|I\rangle = |i_1\rangle |i_2\rangle \cdots |i_m\rangle$

Prepare the EPR state $|\beta_{00}\rangle := \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.
Send Alice the first qubit, send Bob the second.
Alice measures $|0\rangle$ and $|1\rangle$ with equal probability 50%.
If Alice measures $|0\rangle$, then Bob will measure $|0\rangle$ with probability 100%.
The conditional independence statement fails:

$$A \text{ and } B \text{ independent} \quad \Longrightarrow \quad P(B = b | A = a) = P(A = a)P(B = b)$$

$$P(B = |0\rangle \,|A = |0\rangle) = 100\% \neq P(A = |0\rangle)P(B = |0\rangle) = 50\% \cdot 50\%$$

Product states $|\psi_1\rangle \cdots |\psi_m\rangle$ are unentangled, all others are *entangled*.

This Spooky Action at a Distance is a Resource!

Entanglement enables quantum teleportation, quantum channels...

We want to be able to detect and compare entanglement types, and find the states with the most entanglement.

Try to classify entanglement as a group orbit classification:

- Single-particle Local Unitary operators (LU): $U_d \times \cdots \times U_d$.
- Stochastic Local Operations with Classical Communication (SLOCC): $SL_d \times \cdots \times SL_d$.
- Entanglement types are not typically linearly ordered, but can be compared by inclusion of orbit closure.

Classifying Orbits of a Linear Group Action

Dynkin[1]: A classification of representations is a set of "characteristics":

- Invariant: The characteristics of equivalent representations coincide.
- Complete: Same characteristics $\implies$ equivalent representations.
- Compact and easy to compute.

The trichotomy for orbit classifications:

- Finitely many orbits (must have $\dim V \leq \dim G$).
- Tame orbits, classified using finitely many parameters.
- Wild orbit structure.

Attempt to give normal forms as Dynkin's characteristics.

[1] E. B. Dynkin, *Semisimple subalgebras of semisimple Lie algebras*, Am. Math. Soc. Trans. 1960

Singular Value Decomposition and Generalizations

SVD is a Classifier for 2 Particles

Input: 2-particle state $|\psi\rangle \in \mathcal{H} = \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2}$. Output: normal form Σ.

For classical SVD, Σ is quasi-diagonal with non-increasing diagonal.

Algorithm:

- View $|\psi\rangle$ as a map $A : (\mathbb{C}^{n_1})^* \to \mathbb{C}^{n_2}$ and compute eigenpairs of $A^\dagger A$ and of $AA^\dagger$. The eigenvectors give orthonormal bases U_1 of $\mathbb{C}^{n_1}$ and U_2 of $\mathbb{C}^{n_2}$ so that $U_1^\dagger |\psi\rangle U_2 = \Sigma$.
- The singular values are the square roots of the eigenvalues. (unique up to a finite group action).

[Ekhart-Young]: Truncation gives the best low-rank approximation.

Higher-Order Singular Value Decomposition

- Start: $\Phi = a_0 b_0 c_0 - a_1 b_0 c_0 + a_1 b_1 c_0 + a_0 b_0 c_1 + a_1 b_0 c_1 - a_0 b_1 c_1$
- Matrix flattenings:

$$\Phi_A = \begin{array}{c c} & \begin{array}{cccc} b_0 c_0 & b_0 c_1 & b_1 c_0 & b_1 c_1 \end{array} \\ \begin{array}{c} a_0 \\ a_1 \end{array} & \begin{bmatrix} 1 & 1 & 0 & -1 \\ -1 & 1 & 1 & 0 \end{bmatrix} \end{array}, \quad \Phi_B = \begin{array}{c c} & \begin{array}{cccc} a_0 c_0 & a_0 c_1 & a_1 c_0 & a_1 c_1 \end{array} \\ \begin{array}{c} b_0 \\ b_1 \end{array} & \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix} \end{array}, \quad \Phi_C = \begin{array}{c c} & \begin{array}{cccc} a_0 b_0 & a_0 b_1 & a_1 b_0 & a_1 b_1 \end{array} \\ \begin{array}{c} c_0 \\ c_1 \end{array} & \begin{bmatrix} 1 & 0 & -1 & 1 \\ 1 & -1 & 1 & 0 \end{bmatrix} \end{array}$$

