The stochastic heat equation with multiplicative Lévy noise: Existence, moments, and intermittency

Carsten Chong
Department of Statistics

Joint work with Quentin Berger (Sorbonne) and Hubert Lacoin (IMPA)

https://arxiv.org/abs/2111.07988

Frontier Probability Days 2021

Columbia University
in the City of New York

Heat equation with Lévy noise

Stochastic heat equation with Lévy noise:

$$\begin{aligned} \partial_t u(t,x) &= \tfrac{1}{2}\Delta u(t,x) + \beta u(t,x)\dot{L}(t,x), \quad t \geq 0,\, x \in \mathbb{R}^d, \\ u(0,\cdot) &= u_0 \end{aligned} \tag{SHE}$$

where $\dot{L}$ is a Lévy space-time white noise and u_0 is some initial condition.

Heat equation with Lévy noise

Stochastic heat equation with Lévy noise:

$$\begin{aligned} \partial_t u(t,x) &= \tfrac{1}{2}\Delta u(t,x) + \beta u(t,x)\dot{L}(t,x), \quad t \geq 0,\ x \in \mathbb{R}^d, \\ u(0,\cdot) &= u_0 \end{aligned} \tag{SHE}$$

where $\dot{L}$ is a Lévy space-time white noise and u_0 is some initial condition.

In this talk: $\dot{L}$ is a pure-jump Lévy noise:

$$L(\mathrm{d}t,\mathrm{d}x) = \lim_{a\to 0} L_a(\mathrm{d}t,\mathrm{d}x) = \lim_{a\to 0}\left(\sum_{(t,x,z)\in\omega,|z|\geq a} z\delta_{(t,x)} - \kappa_a \,\mathrm{Leb}\right),$$

where

- ω is a Poisson point process with intensity $\mathrm{d}t\,\mathrm{d}x\,\lambda(\mathrm{d}z)$;
- λ is a Lévy measure (i.e., $\int_{\mathbb{R}}(1\wedge z^2)\,\lambda(\mathrm{d}z) < \infty$);
- $\kappa_a = \int_{a\leq|z|<1} z\,\lambda(\mathrm{d}z)$.

Motivation

Theorem (Berger & Lacoin 2021b):

Consider a heavy-tailed environment of iid variables $(\eta_{i,x})_{i\in\mathbb{N},x\in\mathbb{Z}^d}$ with

$$\mathbb{E}[\eta_{1,0}] = 0, \quad \mathbb{P}(\eta_{i,x} \geq -1) = 1, \quad \mathbb{P}(\eta_{i,x} > z) = z^{-\alpha}(1 + o(1))$$

for some $\alpha \in (0, 2 \wedge 1 + 2/d)$. Given an independent SSRW on $\mathbb{Z}^d$ with law P, the partition function

$$Z_{N,\beta} = E\left[\prod_{i=1}^{N}(1 + \beta\eta_{i,S_i})\right]$$

satisfies

$$Z_{N,\beta_N} \xrightarrow{\mathrm{d}} U(t) = \int_{\mathbb{R}^d} u(t,x)\,\mathrm{d}x$$

where $\beta_N = C(\beta,\alpha,d)N^{-d(1+2/d-\alpha)/(2\alpha)}$ and u is the solution to the SHE with an α-stable noise.

Motivation

Theorem (Berger & Lacoin 2021b):

Consider a heavy-tailed environment of iid variables $(\eta_{i,x})_{i\in\mathbb{N},x\in\mathbb{Z}^d}$ with

$$\mathbb{E}[\eta_{1,0}] = 0, \quad \mathbb{P}(\eta_{i,x} \geq -1) = 1, \quad \mathbb{P}(\eta_{i,x} > z) = z^{-\alpha}(1 + o(1))$$

for some $\alpha \in (0, 2 \wedge 1 + 2/d)$. Given an independent SSRW on $\mathbb{Z}^d$ with law P, the partition function

$$Z_{N,\beta} = E\left[\prod_{i=1}^{N}(1 + \beta\eta_{i,S_i})\right]$$

satisfies

$$Z_{N,\beta_N} \xrightarrow{\mathrm{d}} U(t) = \int_{\mathbb{R}^d} u(t,x)\,\mathrm{d}x$$

where $\beta_N = C(\beta,\alpha,d)N^{-d(1+2/d-\alpha)/(2\alpha)}$ and u is the solution to the SHE with an α-stable noise.

