Large deviations for slow-fast systems driven by fractional Brownian motion

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Outline

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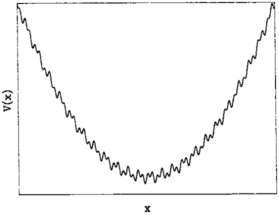
[1]

I: Averaging/homogenization of SDEs: $H = 1/2$



[1] Zwanzig, R., 1988. Diffusion in a rough potential. Proceedings of the National Academy of Sciences, 85(7), pp.2029-2030.

Averaging/homogenization of SDEs: $H = 1/2$

Let $(B^{1/2}, W)$ be independent Brownian motions. Consider the coupled SDEs:

$$\begin{cases} dX_t^{\epsilon,\eta} = \frac{\sqrt{\epsilon}}{\sqrt{\eta}} b(Y_t^\eta)dt + c(X_t^{\epsilon,\eta}, Y_t^\eta)dt + \sqrt{\epsilon}\sigma(X_t^{\epsilon,\eta}, Y_t^\eta)dB_t^{1/2} \\ dY_t^\eta = \frac{1}{\eta} f(Y_t^\eta)dt + \frac{1}{\sqrt{\eta}}\tau(Y_t^\eta)dW_t \\ X_0^{\epsilon,\eta} = x_0 \in \mathbb{R}^m := \mathcal{X}, \quad Y_0^\eta = y_0 \in \mathbb{R}^{d-m} := \mathcal{Y} \end{cases}$$

$\epsilon \ll 1, \eta = \eta(\epsilon) \to 0, \epsilon \to 0$ and $\sqrt{\eta}/\sqrt{\epsilon} \to 0$.
Goal: Find "effective" dynamics of $X^{\epsilon,\eta}$ as $\epsilon, \eta \to 0$.
Under dissipativity asumptions on f, Y^1 is ergodic with unique invariant measure μ. Under the centering condition $\int_{\mathcal{Y}} b(y)d\mu(y) = 0$, sufficiently regular b, f, τ, and uniformly non-degenerate τ, the Poisson equation

$$\begin{cases} \mathcal{L}\Psi(y) := \frac{1}{2}[D^2\Psi(y) : (\tau\tau^T)](y) + \nabla\Psi(y)(y)f(y) = -b(y) \\ \int_{\mathcal{Y}} \Psi(y)d\mu(y) = 0 \end{cases}$$

has a unique solution $\Psi \in C^2(\mathcal{Y})$ with polynomial growth (*corrector* function).

Averaging/homogenization of SDEs: H=1/2

- Averaging principle: For all $T > 0$, $X^{\epsilon,\eta} \underset{\epsilon,\eta\to 0}{\longrightarrow} \bar{X}$ in $L^p(\Omega; C([0,T];\mathcal{X}))$, where

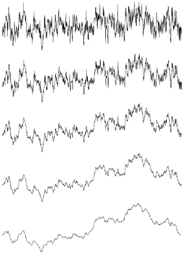
$$\frac{dX_t^0}{dt} = \bar{c}(X_t^0) := \int_{\mathcal{Y}} c(X_t^0, y)d\mu(y)\ ,\ X_0^0 = x_0$$

- Homogenization: Set $\epsilon = 1$ (larger time scale). For all $T > 0$, $X^{1,\eta} \underset{\eta\to 0}{\longrightarrow} \tilde{X}$ *weakly* in $C([0,T];\mathcal{X})$ where $\tilde{X}$ is equal in law to the solution of

$$d\tilde{X}_t = \bar{c}(\tilde{X}_t)dt + \sqrt{\bar{Q}_{1/2}(\tilde{X}_t)}d\tilde{W}_t\ ,\ \tilde{X}_0 = x_0\ ,\ \tilde{W} \text{ is a Bm,}$$

$\bar{Q}_{1/2}(x) := (\ \overline{\sigma(x)\sigma^T(x)} + \overline{(\nabla\Psi_\tau)(\nabla\Psi_\tau)^T}\ )$ is the *effective diffusivity*.
(fast dynamics contribution to the noise captured by corrector function)

II: Averaging/homogenization of SDEs: $H > 1/2$



- fBm: centered Gaussian process with stationary increments and covariance given by $\mathbb{E}[B_t^H B_s^H] = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right)$. A.a. sample paths locally θ-Hölder continuous for any $\theta < H$. For $H = 1/2$, $B^{1/2}$ is a standard Bm. For $H > 1/2$, increments of B^H are positively correlated.

