Invariant measures for stochastic 2D damped Euler equations

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Outline

The Euler equations describe the motion of inviscid fluids

$$\begin{cases} \partial_t u + (u \cdot \nabla) u + \nabla p = f \\ \nabla \cdot u = 0 \end{cases}$$

Without forcing term, it is known that there are many constants of motion
(energy $\frac{1}{2} \int |u(t,x)|^2 \, dx$
when $d = 2$: enstrophy $\frac{1}{2} \int |\text{curl } u(t,x)|^2 \, dx$
g-functionals of the vorticity $\frac{1}{2} \int g(\text{curl } u(t,x)) \, dx$
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Let us consider the bidimensional Euler equations, so that they can be written in terms of the vorticity $\xi = \text{curl } u = \nabla^\perp \cdot u$ as

$$\begin{cases} \partial_t \xi + u \cdot \nabla \xi = 0 \\ \nabla \cdot u = 0 \end{cases}$$

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In particular we consider the stochastic damped 2D Euler equations

$$\begin{cases} \partial_t \xi + [\gamma \xi + u \cdot \nabla \xi] \, dt = dW \\ \nabla \cdot u = 0 \end{cases} \tag{1}$$

$\gamma > 0$ is the sticky viscosity (see Gallavotti)
Known results (for any $\gamma \geq 0$):

- global existence for $\xi(0) = \chi \in L^p$ when $p < \infty$
- uniqueness for $\xi(0) = \chi \in L^\infty$ (as in the deterministic case: see Wolibner, Yudovich)

This is true when the spatial domain is the torus or a smooth bounded domain.
We assume that the noise is sufficiently regular in space (W with paths in $C(\mathbb{R}; H^s)$ with $s = 2$ or $s = 3$.)

Assumptions on the Noise

On $(\Omega, \mathcal{F}, \mathbb{P})$, we define $\{\tilde{\beta}_j(t); t \geq 0\}_{j\in\mathbb{N}}$ of independent standard 1-dimensional Wiener processes defined on it. Then we consider a new sequence of i.i.d. Wiener processes defined for any time $t \in \mathbb{R}$:

$$\beta_i(t) = \begin{cases} \tilde{\beta}_{2i-1}(t) & \text{for } t \geq 0 \\ \tilde{\beta}_{2i}(-t) & \text{for } t \leq 0 \end{cases}$$

$$W(t,x) = \sum_{i\in\mathbb{N}} c_i \beta_i(t) e_i(x) \tag{2}$$

for some $c_i \in \mathbb{R}$, where $\{e_i\}_i$ is a complete orthonormal system of L^2.

$$\sum_i c_i^2 \|e_i\|_{H^s}^2 < \infty. \tag{3}$$

Stationary solutions

Existence of stationary solutions has been proved when $\gamma > 0$ (see Bessaih 2008)
$\rightsquigarrow$ stationary process ξ solving the Euler equation (1); the paths are in

$$C_w([0,\infty); L^p) \cap L^\infty_{loc}(0,\infty; L^p) \cap C([0,\infty); L^2)$$

for any $p < \infty$.
This is obtained as vanishing viscosity limit (as the kinetic viscosity $\nu \to 0$) of the stochastic damped Navier-Stokes equation

$$\begin{cases} \partial_t \xi + [-\nu \Delta \xi + \gamma \xi + u \cdot \nabla \xi] \, dt = dW \\ \nabla \cdot u = 0 \end{cases}$$

The limit process is a stationary process in L^p.

