Invariant Theory in Quantum Information

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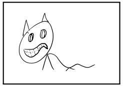
Quantum Entanglement

Basic states for single particles are represented by orthonormal "ket" vectors:

$$\{|0\rangle, |1\rangle, \ldots\} = \text{the measurement basis.}$$

A state for a single particle is a unit vector in a Hilbert space $\mathcal{H}$.



The state of a qubit ($\mathcal{H} = \mathbb{C}^2$) is in superposition: $|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$.
Probabilities are obtained via (Hermitian) measurement operators: $\widehat{A} |v\rangle = \lambda |v\rangle$:

Eigenvalues and eigenvectors are invariants of the operator!

Properties: $P(|\psi\rangle = |0\rangle) = |\langle 0|\psi\rangle|^2 = |\alpha|^2$, $P(|\psi\rangle = |1\rangle) = |\langle 1|\psi\rangle|^2 = |\beta|^2$,
and $|\langle\psi|\psi\rangle| = |\alpha|^2 + |\beta|^2 = 1$.

Think: $|\psi\rangle$ is a light wave. Measurement is passing through a polarizing filter.

Multiple Particle Quantum States

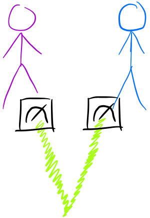
Basic states for multiple particles are represented by tensor products of "ket" vectors like $|00\rangle = |0\rangle \otimes |0\rangle$, $|01\rangle = |0\rangle \otimes |1\rangle$, $|10\rangle = |1\rangle \otimes |0\rangle$, $|11\rangle = |1\rangle \otimes |1\rangle$.

In general, a state for a multi-particle system is a unit vector in a tensor product of Hilbert spaces $\mathcal{H} \otimes \cdots \otimes \mathcal{H}$

A 2-qubit state for particles ψ_0 and ψ_1: $|\psi\rangle = \alpha_{00}\,|00\rangle + \alpha_{01}\,|01\rangle + \alpha_{10}\,|10\rangle + \alpha_{11}\,|11\rangle$.

Single particle projectors: $\langle i| \otimes \widehat{I} =$ project the first particle onto state $|i\rangle$,
and $\widehat{I} \otimes \langle j| =$ project the second particle onto state $|j\rangle$.

Joint probabilities: $P(|\psi_0\rangle = |i\rangle \,\&\, |\psi_1\rangle = |j\rangle) = |\,\langle ij|\psi\rangle\,|^2 = |\alpha_{ij}|^2$



Single particle probabilities: $P(|\psi_1\rangle = |0\rangle) = |\widehat{I} \otimes \langle 1|\psi\rangle\,|^2 = |\alpha_{00}|^2 + |\alpha_{10}|^2$ and $\sum_{ij} |\alpha_{ij}|^2 = 1$.

Entangled and Degenerate States

Example (Unentangled States)

Take $|\varphi\rangle = |00\rangle$. The probabilities are: $P(|\varphi_1\rangle = |0\rangle) = 100\%$, $P(|\varphi_1\rangle = |1\rangle) = 0\%$,
$P(|\varphi_2\rangle = |0\rangle) = 100\%$, $P(|\varphi_2\rangle = |1\rangle) = 0\%$,
The conditional probabilities are: $P(\varphi_2 = |0\rangle \mid \varphi_1 = |0\rangle) = 100\%$ and
$P(\varphi_2 = |0\rangle \mid \varphi_1 = |1\rangle) = 0\%$.
The independence condition $P(\varphi_2 = i \mid \varphi_1 = j) = P(\varphi_2 = i)P(\varphi_1 = j)$ holds for all i, j.

Example (Entangled States)

Take $\Phi = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. The probabilities are:
$P(|\varphi_1\rangle = |0\rangle) = P(|\varphi_1\rangle = |1\rangle) = 50\%, \quad P(|\varphi_2\rangle = |0\rangle) = P(|\varphi_2\rangle = |1\rangle) = 50\%.$
The conditional probabilities are: $P(|\varphi_2\rangle = |0\rangle \mid |\varphi_1\rangle = |0\rangle) = 100\%$ and
$P(|\varphi_2\rangle = |0\rangle \mid |\varphi_1\rangle = |1\rangle) = 0\%$.
The independence condition $P(|\varphi_2\rangle = i \mid |\varphi_1\rangle = j) = P(|\varphi_2\rangle = i)P(|\varphi_1\rangle = j)$ fails.