- Reductions: $\Phi_A \Phi_A^* = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}, \quad \Phi_B \Phi_B^* = \begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix}, \quad \Phi_C \Phi_C^* = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$
- Eigenpairs: $l_A, U_A = \begin{Bmatrix} 3 \\ 3 \end{Bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; l_B, U_B = \begin{Bmatrix} 5.236 \\ .7639 \end{Bmatrix}, \begin{bmatrix} .8507 & .5257 \\ -.5257 & .8507 \end{bmatrix}, \; l_C, U_C = \begin{Bmatrix} 3 \\ 3 \end{Bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
- Normal form (stationary point for HOSVD):

$$\tilde{\Phi} = (U_A, U_B, U_C).\Phi = \begin{array}{l} .85\, a_0 b_0 c_0 - 1.38\, a_1 b_0 c_0 + .53\, a_0 b_1 c_0 + .32\, a_1 b_1 c_0 \\ +1.38\, a_0 b_0 c_1 + .85\, a_1 b_0 c_1 - .32\, a_0 b_1 c_1 + .53\, a_1 b_1 c_1 \end{array}$$

A Uniform View of Singular Value Decompositions

Key ingredients from [Oeding-Tan 2025]:

- Reduction maps
- Pulling back orbits from smaller classifications
- Computing stabilizers

We consider spaces of tensors $V_1 \otimes \cdots \otimes V_m$ with $V_i \cong \mathbb{C}^{d_i}$ and an action of $G = G_1 \times \cdots \times G_m$, with $G_i \leq \mathrm{GL}(V_i)$, defined on rank-1 elements:

$$(g_1, \ldots, g_m).(v_1 \otimes \cdots \otimes v_m) = (g_1.v_1) \otimes \cdots \otimes (g_m.v_m)$$

A **reduction map** is a function from a G-set $\mathcal{S}$ to a G_i set $\mathcal{S}_i$

$$\pi \colon \mathcal{S} \to \mathcal{S}_i \quad \textit{with} \quad \pi((g_1, \ldots, g_m).x) = g_i.\pi(x).$$

Normal forms for G_i acting on $\mathcal{S}_i$ pull back to orbits for G acting on $\mathcal{S}$.

Examples of Reduction Maps

- Take V_i a representation of a group $G_i \subset \mathrm{GL}(V_i)$. **Projection**:

$$V_1 \oplus \cdots \oplus V_m \to V_i$$

- Take $\mathcal{S} = \mathbb{C}^d \otimes \mathbb{C}^e$, $G_1 = \mathrm{U}_d$, $G_2 = \mathrm{U}_e$, and $\mathcal{S}_1$ the $d \times d$ real Hermitian matrices. **Reduced density matrices**:

$$\begin{array}{rcl} \pi\colon V \otimes W & \to & \mathcal{S}_1 \\ \Phi & \mapsto & \Phi_{(1)}\Phi_{(1)}^{\dagger} \end{array}$$

- Our reductions make use of flattenings.
For a tensor $\Phi \in V_1 \otimes \cdots V_m$, the i^{th} 1-**flattening** is a linar map:

$$\Phi_{(i)}\colon (V_1 \otimes \cdots V_{i-1} \otimes V_{i+1} \otimes \cdots \otimes V_m)^* \to V_i.$$

