Numerical Approximation of Nonlinear Stochastic Differential Equations with Continuously Distributed Delay

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Overview

Basic Results

Stochastic Delay Differential Equation-SDDE

An SDDE is a stochastic differential equation that depends on the history (time lag) of the process. An n-dimensional SDDE with discrete delay can be written as

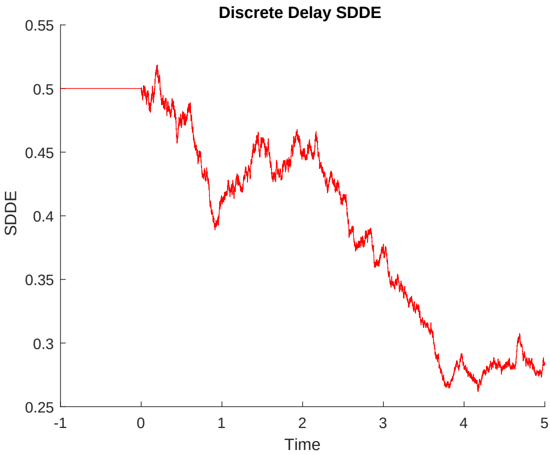
$$dx(t) = f\left(t, x(t), x(t-\tau)\right) dt + g\left(t, x(t), x(t-\tau)\right) dB(t) \qquad (1)$$

with the initial condition

$$\{x(\theta) = \psi(\theta) : -\tau \leq \theta \leq 0\} \qquad (2)$$

where $\psi \in \mathcal{C}\left([-\tau, 0]; \mathbb{R}^n\right)$, $\tau > 0$ is the fixed delay and $f : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n \times m}$.

SDDE Examples



$$dx(t) = x(t)[-0.5 - 0.25x(t) + 0.025x(t-1)]\,dt + 0.1x(t)\,dB(t)$$

with initial condition is $\psi(\theta) = 0.5$ for $\theta \in [-1, 0]$ and $T = 5$.

Basic Results

Theorem (Classical Existence and Uniqueness theorem for SDDE -Mao and Yuan, 2006)

Suppose that the functions f and g satisfies the following conditions:

1. *(Linear growth condition)*

$$|f(t,x,y)|^2 \vee |g(t,x,y)|^2 \leq K\left(1+|x|^2+|y|^2\right); \text{ for some } K>0. \tag{3}$$

2. *(Local Lipschitz condition) For each integer $i \geq 1$ there is a positive constant K_i such that*

$$|f(t,x,y)-f(t,\bar{x},\bar{y})|^2 \vee |g(t,x,y)-g(t,\bar{x},\bar{y})|^2 \leq K_i\left(|x-\bar{x}|^2+|y-\bar{y}|^2\right); \tag{4}$$

for $x, y, \bar{x}, \bar{y} \in \mathbb{R}^n$ with $|x| \vee |y| \vee |\bar{x}| \vee |\bar{y}| \leq i$.

Then the SDDE(1) has a unique solution $x(t)$ on $t \in [-\tau, T]$. Moreover, the solution has the property that $\mathbb{E}\left(\sup_{-\tau \leq t \leq T} |x(t)|^2\right) < \infty$.

SDDE with continuously distributed delay Problem

The following is one of the problems that we are considering with continuously distributed delay.

Problem

Consider a nonlinear SDDE of the form :

$$dx(t) = f\left(x(t), \frac{1}{\tau}\int_0^\tau x(t-s)\,ds\right)dt + g\left(x(t), \frac{1}{\tau}\int_0^\tau x(t-s)\,ds\right)dB(t) \tag{5}$$

for all $t \in [0, T]$, *with the initial condition*

$$\{x(\theta) = \psi(\theta) : -\tau \le \theta \le 0\} \tag{6}$$

where $\psi \in \mathcal{C}([-\tau, 0]; \mathbb{R}^n)$, $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ *and* $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ *and functions* f *and* g *satisfy the local Lipschitz condition but not the linear growth condition.*

Existence and Uniqueness

We define the operator L which, given $V \in \mathcal{C}^2(\mathbb{R}^n; \mathbb{R}_+)$, yields the function $LV : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, where

$$LV(x, y) = V_x(x) f(x, y) + \frac{1}{2} \text{trace} \left[g^T(x, y) V_{xx}(x) g(x, y) \right].$$

Assumption (Local Lipschitz Condition)

For each integer $i \geq 1$ there is a positive constant K_i such that

$$|f(x, y) - f(\bar{x}, \bar{y})|^2 \vee |g(x, y) - g(\bar{x}, \bar{y})|^2 \leq K_i \left(|x - \bar{x}|^2 + |y - \bar{y}|^2 \right); \tag{7}$$

for $x, y, \bar{x}, \bar{y} \in \mathbb{R}^n$ with $|x| \vee |y| \vee |\bar{x}| \vee |\bar{y}| \leq i$.

