Extending HOSVD for Entanglement Classification

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Classifying orbits of a group acting on a vector space

Dynkin[1]: a classification of representations is a set of "characteristics" satisfying:

- The characteristics must be invariant under inner automorphisms so that the characteristics of equivalent representations must coincide.
- They should be complete: If two representations have the same characteristics, they must be equivalent.
- They should be compact and easy to compute.

The trichotomy for obit classifications:

- Finitely many orbits (must have $\dim V \leq \dim G$).
- Tame orbits, classified using finitely (at least $\dim V - \dim G$) many parameters.
- Wild orbit structure.

Attempt to give normal forms as Dynkin's characteristics.

[1] E. B. Dynkin, *Semisimple subalgebras of semisimple Lie algebras*, Am. Math. Soc. Trans. 1960

Orbit Classification for Quantum Information

Quantum Information is interested in two different group actions on $\mathcal{H} = \mathbb{C}^d \otimes \cdots \otimes \mathbb{C}^d$:

- Local Unitary (LU) $= \mathrm{SU}_d \times \cdots \times \mathrm{SU}_d$.
- Stochastic Local Operations and Classical Communication (SLOCC) $= \mathrm{SL}_d \times \cdots \times \mathrm{SL}_d$.
- Tensors represent quantum states of systems of particles.
- Orbits represent equivalence classes of entanglement types.

LU Orbit Classification

Orbits for $\mathrm{SU}_{d_1} \times \cdots \times \mathrm{SU}_{d_n}$ acting on $\mathbb{C}^{d_1} \otimes \cdots \otimes \mathbb{C}^{d_n}$:

- $n = 2$ matrix case: SVD classifies orbits via parameters (singular values).
- $n = 3$ generalized Schmidt decomposition, [Acín et al., 2000]
- $n \geq 4$ unknown even for qubits.
- $n \geq 4$ for *general* states [Krauss 2010] using HOSVD.

SLOCC Orbit Classification

Orbits for $\mathrm{SL}_{d_1} \times \cdots \times \mathrm{SL}_{d_n}$ acting on $\mathbb{C}^{d_1} \otimes \cdots \otimes \mathbb{C}^{d_n}$:

- $n = 2$ matrix case: rank classifies orbits.
- $n = 3$ qubits [classical, or GKZ 1994, or Dür et al. 2000]
- $n = 3$ qutrits [Thrall-Chanler 1938, Ng 1995, Nurmiev 2000, Di Trani et al. 2023]
- $n = 4$ qubits [Chterental and Djokovic 2007, Dietrich et al. 2022]
- $n \geq 5$ *general* qubits [Oeding-Tan 2025]
- otherwise, wildly open.

Our Approach

- We cast HOSVD in the more general context of normal form algorithms via reduction maps in the Hermitian, $\mathbb{C}$-orthogonal and $\mathbb{C}$-symplectic cases.
- We introduce the complex orthogonal HOSVD.
- We introduce a pair of algorithms (in the even and odd cases) for qubit classification for general states.

SVD

Given $A \in \mathbb{C}^{m \times n}$. Then A^*A and AA^* are Hermitian, hence unitarily diagonalizable: There exists $U \in \mathcal{U}(m)$ and $V \in \mathcal{U}(n)$ such that $UA^*AU^* = \Sigma$ and $V^*AA^*V = \Sigma'$, with Σ, Σ' quasi-diagonal with the same non-zero entries, and

$$A = (U, V).\Sigma = U\Sigma V^*$$

The matrix Σ of singular values $\sigma_1 \geq \cdots \sigma_r \geq 0$ is the normal form (used $\mathfrak{S}_n$).
SVD algorithm: Input $A \in \mathbb{C}^{m \times n}$. Output $A = U\Sigma V^*$ as above.

- Compute ONB eigenvectors and eigenvalues of the smaller of AA^* and A^*A to get unitary U or V, and $\sigma_i = \sqrt{\lambda}_i$.
- If you have A, U, then $A = U\Sigma V^*$ gives $U^*A = \Sigma V^*$, solve for V.
- If you have A, V, then $A = U\Sigma V^*$ gives $AV = U\Sigma$, solve for U.