One can work directly on the invariant measures of the stochastic damped Navier-Stokes equation (for which it is "easy" to prove existence and uniqueness) and prove tightness.
Indeed by Itô formula for

$$\begin{cases} \partial_t \xi + [-\nu \Delta \xi + \gamma \xi + u \cdot \nabla \xi] \ dt = dW \\ \nabla \cdot u = 0 \end{cases}$$

working with the stationary solution (with invariant measure μ_ν), we get

$$\gamma \int \|\xi\|_{L^p}^p d\mu_\nu(\xi) \leq (p-1)\, TrQ \int \|\xi\|_{L^{p-2}}^{p-2} d\mu_\nu(\xi)$$

which for $p = 2$ is

$$\int \|\xi\|_{L^2}^2 d\mu_\nu(\xi) \leq \frac{TrQ}{\gamma}$$

The stationary solutions (and the associated measure μ as limit of μ_ν) are important to prove properties in the vanishing viscosity limit, interesting in turbulence theory (see Kupiainen 2011); e.g. there is no anomalous dissipation of enstrophy

$$\lim_{\nu\to 0} \nu \int \|\nabla \xi\|^2_{L^2} d\mu_\nu(\xi) = 0$$

(B-Ferrario: Nonlinearity 2014; the deterministic problem was solved by Constantin and Ramos 2007)

The same for the energy.

$\rightsquigarrow$ stationary solutions
they leave in L^p for $p < \infty$.
what about results in L^∞?

in L^p ($p < \infty$) there is global existence of solutions
in L^∞ there is also uniqueness

L^∞ is the "limit" of L^p as $p \to \infty$ but these spaces are different in some sense.

L^∞ is not separable when we consider the strong topology

Anyway we would like to prove existence of invariant measures for the stochastic damped Euler equation following the Bogoliubov-Krylov's technique.
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Bogoliubov-Krylov's technique

The "classical" version (see Da Prato-Zabczyk) is the following.

Let X be a separable Banach space.
Define the Markov semigroup $P_t : B_b(X) \to B_b(X)$ as

$$P_t\phi(\chi) = \mathbb{E}[\phi(\xi^\chi(t))]$$

If

- (Feller property) $\quad P_t : C_b(X) \to C_b(X)$
- the sequence of measures $\mu_n = \frac{1}{n}\int_0^n P_s^*\delta_0 ds$ is tight in X

then there exists a measure μ on the Borelian subsets of X which is invariant, that is

$$\int P_t\phi \, d\mu = \int \phi \, d\mu \qquad \forall t \geq 0, \phi \in C_b(X).$$

The space L^∞

Let us consider three topologies on L^∞

$\mathcal{T}_n$ the strong (or norm) topology
$\mathcal{T}_{bw\star}$ the bounded weak$\star$ topology
$\mathcal{T}_{w\star}$ the weak$\star$ topology

(the weak$\star$ topology is the weakest topology for which the mappings $\xi \mapsto \langle \xi, g \rangle$ are continuous for any $g \in L^1$)
(the bounded weak$\star$ topology is the finest topology on L^∞ that coincides with the weak$\star$ topology on every norm bounded subset of L^∞)

We have

$$\mathcal{T}_{w\star} \subsetneq \mathcal{T}_{bw\star} \subsetneq \mathcal{T}_n \tag{4}$$

$$C(L^\infty, \mathcal{T}_{w\star}) \subsetneq C(L^\infty, \mathcal{T}_{bw\star}) = SC(L^\infty, \mathcal{T}_{w\star}) \subsetneq C(L^\infty, \mathcal{T}_n).$$

sequentially: $\chi_n \rightharpoonup \chi \Longrightarrow \phi(\chi_n) \to \phi(\chi)$

However for the Borelian subsets of L^∞

$$\mathcal{B}(\mathcal{T}_{w\star}) = \mathcal{B}(\mathcal{T}_{bw\star}) \subsetneq \mathcal{B}(\mathcal{T}_n)$$

(recall that in a separable Banach space X, the Borelian subsets with respect to the strong and the weak topology coincide)

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An important result is that the space C^∞ is dense in L^∞ with respect to the weak$\star$ topology $\mathcal{T}_{w\star}$ but not with respect to the strong topology $\mathcal{T}_n$.

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The spaces $(L^\infty, \mathcal{T}_n)$ and $(L^\infty, \mathcal{T}_{w\star})$ are not Polish spaces.
Which one do we choose?

Let us look at the "ingredients" of Bogoliubov-Krylov's technique.
We start with the tightness.