Enter Invariant Theory

Remark (Invariants distinguish entanglement type)

Represent 2-particle states as matrices. *Matrix rank classifies entanglement.*

$|00\rangle \to \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$ *and* $\det\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right) = 0 \Rightarrow$ *unentangled.*

$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \to \frac{1}{\sqrt{2}}\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ *and* $\det \frac{1}{\sqrt{2}}\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right) = \frac{1}{2} \Rightarrow$ *entangled.*

Rank-1 matrices correspond to *separable* or *unentangled* states.
Represent 2-qudit states as unit norm $d \times d$ complex matrices.

Matrix rank provides a hierarchy of entanglement:

$$\sigma_1 \subset \sigma_2 \subset \cdots \subset \sigma_d$$

A measure of entanglement is maximized when the determinant is maximized.

More particles

An n-qudit system has states in $\mathcal{H}^{\otimes n}$, with $\mathcal{H} = \mathbb{C}^d$.

The separable states are tensor products of states:
$\mathrm{Sep} = \{|\psi_1\rangle \otimes \cdots \otimes |\psi_n\rangle \mid \| |\psi_i\rangle \| = 1\}$.

The most entangled states should be the furthest from the separable states.

Invariants for $(\mathrm{SL}_d)^{\times n}$ can play the role of determinants.

Entanglement types for 3 qubits

$$
\begin{tikzcd}[column sep=small]
& \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \arrow[d] & & \text{GHZ} \\
& \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle) \arrow[dl] \arrow[d] \arrow[dr] & & \text{W} \\
\frac{1}{\sqrt{2}}(|001\rangle + |010\rangle) \arrow[dr] & \frac{1}{\sqrt{2}}(|001\rangle + |100\rangle) \arrow[d] & \frac{1}{\sqrt{2}}(|010\rangle + |100\rangle) \arrow[dl] & \text{biseparable} \\
& |000\rangle & & \text{separable}
\end{tikzcd}
$$

These states are separated by algebraic invariants: determinants and hyperdeterminants.

- *Three qubits can be entangled in two inequivalent ways*, [Dür-Vidal-Cirac (2000)]
- *Discriminants, resultants and multidimensional determinants*, [Gelfand-Kapranov-Zelevinsky (1994)]

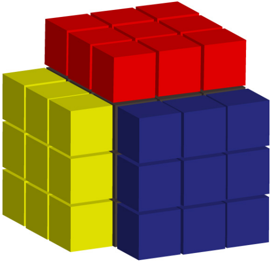
Generalize the determinant to hyperdeterminant

For matrices $A \in \mathcal{H} \otimes \mathcal{H}$ the determinant detects singular (degenerate) matrices:
$\det(A) = 0$ iff $\exists\ \vec{x}, \vec{y} \in \mathcal{H}^*$ so that $\vec{z}^\top A\vec{x} = 0$ and $\vec{y}^\top A\vec{z} = 0$ for all $\vec{z} \in \mathcal{H}^*$.
Up to change of coordinates A has a zero row (and a zero column).

For tensors $A \in \mathcal{H}^{\otimes n}$ the hyperdeterminant detects singular (degenerate) matrices:
$\mathrm{Det}(A) = 0$ iff $\exists\ \vec{x}_i \in \mathcal{H}^*$ for $i = 1..n$ so that

$$A(\vec{x}_1, \ldots, \vec{x}_i, \vec{z}_i, \vec{x}_{i+1}, \ldots, \vec{x}_n) = 0$$

for all i and all $\vec{z}_i \in \mathcal{H}^*$.



The hyperdeterminant becomes quite complicated,
but for small enough formats we can compute them
(like $2 \times 2 \times 2$, $2 \times 2 \times 2 \times 2$, and $3 \times 3 \times 3$).
Try to find states that maximize the absolute value of the hyperdeterminant.
Maximization requires being able to evaluate a function and its derivatives.