Existence and Uniqueness

Assumption (Khasminskii-type condition - Xuerong Mao, 2011)

There are two functions $V \in \mathbb{C}^2(\mathbb{R}^n; \mathbb{R}_+)$ *and* $U \in \mathbb{C}(\mathbb{R}^n; \mathbb{R}_+)$ *and two positive constants* λ_1 *and* λ_2 *s.t.* $\lim_{|x| \to \infty} V(x) = \infty$, *and*

$$LV(x, y) \leq \lambda_1[1 + V(x) + V(y) + U(y)] - \lambda_2 U(x) \tag{8}$$

for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$.

The following assumption is a modification from [Mao and Rassias, 2005].

Assumption (Khasminskii-type condition (2))

There are two non-negative and convex functions $V \in \mathcal{C}^2(\mathbb{R}^n; \mathbb{R}_+)$ *and* $U \in \mathcal{C}^2(\mathbb{R}^n; \mathbb{R}_+)$ *such that* $\lim_{|x| \longrightarrow \infty} V(x) = \infty$ *and*

$$LV(x, y) \leq \alpha_1 [1 + V(x) + V(y)] - \alpha_2 U(x) + \alpha_3 U(y), \tag{9}$$

for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, *where* $\alpha_1 \geq 0$ *and* $\alpha_2 > \alpha_3 \geq 0$.

Example

$$dx(t) = x(t)\left[-b - Ax(t) + \frac{B}{\tau}\int_0^\tau x(t-s)\,ds\right]\,dt$$
$$+x(t)\left[\beta_1 + \beta_2 x(t) + \frac{\beta_3}{\tau}\int_0^\tau x(t-s)\,ds\right]\,dB(t)$$

with the initial condition $x(t) = \psi(t)$ for all $t \in [-\tau, 0]$ where the coefficients A and b are positive numbers and $x(t) \in \mathbb{R}_+$.

Assume that there is a positive number θ such that

$$-A + \frac{B^2}{4\theta} + \theta + \beta^2 \leq 0$$

where $\beta^2 = \beta_1^2 + \beta_2^2 + \beta_3^2$. Then example satisfies the Khasminskii-type condition (2) with functions $V(x) = (x-1) - \log x$ and $U(x) = x^2$.

Existence and Uniqueness

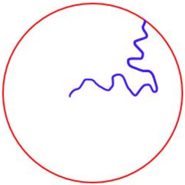
Theorem

Let local Lipschitz and Khasminskii-type condition (2) assumptions hold. Then for any given initial data (6), there is a unique global solution $x(t)$ to stochastic differential equation with continuous delay as in (5) on $t \in [-\tau, \infty)$. Moreover the solution has the properties that

$$\mathbb{E}V(x(t)) < \infty \text{ and } \mathbb{E}\int_0^t U(x(u))\,du < \infty \tag{10}$$

for any $t \geq 0$.

Let's define the stopping time $\sigma_i = \inf\{t \geq 0 : |x(t)| \geq i\}$.



Existence and Uniqueness

Theorem

Let local Lipschitz and Khasminskii-type condition (2) *assumptions hold. Let the unique global solution to the SDDE* (5) *with the initial data* (6)*, be* $x(t)$ *on* $t \in [-\tau, \infty)$*.*
Let $i_0 > 0$ *be sufficiently large so that* $\max_{-\tau \le t \le 0} |x(t)| < i_0$*. Let* $i \ge i_0$ *be any integer and define the stopping time* $\sigma_i = \inf \{t \ge 0 : |x(t)| \ge i\}$*. Set* $\inf \emptyset = \infty$*. Then for any integer* $j \ge 1$*,*

$$\mathbb{E} V(x(t \wedge \sigma_i)) \le K_j e^{2\alpha_1 (j\tau)}, \quad t \in [0, j\tau] \tag{11}$$

and

$$\mathbb{E} \int_0^{j\tau} U(x(t))\, dt \le \frac{K_j}{\alpha_2 - \alpha_3} \left[1 + 2\alpha_1 (j\tau) e^{2\alpha_1 (j\tau)} \right] \tag{12}$$

where $K_j = V(x(0)) + \alpha_1 (j\tau) + \int_{-\tau}^{0} [\alpha_1 V(x(t)) + \alpha_3 U(x(t))]\, dt < \infty$*.*