Recall: $\beta_N = CN^{-1/4}$ for Gaussian noise in dimension 1 (Alberts, Khanin and Quastel, 2004).

What is known so far?

- **Strong moment condition:** Existence and uniqueness of solutions if

$$\exists p \in (1, 1 + \tfrac{2}{d}) : \qquad \int_{\mathbb{R}} |z|^p \, \lambda(\mathrm{d}z) < \infty. \qquad \text{(GLOBAL-p)}$$

Saint Loubert Bié (1998), Applebaum & Wu (2000), Peszat & Zabczyk (2006)

What is known so far?

- **Strong moment condition:** Existence and uniqueness of solutions if

$$\exists p \in (1, 1 + \tfrac{2}{d}) : \qquad \int_{\mathbb{R}} |z|^p \, \lambda(\mathrm{d}z) < \infty. \qquad \text{(GLOBAL-p)}$$

  Saint Loubert Bié (1998), Applebaum & Wu (2000), Peszat & Zabczyk (2006)

- **Stable noise:**
  - Bounded domains: Balan (2014)

What is known so far?

- **Strong moment condition:** Existence and uniqueness of solutions if

$$\exists p \in (1, 1 + \tfrac{2}{d}) : \qquad \int_{\mathbb{R}} |z|^p \, \lambda(\mathrm{d}z) < \infty. \qquad \text{(GLOBAL-p)}$$

  Saint Loubert Bié (1998), Applebaum & Wu (2000), Peszat & Zabczyk (2006)

- **Stable noise:**
  - Bounded domains: Balan (2014)
  - Non-Lipschitz nonlinearity: Mueller (1991), Mytnik (2002)

What is known so far?

- **Strong moment condition:** Existence and uniqueness of solutions if

$$\exists p \in (1, 1+\tfrac{2}{d}): \qquad \int_{\mathbb{R}} |z|^p \, \lambda(\mathrm{d}z) < \infty. \qquad \text{(GLOBAL-p)}$$

  Saint Loubert Bié (1998), Applebaum & Wu (2000), Peszat & Zabczyk (2006)

- **Stable noise:**
  - Bounded domains: Balan (2014)
  - Non-Lipschitz nonlinearity: Mueller (1991), Mytnik (2002)
  - Lipschitz nonlinearity (C., 2017): Existence of solutions if

$$\exists \tfrac{p}{2+2/d-p} < q < p < 1+\tfrac{2}{d}: \quad \int_{[-1,1]} z^p \, \lambda(\mathrm{d}z) + \int_{\mathbb{R}\setminus[-1,1]} z^q \, \lambda(\mathrm{d}z) < \infty$$

What is known so far?

Theorem (Berger & Lacoin 2021a):

Assume that

$$\lambda((-\infty,0)) = 0, \qquad u_0 = \delta_0. \tag{POSITIVE}$$

1. The SHE with truncated Lévy noise L_a has a finite solution u_a if and only if

$$\int_{[1,\infty)} (\log z)^{\frac{d}{2}}\, \lambda(\mathrm{d}z) < \infty. \tag{LARGE}$$

What is known so far?

Theorem (Berger & Lacoin 2021a):

Assume that

$$\lambda((-\infty,0)) = 0, \qquad u_0 = \delta_0. \tag{POSITIVE}$$

1. The SHE with truncated Lévy noise L_a has a finite solution u_a if and only if

$$\int_{[1,\infty)} (\log z)^{\frac{d}{2}}\, \lambda(\mathrm{d}z) < \infty. \tag{LARGE}$$

2. Under (LARGE), if there is $p \in (1, 1+\frac{2}{d})$ such that

$$\int_{(0,1)} z^p\, \lambda(\mathrm{d}z) < \infty, \tag{SMALL-p}$$

   then $u_a \to u$ a.s. for some finite and strictly positive limit u.