Examples of Reduction Maps

$$\pi : \mathcal{S} \to \mathcal{S}_1, \quad \text{with} \quad \pi((g_1, \ldots, g_n).x) = g_i.\pi(x).$$

reduction map	source and target	groups
$\pi : \Phi \mapsto \Phi_{(1)}\Phi_{(1)}^{\dagger}$	$\mathbb{C}^d \otimes \mathbb{C}^e \to \mathbb{C}^d \otimes \mathbb{C}^{d*}$	$\mathrm{U}_d \times \mathrm{U}_e$; U_d
$\pi : \Phi \mapsto \Phi_{(1)}\Phi_{(1)}^{\top}$	$\mathbb{C}^d \otimes \mathbb{C}^e \to S^2\mathbb{C}^d$	$\mathrm{O}_d \times \mathrm{O}_e$; O_d
$\pi_i : \Phi \mapsto \Phi_{(i)}J^{\otimes(n-1)}\Phi_{(i)}^{\top}$	$V \to S^2\mathbb{C}^2$ or $\bigwedge^2\mathbb{C}^2$	$\mathrm{SL}_2^{\times n}$; SL_2
$\pi_{ij} : \Phi \mapsto T\Phi_{(ij)}J^{\otimes(n-2)}\Phi_{(ij)}^{\top}T^{\top}$	$V \to S^2\mathbb{C}^4$ or $\bigwedge^2\mathbb{C}^4$	$\mathrm{SL}_2^{\times n}$; $\mathrm{SL}_2 \times \mathrm{SL}_2 \to \mathrm{SO}_4$

$$\text{with } J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \text{ and } T := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & \mathbf{i} & \mathbf{i} & 0 \\ 0 & -1 & 1 & 0 \\ \mathbf{i} & 0 & 0 & -\mathbf{i} \end{pmatrix}$$

Existence and Uniqueness of Core Elements

For G acting on a set $\mathcal{S}$, a **Normal Form Function** $F\colon \mathcal{S} \to \mathcal{S}$ has

- x and $F(x)$ are in the same G-orbit, and
- $F(x) = F(y)$ if x and y are in the same G-orbit.

Lemma (Oeding-Tan 2025)

If for each i, $\pi_i\colon \mathcal{S} \to \mathcal{S}_i$ is a reduction map and $F_i\colon \mathcal{S}_i \to \mathcal{S}_i$ is a normal form function, then for each $x \in \mathcal{S}$ there exists a ***core element*** *$\omega \in \mathcal{S}$ defined by:*

- *$x = (g_1, \ldots, g_n).\omega$ for some $g_i \in G_i$, and*
- *$\pi_i(\omega) = F_i(\pi_i(\omega))$ for all $1 \leq i \leq n$.*

Moreover, the core element is unique up to the action of the product of the stabilizers $H_i = Stab_{G_i}(\pi_i(\omega))$.

Revisiting HOSVD

Classical HOSVD is a consequence of the Lemma:

Theorem (HOSVD, De Lathauwer 2000 (new rephrasing))

For $1 \leq i \leq n$ let V_i be a d_i-dimensional complex vector space. Then for each tensor $\Phi \in V_1 \otimes \cdots \otimes V_n$ there exists a core tensor Ω such that

- $\Phi = (U_1 \otimes \cdots \otimes U_n)\Omega$ *for some* $U_i \in \mathcal{U}_{d_i}$, *and*
- $D_i = \Omega_{(i)}\Omega^*_{(i)}$ *is real diagonal with weakly decreasing diagonal entries for all* $1 \leq i \leq n$.

The core tensor is unique up to the action of $H_1 \times \cdots \times H_n$, *where* $H_i \leq \mathcal{U}_{d_i}$ *is the stabilizer subgroup of* D_i *by the conjugation action.*

Complex Orthogonal HOSVD

Another consequence of the Lemma:

Theorem (Complex Orthogonal HOSVD, [Oeding-Tan 2025])

For $1 \leq i \leq n$ let V_i be a d_i-dimensional complex vector space. Then for each tensor $\Phi \in V_1 \otimes \cdots \otimes V_n$ there exists a core tensor Ω such that

- $\Phi = (U_1 \otimes \cdots \otimes U_n)\Omega$ *for some* $U_i \in \mathrm{SO}_{d_i}$, *and*
- $D_i = \Omega_{(i)}\Omega_{(i)}^\top$ *is a direct sum of symmetrized Jordan blocks in weakly decreasing order for all* $1 \leq i \leq n$.