Over the reals, get U, V real orthogonal.

SVD Notes

Starting from $A \in \mathbb{C}^{m\times n}$ and $(g,h) \in \mathcal{U}(m) \times \mathcal{U}(n)$ acting via $(g,h).A = gAh^*$.
Have a map $\pi\colon \mathbb{C}^{m\times n} \to \mathbb{H}_m$ to Hermitian matrices defined by $\pi(A) = AA^*$.
This map has a homomorphism-like property for $\mathcal{U}_m$:

$$\pi((g,h).A) = \pi(gAh^*) = gAh^*(gAh^*)^* = gAA^*g^* = g.\pi(A)$$

This allows us to pull back the normal form from $\mathbb{H}_m$ to $\mathbb{C}^{m\times n}$.
Also have a map $\pi_{(1)}\colon \mathbb{C}^{m\times n} \to \mathbb{H}_n$ defined by $\pi(A) = A^*A$ with

$$\pi_{(1)}((g,h).A) = \pi_{(1)}(gAh^*) = (gAh^*)^*gAh^* = hA^*Ah^* = h.\pi_{(1)}(A)$$

This allows us to pull back the normal form from $\mathbb{H}_n$ to $\mathbb{C}^{m\times n}$.
They happen to give essentially the same information.

HOSVD

[DeLathauwer, Lim, Qi, and others have introduced several notions of SVD for tensors]
Input: $A \in \mathbb{C}^{d_1 \times \cdots \times d_n}$.
Output: $(U_1, \ldots, U_n) \in \mathcal{U}(d_1) \times \cdots \times \mathcal{U}(d_n)$ and Σ "all-orthogonal" such that
$A = (U_1, \ldots, U_n).\Sigma$

- Flatten (reshape): $A_{(1)} = (A)_{i_1,(i_2,\ldots,i_n)} \in \mathbb{C}^{d_1 \times (d_2 \cdots d_n)}$
- Compute left singular vectors, from $A_{(1)} A_{(1)}^* \in \mathbb{C}^{d_1 \times d_1}$ to produce unitary U_1
- $\vdots$
- Flatten (reshape): $A_{(n)} = (A)_{i_n,(i_1,\ldots,i_{n-1})} \in \mathbb{C}^{d_n \times (d_1 \cdots d_{n-1})}$
- Compute left singular vectors, from eig $A_{(n)} A_{(n)}^* \in \mathbb{C}^{d_n \times d_n}$ to produce unitary U_n
- Compute $\Sigma = (U_1, \ldots, U_n).A$
- $D_i = \Sigma_{(i)} \Sigma_{(i)}^*$ is real diagonal with weakly decreasing diagonal entries for all $1 \leq i \leq n$.

Edge cases?

HOSVD Notes

- Starting from $A \in \mathbb{C}^{d_1 \times \cdots \times d_n}$ and $(U_1, \ldots, U_n) \in \mathcal{U}(d_1) \times \cdots \times \mathcal{U}(d_n)$ acting via modal products $(U_1, \ldots, U_n).A$
- Have maps $\pi_{(i)} \colon \mathbb{C}^{d_1 \times \cdots d_n} \to \mathbb{H}_{d_i}$ defined by $\pi_{(i)}(A) = A_{(i)} A_{(i)}^*$.
- This map has a homomorphism-like property for $\mathcal{U}_{d_i}$ for each i.
- We are essentially pulling back the normal forms from $\mathbb{H}_{d_i}$.

Reduction maps

Let a product of groups $G = G_1 \times \cdots \times G_n$ act on a set $\mathcal{S}$. A reduction map is a function $\pi \colon \mathcal{S} \to \mathcal{S}_i$ if for all $(g_1, \ldots, g_n) \in G$ and all $x \in \mathcal{S}$ we have

$$\pi((g_1, \ldots, g_n).x) = g_i.\pi(x).$$

Example

Direct sum of representations $\mathcal{S} = \mathbb{C}^{d_1} \oplus \cdots \oplus \mathbb{C}^{d_n}$ of $G_i \mapsto \mathrm{GL}(\mathbb{C}^{d_i})$.
Projection $\mathcal{S} \to \mathbb{C}^{d_i}$ is a reduction map.