Generalize the determinant to polynomial invariants

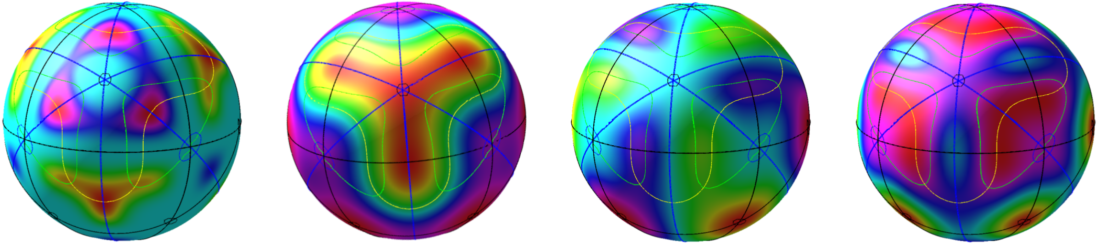
If V is a vector space with the action of a group $G \subset \mathrm{GL}(V)$, one way to separate G-orbits in V (think: entanglement types) is via polynomials.
The ring of G-invariant polynomials, $\mathbb{C}[V]^G$, is particularly useful.

Remark

When $V = \mathcal{H}^{\otimes n}$, we take G to be either the group of local unitary (LU) operations $U_d^{\times n}$, or the SLOCC-group, $\mathrm{SL}_d^{\times n}$.
Non-trivial G-invariants vanish on the set of separable states.

Maximizing polynomial invariants

After a significant variable reduction (Tensor Jordan Decomposition) we can plot the relevant values for the invariants for real 3-qutrit systems $(3 \times 3 \times 3)$ on a sphere:



hdet   deg 6   deg 9   deg 12

[The lines represent level sets for the other invariants.]

We can find the most entangled states for each measure of entanglement $|f(|\psi\rangle)|$ for invariants f.

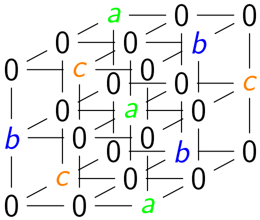
Invariants and Jordan Canonical form

Focus on 3-qutrit states for a moment.
A connection to the exceptional Lie group E_8 allows tensors $T \in (\mathbb{C}^3)^{\otimes 3}$ to have a Jordan decomposition: $T = S + N$, with S semi-simple, N nilpotent, and $[S, N] = 0$.

Generic semi-simple states are:

$$|\psi_S\rangle = a\,|v_1\rangle + b\,|v_2\rangle + c\,|v_3\rangle\,, \text{ with } (a, b, c) \in \mathbb{C}^3, \text{ and } |a|^2 + |b|^2 + |c|^2 = 1.$$

$$\begin{aligned} |v_1\rangle &= \tfrac{1}{\sqrt{3}}\left(|000\rangle + |111\rangle + |222\rangle\right), \\ |v_2\rangle &= \tfrac{1}{\sqrt{3}}\left(|012\rangle + |120\rangle + |201\rangle\right), \\ |v_3\rangle &= \tfrac{1}{\sqrt{3}}\left(|021\rangle + |102\rangle + |210\rangle\right). \end{aligned}$$



Invariants become much simpler since $f(S + N) = f(S)$.

Expressions of invariants on semi-simple elements

The restriction of invariants to normalized states $|\psi_S\rangle$:

$$\begin{aligned}\Delta_{333}(|\psi_S\rangle) &= -\tfrac{4}{3^{18}}\, a^3b^3c^3\,(a+b+c)^3 \times \\ &\quad (a^2+2\,ab-ac+b^2-bc+c^2)^3\,(a^2-ab+2\,ac+b^2-bc+c^2)^3 \times \\ &\quad (a^2-ab-ac+b^2+2\,bc+c^2)^3\,(a^2-ab-ac+b^2-bc+c^2)^3\ ,\end{aligned}$$

$$I_6(\psi_S) = \tfrac{1}{27}\left(a^6-10a^3b^3-10a^3c^3+b^6-10b^3c^3+c^6\right)\ ,$$

$$I_9(\psi_S) = \tfrac{-\sqrt{3}}{243}\,(a-b)\,(a-c)\,(b-c)\left(a^2+ab+b^2\right)\left(a^2+ac+c^2\right)\left(b^2+bc+c^2\right)\ ,$$

$$I_{12}(\psi_S) = \tfrac{1}{729}\left(\begin{array}{l} a^9b^3+a^3b^9+a^9c^3+b^9c^3+a^3c^9+b^3c^9 \\ \qquad -4\,(a^6b^6+a^6c^6+b^6c^6)+2\,(a^6b^3c^3+a^3b^6c^3+a^3b^3c^6)\end{array}\right).$$

New maximally entangled states for 3-qutrits

Theorem (Jaffali-Holweck-Oeding 2024)

