The Variety of Principal Minors of Symmetric Matrices and its Set Theoretic Defining Equations

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Goals

- Let $G \subset GL(V)$, $V$- vector space over $\mathbb{C}$. A variety $X \subset \mathbb{P}V$ is a $G$-variety if $G.X \subset X$.

- Goal 1: Study a prototypical $G$-variety and learn how to study other $G$-varieties which arise in fields such as algebraic statistics, probability theory, signal processing, etc.

- Goal 2: Solve the Holtz-Sturmfels Conjecture (set theoretic version).
Questions

- A principal minor of a matrix $A$ is the determinant of a submatrix formed by striking out the same rows and columns of $A$, i.e. centered on the diagonal.

- Holtz and Schneider, D. Wagner: When is it possible to prescribe the principal minors of a symmetric matrix?

- Equivalently, when can you prescribe all the eigenvalues of a symmetric matrix and all of its principal submatrices?

- For $n \geq 3$ this is an overdetermined problem: $\binom{n+1}{2}$ versus $2^n$. 
Examples: 2 × 2 case

Define a (homogeneous) map:

\[ \varphi : \text{symmetric matrices} \rightarrow \text{principal minors} : \]

\[ \varphi \left( \begin{pmatrix} a & c \\ c & b \end{pmatrix}, t \right) = [t^2, ta, tb, ab - c^2] \]

When can we go backwards? Given \([w, x, y, z]\) is there a \(2 \times 2\) matrix that has these principal minors? Need to solve: (WLOG assume \(t = w = 1\))

\[
\begin{align*}
x &= a \\
y &= b \\
z &= ab - c^2 \implies c = \pm \sqrt{xy - z}
\end{align*}
\]

\[ \varphi \left( \begin{pmatrix} x & \pm \sqrt{xy - z} \\ \pm \sqrt{xy - z} & y \end{pmatrix}, 1 \right) = [1, x, y, z] \]

Conclude: Even in the \(n \times n\) case, the 0 × 0, 1 × 1, and 2 × 2 minors determine a symmetric matrix up to the signs of the off-diagonal terms.
Examples $3 \times 3$:

$$\varphi \left( \left( \begin{array}{ccc} x_{11} & x_{12} & x_{13} \\ x_{12} & x_{22} & x_{23} \\ x_{13} & x_{23} & x_{33} \end{array} \right), t \right)$$

$$= \left[ t^3, t^2 x_{11}, t^2 x_{22}, t(x_{11} x_{22} - x_{12}^2), t^2 x_{33}, t(x_{11} x_{33} - x_{13}^2), t(x_{22} x_{33} - x_{23}^2), x_{11} x_{22} x_{33} + 2 x_{12} x_{13} x_{23} - x_{11} x_{23}^2 - x_{22} x_{13}^2 - x_{33} x_{12}^2 \right]$$

Given $[X^{000}, X^{100}, X^{010}, X^{110}, X^{001}, X^{101}, X^{011}, X^{111}]$ is there a matrix that maps to it?

Count parameters: 7 versus 8 - there must be some relation that holds!
The First Relation

\[ x_{12}^2 = X^{100}X^{010} - X^{110} \]
\[ x_{13}^2 = X^{100}X^{001} - X^{101} \]
\[ x_{23}^2 = X^{001}X^{010} - X^{011} \]

\[ X^{111} = X^{100}X^{010}X^{001} - X^{100}x_{23}^2 - X^{010}x_{13}^2 - X^{001}x_{12}^2 + 2x_{12}x_{13}x_{23} \]

\[ (X^{111} - X^{100}X^{010}X^{001} + X^{100}x_{23}^2 + X^{010}x_{13}^2 + X^{001}x_{12}^2)^2 \]
\[ = 4(x_{12}x_{13}x_{23})^2 \]

\[ \left( \frac{X^{111} - X^{100}X^{010}X^{001} + X^{100}(X^{001}X^{010} - X^{011})}{+X^{010}(X^{100}X^{001} - X^{101}) + X^{001}(X^{100}X^{010} - X^{110})} \right)^2 \]
\[ = 4(X^{100}X^{010} - X^{110})(X^{100}X^{001} - X^{101})(X^{001}X^{010} - X^{011}) \]

\[ 0 = (X^{111})^2 + (X^{100})^2(X^{011})^2 + (X^{010})^2(X^{101})^2 + (X^{110})^2(X^{001})^2 \]
\[ + 4X^{110}X^{101}X^{011} + 4X^{100}X^{010}X^{001}X^{111} \]
\[ - 2X^{100}X^{011}X^{111} - 2X^{100}X^{010}X^{011}X^{101} - 2X^{010}X^{101}X^{111} \]
\[ - 2X^{100}X^{001}X^{110}X^{011} - 2X^{001}X^{110}X^{111} - 2X^{001}X^{010}X^{101}X^{110} \]
First result