Reduced Density Matrices

Notation: for $\Phi \in V = V_1 \otimes \cdots \otimes V_n$, a flattening is $\Phi_{(i)} \colon V_{\hat{i}}^* \to V_i$. For $G = G_1 \times \cdots \times G_n$, we have $(g.\Phi)_{(i)} = g_i \Phi_{(i)} \widehat{g_i}^\top$.

Example

$\mathcal{S} = \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ with $G = U_{d_1} \times U_{d_2}$, and $\mathcal{S}_1$ the $\mathbb{R}$-vector space of Hermitian matrices. The map $\pi(\Phi) = \Phi_{(1)}\Phi_{(1)}^*$ is a reduction map since:

$$\pi((U_1, U_2).\Phi) = U_1 \Phi_{(1)} U_2^\top (U_1 \Phi_{(1)} U_2^\top)^* = U_1 \Phi_{(1)} \Phi_{(1)}^* U_1 = U_1.\pi(\Phi).$$

A reduction map for SLOCC and qubits

- $\Phi \in V = V_1 \otimes \cdots \otimes V_n$, each $V_i = \mathbb{C}^2$,
- $G = \mathrm{SL}_2 \times \cdots \times \mathrm{SL}_2$,
- $\mathcal{S}_i = S^2 V_i$ for n-odd, $\mathcal{S}_i = \bigwedge^2 V_i$ for n-even, both with the action $A.M = AMA^\top$ for $A \in \mathrm{SL}_2$ and $M \in \mathcal{S}_i$.
- Set $J = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$,
  - $AJA^\top = \det(A)J$.
  - Note $J^{\otimes k}$ is symmetric (resp. skew) when k is even (odd).

We have reduction maps $\pi_i \colon V \to \mathcal{S}_i$ defined by $\pi_i(\Phi) = \Phi_{(i)} J^{\otimes(n-1)} \otimes \Phi_{(i)}^\top$. since

$$\begin{aligned}
\pi_i((A_1, \dots, A_n).\Phi) &= A_i \Phi_{(i)} \widehat{A_i}^\top J^{\otimes(n-1)} (A_i \Phi_{(i)} \widehat{A_i}^\top)^\top \\
&= A_i \Phi_{(i)} \widehat{A_i}^\top J^{\otimes(n-1)} \widehat{A_i} \Phi_{(i)}^\top A_i^\top \\
&= A_i \Phi_{(i)} J^{\otimes(n-1)} \Phi_{(i)}^\top A_i^\top \\
&= A_i . \pi_i(\Phi)
\end{aligned}$$

A reduction map for SLOCC and qubits

- Set $T := \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & \mathrm{im} & \mathrm{im} & 0 \\ 0 & -1 & 1 & 0 \\ \mathrm{im} & 0 & 0 & -\mathrm{im} \end{pmatrix}$, giving the isomorphism

$$\begin{aligned} \mathrm{SL}_2 \otimes \mathrm{SL}_2 &\longrightarrow \mathrm{SO}_4 \\ A \otimes B &\longmapsto T(A \otimes B)T^* \end{aligned}$$

- $\Phi \in V = V_1 \otimes \cdots \otimes V_n$, each $V_i = \mathbb{C}^2$, and $G = \mathrm{SL}_2 \times \cdots \times \mathrm{SL}_2$,
- For n even, set $\mathcal{S}_{ij} \cong S^2(V_i \otimes V_j)$, the space of 4×4 complex symmetric matrices
- For n odd, set $\mathcal{S}_{ij} \cong \bigwedge^2(V_i \otimes V_j)$, the space of 4×4 complex skew-symmetric matrices
- Set $\Phi_{(ij)}$ the 2-flattening.
- Define the reduction map $\pi_{ij} : V_1 \otimes \cdots \otimes V_n \to \mathcal{S}_{ij}$ by $\pi_{ij}(\Phi) = T\Phi_{(ij)}J^{\otimes(n-2)}\Phi_{(ij)}^\top T^\top$.
- If $\Phi, \Phi' \in V$ are in the same SLOCC orbit, then $\pi_{ij}(\Phi)$ and $\pi_{ij}(\Phi')$ are in the same SO_4-orbit by conjugation. (see [Li 2018] for a weaker claim).