For the equation

$$d\xi + (\gamma\xi + u \cdot \nabla\xi)\, dt = dW$$

we can prove uniform L^∞-bounds in probability (for any

Proposition

Let $\gamma > 0$, then for any $\epsilon > 0$ there exists $R_\epsilon > 0$ such that

$$\inf_{t \geq 0} \mathbb{P}\{\|\xi^0(t)\|_\infty \leq R_\epsilon\} \geq 1 - \epsilon$$

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To prove this result, first, let us note that for any $t_0 < 0$ the random variables $\xi(0; \xi(t_0) = 0)$ and $\xi(-t_0; \xi(0) = 0)$ have the same law (homogeneity). Hence,

$$\mathbf{P}\{|\xi(t; \xi(0) = 0)|_\infty \leq R_\epsilon\} = \mathbf{P}\{|\xi(0; \xi(-t) = 0)|_\infty \leq R_\epsilon\}$$

Proposition

Let $\gamma > 0$. Then, there exists a real random variable r ($\mathbf{P}$-a.s. finite) such that

$$\sup_{t_0 \leq 0} |\xi(0; \xi(t_0) = 0)|_\infty \leq r \qquad \mathbf{P} - a.s. \tag{5}$$

Proof: Sketch
We introduce the linear equation

$$dz_\lambda(t) + \lambda z_\lambda(t)dt = dW(t) \tag{6}$$

for $\lambda > 0$; its stationary solution is

$$z_\lambda(t) = \int_{-\infty}^{t} e^{-\lambda(t-s)} dW(s). \tag{7}$$

$$\mathbf{E}\left[\|z_\lambda(t)\|^2_{H^a}\right] = \frac{1}{2\lambda}\mathbf{E}\left[\|W(1)\|^2_{H^a}\right].$$

Set $\eta_\lambda = \xi - z_\lambda$. Then

$$\frac{\partial \eta_\lambda}{\partial t} + \gamma\eta_\lambda + [K\star(\eta_\lambda + z_\lambda)]\cdot\nabla\eta_\lambda = -[K\star(\eta_\lambda + z_\lambda)]\cdot\nabla z_\lambda + (\lambda-\gamma)z_\lambda.$$

We multiply equation by $|\eta_\lambda|^{p-2}\eta_\lambda, \quad p \geq 2$, and integrate over the spatial domain D; using that $\langle u\cdot\nabla\eta_\lambda, |\eta_\lambda|^{p-2}\eta_\lambda\rangle = 0$.

We get for $p \geq 1$

$$\frac{d}{dt}|\eta_\lambda(t)|_p + \left(\gamma - C|\nabla\zeta_\lambda(t)|_\infty\right)|\eta_\lambda(t)|_p \leq \left(C|\nabla z_\lambda(t)|_\infty + |\lambda-\gamma|\right)|z_\lambda(t)|_p.$$

Now Grönwall's inequality yields on the interval $[t_0, 0]$

$$|\eta_\lambda(0)|_p \leq |\eta_\lambda(t_0)|_p e^{-\int_{t_0}^0 (\gamma - C|\nabla z_\lambda(s)|_\infty) ds}$$
$$+ \int_{t_0}^0 \big(C|\nabla \zeta_\lambda(s)|_\infty + |\lambda - \gamma| \big) |z_\lambda(s)|_p \; e^{-\int_s^0 (\gamma - C|\nabla z_\lambda(r)|_\infty) dr} ds \tag{8}$$

Using that $H^{a-1} \subset L^\infty$ for any $a > 2$ and taking $p \to \infty$, we get that

$$|\eta_\lambda(0)|_\infty \leq |\eta_\lambda(t_0)|_\infty e^{-\int_{t_0}^0 (\gamma - \tilde{C}\|z_\lambda(s)\|_{H^a}) ds}$$
$$+ \int_{t_0}^0 C\big(\|z_\lambda(s)\|_{H^a} + |\lambda - \gamma| \big) \|z_\lambda(s)\|_{H^a} e^{-\int_s^0 (\gamma - \tilde{C}\|\zeta_\lambda(r)\|_{H^a}) dr} ds$$

for some positive constants C and $\tilde{C}$.