The global maximum of the absolute value of the hyperdeterminant $|\Delta_{333}|$*, when restricted to real states is* $\frac{\sqrt{3}}{2^{19}\times 3^{14}}$*. The global max is reached at 12 semi-simple points* $a\,|v_1\rangle + b\,|v_2\rangle + c\,|v_3\rangle$ *with the following values and their permutations:*

$$(a,b,c) = (rs, s, s)\,, \quad \textit{with} \quad r = (1 \pm \sqrt{3}) \quad \textit{and} \quad s = \pm\sqrt{\frac{1}{(r^2+2)}}\,. \tag{1}$$

New maximally entangled states for 3-qutrits

Theorem (Jaffali-Holweck-Oeding 2024)

The global maximum of the absolute value of the fundamental invariants I_6, I_9 *and* I_{12} *restricted to generic 3-qutrits, with maximum values respectively* $\frac{1}{18} = .05\overline{5}$, $\frac{\sqrt{6}}{3888} \simeq 0.00063$ *and* $\frac{1}{7776} \simeq 0.0001286$, *is reached for the real 3-qutrit Aharonov state:*

$$|M_{333,I}\rangle = \frac{1}{\sqrt{2}}\Big(|v_1\rangle - |v_2\rangle \Big) .$$

All real semi-simple states that obtain the maximum are permutations of $|M_{333,I}\rangle$.

4-qubit critical states

Using a variant of HOSVD from [Oeding-Tan (SIAM/ag)2025], with Jordan decomposition, we maximize the (norms of the) fundamental symmetric invariants for tensors of format $2 \times 2 \times 2 \times 2$. We utilize the Julia library HomotopyContinuation.jl.

We find (all Enriquez's and several new) maximally entangled 4-qubit states.

Theorem (Oeding-Tan 2025)

The following states (in Cartan coordinates) maximize $|f_6(|\psi_S\rangle|$:

- $\varphi_1 = (1, 0, 0, 0) \cong |MP\rangle$
- $\varphi_2 = (1, 1, 0, 0) \cong |GHZ\rangle$
- $\varphi_3 = (1, 1, 1, 0)$
- $\varphi_4 = (2, 1, 1, 0)$
- $\vdots$

(We have similar results for $|f_8(|\psi_S\rangle|$.)

- $\varphi_5 = (\sqrt{2}, \mathrm{im}, 0, 0)$
- $\varphi_6 = (\sqrt{2}, \mathrm{im}, \mathrm{im}, 0)$
- $\varphi_7 = (\sqrt{2}, \mathrm{im}, \mathrm{im}, \mathrm{im})$
- $\varphi_8 = (\sqrt{3}, \mathrm{im}, \mathrm{im}, \mathrm{im}) \cong |HS\rangle$
- $\varphi_9 = (1, e^{\pi \,\mathrm{im}/3}, e^{2\pi \,\mathrm{im}/3}, 0) \cong |HD\rangle$
- $\varphi_{10} = (1, 1 - \frac{\sqrt{3}}{\sqrt{2}} + \frac{1}{\sqrt{2}}\,\mathrm{im}, \frac{\sqrt{3}}{\sqrt{2}} - 1 + \frac{1}{\sqrt{2}}\,\mathrm{im}, (\sqrt{3} - \sqrt{2})\,\mathrm{im})$
- $\varphi_{11} = (7, 2\sqrt{7}\,\mathrm{im}, 2\sqrt{7}\,\mathrm{im}, 0)$
- $\varphi_{12} = (18, 11 - \sqrt{203}\,\mathrm{im}, 7 + \sqrt{203}\,\mathrm{im}, 0)$
- $\varphi_{13} = (1, a + b\,\mathrm{im}, -a - b\,\mathrm{im}, 0)$, where

  $$a \approx 0.0933383722, \quad b \approx 0.6221645823$$

  satisfies the following.

  $$1000a^2b^2 - 872b^4 + 85a^2 + 345b^2 - 7 = 0$$
  $$100a^4 - 244b^4 + 45a^2 + 65b^2 + 11 = 0$$

- $\varphi_{14} = (1, a - b\,\mathrm{im}, a + b\,\mathrm{im}, c\,\mathrm{im})$, where

  $$a \approx 0.2166757505, \quad b \approx 0.8300687024, \quad c \approx 0.3661117361$$


ML
Luke
Hamza
May 3
+ Outline
May 30
Network Design for learning
* linear (Subspaces)
* Curves
Experiments
* Seg
* Dual
* r = 2, 3
Context / Application
* Classifier for 4 Qubits ...
vs
PCA
vs
SVM
vs. Homotopy continuation

References

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