Theorem (Holtz-Sturmfels ’07)

All relations among the principal minors of a $3 \times 3$ matrix are generated by ...this beautiful degree 4 homogeneous polynomial:

\[
\begin{align*}
(X^{000})^2(X^{111})^2 + (X^{100})^2(X^{011})^2 \\
+ (X^{010})^2(X^{101})^2 + (X^{110})^2(X^{001})^2 \\
+ 4X^{000}X^{110}X^{101}X^{011} + 4X^{100}X^{010}X^{001}X^{111} \\
- 2X^{000}X^{100}X^{011}X^{111} - 2X^{100}X^{010}X^{011}X^{101} \\
- 2X^{000}X^{010}X^{101}X^{111} - 2X^{100}X^{001}X^{110}X^{011} \\
- 2X^{000}X^{001}X^{110}X^{111} - 2X^{001}X^{010}X^{101}X^{110}
\end{align*}
\]

-Cayley’s hyperdeterminant of format $2 \times 2 \times 2$. Notice: It is invariant under the action of $\mathfrak{S}_3 \ltimes SL(2) \times SL(2) \times SL(2)$!
The Variety of Principal Minors of Symmetric Matrices

The variety of principal minors of $n \times n$ symmetric matrices, $Z_n$, is defined by the following rational map

$$
\varphi : \mathbb{P}(S^2 \mathbb{C}^n \oplus \mathbb{C}) \dasharrow \mathbb{P}(\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2) = \mathbb{P}\mathbb{C}^{2^n}
$$

$$
[A, t] \mapsto [t^n, t^{n-1} \Delta_{[10\ldots0]}(A), t^{n-1} \Delta_{[010\ldots0]}(A), t^{n-2} \Delta_{[110\ldots0]}(A), t^{n-1} \Delta_{[0010\ldots0]}(A), t^{n-2} \Delta_{[1010\ldots0]}(A), t^{n-2} \Delta_{[0110\ldots0]}(A), \ldots, \Delta_{[1\ldots1]}(A)]
$$

where $\Delta_{[I]}(A)$ is the principal minor of $A$ with rows indicated by $I$.

Q: Given a vector $v$ of length $2^n$, how can you tell whether or not it arose in this way?

A: test whether $v$ satisfies all the relations in $\mathcal{I}(Z_n)$. 
Hidden Symmetry

**Theorem (Landsberg,Holtz-Sturmfels)**

$Z_n$ is invariant under the action of $G = \mathfrak{S}_n \ltimes SL(2)^\times n$.

- **Fact:** A variety $X \subset \mathbb{P}^N$ is a $G$-variety $\iff$ the ideal $\mathcal{I}(X)$ is a $G$-module.
- $Z_n$ is a subvariety of $\mathbb{P}(V_1 \otimes \cdots \otimes V_n)$, where each $V_i \simeq \mathbb{C}^2$.
- **KEY POINT:** We must study $\mathcal{I}(Z_n) \subset \text{Sym}(V_1^* \otimes \cdots \otimes V_n^*)$ as a $G$-module!
- **Mantra:** “Each irreducible module is either in or out!”
For non-degenerate $\omega \in \bigwedge^2 \mathbb{C}^n$, the Lagrangian Grassmannian is $Gr_\omega(n, 2n) = \{ E \in Gr(n, 2n) \mid \omega(v, w) = 0 \ \forall v, w \in E \}$.

$Gr_\omega(n, 2n)$ is a homogeneous variety for $Sp(2n)$.