Moreover, $K_{j+1} = \alpha_1 \tau + K_j < \infty$ *where* $j \ge 1$*.*

SDDE with continuously distributed delay

Lemma

Let's consider any non-negative convex function f from $\mathbb{R}_+^n$ to $\mathbb{R}_+$. Then for any $T > 0$ and $\tau > 0$ the following inequality holds.

$$\int_0^T f\left(\int_0^\tau x(t-s)\,d\mu(s)\right)\,dt \leq \int_{-\tau}^0 f(x(t))\,dt + \int_0^T f(x(t))\,dt,$$

where μ is a probability measure on $[0,\tau]$.

$$\begin{aligned}\int_0^T f\left(\int_0^\tau x(t-s)\,d\mu(s)\right)\,dt &\leq \int_0^\tau \int_{-s}^{T-s} f(x(t))\,dt\,d\mu(s)\\ &\leq \int_{-\tau}^0 f(x(t))\,dt + \int_0^T f(x(t))\,dt\end{aligned}$$

Corollary

Let local Lipschitz and Khasminskii-type condition (2) assumptions hold. Let $\epsilon \in (0,1)$ and $T > 0$ be arbitrary. Then there is a sufficiently large integer i^*, depending on ϵ and T, s.t.

$$\mathbb{P}\left(\sigma_i \leq T\right) \leq \epsilon, \;\; \forall i \geq i^*, \tag{13}$$

Euler Maruyama (EM) approximation

Let $N > \tau$ be a positive integer and $\Delta t = \frac{\tau}{N}$ be the step size. Define $t_n = n\Delta t$ for $n = -N, -(N-1), \ldots, 0, 1, 2, \ldots$

The EM numerical scheme applied to SDDE (5) is to compute the discrete-time numerical approximations $X_n \approx x(t_n)$ by setting $X_n = \psi(n\Delta t)$ for $n = -N, -(N-1), \ldots, 0$ and setting

$$X_n = X_{n-1} + f\left(X_{n-1}, \frac{\Delta t}{\tau}\sum_{i=1}^{N} X_{n-1-i}\right)\Delta t + g\left(X_{n-1}, \frac{\Delta t}{\tau}\sum_{i=1}^{N} X_{n-1-i}\right)\Delta B_n \tag{14}$$

for $n = 1, 2, \ldots,$ where $\Delta B_n = B(t_n) - B(t_{n-1})$.

Euler Maruyama (EM) approximation

Continuous-time approximations are similar to the discrete delay case in [Xuerong Mao, 2011].

1. Continuous-time step process on $t \in [-\tau, \infty)$:

$$\bar{X}(t) = \sum_{n=-N}^{\infty} X_n I_{[n\Delta t,(n+1)\Delta t)}(t) \tag{15}$$

2. Continuous-time continuous approximate process on $t \in [-\tau, \infty)$ defined by $X(t) = x(t) = \psi(t)$ for $t \in [-\tau, 0]$ while for $t \geq 0$,

$$X(t) = \bar{X}(0) + \int_0^t f\left(\bar{X}(u), \frac{\Delta t}{\tau}\sum_{j=1}^{N}\bar{X}(u - j\Delta t)\right) du$$

$$+ \int_0^t g\left(\bar{X}(u), \frac{\Delta t}{\tau}\sum_{j=1}^{N}\bar{X}(u - j\Delta t)\right) dB(u) \tag{16}$$

Euler Maruyama (EM) approximation

Lemma

Suppose the local Lipschitz condition holds. Let $T > 0$ be arbitrary and let i be any sufficiently large integer so that $\max_{-\tau \le t \le 0} |x(t)| \le i$. Define the stopping time $\rho_i := \inf\{t \ge 0 : |X(t)| \ge i\}$. Let $p > 1$ be any integer sufficiently large so that

$$\left(\frac{2p}{2p-1}\right)^2 (T+1)^{1/p} < 2 \tag{17}$$

Then

$$\mathbb{E}\left(\sup_{0 \le t \le T \wedge \rho_i} |X(t) - \bar{X}(t)|^2\right) \le C\,(1 + 2mp)\,\Delta t^{(p-1)/p} \tag{18}$$

where $C = 4\left(2i^2 K_i + |f(0,0)|^2 \vee |g(0,0)|^2\right)$ and K_i is the local Lipschitz constant for the coefficients of SDDE (5).