[2] Georgiy Shevchenko, Fractional Brownian motion in a nutshell,7th Jagna International Workshop, 2015

Averaging/homogenization of SDEs: H>1/2

- What if the Bm $B^{1/2}$ is replaced by a fractional Brownian motion (fBm) B^H with Hurst index $H > 1/2$? Consider a pair of independent fBm and Bm (B^H, W) on a filtered p.s. $(\Omega, \mathcal{F}, \mathcal{F}_{t\geq 0}, \mathbb{P})$ and:

$$\begin{cases} dX_t^{\epsilon,\eta} = \frac{\sqrt{\epsilon}}{\sqrt{\eta}} b(Y_t^\eta)dt + c(X_t^{\epsilon,\eta}, Y_t^\eta)dt + \sqrt{\epsilon}\sigma(X_t^{\epsilon,\eta}, Y_t^\eta)dB_t^H \\ dY_t^\eta = \frac{1}{\eta} f(Y_t^\eta)dt + \frac{1}{\sqrt{\eta}}\tau(Y_t^\eta)dW_t \\ X_0^{\epsilon,\eta} = x_0 \in \mathbb{R}^m := \mathcal{X} \ , \quad Y_0^\eta = y_0 \in \mathbb{R}^{d-m} := \mathcal{Y}. \end{cases}$$

$\epsilon \ll 1, \eta = \eta(\epsilon) \to 0, \epsilon \to 0$ and $\sqrt{\eta}/\sqrt{\epsilon} \to 0$.

- First issue: fBm is neither Markovian nor a martingale. We have to abandon the tools of Itô calculus and specify the interpretation of "dB_t^H". Several approaches using divergence, Stratonovich and pathwise Young integration have been established e.g. [DÜ99, Zäh99, Nua].

Averaging/homogenization of SDEs: H>1/2

$$\begin{cases} dX_t^{\epsilon,\eta} = \frac{\sqrt{\epsilon}}{\sqrt{\eta}} b(Y_t^\eta)dt + c(X_t^{\epsilon,\eta}, Y_t^\eta)dt + \sqrt{\epsilon}\sigma(X_t^{\epsilon,\eta}, Y_t^\eta)dB_t^H \\ dY_t^\eta = \frac{1}{\eta} f(Y_t^\eta)dt + \frac{1}{\sqrt{\eta}}\tau(Y_t^\eta)dW_t,\ X_0^{\epsilon,\eta} = x_0 \in \mathcal{X}\ ,\quad Y_0^\eta = y_0 \in \mathcal{Y}, t \in [0,T], T > 0. \end{cases}$$

- Hairer-Li (02/2019)[HL20]: Let $b = 0$, $\epsilon = 1$, dB_t^H = extension of pathwise Young integral. Then $X^{1,\eta} \underset{\eta\to 0}{\longrightarrow} X^0$ *in probability*, X^0 solves the "naïvely" averaged equation

$$dX_t^0 = \bar{c}(X_t^0)dt + \bar{\sigma}(X_t^0)dB_t^H\ , X_0^0 = x_0.$$

- Pei-Inahama-Xu (01/2020)[PIX20]: Let $b = 0, \epsilon = 1, \sigma(x,y) = \sigma(x), dB_t^H =$ pathwise Young integral. Then $X^{1,\eta} \underset{\eta\to 0}{\longrightarrow} X^0$ in $L^2(\Omega; C([0,T]; \mathcal{X}))$, $dX_t^0 = \bar{c}(X_t^0)dt + \sigma(X_t^0)dB_t^H\ , X_0^0 = x_0.$
- Bourguin-Gailus-Spiliopoulos (08/2020)[BGS20]: For suff. small $p \geq 1$, $b \neq 0$, $\sigma(x,y) = \sigma(y), dB^H =$ divergence integral, $X^{\epsilon,\eta} \underset{\epsilon,\eta\to 0}{\longrightarrow} X^0$ in $L^p(\Omega; C([0,T]; \mathcal{X}))$, $dX_t^0 = \bar{c}(X_t^0)\ , X_0^0 = x_0.$

III: Large Deviations from the averaging limit

"*If an unlikely event occurs, it is very likely to occur in the most likely way*"

- Averaging principle=typical behavior of the slow-fast system (LLN).
- Goal: Characterize the decay rates of probabilities of large deviations from the typical behavior. In particular, find a rate function $S : C([0,T];\mathcal{X}) \to [0,\infty]$ that satisfies the Large Deviation Principle (LDP):

$$\forall B \in \mathscr{B}(C([0,T];\mathcal{X})) : \mathbb{P}[X^{\epsilon,\eta} \in B] \approx e^{-\inf_{\phi\in B} S(\phi)/\epsilon} \text{ as } \epsilon \to 0 \quad \text{(LDP)}$$