$Gr_\omega(n, 2n)$ is the image of the rational map:

$$
\psi : \mathbb{P}(S^2 \mathbb{C}^n \oplus \mathbb{C}) \dashrightarrow \mathbb{P} \Gamma_n \cong \mathbb{P}^\left(\binom{2n}{n} - \binom{2n}{n-2} - 1\right)
$$

$\{\text{symmetric matrix}\} \mapsto \{\text{vector of all nonredundant minors}\}$

The connection: $Z_n$ is a linear projection of $Gr_\omega(n, 2n)$.

Can use this projection to find the symmetry group of $Z_n$ as a subgroup of $Sp(2n)$.

Try to find projections of homogeneous varieties to study other $G$-varieties.
Multilinear Algebra

- \( S^d(V_1^* \otimes \cdots \otimes V_n^*) = \) homogeneous degree \( d \) polynomials on \( 2^n \) variables. It is a module for \( G = SL(V_1) \times \cdots \times SL(V_n) \)

- If we choose a basis \( \{ x_{i}^0, x_{i}^1 \} \) of \( V_i^* \simeq \mathbb{C}^2 \) for each \( i \), then \( V_1^* \otimes \cdots \otimes V_n^* \) has the induced basis \( x_{\epsilon}^1 \otimes \cdots \otimes x_{\epsilon}^n =: X^I \).

- Then \( G \) acts on \( V_1^* \otimes \cdots \otimes V_n^* \) by change of basis in each factor: If \( g = (g_1, \ldots, g_n) \in G \), then

  \[
  g.X^I = (g_1.x_{\epsilon}^1) \otimes \cdots \otimes (g_n.x_{\epsilon}^n),
  \]

  and acts on \( S^d(V_1^* \otimes \cdots \otimes V_n^*) \) by the induced action:

  \[
  g.(X^I X^J \cdots X^K) = (g.X^I)(g.X^J) \cdots (g.X^K)
  \]

- We have defined the action on a basis of each module, so we can just extend by linearity to get the action on the whole module.
Want to study $I_d(Z_n) \subset S^d(V_1^* \otimes \cdots \otimes V_n^*)$.

Each irreducible $G_n \ltimes SL(2)^{\times n}$-module in $S^d(V_1^* \otimes \cdots \otimes V_n^*)$ is isomorphic to one indexed by partitions $\pi_i$ of $d$ of the form:

$$S_{\pi_1} S_{\pi_2} \cdots S_{\pi_n} := \bigoplus_{\sigma \in S_n} S_{\pi_{\sigma(1)}} V_1^* \otimes S_{\pi_{\sigma(2)}} V_2^* \otimes \cdots \otimes S_{\pi_{\sigma(n)}} V_n^*$$

Can use the combinatorial information $\pi_1, \ldots, \pi_n$ to construct the module.

If $M$ is an irreducible $G$-module, then $M = \{G.v\}$, some vector $v$ - use this as often as possible.

This gives a finite list of vectors to test for ideal membership!

Also gives a way to produce many polynomials in $I(Z_n)$ from one polynomial.
An Example

The module $S_{(2,2)}V \subset V^\otimes 4$ is one dimensional, and every vector is a scalar multiple of

$$h = 2X^{0011} - X^{1001} - X^{1010} - X^{0101} - X^{0110} + 2X^{1100}$$

To find a polynomial in $S_{(2,2)}V_1 \otimes S_{(2,2)}V_2 \otimes S_{(2,2)}V_3$, we need to compute $h \otimes h \otimes h$ in $V_1^\otimes 4 \otimes V_2^\otimes 4 \otimes V_3^\otimes 4$, but we want a polynomial in $S^4(V_1 \otimes V_2 \otimes V_3)$, so we just permute

$$V_1^\otimes 4 \otimes V_2^\otimes 4 \otimes V_3^\otimes 4 \rightarrow (V_1 \otimes V_2 \otimes V_3)^\otimes 4$$

and symmetrize

$$(V_1 \otimes V_2 \otimes V_3)^\otimes 4 \rightarrow S^4(V_1 \otimes V_2 \otimes V_3)$$
Finally, we get the result

\[
(X^{000})^2(X^{111})^2 + (X^{100})^2(X^{011})^2 \\
+ (X^{010})^2(X^{101})^2 + (X^{110})^2(X^{001})^2 \\
+ 4X^{000}X^{110}X^{101}X^{011} + 4X^{100}X^{010}X^{001}X^{111} \\
- 2X^{000}X^{100}X^{011}X^{111} - 2X^{100}X^{010}X^{011}X^{101} \\
- 2X^{000}X^{010}X^{101}X^{111} - 2X^{100}X^{001}X^{110}X^{011} \\
- 2X^{000}X^{001}X^{110}X^{111} - 2X^{001}X^{010}X^{101}X^{110}
\]

