Units

1 Example 1 with units

How should one deal with units? Consider the example

\[ A := \mathbb{F}_2[y, x]/\langle y^3x^3 - y^2 - yx - x^2 \rangle. \]

The integral closure is

\[ C(A, Q(A)) = \mathbb{F}_2[u, u^{-1}]/(uu^{-1} - 1) \]

with \( x = u^3 + u^2 + u \) and \( y = u^{-1} + u^{-2} + u^{-3} \).

Let’s see how this fares using the various implementations of integral closure algorithms.
If I computed correctly, $T(2) = u^9 + u^8$ and $T(1) = u^6 + u^4$. But to get $u$ I would need to compute $T(2) + x^3 + x^2 + x$. Why would I think to do this, especially not knowing that the unit $u$ exists?
ring r=2,(y,x),dp;
> ideal i=y3x3-y2-yx-x2;
> int time=timer;
> list norp=normalP(i,"withRing");norp;
// characteristic : 2
// number of vars : 3
// block 1 : ordering dp
// : names T(2)
// block 2 : ordering dp
// : names y x
// block 3 : ordering C
[2]:
  [1]:
    _[1]=yx3+x4+yx+x2
    _[2]=x5+x4+x3+x2
    _[3]=y2+x2
> def Rp=norp[1][1];
> setring Rp;
> option(redSB);
> ideal sp=std(norid);sp;
sp[1]=y^3*x^3+y^2+y*x+x^2
sp[2]=T(2)*x^2+T(2)*x+y^2*x^3+x^5+y+x
sp[3]=T(2)*y*x+T(2)*y+y*x^4+x
sp[4]=T(2)*y^2+T(2)*x+y^2*x^3+x^4+x^3+x^2+y+x
sp[5]=T(2)^2*x^2+T(2)^2+T(2)*y*T(2)+x^7+x^6+x^5+y*x^3+x^4+x^3+x+1
sp[6]=T(2)^2*y^2+T(2)^2+T(2)*y*T(2)+x^6+y*x^6+y*x^4+y*x^3+x^4+x^3+x+1
sp[7]=T(2)^3+y^2*x^7+x^9+y^2*x^6+x^8+x+1

This is the same with $T(1) = T(2)^2 + x^6 + x^4$ eliminated.
Macaulay2, version 1.4
with packages: ConwayPolynomials, Elimination, IntegralClosure, LLLBases,
PrimaryDecomposition, ReesAlgebra, TangentCone

i1 : A=ZZ/2[y,x]/(y^3*x^3-y^2-y*x-x^2);
i2 : time icf=icFractions A;
    -- used 0.158133 seconds
i3 : toString icf
o3 = {(y^2*x^3+y*x^4+x^4+x^3+x^2+y)
    /(y^2+y*x+y+x),
    y,
    x}
i4 : time icp=icFracP A;
    -- used 0.0635558 seconds
i5 : toString icp
o5 = {y,
    (y*x^2+y)
    /(y+x),
    (y*x^2+y*x+x+1)
    /(y^2+1)}
i6 : G=transpose gens gb presentation integralClosure A
o6 = {-6} | y3x3+y2+yx+x2 |
    {-5} | w_(1,1)x2+w_(-1,1)x+y2x3+x5+y+x |
    {-5} | w_(-1,1)yx+w_(-1,1)y+yx4+x |
    {-5} | w_(-1,1)y2+w_(-1,1)x+y2x3+x4+x3+x2+y+x |
    {-7} | w_(-1,1)^2x+w_(-1,1)^2+2+w_(-1,1)y+w_(-1,1)+x7+x6+x5+yx3+x4+1 |
    {-7} | w_(-1,1)^2y+w_(-1,1)^2+2+w_(-1,1)y+w_(-1,1)+yx6+yx4+yx3+x4+x3+x+1 |
    {-9} | w_(-1,1)^3+y2x7+x9+y2x6+x8+x+1 |

Again even knowing that \((y^2x^3 + yx^4 + x^3 + x^2 + y)/(y^2 + yx + y + x)\)
should be \(u^9 + u^8\), what is \(u\)? And since \(y(x+1)^2/(y+x)\) should be \(1 + u^2\) and
\((yx^2 + yx + x + 1)/(y^2 + 1)\) should be \(u^{-2} + u^{-1}\), what is \(u\)?
4 Magma

\begin{verbatim}
F:=GF(2);
P<y,x>:=PolynomialRing(F,2);
I:=ideal<P|y^3*x^3-y^2-y*x-x^2>;
N:=Normalisation(I);
y@N[1][2];
x@N[1][2];
G:=GroebnerBasis(N[1][1]);G;
$.1^3 + $.1^2*$.2 + $.1*$.2
$.1^2*$.2 + $.1*$.2 + $.1
[ $.1*$.2 + $.2^2 + $.2 + 1
]
\end{verbatim}