Since $\xi(t_0) = 0$, we have

$$|\eta_\lambda(0)|_\infty \le C\|\zeta_\lambda(t_0)\|_{H^a} e^{-\int_{t_0}^0(\gamma - \tilde{C}\|\zeta_\lambda(s)\|_{H^a})ds}$$
$$+ \int_{t_0}^0 C\left(\|\zeta_\lambda(s)\|_{H^a} + |\lambda - \gamma|\right)\|\zeta_\lambda(s)\|_{H^a} e^{-\int_s^0(\gamma - \tilde{C}\|\zeta_\lambda(r)\|_{H^a})dr} ds \tag{9}$$

Since z_λ is an ergodic process, we have

$$\lim_{t_0 \to -\infty} \frac{1}{-t_0} \int_{t_0}^0 \|z_\lambda(s)\|_{H^a} ds = \mathbf{E}\|z_\lambda(0)\|_{H^a} \qquad \mathbf{P} - a.s.$$

We choose λ large enough such that

$$\tilde{C}\mathbf{E}\|z_\lambda(0)\|_{H^a} \le \frac{\tilde{C}}{\sqrt{2\lambda}} \sqrt{\mathbf{E}\|W(1)\|_{H^a}^2} < \frac{\gamma}{2} \tag{10}$$

$$\lim_{t_0 \to -\infty} \frac{1}{-t_0} \int_{t_0}^0 \tilde{C} \|z_\lambda(s)\|_{H^a} ds < \frac{\gamma}{2} \qquad \mathbf{P} - a.s.$$

Hence

$$e^{-\int_{t_0}^0 (\gamma - \tilde{C}\|z_\lambda(s)\|_{H^a}) ds}$$

is (pathwise) uniformly bounded for $t_0 < 0$ and vanishes exponentially fast as $t_0 \to -\infty$.

Thus, there exists a random variable r_3 (**P**-a.s. finite) such that

$$\sup_{t_0 \leq 0} |\eta_\lambda(0; \eta(t_0) = -z_\lambda(t_0))|_\infty \leq r_3 \qquad \mathbf{P} - a.s.$$

Since $\xi = \eta_\lambda + z_\lambda$, we obtain (5).

Since the balls in L^∞ are compact for the weak$\star$ topology (and for the bounded weak$\star$ topology), from that bound we get **tightness** of the sequence of measures

$$\mu_n = \frac{1}{n}\int_0^n \mathcal{L}(\xi^0(s))ds$$

with respect to the weak $\star$ topology.

So we avoid to work with the strong topology on L^∞.

See the paper by Maslowski and Seidler (1999) for the idea to **use weak topologies**. But they worked in a separable Hilbert space!

Looking for the transition semigroup

We can prove a weak form of continuous dependence on the initial data.

Proposition

Let $\gamma \geq 0$.
Given a sequence $\{\chi^n\}_n \subset L^\infty$ which converges weakly$\star$ in L^∞ to $\chi \in L^\infty$, we have that, $\mathbb{P}$-a.s., for every $t > 0$ the sequence $\left\{\xi^{\chi^n}(t)\right\}_n$ converges weakly$\star$ in L^∞ to $\xi^\chi(t)$.

Therefore we have a "weak Feller" property for the operator P_t defined as

$$P_t\phi(\chi) = \mathbb{E}[\phi(\xi^\chi(t))].$$

Proposition

The operator P_t is sequentially weakly$\star$ Feller in L^∞, that is

$$P_t : SC_b(L^\infty, \mathcal{T}_{w\star}) \to SC_b(L^\infty, \mathcal{T}_{w\star}) \tag{11}$$

for any $t \geq 0$.