The core tensor is unique up to the action of $H_1 \times \cdots \times H_n$, where $H_i \leq \mathrm{SO}_{d_i}$ is the stabilizer subgroup of D_i by the conjugation action.

LU Orbit Classification

Orbits for $\mathrm{SU}_{d_1} \times \cdots \times \mathrm{SU}_{d_n}$ acting on $\mathbb{C}^{d_1} \otimes \cdots \otimes \mathbb{C}^{d_n}$:

- $n = 2$ matrix case: SVD classifies orbits via singular values.
- $n = 3$ generalized Schmidt decomposition, [Acín et al., 2000]
- $n \geq 4$ for *general* states [Krauss 2010] using HOSVD.
- otherwise, wildly open.

[Chterental-Djokovic 2007] Method for 4 Qubits

The double cover $T\colon \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SO}_4(\mathbb{C})$ lets us :

- View $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ with the $(\mathrm{SL}_2)^{\times 4}$-action as $\mathbb{C}^4 \otimes \mathbb{C}^4$ with the SO_4^2-action.
- Use Complex Orthogonal SVD to give the SO_4^2-orbits.
- Pull these SO_4^2-orbits back to the $\mathrm{SL}_2^{\times 4}$-orbits.
- Our work: view $(\mathbb{C}^2)^n \cong \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ with $\mathrm{SL}_2^{\times n}$ action as $(\mathbb{C}^2 \otimes \mathbb{C}^2) \otimes (\mathbb{C}^2 \otimes \mathbb{C}^2) \otimes \cdots$ with $\mathrm{SO}_4^{\times \lfloor \frac{n}{2} \rfloor}$ action (even/odd cases are different).

SLOCC Orbit Classification

Orbits for $\mathrm{SL}_{d_1} \times \cdots \times \mathrm{SL}_{d_n}$ acting on $\mathbb{C}^{d_1} \otimes \cdots \otimes \mathbb{C}^{d_n}$:

- $n = 2$ matrix case: rank classifies orbits.
- $n = 3$ qubits [classical, or GKZ 1994, or Dür et al. 2000]
- $n = 3$ qutrits [Thrall-Chanler 1938, Ng 1995, Nurmiev 2000, Di Trani et al. 2023]
- $n = 4$ qubits [Chterental and Djokovic 2007, Dietrich et al. 2022]
- $n \geq 5$ *general* qubits [Oeding-Tan 2025]
- otherwise, wildly open.

Maximizing Invariants

Maximizing Attendance

[SIAM News October 2023]

Luke Oeding and his children explore the campus of the Eindhoven University of Technology in a cargo bike during a break from the technical sessions at the 2023 SIAM Conference on Applied Algebraic Geometry, which took place in July 2023. Image courtesy of Michael Burr.

Enter Invariant Theory

Remark (Invariants distinguish entanglement type)

Represent 2-particle states as matrices. Rank classifies entanglement.

$$|00\rangle \to \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \textit{ and } \det\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) = 0 \Rightarrow \textit{unentangled.}$$

$$\tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \to \tfrac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \textit{ and } \det \tfrac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) = \tfrac{1}{2} \Rightarrow \textit{entangled.}$$

Rank-1 matrices correspond to *separable* or *unentangled* states.

Matrix rank (determinants) classify entanglement for 2 particles.

$$\sigma_r = \{A \in \mathbb{C}^{d\times d} \mid \operatorname{rank}(A) \leq r\}: \qquad \sigma_1 \subset \sigma_2 \subset \cdots \subset \sigma_d.$$

A measure of entanglement is maximized when $\det(A)$ is maximized.

Entanglement Measures

[Osterloh and Siewert 2010] Notion of entanglement measure:

$$E\colon \mathcal{H}_n \to [0,\infty), \qquad E(\varphi) = |f(\varphi)|^{1/m}$$

with f a homogeneous symmetric SLOCC invariant polynomial of degree $m > 0$. It satisfies:

- $E(\varphi) = 0$ when $\varphi \in \mathcal{H}_n$ is separable (unentangled).
- E is invariant for the LU group.
- E is invariant for the symmetric group.
- E is an entanglement monotone (doesn't increase, on average, under local operations, [Verstraete et al. 2003].