What is known so far?

Theorem (Berger & Lacoin 2021a):

Assume that

$$\lambda((-\infty,0)) = 0, \qquad u_0 = \delta_0. \tag{POSITIVE}$$

1. The SHE with truncated Lévy noise L_a has a finite solution u_a if and only if

$$\int_{[1,\infty)} (\log z)^{\frac{d}{2}} \, \lambda(\mathrm{d}z) < \infty. \tag{LARGE}$$

2. Under (LARGE), if there is $p \in (1, 1 + \frac{2}{d})$ such that

$$\int_{(0,1)} z^p \, \lambda(\mathrm{d}z) < \infty, \tag{SMALL-p}$$

   then $u_a \to u$ a.s. for some finite and strictly positive limit u.

3. Conversely, still under (LARGE), we have that $u_a \to 0$ a.s. if

$$\int_{(0,1)} z^2 |\log z| \, \lambda(\mathrm{d}z) = \infty \quad (d = 2) \quad \textit{or} \quad \int_{(0,1)} z^{1+\frac{2}{d}} \, \lambda(\mathrm{d}z) = \infty \quad (d \geq 3).$$

Main results: Existence vs non-existence

Theorem (Berger, C. & Lacoin 2021)

Assume (POSITIVE) and (LARGE).

1. If $d = 2$, u_a converges to a finite strictly positive limit u **if and only if**

$$\int_{(0,1)} z^2 |\log z| \, \lambda(\mathrm{d}z) < \infty.$$

Main results: Existence vs non-existence

Theorem (Berger, C. & Lacoin 2021)

Assume (POSITIVE) and (LARGE).

1. If $d = 2$, u_a converges to a finite strictly positive limit u **if and only if**

$$\int_{(0,1)} z^2 |\log z| \, \lambda(\mathrm{d}z) < \infty.$$

2. If $d \geq 3$, u_a converges to a finite strictly positive limit u if

$$\int_{(0,1)} z^{1+2/d} |\log z| \, \lambda(\mathrm{d}z) < \infty; \qquad \text{(SMALL)}$$

u_a converges to 0 if, for some $\varepsilon > 0$,

$$\int_{(0,1)} z^{1+2/d} |\log z|^{1-\varepsilon} \, \lambda(\mathrm{d}z) = \infty.$$

Main results: Existence vs non-existence

Theorem (Berger, C. & Lacoin 2021)

Assume (POSITIVE) and (LARGE).

1. If $d = 2$, u_a converges to a finite strictly positive limit u **if and only if**

$$\int_{(0,1)} z^2 |\log z| \, \lambda(\mathrm{d}z) < \infty.$$

2. If $d \geq 3$, u_a converges to a finite strictly positive limit u if

$$\int_{(0,1)} z^{1+2/d} |\log z| \, \lambda(\mathrm{d}z) < \infty; \qquad \text{(SMALL)}$$

   u_a converges to 0 if, for some $\varepsilon > 0$,

$$\int_{(0,1)} z^{1+2/d} |\log z|^{1-\varepsilon} \, \lambda(\mathrm{d}z) = \infty.$$

3. Under (SMALL), the limit u is indeed a mild solution to (SHE).

Main results: Moments

Theorem (Berger, C. & Lacoin 2021)

Under (SMALL), if

$$\int_{(1,\infty)} z^p \, \lambda(\mathrm{d}z) < \infty, \qquad \text{(LARGE-p)}$$

for some $p \in (1, 1 + 2/d)$, we further have

$$\mathbb{E}[u(t,x)^p] < \infty$$

for all $(t,x) \in (0,T] \times \mathbb{R}^d$.

Main results: Moments

Theorem (Berger, C. & Lacoin 2021)

Under (SMALL), if

$$\int_{(1,\infty)} z^p \,\lambda(\mathrm{d}z) < \infty, \qquad \text{(LARGE-p)}$$

for some $p \in (1, 1 + 2/d)$, we further have

$$\mathbb{E}[u(t,x)^p] < \infty$$

for all $(t, x) \in (0, T] \times \mathbb{R}^d$.