Summary of reduction maps

reduction map	source and target	groups
$\pi : \Phi \mapsto \Phi_{(1)}\Phi^*_{(1)}$	$\mathbb{C}^d \otimes \mathbb{C}^e \to \mathbb{C}^d \otimes \mathbb{C}^{d*}$	$\mathcal{U}_d \times \mathcal{U}_e$; $\mathcal{U}_d$
$\pi : \Phi \mapsto \Phi_{(1)}\Phi^\top_{(1)}$	$\mathbb{C}^d \otimes \mathbb{C}^e \to S^2\mathbb{C}^d$	$\mathrm{O}_d \times \mathrm{O}_e$; O_d
$\pi_i : \Phi \mapsto \Phi_{(i)}J^{\otimes(n-1)}\Phi^\top_{(i)}$	$V \to S^2\mathbb{C}^2$ or $\bigwedge^2\mathbb{C}^2$	$\mathrm{SL}_2^{\times n}$; SL_2
$\pi_{ij} : \Phi \mapsto T\Phi_{(ij)}J^{\otimes(n-2)}\Phi^\top_{(ij)}T^\top$	$V \to S^2\mathbb{C}^4$ or $\bigwedge^2\mathbb{C}^4$	$\mathrm{SL}_2^{\times n}$; $\mathrm{SL}_2 \times \mathrm{SL}_2 \to \mathrm{SO}_4$

Table: Some Reduction Maps

Core Elements

Recall $G = G_1 \times \cdots \times G_n$ acting on $\mathcal{S}$. A reduction map is a function $\pi \colon \mathcal{S} \to \mathcal{S}_i$ if for all $(g_1, \ldots, g_n) \in G$ and all $x \in \mathcal{S}$ we have

$$\pi((g_1, \ldots, g_n).x) = g_i.\pi(x).$$

Definition

A normal form function is a map $F \colon \mathcal{S} \to \mathcal{S}$ so that x and $F(x)$ are in the same G-orbit, and for $x, y \in V$, $F(x) = F(y)$ implies that x and y are in the same orbit.

Core Elements

Lemma (Existence and Uniqueness of Core Elements)

Let $G_1, \ldots, G_n$ be groups. Suppose $G_1 \times \cdots \times G_n$ acts on a set $\mathcal{S}$ and for each $1 \leq i \leq n$ there exists a reduction map $\pi_i \colon \mathcal{S} \to \mathcal{S}_i$ to a G_i-set $\mathcal{S}_i$:

$$\pi_i((g_1, \ldots, g_n).x) = g_i.\pi_i(x) \quad \text{for all } g_i \in G_i, x \in \mathcal{S}. \tag{1}$$

Fix a normal form function $F_i \colon \mathcal{S}_i \to \mathcal{S}_i$ for the G_i-action on $\mathcal{S}_i$. Then for each $x \in \mathcal{S}$ there exists a ***core element*** *$\omega \in \mathcal{S}$, which is defined by the properties:*

- *$x = (g_1, \ldots, g_n).\omega$ for some $g_i \in G_i$, and*
- *$\pi_i(\omega) = F_i(\pi_i(\omega))$ for all $1 \leq i \leq n$.*

Moreover, the core element is unique up to the action of $H_1 \times \cdots \times H_n \leq G_1 \times \cdots \times G_n$, where $H_i = \{g \in G_i : g.\pi_i(\omega) = \pi_i(\omega)\}$ is the stabilizer subgroup of $\pi_i(\omega)$.

Note the Core Lemma implies HOSVD.