This is similar to the lemma given in [Xuerong Mao, 2011] for the discrete delay case.

Euler Maruyama (EM) approximation

Lemma

Let local Lipschitz and Khasminskii-type condition (2) *assumptions hold. Then for any pair of $T > 0$ and $\epsilon \in (0,1)$, there is a sufficiently large i^* and sufficiently small Δt^* s.t.*

$$\mathbb{P}(\rho_{i^*} \leq T) \leq \epsilon, \ \ \forall \, \Delta t \leq \Delta t^* \tag{19}$$

where $\rho_{i^*} := \inf \{ t \geq 0 : |X(t)| \geq i^* \}$

Proof follows the similar methods in [Xuerong Mao, 2011] adapted to our problem.

Convergence in Probability

Lemma

For all sufficiently large integer i, we have

$$\mathbb{E}\left(\sup_{0\le t\le T}|x(t\wedge\nu_i)-X(t\wedge\nu_i)|^2\right)=O\left(\Delta t^{2/3}\right) \tag{20}$$

where $\nu_i=\sigma_i\wedge\rho_i$.

Theorem

For any $T>0$,

1. $\lim_{\Delta t\to 0}\left(\sup_{0\le t\le T}|x(t)-X(t)|\right)=0$ *in probability.*
2. $\lim_{\Delta t\to 0}\left(\sup_{0\le t\le T}|x(t)-\bar{X}(t)|\right)=0$ *in probability.*

Theorem proof follows the similar methods in [Xuerong Mao, 2011] adapted to our problem.

Extensions of Results

Let G be an open subset of $\mathbb{R}^n$ such that there is an sequence of $\{G_i\}_{i\geq 1}$ of increasing compact subsets of $\mathbb{R}^n$ such that $\lim_{i\to\infty} G_i = \bigcup_{i=1}^{\infty} G_i = G$. In particular, take $G = \mathbb{R}_+^n$ and $G_i = \left\{x \in \mathbb{R}_+^n : 1/i \leq x_j \leq i, 1 \leq j \leq n\right\}$. A stopping time can be defined as $\sigma_i = \inf\left\{t \geq 0 : x(t) \notin G_i\right\}$.

Problem

Consider a nonlinear SDDE

$$dx(t) = f\left(x(t), \frac{1}{\tau}\int_0^\tau x(t-s)\,ds\right)dt + g\left(x(t), \frac{1}{\tau}\int_0^\tau x(t-s)\,ds\right)dB(t) \tag{21}$$

with the initial condition $\{x(\theta) = \psi(\theta) : -\tau \leq \theta \leq 0\}$ *where* $\psi \in \mathcal{C}\left([-\tau, 0]; G\right)$, $f : G \times G \to \mathbb{R}^n$ *and* $g : G \times G \to \mathbb{R}^{n\times m}$.

Similar results as above can be shown for processes on such G.

Extensions of Results

Problem

Consider the nonlinear continuously distributed delay SDDE,

$$dx(t) = f\left(x(t), \int_0^\tau x(t-s)\,d\mu(s)\right)\,dt$$

$$+g\left(x(t), \int_0^\tau x(t-s)\,d\mu(s)\right)\,dB(t)$$

for certain probability measure μ *on* $[0,\tau]$*. Where the initial data given by* $\{x(t) = \psi(\theta) : -\tau \leq t \leq 0\}$.

Future Work

We have established some sufficient conditions on the coefficients in our non-linear SDDEs to avoid the explosion and extinction in a finite time. Thus, in future, I'm planning to research the dependency on coefficients of non-extinction and other almost sure results.

I am interested to study theoretically and numerically, for example, with MATLAB simulations, about behavior changes of the trajectory of non-linear SDDEs with continuously distributed delay.

References

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THANK YOU