Note: This is nontrivial unless $\int_{(0,\infty)} z^p \,\lambda(\mathrm{d}z) < \infty$! In fact, for the SHE driven by an α-stable noise with $\alpha \in (1, 1 + \frac{2}{d})$, it was open until now whether the solution has a finite pth moment for $p \in (1, \alpha)$.

Main results: Intermittency

Moment Lyapunov exponents:

Assume (LARGE-p) for some $p \in (1, 1 + 2/d)$ and define

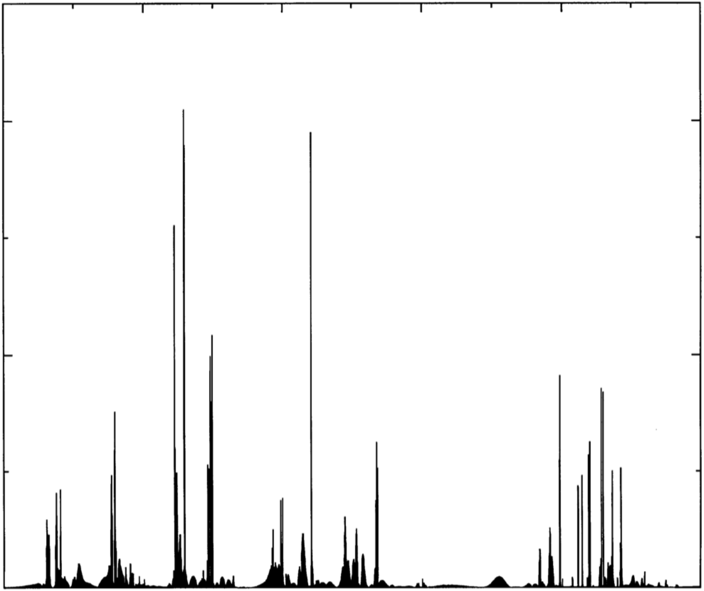
$$\gamma_\beta(p) := \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}[\overline{u}(t,x)^p] > 0,$$

where $\overline{u}(t,x) = e^{-\mu t} u(t,x)$, μ is the mean of the noise and u is the solution to SHE with initial condition 1.

Question: Do we have

$$0 < |\gamma_\beta(p)| < \infty?$$

If so, we say that u is **intermittent of order** p.



Known results

- **Discrete-space Gaussian/finite variance Lévy noise:**
  - $d = 1, 2$: Intermittency for all integer $p \geq 2$ and all β
  - $d \geq 3$: Intermittency for integer $p \geq 2$ if and only if $\beta > \beta_c > 0$.

Carmona & Molchanov (1994), Ahn et al. (1992)

Known results

- **Discrete-space Gaussian/finite variance Lévy noise:**
  - $d = 1, 2$: Intermittency for all integer $p \geq 2$ and all β
  - $d \geq 3$: Intermittency for integer $p \geq 2$ if and only if $\beta > \beta_c > 0$.

  Carmona & Molchanov (1994), Ahn et al. (1992)
- **Continuous-space Gaussian noise:**
  - $d = 1$:
  $$\gamma_\beta(p) = \frac{\beta^4 p(p^2 - 1)}{24}, \qquad p \in (0, \infty).$$
  Bertini & Cancrini (1995), ..., Ghosal & Lin (2020).

Known results

- **Discrete-space Gaussian/finite variance Lévy noise:**
  - $d = 1, 2$: Intermittency for all integer $p \geq 2$ and all β
  - $d \geq 3$: Intermittency for integer $p \geq 2$ if and only if $\beta > \beta_c > 0$.

  Carmona & Molchanov (1994), Ahn et al. (1992)
- **Continuous-space Gaussian noise:**
  - $d = 1$:
$$\gamma_\beta(p) = \frac{\beta^4 p(p^2 - 1)}{24}, \qquad p \in (0, \infty).$$
    Bertini & Cancrini (1995), ..., Ghosal & Lin (2020).
  - $d \geq 2$ (with a compactly supported spatial noise correlation):
    - $d = 1, 2$: Intermittency for all integer $p \geq 2$ and all β
    - $d \geq 3$: Intermittency for integer $p \geq 2$ iff $\beta > \beta_c > 0$.