Here $S.2 = u$ and $S.1 = u + 1 + u^{-1}$, so this implementation does really well.
FF<x>::FunctionField(F);
PP<y>::PolynomialRing(FF);
f:=y^3*x^3-y^2-y*x-x^2;
RER<y>::RationalExtensionRepresentation(FunctionField(f));
C<X>::CoefficientRing(RER);
INT:=Integers(C);
IC:=IntegralClosure(INT,RER);
B:=Basis(IC);B;
[ 1, X^3/(X + 1)*Y + 1/(X + 1), X^2/(X^2 + 1)*Y^2 + (X^2 + 1)/X*Y + 1/(X^2 + 1) ]

But this doesn’t! $B[2]$ seems to be $u^5 + u^4$ while $B[3]$ seems to be $u^2 + 1$. 
5 Integral extensions

Had I used $z := yx^3$ to get a related integral extension problem:

$$F_2[z, x]/(z^3 + z^2 + zx^4 + x^8)$$

I could have used $s^4 := z$ and $u := x/s$ to get the required integral closure and parameterization. But, in general, non-linear transformations to produce integral extensions don’t necessarily give the whole integral closure when mapped back to the original problem.
> ring r=0,(y,x),dp;
> ideal i=y3x3-y2-yx-x2;
> list nor=normal(i);nor;
> list nor=normal(i);nor;
// characteristic : 0
// number of vars : 3
// block 1 : ordering dp
// : names T(1)
// block 2 : ordering dp
// : names y x
// block 3 : ordering C
[2]:
  [1]:
    _[1]=y
    _[2]=x
> def R=nor[1][1];
> setring R;
> option(redSB);
> ideal s=std(norid);s;
  s[1]=y^3*x^3-y^2-y*x-x^2
  s[2]=T(1)*x-y
  s[3]=T(1)*y-y^3*x^2+y+x
  s[4]=T(1)^2*T(1)-y^3*x+1

> ring r=23,(y,x),dp;
> ideal i=y3x3-y2-yx-x2;
> list norp=normalP(i,"withRing");norp;
// characteristic : 23
// number of vars : 2
// block 1 : ordering dp
// : names T(1)
// block 2 : ordering dp
// : names x
// block 3 : ordering C
[2]:
  [1]:
    _[1]=y
    _[2]=x
> def Rp=norp[1][1];
> setring Rp;
> option(redSB);
> ideal sp=std(norid);sp;
  sp[1]=T(1)^3*x^4-T(1)^2-T(1)-1
Does it really make sense to have $T(1) := y/x$ integral in terms of $y$ and $x$ in the first? Well, the second shows what one gets by replacing $y$ by $T(1)x$. The result is not really integral any more. Yet it is theoretically correct. Hmm....

Well, if we treat $u^{-1} := T(1)$ as a unit, and work over

$$\mathbf{F}\{u\} := \mathbf{F}[u, u^{-1}]/(uu^{-1} - 1)$$

then

$$C(A, Q(A)) = \mathbf{F}\{u\}[x]/(x^4 - u^3 - u^2 - u)$$

for characteristic $q \neq 2$. 
7 Example 2 with units

Consider something as simple as the Klein quartic:

\[ A := \mathbb{F}_2[y, x]/(y^3 + x^3 y + x) \]

in its (standard) two point form in that the divisors are

\[ ((y)) = -3 \cdot P + 2 \cdot Q + 1 \cdot R \]

and

\[ ((x)) = -2 \cdot P - 1 \cdot Q + 3 \cdot R \]

with two points at which there are poles.

This is integrally closed. But it contains the unit \( u := y^3/x = 1 + yx^2 \), with \( u^{-1} = 1 + yx^2 + y^2x^4 + x^7 \). So perhaps this should be viewed as

\[ A = C(A, Q(A)) = \mathbb{F}_2[u][x]/(x^7 + u^2 + u + 1 + u^{-1}). \]

If so, then this raises the question of how one detects the existence of units in a problem, since it was only clear to me in this problem from the divisors,

\[ ((u)) = -7 \cdot P + 7 \cdot Q \]

and hence

\[ ((u^{-1})) = 7 \cdot P - 7 \cdot Q \]

information not usually used in computing an integral closure.