Complex Orthogonal SVD

Consider $\mathcal{S}_i = S^2 V_i$ as a representation of $\mathrm{SO}(V_i)$: Define

$$\pi_i \colon V_1 \otimes \cdots \otimes V_n \to \mathcal{S}_i, \quad \text{where} \quad \pi_i(\Phi) = \Phi_{(i)} \Phi_{(i)}^\top$$

for $\Phi \in V_1 \otimes \cdots \otimes V_n$. Then π_i is a reduction map.

Theorem (Complex Orthogonal Symmetric SVD)

Every $M \in = S^2\mathbb{C}^n$ can be factored as $M = UDU^\top$ where $U \in \mathrm{SO}_n$ and $D = J_{k_1}(\lambda_1) \oplus \cdots \oplus J_{k_r}(\lambda_r)$ is a direct sum of symmetrized Jordan blocks.
We can uniquely specify D by ordering blocks.

Can use this to pull back normal forms to space of tensors via the reduction maps.

Symmetrized Jordan Blocks

$$J_k(\lambda) = \begin{pmatrix} \lambda & 1 & \cdots & \cdots & 0 \\ 1 & \lambda & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \lambda & 1 \\ 0 & \cdots & \cdots & 1 & \lambda \end{pmatrix} + \mathrm{i}m \begin{pmatrix} 0 & \cdots & \cdots & 1 & 0 \\ \vdots & & \iddots & 0 & -1 \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ 1 & 0 & \iddots & & \vdots \\ 0 & -1 & \cdots & \cdots & 0 \end{pmatrix}.$$

Complex Orthogonal HOSVD

Theorem ((Orthogonal HOSVD), Oeding-Tan 2025)

For each tensor $\Phi \in V_1 \otimes \cdots \otimes V_n$ there exists a core tensor Ω such that

- $\Phi = (U_1 \otimes \cdots \otimes U_n)\Omega$ *for some* $U_i \in \mathrm{SO}_{d_i}$, *and*
- $D_i = \Omega_{(i)}\Omega_{(i)}^\top$ *is a direct sum of symmetrized Jordan blocks in weakly decreasing order for all* $1 \leq i \leq n$.

The core tensor is unique up to the action of $H_1 \times \cdots \times H_n$, *where* $H_i \leq \mathrm{SO}_{d_i}$ *is the stabilizer subgroup of* D_i *by the conjugation action.*

Proof.

Apply Core Lemma. □

Complex Orthogonal HOSVD

Input: A tensor $\Phi \in \mathbb{C}^{d_1} \otimes \cdots \otimes \mathbb{C}^{d_n}$ such that for each $1 \leq i \leq n$, $\Phi_{(i)}\Phi_{(i)}^\top$ has distinct eigenvalues.
Output: Core tensor Ω for Orthogonal HOSVD.

1. For $1 \leq i \leq n$ use Complex Orthogonal Symmetric SVD to factorize $\Phi_{(i)}\Phi_{(i)}^\top = U_i D_i U_i^\top$, where $U_i \in \mathrm{SO}_{d_i}$ and D_i is diagonal with decreasing diagonal entries.
2. Set $\Omega \leftarrow (U_1^\top \otimes \cdots \otimes U_n^\top)\Phi$.

Normal forms for general qubits

We present a pair of algorithms in the even and odd cases that give normal forms for $n \geq 5$ general qubits for the SLOCC action.
These algorithms build on the Core Lemma, and repeatedly use reduction maps to pull back normal forms.

Simple normal form under unitary stabilizers

Input: $\Omega \in \mathcal{H}_n = \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$ such that $\Omega_{\mathbf{v}} \neq 0$ whenever $\mathbf{v} \in \mathcal{B} \cup \{\mathbf{0}\}$.
Output: The unique Ω' in the $H^{\times n}$-orbit of Ω such that each entry $\Omega'_{\mathbf{v}}$ is real and positive whenever $\mathbf{v} \in \mathcal{B} \cup \{0\}$.