In fact, this is Cayley’s hyperdeterminant of format $2 \times 2 \times 2$!
It’s an irreducible degree 4 polynomial on 8 variables.
It is invariant under the action of $\mathfrak{S}_3 \times SL(2) \times SL(2) \times SL(2)$.
It generates the module $S_{(2,2)}S_{(2,2)}S_{(2,2)}$.
It is the single equation defining the hypersurface $Z_3$. 
Rephrasing of Previous Results

Theorem (Holtz-Sturmfels)

\( \mathcal{I}(\mathbb{Z}_3) \) is generated in degree 4 by \( S_{(2,2)}S_{(2,2)}S_{(2,2)} \) (Cayley’s Hyperdeterminant of format \( 2 \times 2 \times 2 \)).

Theorem (H-S)

\( \mathcal{I}(\mathbb{Z}_4) \) is generated in degree 4 by \( S_{(4)}S_{(2,2)}S_{(2,2)}S_{(2,2)} \) (A hyperdeterminantal module).

Remark: \( S_{(4)}S_{(2,2)}S_{(2,2)}S_{(2,2)} \) is the span of the \( G \)-orbit of the \( 2 \times 2 \times 2 \) hyperdeterminant on the variables \( X^{[***0]} \).

Conjecture (H-S)

\( \mathcal{I}(\mathbb{Z}_n) \) is generated in degree 4 by \( S_{(4)} \ldots S_{(4)}S_{(2,2)}S_{(2,2)}S_{(2,2)} \) (the hyperdeterminantal module).
A Limit of the Computer’s Usefulness

• For \( n = 3 \): A single irreducible degree 4 polynomial on 8 variables cuts out the irreducible hypersurface in \( \mathbb{P}^7 \).
• For \( n = 4 \): 20 degree 4 polynomials on 16 variables. Macaulay2 \( \Rightarrow \) the ideal is prime and has the correct dimension. But \( \mathbb{Z}_4 \) is an irreducible variety + some facts from comm. alg. \( \Rightarrow \) done.
• For \( n = 5 \): 250 degree 4 polynomials on 32 variables. Sadly, the computer has not yet told me whether or not this ideal is prime.
• For \( n = 6 \): 2500 degree 4 polynomials on 64 variables. :-(
• For \( n = n \): \( \binom{n}{3} 5^{n-3} \) degree 4 polynomials on \( 2^n \) variables. What can we say in general without the computer?
New Results

**Theorem (-)**

Let $HD := \{\mathfrak{S}_n \times SL(2)^\times n.hyp_{123}\} = S_{(4)} \cdots S_{(4)}S_{(2,2)}S_{(2,2)}S_{(2,2)}$. The variety $Z_n$ is cut out set theoretically by the hyperdeterminantal module.

$$\mathcal{V}(HD) = Z_n.$$ 

- To prove that $Z_n \subset \mathcal{V}(HD)$, show that $hyp$, a highest weight vector for the irreducible module $M$, vanishes on every point of $Z_n$. Follows from $3 \times 3$ case.

- To prove that $Z_n \supset \mathcal{V}(HD)$, a more geometric understanding of the zero set, $\mathcal{V}(HD)$, is needed.
Outline of proof of main theorem

- Want to show $\mathcal{V}(HD) \subset Z_n$ - do induction on $n$.

- Give a geometric characterization of $\mathcal{V}(HD)$.

- Attempt to construct a matrix $A \in S^2 \mathbb{C}^n$ that maps to $z \in \mathcal{V}(HD)$.

- Identify possible obstructions as $G$-modules.

- Identify the space of obstructions geometrically.

- Show $\mathcal{V}(HD)$ also contains the space of obstructions.
Applications outside of geometry

- Spectral graph theory.
- Probability theory - covariance of random variables.
- Statistical physics - determinantal point processes.
- Matrix theory - $P$-matrices, GKK-$\tau$ matrices.
Spectral Graph Theory

Let $\Gamma$ be a graph with

- vertex set $Q_0 = \{v_1, \ldots, v_n\}$
- edge set $Q_1 = \{e_{i,j} \mid \overrightarrow{v_i v_j} \in \Gamma\}$.