Since $C(L^\infty, \mathcal{T}_{bw\star}) = SC(L^\infty, \mathcal{T}_{w\star})$, this is equivalent to be Feller with respect to the bounded weak$\star$ topology

$$P_t : C_b(L^\infty, \mathcal{T}_{bw\star}) \to C_b(L^\infty, \mathcal{T}_{bw\star})$$

REMARK: Since the weak topologies are not metrizable

sequential continuity $\neq$ continuity

Markov property

We want to prove that
for every $\phi \in SC_b(L^\infty, \mathcal{T}_{w\star})$, $\chi \in L^\infty$ and $t, s > 0$

$$\mathbb{E}\left[\phi\left(\xi^\chi(t+s)\right)|\mathcal{F}_t\right] = \left(P_s\phi\right)\left(\xi^\chi(t)\right) \qquad \mathbb{P} - a.s. \qquad (12)$$

We have an auxiliary result

Lemma (easier since $W^{1,4}$ is separable)

Let $\gamma \geq 0$.
For every $\phi \in SC_b(L^\infty, \mathcal{T}_{w\star})$, $\chi \in W^{1,4}(D)$ and $t, s > 0$ we have

$$\mathbb{E}\left[\phi\left(\xi^\chi(t+s)\right)|\mathcal{F}_t\right] = \left(P_s\phi\right)\left(\xi^\chi(t)\right) \qquad \mathbb{P}\text{-}a.s.. \qquad (13)$$

We need to show that the Euler equations are well posed in the space $W^{1,4}$.

To get $\nabla\xi \in L^4$ we need to analyse the gradient of equation (1):

$$d\nabla\xi + \gamma\nabla\xi + \nabla(u \cdot \nabla\xi)\ dt = d\nabla W.$$

Proposition

Let $\gamma \geq 0$.
If $\xi_0 \in W^{1,4}$, then $\xi \in L^\infty_{loc}(0,\infty; W^{1,4}) \cap C_w([0,\infty); W^{1,4})$ $\mathbb{P}$-a.s..

REMARK: here we loose good uniform estimates. We can get them only in the L^∞-norm.
But there is Markov property in $W^{1,4}$, since this is a separable Banach space (usual techinques work).

1. If $\phi \in SC_b(L^\infty, \mathcal{T}_{w\star})$, then $\phi|_{W^{1,4}} \in C_b(W^{1,4})$.
2. $$\mathbb{E}\left[\phi\left(\xi_{t,t+s}^{\eta}\right) Z\right] = \mathbb{E}\left[\left(P_s\phi\right)(\eta)\, Z\right]$$
   for every bounded $\mathcal{F}_t$-measurable r.v. Z and $\eta \in W^{1,4}$.
3. The same for every random variable $\eta = \sum_{i=1}^{k} \eta^{(i)} 1_{A^{(i)}}$ with $\eta^{(i)} \in W^{1,4}$, $A^{(i)} \in \mathcal{F}_t$
   $\{A^{(1)}, A^{(2)}, \dots, A^{(k)}\}$ a partition of Ω.
4. Pass to the limit as $k \to \infty$, using that the strong convergence of η_k in $W^{1,4}$ implies the weak$\star$ convergence in L^∞, so $\left(P_s\phi\right)(\eta_k)$ converges $\mathbb{P}$-a.s. to $(P_s\phi)(\eta)$ and $\xi_{t,t+s}^{\eta_k}$ converges weakly$\star$ in L^∞ to $\xi_{t,t+s}^{\eta}$, so $\phi\left(\xi_{t,t+s}^{\eta_k}\right)$ also converges to $\phi\left(\xi_{t,t+s}^{\eta}\right)$ $\mathbb{P}$-a.s.
5. Taking $\eta = \xi_t^{\chi}$, by uniqueness ($\xi_{t+s}^{\chi} = \xi_{t,t+s}^{\xi_t^{\chi}}$) we get the formula in the Lemma.

Then we use that $W^{1,4}$ is dense in L^∞ with respect to the weak$\star$ topology $\mathcal{T}_{w\star}$, to get the Markov property:

$$\mathbb{E}\left[\phi\left(\xi^{\chi}(t+s)\right)|\mathcal{F}_t\right] = \left(P_s\phi\right)\left(\xi^{\chi}(t)\right) \qquad \mathbb{P}-a.s.$$

for every $\phi \in SC_b(L^\infty, \mathcal{T}_{w\star})$, $\chi \in (L^\infty, \mathcal{T}_{w\star})$ and $t, s > 0$.