1. Update $\Omega \leftarrow e^{\mathrm{im}\, t}\Omega$,where $t \in \mathbb{R}$ such that $e^{\mathrm{im}\, t}\Omega_{\mathbf{0}}$ is real and positive.
2. For $1 \leq i \leq n$ choose $t_i \in \mathbb{R}$ so that $e^{\mathrm{im}\, t_i \Omega_{\mathbf{v}^i}}$ is real and positive.
3. Compute $\Omega' = \begin{pmatrix} 1 & \\ & e^{\mathrm{im}\, t_1} \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 1 & \\ & e^{\mathrm{im}\, t_n} \end{pmatrix} \Omega.$

Kraus's algorithm

Input: $\Omega \in \mathcal{H}_n = \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$ such that $\Omega_{\mathbf{0}} \neq 0$.

Output: The unique Ω' in the $H^{\times n}$-orbit of Ω such that each entry $\Omega'_{\mathbf{v}}$ is real and positive whenever $\mathbf{v} \in \mathcal{B} \cup \{0\}$, where $\mathcal{B} \subset \mathrm{supp}(\Omega)$ is constructed from $\mathrm{supp}(\Omega)$ by the algorithm.

1. Update $\Omega \leftarrow e^{\mathrm{im}\, t}\Omega$, where $t \in \mathbb{R}$ such that $e^{\mathrm{im}\, t}\Omega_{\mathbf{0}}$ is real and positive.
2. Construct $\mathcal{B} = \{\mathbf{v}^1, \ldots, \mathbf{v}^m\}$ as follows. First set $\mathcal{B} \leftarrow \emptyset$. Then, going over elements $\mathbf{v} \in \mathrm{supp}(\Omega) \setminus \{\mathbf{0}\}$ in increasing lex order append $\mathbf{v}$ to $\mathcal{B}$ if $\mathbf{v}$ is linearly independent over $\mathbb{R}$ from the vectors already in $\mathcal{B}$. Stop once $\mathcal{B}$ spans the same space as $\mathrm{supp}(\Omega)$.
3. Compute any row vector $\mathbf{t} = (t_1, \ldots, t_n) \in \mathbb{R}^n$ satisfying the system
$$\begin{pmatrix} t_1 & \ldots & t_n \end{pmatrix} \begin{pmatrix} \mathbf{v}^1 & \ldots & \mathbf{v}^m \end{pmatrix} = -\begin{pmatrix} \arg(\Omega_{\mathbf{v}^1}) & \ldots & \arg(\Omega_{\mathbf{v}^m}) \end{pmatrix}.$$
4. Compute $\Omega' = \begin{pmatrix} 1 & 0 \\ 0 & e^{\mathrm{im}\, t_1} \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 1 & 0 \\ 0 & e^{\mathrm{im}\, t_n} \end{pmatrix} \Omega.$

Simple normal form under orthogonal stabilizers

Input: $\Omega \in \mathcal{H}_n$ such that $\mathrm{Re}(\Omega_{\mathbf{v}}) \neq 0$ whenever $\mathbf{v} \in \mathcal{B} \cup \{\mathbf{0}\}$.
Output: The unique Ω' in the $\mathcal{T}^{\times n}$-orbit of Ω such that $\mathrm{Re}(\Omega'_{\mathbf{v}}) > 0$ whenever $\mathbf{v} \in \mathcal{B} \cup \{0\}$.

1. Update $\Omega \leftarrow (-1)^t \Omega$, where $t \in \{0, 1\}$ such that $(-1)^t \mathrm{Re}(\Omega_{\mathbf{0}})$ is positive.
2. For $1 \leq i \leq n$ choose $t_i \in \{0, 1\}$ such that $(-1)^{t_i} \mathrm{Re}(\Omega_{\mathbf{v}^i})$ is positive.
3. Compute $\Omega' \leftarrow \begin{pmatrix} 1 & \\ & (-1)^{t_1} \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 1 & \\ & (-1)^{t_n} \end{pmatrix} \Omega$.

