4.7 Time reversibility

Time reversed Markov chain (discrete time): Let $X_n$ be a discrete time irreducible Markov chain under stationary distribution $\pi_j$. Fix a large time $N > 0$. The time reversed process $X_n^*$ obtained by time reversing $X_n$ at time $T$ is defined by $X_n^* = X_{N-n}$ for $n = 0, 1, \ldots, N$. Then for $n < N$,

$$P(X_{n+1}^* = j | X_0^* = i_0, X_1^* = i_1, \ldots, X_n^* = i_{n-1}, X_n^* = i)$$

$$= \frac{P(X_N = i_0, X_{N-1} = i_1, \ldots, X_{n+1} = i_{n-1}, X_n = i, X_{n-1} = j)}{P(X_N = i_0, X_{N-1} = i_1, \ldots, X_{n+1} = i_{n-1}, X_n = i)}$$

$$= \frac{\pi_j P_{ji}}{\pi_i}.$$

This shows that the time reversed process $X_n^*$ is also a Markov chain with transition probabilities

$$P_{ij}^* = \frac{\pi_j}{\pi_i} P_{ji}.$$

Time reversible Markov chain (discrete time): The process $X_n$ is called time reversible if $P^* = P$, that is, the time reversed process has the same transition probabilities as the original process. The time reversibility condition $P^* = P$ can also be written in the following more convenient form: for all states $i$ and $j$,

$$\pi_i P_{ij} = \pi_j P_{ji}.$$

Note that if $\{\pi_j\}$ is a distribution (that is, a collection of nonnegative numbers, one for each state and summing to 1) that satisfies the above equation, then $\pi_j$ is the stationary distribution of $X_n$ and $X_n$ is time reversible. To show this, one just need to sum over $i$ in the above equation to obtain $\sum_i \pi_i P_{ij} = \pi_j$.

Kolmogorov criterion (discrete time): A discrete time Markov chain is time reversible if and only if the product of transition probabilities along any loop is the same as that for the reversed loop, that is, for any states $i, i_1, i_2, \ldots, i_k$,

$$P_{i_1 i_2} \cdots P_{i_k i} = P_{i_k i_1} P_{i_1 i_2} \cdots P_{i_k i}.$$

Note that this criterion does not require the knowledge of the stationary distribution.

Proof: Assume time reversible with stationary distribution $\pi_j$. It is trivial for a loop of two states. For a loop of three states $i, j, k$, $\pi_i P_{ij} P_{jk} P_{ki} = \pi_j P_{ji} P_{jk} P_{ki}$, hence $P_{ji} P_{kj} \pi_i P_{ik} = \pi_j P_{ik} P_{kj}$. This proves the desired equality because $\pi_i > 0$. Similarly for a loop of any number of states. Next, assume the equality holds for any loop. For any
two states $i$ and $j$, $P_{i i} P_{i i^2} \cdots P_{i j} P_{j i} = P_{i j} P_{j i^k} \cdots P_{i i}$. Summing over $i_1, i_2, \ldots, i_k$ yields $P_{i j}^{(k)} P_{j i} = P_{i j} P_{j i}^{(k)}$. Letting $k \to \infty$, we obtain $\pi_j P_{j i} = P_{ij} \pi_i$.

Example 1: A random walk on $0, 1, 2, \ldots, N$ with reflecting boundary is time reversible, because each segment in a loop will be traversed twice in opposite directions and hence the transition probabilities are identical to those of the reversed loop.

Time reversed Markov chain (continuous time): Let $X(t)$ be a continuous time irreducible Markov chain under stationary distribution $\pi_i$. Let $q_i$ and $P_{ij}$ be respectively the transition rates from state $i$ and the transition probabilities. The time reversed process from time $T$ is defined by $X^*(t) = X(T - t)$ for $0 \leq t \leq T$. We now show that $X^*(t)$ is also a continuous time Markov chain with same transition rates $q_i$ and its transition probabilities are $P_{ij}^* = (\pi'_j / \pi'_i) P_{ji}$, where $\{\pi'_j\}$ is the stationary distribution of the embedded discrete time Markov chain.