Entanglement Types for 3 Qubits

$$\begin{array}{ccc} & \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) & \\ & \downarrow & \\ & \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle) & \\ \swarrow & \downarrow & \searrow \\ \frac{1}{\sqrt{2}}(|001\rangle + |010\rangle) & \frac{1}{\sqrt{2}}(|001\rangle + |100\rangle) & \frac{1}{\sqrt{2}}(|010\rangle + |100\rangle) \\ \searrow & \downarrow & \swarrow \\ & |000\rangle & \end{array}$$

GHZ, $\quad \mathrm{Hdet} \neq 0$

W, $\quad \mathrm{Hdet} = 0$

biseparable, $\det_i = 0$

separable, $\det_i = 0, \forall i$

States separated by algebraic invariants: determinants and hyperdeterminants.

- *Three qubits can be entangled in two inequivalent ways*, [Dür-Vidal-Cirac (2000)]
- *Discriminants, resultants and multidimensional determinants*, [Gelfand-Kapranov-Zelevinsky (1994)]

Generalize the Determinant to Polynomial Invariants

Remark

$V = \mathcal{H}^{\otimes n}$. Take G to be $U_d^{\times n}$, or $\mathrm{SL}_d^{\times n}$.

Non-trivial G-invariants vanish on the set of unentangled states.

The states $\Phi \in V$ that maximize $|F(\Phi)|$ for an invariant F should have a maximal amount of entanglement.

The $2 \times 2 \times 2 \times 2$ hyperdeterminant is a polynomial of degree 24 in 16 unknowns with 2,894,276 terms [Huggins-Sturmfels-Yu-Yuster 2008].

The $3 \times 3 \times 3$ fundamental invariants of degree 6, 9, 12 respectively have 1152, 9216, and 209061 terms, and the hyperdeterminant may have up to $\sim 10^{13}$ terms [See Ottaviani's *Intro to the Hyperdeterminant* 2013]

Invariants and Jordan Canonical Form, 3-Qutrit Case

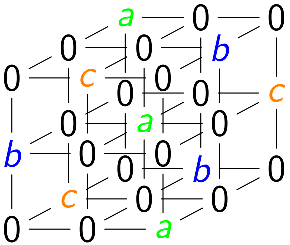
The exceptional Lie group E_8 gives $T \in (\mathbb{C}^3)^{\otimes 3}$ a Jordan decomposition: $T = S + N$, with S semi-simple, N nilpotent, and $[S, N] = 0$.

Invariants are simpler on the semi-simple states! $f(S+N) = f(S)$

Generic semi-simple states are:

$|\psi_S\rangle = a\,|v_1\rangle + b\,|v_2\rangle + c\,|v_3\rangle$, with $(a, b, c) \in \mathbb{C}^3$, and $|a|^2 + |b|^2 + |c|^2 = 1$.

$$|v_1\rangle = \frac{1}{\sqrt{3}}\left(|000\rangle + |111\rangle + |222\rangle\right),$$
$$|v_2\rangle = \frac{1}{\sqrt{3}}\left(|012\rangle + |120\rangle + |201\rangle\right),$$
$$|v_3\rangle = \frac{1}{\sqrt{3}}\left(|021\rangle + |102\rangle + |210\rangle\right).$$



Expressions of Invariants on Semi-Simple Elements

The restriction of invariants to semi-simple states $|\psi_S\rangle$:

$$I_6(|\psi_S\rangle) = \tfrac{1}{27}\left(a^6 - 10a^3b^3 - 10a^3c^3 + b^6 - 10b^3c^3 + c^6\right)$$

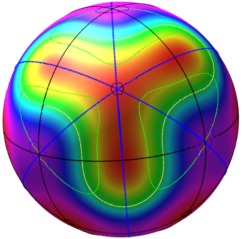
$$I_9(|\psi_S\rangle) = \tfrac{-\sqrt{3}}{243}(a-b)(a-c)(b-c)\left(a^2+ab+b^2\right)\left(a^2+ac+c^2\right)\left(b^2+bc+c^2\right)$$