semigroup

Taking the expectation in

$$\mathbb{E}\left[\phi\left(\xi^\chi(t+s)\right)|\mathcal{F}_t\right] = \left(P_s\phi\right)\left(\xi^\chi(t)\right)$$

we get

$$\mathbb{E}\left[\phi\left(\xi^\chi(t+s)\right)\right] = \mathbb{E}\left[\left(P_s\phi\right)\left(\xi^\chi(t)\right)\right]$$

which can be rewritten as

$$(P_{t+s}\phi)(\chi) = \left(P_t(P_s\phi)\right)(\chi).$$

Hence we have $P_{t+s} = P_t P_s$ on $SC_b(L^\infty, \mathcal{T}_{w\star})$.

Summing up: we have

- a Markov semigroup $\{P_t\}_t$ acting on $C_b(L^\infty, \mathcal{T}_{bw\star})$

- $\{P_t\}_t$ is Feller in $(L^\infty, \mathcal{T}_{bw\star})$

- a tight sequence of measures μ_n with respect to the bounded weak$\star$ topology $\mathcal{T}_{bw\star}$

We are ready to use the Bogoliubov-Krylov's technique to get existence of invariant measures.

Existence of invariant measures

We apply Prokhorov's theorem in the version given by Jakubowski (1997) so to work in non metric spaces.
This requires that the space L^∞ with the bounded weak$\star$ topology $\mathcal{T}_{bw\star}$ is countably separated. This is our case, since L^1 is separable.

Hence
$\exists$ a subsequence $\{\mu_{n_k}\}_k$ and a probability measure μ on $\mathcal{B}(\mathcal{T}_{bw\star})$ such that μ_{n_k} converges narrowly to μ as $k \to \infty$ ($n_k \to \infty$), that is

$$\int \phi \, d\mu_{n_k} \to \int \phi \, d\mu$$

for any $\phi \in C_b(L^\infty, \mathcal{T}_{bw\star})$.

Write $\int \phi \, d\mu = \langle \phi, \mu \rangle$. We have

$$\langle \phi, \mu_n \rangle = \frac{1}{n} \int_0^n \langle \phi, \mathcal{L}(\xi^0(s)) \rangle ds$$

So

$$\langle P_t \phi, \mu_{n_k} \rangle = \langle \phi, \mu_{n_k} \rangle + \frac{1}{n_k} \int_{n_k}^{t+n_k} \langle \phi, \mathcal{L}(\xi^0(s)) \rangle ds - \frac{1}{n_k} \int_0^t \langle \phi, \mathcal{L}(\xi^0(s)) \rangle ds$$

Letting $k \to \infty$, the two latter terms vanish.
From the Feller property in the weak form (11), we know that $P_t \phi \in C_b(L^\infty, \mathcal{T}_{bw\star})$ if $\phi \in C_b(L^\infty, \mathcal{T}_{bw\star})$. Hence in the limit we obtain

$$\langle P_t \phi, \mu \rangle = \langle \phi, \mu \rangle$$

for each $\phi \in C_b(L^\infty, \mathcal{T}_{bw\star})$ and each $t \geq 0$.

This proves the following

Theorem (∃ invariant measure)

Let $\gamma > 0$.
Then there exists at least one invariant measure μ for the stochastic damped Euler equation.

This is a measure on the Borel subsets $\mathcal{B}(\mathcal{T}_{bw\star}) = \mathcal{B}(\mathcal{T}_{w\star})$ such that

$$\int P_t\phi \; d\mu = \int \phi \; d\mu$$

for all $t \geq 0$ and $\phi \in C_b(L^\infty, \mathcal{T}_{bw\star})$.

Work in progress

- Uniqueness of the invariant measure (difficult!):

  Usual techniques don't work.

- Inviscid limit problem:

  If $\mu^{\nu,\gamma}$ is the invariant measure for the stochastic damped 2D Navier-Stokes equations. What happens when the viscosity $\nu \to 0$.

- Other limit problems:

  What happens to the measure $\mu^{0,\gamma}$ when the damping coefficient $\gamma \to 0$?

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THANK YOU