The graph Laplacian of an undirected graph is a matrix

$$\Delta(\Gamma)_{i,j} = \begin{cases} 
-1 & \text{if } i \neq j \text{ and } e_{i,j} \in Q_1 \\
0 & \text{if } i \neq j \text{ and } e_{i,j} \notin Q_1 \\
\deg(v_i) & \text{if } i = j 
\end{cases}$$

The principal minors of $\Delta(\Gamma)$ are invariants of the graph, in fact:

**Theorem (Kirchoff’s Matrix-Tree theorem (∼1850’s))**

Any $(n - 1) \times (n - 1)$ principal minor of $\Delta(\Gamma)$ counts the number of spanning trees of $\Gamma$. 

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There are many generalizations of the Matrix-Tree Theorem, such as

**Theorem (Matrix-Forest Theorem)**

\[ \Delta(\Gamma)^S = \text{number of spanning forests of } \Gamma \text{ rooted at vertices indexed by } S, \text{ where } \Delta(\Gamma)^S \text{ is the principal minor of } \Delta(\Gamma) \text{ indexed by } S. \]

The \( \Delta(\Gamma)^S \) are graph invariants. The relations among principal minors are then also relations among these graph invariants.

**Corollary (Corollary to Main Theorem)**

There exists an undirected weighted graph \( \Gamma \) with invariants \([v] \in \mathbb{P}^{2n-1}\) specified by the principal minors of a symmetric matrix \( \Delta_{wt}(\Gamma) \) if and only if \([v]\) is a zero of all the polynomials in the hyperdeterminant module.
Concluding Remarks

- This problem shows how representation theory and geometry can be used to prove exciting new results.

- We resolved the set theoretic version of the Holtz-Sturmfels conjecture, but more work needs to be done in order to prove the ideal theoretic version.

- Thank you for attending! Special thanks to my thesis advisor, J.M. Landsberg.
Characterizing the zero set of $\mathcal{V}(HD)$ via augmentation

- Notice that $HD_n = S(4) \cdots S(4) S_{(2,2)} S_{(2,2)} S_{(2,2)}$ and $HD_{n+1} = S(4) \cdots S(4) S_{(2,2)} S_{(2,2)} S_{(2,2)}$ is still degree 4.

- What can we say about zero set of an augmented ideal $\mathcal{V}(\mathcal{I}_d(X) \otimes S^d V^*)$ based on $\mathcal{V}(\mathcal{I}_d(X))$?

Lemma (inspired by Landsberg-Manivel lemma regarding prolongation)

Let $X \subset \mathbb{P}W$ and let $\tilde{X} = \mathcal{V}(\mathcal{I}_d(X))$ (notation).

$$\mathcal{V}(\mathcal{I}_d(X) \otimes S^d V^*) = \text{Seg}(\tilde{X} \times \mathbb{P}V) \cup \bigcup_{L \subset \tilde{X}} \mathbb{P}(L \otimes V),$$

where $L \subset \tilde{X}$ are linear subspaces.
What does this buy us?

Consequence

Assume that $HD = \bigoplus_i HD_i \otimes S^d V_i \subset S^d (V_1 \otimes \cdots \otimes V_n)$ and $V_i \simeq \mathbb{C}^2$, then

$$\mathcal{V}(HD) = \bigcap_{i=1}^n \left( \bigcup_{L \subset \mathcal{V}(HD_i)} \mathbb{P}(L \otimes V_i) \right).$$

- Suppose $z \in \mathcal{V}(HD) = \mathcal{V}(\bigoplus_i HD_i \otimes S^d V_i)$, and assume for induction that $\mathcal{V}(HD_i) \simeq \mathbb{Z}_{n-1}$.
- Then our geometric realization gives $n$ different expressions for $z$,

\[ z = \varphi([A^{(i)}, t^{(i)}]) \otimes x_0^0 + \varphi([B^{(i)}, s^{(i)}]) \otimes x_1^1, \quad \text{(no summation)} \]

where $A^{(i)}, B^{(i)}$ are $n-1 \times n-1$ symmetric matrices, and

\[ \{x_0^i, x_1^i\} = V_i. \]

