

ELECTRIC NETWORK SYNTHESIS

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This document summarizes our understanding of the problem presented by Rudolf Kalman in his lecture in the Berkeley Algebraic Statistics Seminar on October 26, 2011.

The kind of electrical networks we are considering can be defined mathematically as follows. An *RLC network* is an undirected connected graph $\Gamma = (V, E)$ satisfying the following properties:

- There are no loops but parallel edges are allowed.
- Two of the nodes are distinguished.
- Every edge is labeled by an unknown real parameter times either s^{-1} , s^0 or s^1 .

In practice Γ is a network of resistors (R), capacitors (C) and inductors (L), and one is interested in the impedance (defined below) between the two distinguished nodes. Two basic questions in electrical network theory are the following:

- (1) For a given network, what is the impedance between the two distinguished nodes? (prediction)
- (2) Given an impedance function, does there exist a network with that impedance, and if so what is the minimal such network? (synthesis)

If Z is the impedance of a component (or network), $Y = \frac{1}{Z}$ is called the *admittance*. Consider the simplest network with precisely two components of the same type. There are only two graphs – either a loop (two components in parallel) or a chain (two components in series). There are only two rules for how to compute the impedance Z of the network, depending on the type of component. For resistors and inductors impedance adds in series; $Z = Z_1 + Z_2$, and admittance adds in parallel; $\frac{1}{Z} = \frac{1}{Z_1} + \frac{1}{Z_2}$. For capacitors the reverse is true, namely the impedance of capacitors adds in parallel and the admittance adds in series.

If a network is inductively constructed from sub-networks that are all either in series or parallel, we call it a *series-parallel* network. Such networks are nice because their impedance can be computed inductively. However not all networks are series-parallel: the first example of such a network is called a bridge network, and we consider one such bridge network in Example 1 below.

If a network is not series-parallel, we will define the impedance by considering all paths in the underlying graph, (of which there are infinitely many) and say that the impedance between the two distinguished nodes is the sum of the impedance of each path in the graph from one distinguished node to the other. This infinite sum may be undesirable, so we do something slightly different.

Following is a mathematical description of an electrical network and how to compute its impedance. (*Professor Kalman asks if there might be an easy proof that this is equivalent to the usual definition, but what is the usual definition?*) Consider a graph $\Gamma = (V, E)$ associated to an RLC-network as above. Set $V = \{1, 2, \dots, d\}$, assume that the distinguished nodes are 1 and 2, and represent the network Γ by its Laplacian matrix Λ_Γ . This is the symmetric

$d \times d$ -matrix of rank $d - 1$ whose columns sum to zero and whose off-diagonal entries are the labels of the edges. By the Matrix Tree Theorem, all $(d - 1) \times (d - 1)$ -subdeterminants of Λ_Γ are equal up to sign. Explicitly, that common comaximal minor $f_\Gamma(s)$ is the sum over all spanning trees T of the products of the edge labels in T . Thus $f_\Gamma(s)$ is a Laurent polynomial in s whose coefficients are sums of monomials in the $|E|$ positive real parameters. Let $\Gamma[1 = 2]$ denote the graph obtained from Γ by identifying the two distinguished nodes. Thus $\Gamma[1 = 2]$ is the graph on $d - 1$ nodes whose Laplacian $\Lambda_{\Gamma[1=2]}$ is obtained from Λ_Γ by adding the first and second rows. We write $f_{\Gamma[1=2]}(s)$ for the Laurent polynomial that is obtained by taking any comaximal subdeterminant of Λ_Γ . The *impedance* of the network Γ is the rational function

$$z_\Gamma(s) = \frac{a_\Gamma(s)}{b_\Gamma(s)} = \frac{f_{\Gamma[1=2]}(s)}{f_\Gamma(s)}.$$

Here $a_\Gamma(s)$ and $b_\Gamma(s)$ are relatively prime polynomials in s . The pair of nonnegative integers

$$(\alpha_\Gamma, \beta_\Gamma) = (\text{degree}_s(a_\Gamma), \text{degree}_s(b_\Gamma))$$

is called the *degree* of the network Γ .

The set of all rational functions with that given numerator degree and denominator degree is the projective space $\mathbb{P}^{\alpha_\Gamma + \beta_\Gamma + 1}$. The space of parameters for the network Γ is $\mathbb{R}_{>0}^E$. What we are interested in is the polynomial map that takes the parameters to the impedance:

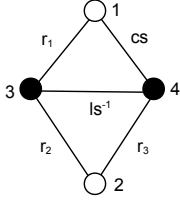
$$Z_\Gamma : \mathbb{R}^E \rightarrow \mathbb{P}^{\alpha_\Gamma + \beta_\Gamma + 1}, \text{ edge weights } \mapsto \frac{a_\Gamma(s)}{b_\Gamma(s)}$$

The network Γ is *minimal* if the map Z_Γ is finite-to-one. A necessary condition for minimality is that $|E| = \alpha_\Gamma + \beta_\Gamma + 1$. Here is the problem we wish to solve for any fixed degree (α, β) :

- Problem 1.** (1) *Compile the finite list of all networks Γ that satisfy the necessary condition.*
- (2) *Identify all networks that are minimal and compute their impedances $z_\Gamma(s)$.*
- (3) *For each minimal network Γ , characterize the set $Z_\Gamma(\mathbb{R}_{>0}^E)$, i.e. the image of the orthant of positive parameter values, as a semi-algebraic subset of $\mathbb{P}^{\alpha_\Gamma + \beta_\Gamma + 1}$.*
- (4) *By intersecting the various $Z_\Gamma(\mathbb{R}_{>0}^E)$, determine the stratification of the space $\mathbb{P}_{>0}^{\alpha_\Gamma + \beta_\Gamma + 1}$ of positive rational functions according to sets of networks that realize them.*
- (5) *For each minimal Γ , find the algebraic degree of the map ψ_Γ , and compute the inverse as an algebraic function.*
- (6) *Open question: Is the inverse always rational?*
- (7) *Conjecture: That degree is 1 if and only if Γ is series-parallel.*

This latex file is looking for a graduate student who will solve the problems for $(\alpha, \beta) = (3, 3)$.