General normal form under orthogonal stabilizers (NFOS)

Input: A tensor $\Omega \in \mathcal{H}_n$.
Output: The unique Ω' in the $\mathcal{T}^{\times n}$-orbit of Ω such that $s(\Omega'_{\mathbf{v}}) = 0$ whenever $\mathbf{v} \in \mathcal{B} \cup \{0\}$, where $\mathcal{B} \subset \mathrm{supp}(\Omega)$ is constructed from $\mathrm{supp}(\Omega)$ by the algorithm.

1. Update $\Omega \leftarrow (-1)^t \Omega$, where $t \in \{0, 1\}$ such that $s((-1)^t \Omega_{\mathbf{0}}) = 0$.
2. Construct $\mathcal{B} = \{\mathbf{v}^1, \ldots, \mathbf{v}^m\}$ as follows. First set $\mathcal{B} \leftarrow \emptyset$. Then, going over elements $\mathbf{v} \in \mathrm{supp}(\Omega) \setminus \{\mathbf{0}\}$ in increasing lex order append $\mathbf{v}$ to $\mathcal{B}$ if $\mathbf{v}$ is linearly independent over $\mathbb{F}_2$ from the vectors already in $\mathcal{B}$. Stop once $\mathcal{B}$ spans the same space as $\mathrm{supp}(\Omega)$.
3. Compute any row vector $(t_1, \ldots, t_n)$ over $\mathbb{F}_2$ satisfying the system
$$\begin{pmatrix} t_1 & \ldots & t_n \end{pmatrix} \begin{pmatrix} \mathbf{v}^1 & \ldots & \mathbf{v}^m \end{pmatrix} = \begin{pmatrix} s(\Omega_{\mathbf{v}^1}) & \ldots & s(\Omega_{\mathbf{v}^m}) \end{pmatrix}.$$
4. Compute $\Omega' \leftarrow \begin{pmatrix} 1 & \\ & (-1)^{t_1} \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 1 & \\ & (-1)^{t_n} \end{pmatrix} \Omega.$

SLOCC normal form for general qubits, even case

Input: A tensor $\Phi \in \mathcal{H}_{2k} \cong (\mathbb{C}^2 \otimes \mathbb{C}^2)^{\otimes k}$ with $k > 1$ such that for each $1 \leq i \leq k$ the matrix $(T^{\otimes k}\Phi)_{(i)}(T^{\otimes k}\Phi)_{(i)}^{\top}$ has distinct eigenvalues.
Output: Normal form Ω in the SLOCC orbit of Φ.

1. Set $\Phi' \leftarrow T^{\otimes k}\Phi$.
2. Use OHOSVD Algorithm to compute a core tensor Ω' for Φ'.
3. Use NFOS Algorithm to compute the normal form Ω'' in the $\mathcal{T}^{\times 2k}$-orbit of Ω'.
4. Set $\Omega \leftarrow T^{*\otimes k}\Omega''$.

SLOCC normal form for general qubits, odd case

Input: A tensor $\Phi \in \mathcal{H}_n$ general with $n \geq 5$ odd.
Output: Normal form Ω in the SLOCC orbit of Φ.

1. For $1 \leq i \leq n$ compute $L_i \in \mathrm{SL}_2$ such that $L_i \pi_i(\Phi) L_i^\top = \sqrt{\delta_i} I_2$, where $\delta_i = \det(\pi_i(\Phi))$.
2. Set $\Psi \leftarrow (L_1 \otimes \cdots \otimes L_n)\Phi$ so that $\pi_i(\Psi) = \sqrt{\delta_i} I_2$ for all i.
3. Update $\Psi \leftarrow (A_1 \otimes \cdots \otimes A_n)\Psi$, where A_i equals K if $\pi_i(\Psi) = \sqrt{\delta_i} I_2$ is not in normal form, i.e. if $\sqrt{\delta_i} < -\sqrt{\delta_i}$ in lex order, otherwise $A_i = I_2$.
4. Use OHOSVD Algorithm to compute a core tensor Ω for Ψ.
5. If the first nonzero entry $a \in \mathbb{C}$ of Ω is less than $-a$ in lex order, update $\Omega \leftarrow -\Omega$.