Because the time reversed embedded discrete time Markov chain is known to be a Markov chain with transition probabilities $P_{ij}^*$, it suffices to show that the time $\tau_i^*$ spent in state $i$ by $X^*(t)$ is exponential of rate $q_i$. This follows from

$$P[\tau_i^* > t \mid X^*(0) = i] = P[X^*(s) = i \text{ for } 0 \leq s \leq t \mid X^*(0) = i]$$

$$= P[X(s) = i \text{ for } T - t \leq s \leq T \mid X(T) = i] = \frac{P[X(s) = i \text{ for } T - t \leq s \leq T]}{P[X(T) = i]}$$

$$= \frac{P[X(T - t) = i] P[X(s) = i \text{ for } T - t \leq s \leq T \mid X(T - t) = i]}{P[X(T) = i]} = \frac{P[X(T - t) = i] e^{-q_i t}}{P[X(T) = i]}$$

$$= e^{-q_i t}.$$ 

Time reversible Markov chain (continuous time): The process $X(t)$ is called time reversible if it has the same distribution as the reversed process. Since they have the same rates $q_i$, so $X(t)$ is time reversible if and only if the embedded discrete time Markov chain is time reversible, that is, if and only if $P^*_{ij} = P_{ij}$. Recall $\pi_j = (\pi'_j / q_j) / (\sum_i \pi'_i / q_i)$, we see that $X(t)$ is time reversible if and only if

$$\pi_i q_{ij} = \pi_j q_{ji},$$

where $q_{ij} = q_i P_{ij}$ is the transition rate from $i$ to $j$ as before.

In general, given an irreducible $X(t)$, it is time reversible if and only if there is a distribution $\{\pi_j\}$ satisfying the above equation. In this case, $\{\pi_j\}$ is also the stationary distribution.

Kolmogorov reversibility criterion (continuous time): A continuous time Markov chain is time reversible if and only if the product of transition rates $q_{ij}$ along any loop is as that of the reversed loop, that is,

$$q_{i_1} q_{i_1 i_2} \cdots q_{i_k} = q_{i_1} q_{i_k i_{k-1}} \cdots q_{i_1 i}. $$
The proof is almost identical to the discrete time case with $P_{ij}$ replaced by $q_{ij}$.

Example 2: A BD (birth and death) process in steady state is time reversible. This follows directly from Kolmogorov criterion in the same way as the time reversibility of a reflecting random walk.

Example 3 (M/M/s queue): Recall the queue length $X(t)$ for m/m/s queue at time $t$ (which includes the number of arrivals currently in service) is a BD process, hence is time reversible if the system is stable. It follows that the departure process $Y(t)$, defined as the number of departures by time $t$, has the same distribution as the arrival process, that is, a Poisson process of rate $\lambda$ (arrival rate).

Example 4 (m/m/2 with heterogeneous servers): An m/m/2 system has arrival rate $\lambda$ and two different service rates $\mu_1$ and $\mu_2$ for the two servers. The system service rate is $\mu = \mu_1 + \mu_2$. Assume $\lambda < \mu$, so the system is stable. Assume also that an arrival at an empty system will be served by server $i$ with probability $p_i$ for $i = 1, 2$ ($p_1 + p_2 = 1$). At time $t$, let $X(t) = 0$ if the system is empty, let $X(t) = 1A$ (resp. $1B$) if there is 1 arrival in system who is served by server 1 (resp. 2), and let $X(t) = n$ if there are $n \geq 2$ arrivals in system. The process is a continuous time Markov chain with transition rates given by

$$q_{01A} = \lambda p_1, \quad q_{01B} = \lambda p_2, \quad q_{1A0} = \mu_1, \quad q_{1B0} = \mu_2, \quad q_{1A2} = q_{1B2} = \lambda, \quad q_{21A} = \mu_2, \quad q_{21B} = \mu_1,$$

$q_{nn+1} = \lambda$ and $q_{n+1n} = \mu$ for $n \geq 2$, and all other $q_{ij} = 0$. By drawing a transition graph, it is easy to see that to use Kolmogorov’s criterion to check time reversibility, it suffices to look at the loop $0 \rightarrow 1A \rightarrow 2 \rightarrow 1B \rightarrow 0$. The corresponding equation is $(\lambda p_1)\lambda \mu_1 \mu_2 = (\lambda p_2)\lambda \mu_2 \mu_1$. It follows the this system is time reversible if and only if $p_1 = p_2$.

Example 5 (Two m/m/s queues in tandem): Consider two m/m/s queues in tandem in steady state, where there are $s_1$ servers in queue 1 of same service rate $\mu_1$ and $s_2$ servers in queue 2 of same service rate $\mu_2$. Let $X_1(t)$ and $X_2(t)$ be respectively the numbers of customers in queue 1 and queue 2 at time $t$, and let $W_1$ and $W_2$ be respectively the times spent in queue 1 and queue 2 of some customer. Then at any time $t$, $X_1(t)$ and $X_2(t)$ are independent, and $W_1$ and $W_2$ are also independent.

To see this, note that $X_2(t)$ is determined by the departures from queue 1 before time $t$, but in reversed time, those departures become arrivals after time $t$ and so are independent of $X_1(t)$. Similarly, $W_2$ is determined by the departures from queue 1 during the time spent in queue 1 which in reversed time are the arrivals and so are independent of the time spent in queue 1.