    Chen & Kim (2019), Lacoin (2011)

Known results

- **Discrete-space Gaussian/finite variance Lévy noise:**
  - $d = 1, 2$: Intermittency for all integer $p \geq 2$ and all β
  - $d \geq 3$: Intermittency for integer $p \geq 2$ if and only if $\beta > \beta_c > 0$.

  Carmona & Molchanov (1994), Ahn et al. (1992)
- **Continuous-space Gaussian noise:**
  - $d = 1$:
    $$\gamma_\beta(p) = \frac{\beta^4 p(p^2 - 1)}{24}, \qquad p \in (0, \infty).$$
    Bertini & Cancrini (1995), ..., Ghosal & Lin (2020).
  - $d \geq 2$ (with a compactly supported spatial noise correlation):
    - $d = 1, 2$: Intermittency for all integer $p \geq 2$ and all β
    - $d \geq 3$: Intermittency for integer $p \geq 2$ iff $\beta > \beta_c > 0$.

    Chen & Kim (2019), Lacoin (2011)
- **Continuous-space Lévy noise:** Under (GLOBAL-p),
  - $d = 1$: Intermittency for all $p \in (1, 3)$ and $\beta > 0$
  - $d \geq 2$: Intermittency for **large** p (= close to $1 + 2/d$) or **large** β.

  C. & Kevei (2019)

Open questions in the Lévy case

- What happens in $d \geq 2$ if both $p > 1$ and $\beta > 0$ are small?
- What happens with $p \in (0, 1)$?
- What happens if (GLOBAL-p) is not satisfied (e.g., stable noise)?

Open questions in the Lévy case

- What happens in $d \geq 2$ if both $p > 1$ and $\beta > 0$ are small?
- What happens with $p \in (0, 1)$?
- What happens if (GLOBAL-p) is not satisfied (e.g., stable noise)?

First guess: Based on the previous results, one may conjecture:

- There is intermittency for all $p \in (0, 1 + \frac{2}{d})$ [for which the noise has a pth moment] and for all β if $d = 1, 2$;
- There is intermittency for all $p \in (0, 1 + \frac{2}{d})$ [for which the noise has a pth moment] in $d \geq 3$ if and only if β is larger than a critical value.

Open questions in the Lévy case

- What happens in $d \geq 2$ if both $p > 1$ and $\beta > 0$ are small?
- What happens with $p \in (0, 1)$?
- What happens if (GLOBAL-p) is not satisfied (e.g., stable noise)?

First guess: Based on the previous results, one may conjecture:

- There is intermittency for all $p \in (0, 1 + \frac{2}{d})$ [for which the noise has a pth moment] and for all β if $d = 1, 2$;
- There is intermittency for all $p \in (0, 1 + \frac{2}{d})$ [for which the noise has a pth moment] in $d \geq 3$ if and only if β is larger than a critical value.

This turns out to be a fallacy.

Main results: Intermittency

Theorem (Berger, C. & Lacoin 2021)

With any non-trivial Lévy noise, we have intermittency

- **for all** $p \in (1, 1 + \frac{2}{d}) \setminus \{1\}$ [for which the noise has a pth moment]
- **for all** $\beta > 0$
- **for all** dimensions $d \geq 1$.

Main results: Intermittency

Theorem (Berger, C. & Lacoin 2021)

With any non-trivial Lévy noise, we have intermittency

- **for all** $p \in (1, 1 + \frac{2}{d}) \setminus \{1\}$ [for which the noise has a pth moment]
- **for all** $\beta > 0$
- **for all** dimensions $d \geq 1$.

Remarks:

- We could also show $|\gamma_\beta(p)| > 0$ for $p \in (0, 1)$.
- This fact, which we call **strong intermittency** automatically implies *full intermittency* (i.e., $\gamma_\beta(p) > 0$ for all $p > 1$).
- In addition, **strong intermittency** plus ergodicity implies *physical intermittency* of the solution.