Example 2. Consider the following RCL network Γ with real parameters r_1, r_2, r_3, l, c . The white nodes represent the distinguished nodes 1 and 2. One can verify that the impedance for this network is as given below.



$$z_{\Gamma}(s) = \frac{a_2 s^2 + a_1 s + a_0}{b_2 s^2 + b_1 s + b_0}$$

$$\begin{aligned} a_2 &= c(r_1 + r_2) \\ a_1 &= cl + r_3(r_1 + r_2) \\ a_0 &= l(r_1 + r_2 + r_3) \\ b_2 &= c(r_1 r_2 + r_2 r_3 + r_1 r_3) \\ b_1 &= cl(r_2 + r_3) + r_1 r_2 r_3 \\ b_0 &= lr_1(r_2 + r_3) \end{aligned}$$

This gives us the map

$$Z_{\Gamma} : \mathbb{R}^5 \rightarrow \mathbb{P}^5, \quad (r_1, r_2, r_3, l, c) \mapsto (a_2 : a_1 : a_0 : b_2 : b_1 : b_0).$$

The resultant is easily computed

$$\text{Res}(a, b) = lc(lcr_2 r_3 + r_1 r_2 r_3^2 + r_2^2 lc + r_1^2 r_3^2)^2.$$

This resultant is not a product of monomials in r, l, c , so according to the Theorem below, this impedance is not realizable as a series-parallel network.

There are three trivial regions we want to avoid, namely the trivial solutions to the equations above:

ideal (c, r3, r2, r1),
ideal (c, l, r3),
ideal (l, r2, r1)

Finally, if we choose random values ($a_2 = 1, a_1 = 2, a_0 = 3, b_2 = 5, b_1 = 7, b_0 = 11$, for instance), compute the ideal of 2-minors of the matrix given by the 6 above equations, the nontrivial component corresponds to the following zero-dimensional degree 4 ideal:

ideal(1106*r2-3818*r3-1710*1+5942*C+167, 316*r1-982*r3-1197*1+2690*C-49, 212*1*C-552*C^2-1881*1+4896*C+1, 36252*1^2-245768*C^2-1536777*1+4000032*C+817, -15484*C^2+101177*r3-562590*1+1301509*C+275, 404708*r3*C-157892*C^2+176301*1-1071906*C-8202354*r3*1-205576*C^2-815975*1+1325944*C-557, 404708*r3^2-155428*C^2-3238227*1+7202652*

Without making random choices for the (positive) values of a_i and b_i , eliminating a_i and b_i from the ideal (computed in Macaulay2) yields the zero ideal. So the map generically has degree 4, and therefore the network is minimal.

We would like to find out what is the inverse map, and of course we only care about possible inverses restricted to a semi-algebraic set defined by inequations (all the parameters are positive) and equations (the resultant is non-negative). Professor Kalman claimed that the inverse map can be computed by using some of the expressions that occur in the inverse of the Sylvester matrix that computes the resultant of a and b . We haven't yet worked out any of this.

Remark 3. In applications [2–4], edges labeled by some cs, ls^{-1} and r correspond to electrical components with capacitance $C = c$, inductance $L = l^{-1}$ and resistance $R = r^{-1}$ respectively.

Following are two Theorems* from Professor Kalman's talk, which we should regard as true for $n = 2$ (proof by exhaustion), and as conjectures with overwhelming evidence for $n \geq 3$.

Theorem 4. *The following are equivalent for a RLC network Σ with associated graph Γ_{Σ} :*

- (1) Γ_{Σ} is simple series parallel
- (2) $\text{Res}(Z(\Sigma))$ is a product of monomials in RCL

- (3) *Each coefficient of the numerator and denominator of Z is expressible as a ratio of “invariants” (entries of the adjoint of the Sylvester matrix).*

Note *simple series parallel* is the condition that the maximum length of an elementary cycle Γ_Σ is 2.

Theorem 5. *Suppose Γ_Σ is a simple series parallel network, with impedance Z , and let $\mathbb{B}$ be the invariant... (not defined in the talk) Then*

$$\begin{aligned} \text{Res}(a, b) > 0, \mathbb{B} > 0 &\text{ iff } Z \text{ is RC-realizable,} \\ \text{Res}(a, b) > 0, \mathbb{B} < 0 &\text{ iff } Z \text{ is RL-realizable.} \end{aligned}$$

The general realizability question has been solved also.

Theorem 6. *The impedance function Z is RLC-realizable if and only if the roots of a and b are real and interlace.*

Note, Lemma 5 [1] implies that the hypotheses on a and b are satisfied if and only if the Bezoutian is positive definite.

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