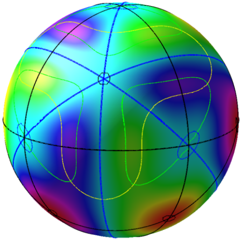
$$I_{12}(|\psi_S\rangle) = \tfrac{1}{729}\left(\begin{array}{r} a^9b^3 + a^3b^9 + a^9c^3 + b^9c^3 + a^3c^9 + b^3c^9 \\ -4\left(a^6b^6 + a^6c^6 + b^6c^6\right) + 2\left(a^6b^3c^3 + a^3b^6c^3 + a^3b^3c^6\right) \end{array}\right)$$

$$\begin{aligned}\Delta_{333}(|\psi_S\rangle) \;=\; & -\tfrac{4}{3^{18}}\,a^3b^3c^3(a+b+c)^3 \times \\ & (a^2+2\,ab-ac+b^2-bc+c^2)^3(a^2-ab+2\,ac+b^2-bc+c^2)^3 \times \\ & (a^2-ab-ac+b^2+2\,bc+c^2)^3(a^2-ab-ac+b^2-bc+c^2)^3\end{aligned}$$

Maximizing Polynomial Invariants

Variable reduction by Tensor Jordan Decomposition: we can plot the values for the invariants for real 3-qutrit systems $(3 \times 3 \times 3)$ on a sphere:


deg 6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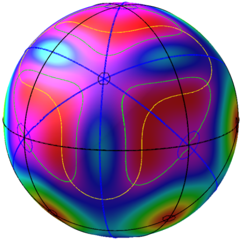
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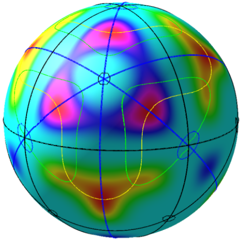
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[The lines represent level sets for the other invariants.]

Find the maximizing states for each measure of entanglement $|f(|\psi\rangle)|$.

New Maximally Entangled States for 3-Qutrits

Theorem (Jaffali-Holweck-Oeding 2024)

The global maximum of the absolute value of the hyperdeterminant $|\Delta_{333}|$, *when restricted to real states is* $\frac{\sqrt{3}}{2^{19}\times 3^{14}}$.

The global max is reached at 12 semi-simple points $a\,|v_1\rangle + b\,|v_2\rangle + c\,|v_3\rangle$ *with the following values and their permutations:*

$$(a, b, c) = (rs, s, s)\,, \quad \text{with} \quad r = (1 \pm \sqrt{3}) \quad \text{and} \quad s = \pm\sqrt{\frac{1}{(r^2+2)}}\,.$$

New Maximally Entangled States for 3-Qutrits

Theorem (Jaffali-Holweck-Oeding 2024)

The global maxima of the absolute value of the fundamental invariants I_6, I_9, and I_{12} restricted to generic 3-qutrits, with maximum values respectively $\frac{1}{18} = .05\overline{5}$, $\frac{\sqrt{6}}{3888} \simeq 0.00063$ and $\frac{1}{7776} \simeq 0.0001286$, are simultaneously reached for the real 3-qutrit Aharonov state:

$$|M_{333,I}\rangle = \frac{1}{\sqrt{2}}\Big(|v_1\rangle - |v_2\rangle\Big) .$$

All real semi-simple state maximizers are permutations of $|M_{333,I}\rangle$.

How to Solve Polynomial Systems, Professionally

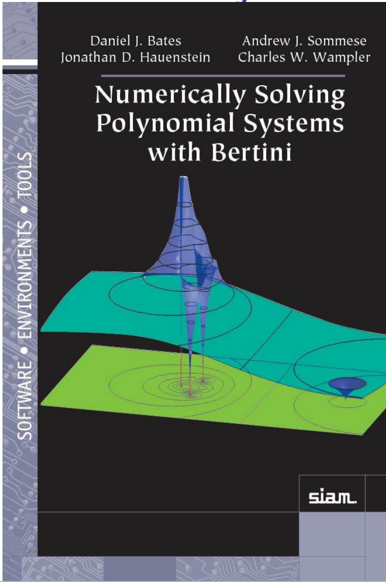
$$\text{Maximize} \quad |F(x)|^2 \quad \text{subject to} \quad |x|^2 = 1.$$

Off the shelf optimization finds one extremum.