- If we can use this information to build an $n \times n$ matrix $A$ so that $\varphi([A, t]) = z$, we will have proved the theorem.
Building a matrix

We have \( n \) expressions

\[
z = \varphi([A^{(i)}, t^{(i)}]) \otimes x_i^0 + \varphi([B^{(i)}, s^{(i)}]) \otimes x_i^1,
\]

and the term \( \varphi([A^{(1)}, t^{(1)}]) \otimes x_1^0 \) can be thought of as the principal minors (not involving the first row and column) of the matrix

\[
A(\overrightarrow{x_1}) = \begin{pmatrix}
  x_{1,1} & x_{1,2} & x_{1,3} & \ldots & x_{1,n} \\
  x_{1,2} & a_{1,2}^{(1)} & a_{2,2}^{(1)} & \ldots & a_{2,n}^{(1)} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  x_{1,n} & a_{1,n}^{(1)} & a_{2,n}^{(1)} & \ldots & a_{n,n}^{(1)}
  \end{pmatrix},
\]

where \( x_{1,i} \) are variables, and the entries of \( A^{(1)} = (a_{i,j}^{(1)}) \), are fixed. The other expressions \( \varphi([A^{(i)}, t^{(i)}]) \otimes x_i^0 \) have a similar interpretation.
Building a matrix

- The $1 \times 1$ principal minors determine the diagonal entries and the $2 \times 2$ principal minors are all of the form $a_{i,i}a_{j,j} - a_{i,j}^2$.
- We know that the principal minors $\Delta_I(A(\vec{x}_i))$ and $\Delta_I(A(\vec{x}_j))$ agree whenever $i, j \notin I$.
- Our question comes down to whether we can make consistent choices so that the matrices $A(\vec{x}_i)$ agree.
- It suffices to prove that if we fix $A^{(1)}$, that we can choose $\vec{x}_1$ and $A^{(i)}$ so that all of the principal minors agree where the matrices overlap.
• Construct $A(\overrightarrow{x_i})^{(j)}$, by deleting the $j^{th}$ row and column.

• By induction, it suffices to consider

$$A(x_{1,2}) = \begin{pmatrix}
a_{1,1} & x_{1,2} & a_{1,3} & \cdots & a_{1,n} \\
x_{1,2} & a_{2,2} & \cdots & \cdots & a_{2,n} \\
a_{1,3} & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{1,n} & a_{2,n} & \cdots & a_{n,n}
\end{pmatrix},$$

and show that we can pick $x_{1,2}$ so that all of the principal minors of $A(\overrightarrow{x_i})^{(j)}$ agree.

• We will have only determined that the matrix $A(x_{1,2})$ has all the correct principal minors (matching our point $z \in \mathcal{V}(HD)$) except possibly the determinant.
Lemma (The Almost Lemma, $n \geq 4.$)

Suppose $[z] = [z_I X^I] \in \mathcal{V}(HD)$, and $[v_A] = [v_{A,I} X^I] = [\varphi([A, t])] \in \mathbb{Z}_n$ are such that $z_I = v_{A,I}$ for all $I \neq [1, \ldots, 1]$. If $z_{[1,\ldots,1]} \neq v_{A,[1,\ldots,1]}$, then

$$[z] \in \bigcup_{|I_s| \leq 2} (\text{Seg}(\mathbb{P}V_{I_1} \times \cdots \times \mathbb{P}V_{I_m})) \subset \mathbb{Z}_n.$$ 

We have essentially made a reduction to a problem in a single variable. Once the obstructions to solving this problem are identified as a $G$-module, the proof of this lemma is an application of the geometric characterization above.
Almost...but what does this buy me?

The lemma says that \( \text{Seg}(\mathbb{P}V_{I_1} \times \cdots \times \mathbb{P}V_{I_m}) \subset \mathbb{Z}_n \).

- In fact, every point in \( \text{Seg}(\mathbb{P}V_{I_1} \times \cdots \times \mathbb{P}V_{I_m}) \subset \mathbb{Z}_n \) comes from a block diagonal matrix with only \( 1 \times 1 \) and \( 2 \times 2 \) blocks.

- Such a matrix is a special case of a symmetric tri-diagonal matrix, and it’s a fact that none of its principal minors depend on the sign of the off diagonal terms.

- We use this fact iteratively in our induction for the proof of the final lemma.