THANK YOU!

Theorem (Berger, C. & Lacoin 2021)

In fact, we can show:

1. $d = 1$: for small β, if $\int_{[1,\infty)} z^2 \, \lambda(\mathrm{d}z) < \infty$,
$$C'_p \beta^4 \le \gamma_\beta(p) \le C_p \beta^4$$
2. $d \ge 2$: if $\int_{[1,\infty)} z^{1+2/d} \, \lambda(\mathrm{d}z) < \infty$, then
$$\lim_{\beta \to 0} \frac{\log|\log \gamma_\beta(p)|}{|\log \beta|} = 1 + \tfrac{2}{d}.$$
3. For any d, if $\lambda((u,\infty)) \sim u^{-\alpha}$ (as $u \to \infty$) where $\alpha \in (1, 2 \wedge (1 + \frac{2}{d}))$,
$$\lim_{\beta \to 0} \frac{\log \gamma_\beta(p)}{\log \beta} = \frac{\alpha}{1 - \frac{d}{2}(\alpha - 1)}.$$

A word or two about the proofs

Upper bounds:

- Chaos decomposition: analysis of multiple Poisson integrals
- Main tool: Decoupling for Poisson stochastic integrals:

$$\mathbb{E}\left[\left(\int\cdots\int_{t_1<\cdots<t_k} f(t_1,\ldots,t_k)\,L(\mathrm{d}t_1)\cdots L(\mathrm{d}t_k)\right)^p\right]$$
$$\approx \mathbb{E}\left[\left(\int\cdots\int_{t_1<\cdots<t_k} f(t_1,\ldots,t_k)\,L_1(\mathrm{d}t_1)\cdots L_k(\mathrm{d}t_k)\right)^p\right]$$

  where $L_1,\ldots,L_k$ are independent copies of L.
- After decoupling, we are free to choose the order of integration (without losing important properties of Itô integrals)!
- For us, $f(t_1,\ldots,t_k)=\rho(t_k-t_{k-1})\cdots\rho(t_2-t_1)\rho(t_1)$
- Main difficulty: identify an optimal order of integration!

A word or two about the proofs

Lower bounds:

- Size-bias representation (Berger & Lacoin 2021): under the size-biased measure
$$\widetilde{\mathbb{P}}_{\beta,t}(\omega \in A) := \mathbb{E}[\overline{u}(t,0)\mathbf{1}_A],$$
the distribution of ω (the point process) is the same as $\widehat{\omega}$, which is obtained by adding to ω an independent point process ω' with intensity $\beta z \,\mathrm{d}t\, \lambda(\mathrm{d}z)$ on the path of an independent standard Brownian motion.

A word or two about the proofs

Lower bounds:

- One can prove intermittency (or degeneracy of the limit u) if one can find a random variable $f(\omega)$ such that

$$\widetilde{\mathbb{E}}_{\beta,t}[f(\widehat{\omega})] \geq C_d \max\{\mathrm{Var}(f)^{1/2},\ \widetilde{\mathrm{Var}}_{\beta,t}(f)^{1/2}\}$$

  for some constant C_d that only depends on d.

- A typical f:

$$f(\omega) = \int_{([0,T]\times\mathbb{R}^d\times[a,b))^k} \mathbf{1}_{\{t_1\leq\frac{T}{2},\ \forall i=2,\dots,k-1:\ \Delta t_i\in[T^{-\kappa'},T^{-\kappa}]\}} \times \mathbf{1}_{\{\|x_1\|_\infty\leq R\sqrt{T},\ \forall i=2,\dots,k:\ \|\Delta x_i\|_\infty\leq R\sqrt{\Delta t_i}\}} \times \mathbf{1}_{\{T^{-d/2}\prod_{i=2}^k(t_i-t_{i-1})^{-d/2}\geq M\beta^{-k}\}} \times \prod_{i=1}^k \delta_\omega(\mathrm{d}t_i,\mathrm{d}x_i,\mathrm{d}z_i).$$