We can use Lagrange multipliers and solve a polynomial system of equations.

Symbolic methods can be used for small systems, but scale unfavorably.

Homotopy Continuation utilizes parallelism to numerically solve polynomial systems, and have guarantees that we found all solutions.



Sponsored ⓘ

4-Qubit Critical States

Use Tensor Jordan Decomposition (using E_7) for variable reduction, $\mathbb{S}^{2n-1} \subset \mathbb{R}^{2n} \cong \mathbb{C}^n$, dehomogenization, and HomotopyContinuation.jl to find all extrema for the $2 \times 2 \times 2 \times 2$ fundamental invariants:

4-Qubit Critical States

Use Tensor Jordan Decomposition (using E_7) for variable reduction, $\mathbb{S}^{2n-1} \subset \mathbb{R}^{2n} \cong \mathbb{C}^n$, dehomogenization, and HomotopyContinuation.jl to find all extrema for the $2 \times 2 \times 2 \times 2$ fundamental invariants:

$$
\begin{bmatrix}
-2.4142135623730954 & -2.414213562373096 & -1.0000000000000009 & -0.4142135623709653 & 1.00 \\
-2.414213562373097 & -2.4142135623730976 & -1.0000000000000013 & -0.41421356237309 66 & 1.00 \\
2.4142135623730985 & -2.4142135623730985 & 1.000000000000002 & 0.4142135623709664 & 1.00 \\
2.414213562373094 & 2.414213562373094 & -0.9999999999999992 & 0.41421356237309 54 & 1.00 \\
-2.4142135623730945 & -2.4142135623730945 & -0.9999999999999997 & 0.4142135623709476 & -0.99 \\
2.414213562373094 & -2.414213562373094 & 0.9999999999999997 & -0.41421356237309 53 & -1.00 \\
2.4142135623730945 & 2.4142135623730927 & -0.9999999999999976 & -0.41421356237309 56 & -1.0 \\
-2.414213562373096 & 2.4142135623730945 & 0.9999999999999998 & 0.41421356237309 52 & -0.99 \\
-2.414213562373093 & 2.4142135623730936 & 0.9999999999999994 & -0.4142135623709476 & 1.00 \\
-2.4142135623730954 & 2.4142135623730945 & 1.0 & -0.41421356237309 47 & 0.99 \\
2.414213562373097 & 2.4142135623730967 & -1.0000000000000016 & 0.41421356237309 54 & 0.99 \\
2.4142135623730936 & -2.414213562373095 & 0.9999999999999994 & 0.4142135623709576 & 1.00 \\
-2.414213562373095 & 2.4142135623730945 & 0.9999999999999996 & 0.41421356237309 66 & -1.00 \\
2.4142135623730945 & 2.414213562373094 & -0.9999999999999991 & -0.4142135623709465 & -1.00 \\
2.414213562373097 & -2.414213562373097 & 1.000000000000001 & -0.4142135623709637 & -1.00 \\
-2.414213562373093 & -2.4142135623730936 & -0.9999999999999993 & 0.4142135623709487 & -1.00 \\
1.4281392945149085 & 0.8114939752095733 & -0.8714428811730411 & -2.622987950419146 2 & 0.31 \\
1.4281392945149092 & 0.8114939752095733 & -0.8714428811730395 & -2.622987950419145 3 & 0.31
\end{bmatrix}
$$

4-Qubit Critical States

We clean and re-verify the solutions.
We compare normal forms using a variant of HOSVD [Oeding-Tan 2025].
We find maximally entangled 4-qubit states [see Enriquez (2016)].
Here's an example for the degree 6 invariant:

Theorem (Oeding-Tan (Journal of Physics A 2025))

The following states (in Cartan coordinates) maximize $|f_6(|\psi_S\rangle|$*:*

$$\varphi_1 = (1,0,0,0) \cong |\mathsf{MP}\rangle \quad \varphi_2 = (1,1,0,0) \cong |\mathsf{GHZ}\rangle \quad \varphi_3 = (1,1,1,0)$$
$$\varphi_4 = (2,1,1,0) \quad \varphi_5 = (\sqrt{2},\mathbf{i},0,0) \quad \varphi_6 = (\sqrt{2},\mathbf{i},\mathbf{i},0)$$
$$\varphi_7 = (\sqrt{2},\mathbf{i},\mathbf{i},\mathbf{i}) \quad \varphi_8 = (\sqrt{3},\mathbf{i},\mathbf{i},\mathbf{i}) \cong |\mathsf{HS}\rangle \quad \varphi_9 = (1,e^{\frac{\pi\mathbf{i}}{3}},e^{\frac{2\pi\mathbf{i}}{3}},0) \cong |\mathsf{HD}\rangle$$
$$\varphi_{10} = \left(1, 1-\frac{\sqrt{3}}{\sqrt{2}}+\frac{1}{\sqrt{2}}\mathbf{i}, \frac{\sqrt{3}}{\sqrt{2}}-1+\frac{1}{\sqrt{2}}\mathbf{i}, (\sqrt{3}-\sqrt{2})\mathbf{i}\right) \quad \varphi_{11} = (7, 2\sqrt{7}\mathbf{i}, 2\sqrt{7}\mathbf{i}, 0)$$
$$\varphi_{12} = (18, 11-\sqrt{203}\mathbf{i}, 7+\sqrt{203}\mathbf{i}, 0) \ \ldots$$

$\varphi_{13} = (1, a + b\mathbf{i}, -a - b\mathbf{i}, 0)$, where

$$a \approx 0.0933383722, \quad b \approx 0.6221645823$$

satisfies the following: $\begin{aligned} 0 &= 1000a^2b^2 - 872b^4 + 85a^2 + 345b^2 - 7 \\ 0 &= 100a^4 - 244b^4 + 45a^2 + 65b^2 + 11 \end{aligned}$

$\varphi_{14} = (1, a - b\mathbf{i}, a + b\mathbf{i}, c\mathbf{i})$, where

$$a \approx 0.2166757505, \quad b \approx 0.8300687024, \quad c \approx 0.3661117361$$

satisfies the following.

$$\begin{aligned} 0 &= 5a^4c - 30a^2b^2c + 5b^4c - 5a^2c^3 + 5b^2c^3 + 2c^5 + 5a^2c - 5b^2c - 5c^3 + 2c \\ 0 &= 10a^4b - 40a^2b^3 + 14b^5 - 15a^2bc^2 + 5b^3c^2 + 5bc^4 + 25a^2b - 15b^3 - 5bc^2 + 4b \\ 0 &= 4a^5 + 20a^3b^2 - 5a^3c^2 + 15ab^2c^2 - 5a^3 - 5ab^2 + 5ac^2 + a \end{aligned}$$

Conclusions

- Higher Order Singular Value Decomposition and generalizations yield normal form algorithms that classify almost all n-qubit states.
- Invariant Theory combined with Polynomial Optimization via Homotopy Continuation allow us to find states that maximize algebraic measures of entanglement. In the case of 3 qutrits and 4 qubits this method found all of the known maximally entangled states, and some new ones.
- Nonlinear Algebra contains beautiful tools with nice applications!
- Do you have large systems of polynomials to solve?
  Ask someone in SIAM(ag)^2 to help!


ML
Luke
Hamza
May 3
+ Outline
May 30
Network Design for learning
* linear (Subspaces)
* Curves
Experiments
* Seg
* Dual
* r = 2, 3
vs. All Interpolation
V.S.
Homotopy continuation
Context / Application
* Classifier for 4 cubits ...
vs
PCA
vs
SVM
M: V → V
M ∈ V*⊗V

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