Assuming that S has compact sub-level sets, the latter is equivalent to the Laplace Principle (LP): $\forall h \in C_b(C([0,T];\mathcal{X});\mathbb{R})$:

$$\lim_{\epsilon\to 0} \epsilon \log \mathbb{E}[e^{-h(X^{\epsilon,\eta})/\epsilon}] = -\inf_{\phi\in C([0,T];\mathcal{X})} \left[S(\phi) + h(\phi)\right] \quad \text{(LP)}$$

Large Deviations: The weak convergence method

Using a variational representation on abstract Wiener spaces (X.Zhang [Zha09], see also Budhiraja-Song [BS20]) we express the Laplace functional as follows:

$$-\epsilon \log \mathbb{E} e^{-h(X^{\epsilon,\eta})/\epsilon} = \inf_{u=(u_1,u_2)\in\mathcal{A}_b} \mathbb{E}\left\{ \frac{1}{2}\left(\|u_1\|^2_{\mathcal{H}_H} + \|u_2\|^2_{\mathcal{H}_{1/2}} \right) + h\left(X^{\epsilon,\eta,u}\right) \right\}, \tag{1}$$

$\mathcal{H}_H \oplus \mathcal{H}_{1/2}$: Cameron-Martin space of mixed noise, $\mathcal{H}_H := K_H[L^2([0,T])]$,

$$K_H = \text{"}(H+1/2)-\textit{integral}\text{"},$$

inner product $\langle f, g\rangle_{\mathcal{H}_H} := \langle K_H^{-1} f, K_H^{-1} g\rangle_{L^2}$, $\mathcal{A}_b$: a.s. bounded stochastic controls, adapted to common filtration,

$$\begin{cases} dX_t^{\epsilon,\eta,u} = \frac{\sqrt{\epsilon}}{\sqrt{\eta}} b(Y_t^{\epsilon,\eta,u})dt + c(X_t^{\epsilon,\eta,u}, Y_t^{\epsilon,\eta,u})dt + \sigma(X_t^{\epsilon,\eta,u}, Y_t^{\epsilon,\eta,u})du_1(t) \\ \qquad\qquad + \sqrt{\epsilon}\sigma(X_t^{\epsilon,\eta,u}, Y_t^{\epsilon,\eta,u})dB_t^H \\ dY_t^{\epsilon,\eta,u} = \frac{1}{\eta} f(Y_t^{\epsilon,\eta,u})dt + \frac{1}{\sqrt{\epsilon\eta}}\tau(Y_t^{\epsilon,\eta,u})du_2(t) + \frac{1}{\sqrt{\eta}}\tau(Y_t^{\epsilon,\eta,u})dW_t \end{cases}$$

- Finding $\lim_{\epsilon\to 0}$ (1) $\Longrightarrow$ LP

Large Deviations: Tightness and limiting behavior

$W_0^{\alpha,\infty}([0,T];\mathcal{X})$: Banach space of measurable paths s.t.

$$\|X\|_{\alpha,\infty} := \sup_{t\in[0,T]}\left[|X_t| + \int_0^t \frac{|X_t - X_r|}{(t-r)^{\alpha+1}}dr\right] < \infty.$$

Proposition (Gailus, G.)

Let $T>0$ and assume that $\sqrt{\epsilon}/\sqrt{\eta}\to\infty$. If one of the following holds:
(H1) $\sigma=\sigma(x,y), H\in(\frac{3}{4},1)$ and $\exists\beta\in(2(1-H),\frac{1}{2})$ s.t. $\sqrt{\epsilon}/\eta^{\beta}\to 0$
(H2) $\sigma=\sigma(x), H\in(\frac{1}{2},1)$,
then $\exists\ \alpha\in(1-H,\frac{1}{4})$ (resp. $\alpha\in(1-H,\frac{1}{2})$), $\epsilon_0>0$ and $C=C_{x_0,y_0,T}>0$ s.t. $\sup_{\epsilon<\epsilon_0,u\in\mathcal{A}_b}\mathbb{E}\left\|X^{\epsilon,\eta,u}\right\|_{\alpha,\infty}\leq C$. In particular, since $W_0^{\alpha,\infty}\subset C^{\alpha-}$, the family $\{X^{\epsilon,\eta,u};\epsilon<\epsilon_0,u\in\mathcal{A}_b\}$ of $C([0,T];\mathcal{X})$ random elements is tight.

- Intuition for (H_1): Fast scale feeds back into $\sigma(X^{\epsilon,\eta,u},Y^{\epsilon,\eta,u})dB^H$ at a rate of $O(\eta^{-\frac{\alpha}{\theta}})$, θ : Hölder exponent of $Y^{\epsilon,\eta,u}$. As $H\uparrow$ we can integrate "rougher" integrands wrt B^H i.e. $\alpha\downarrow$ and $\eta^{-\frac{\alpha}{\theta}}\downarrow$.
- Scaling $\sqrt{\eta}\lesssim\sqrt{\epsilon}\lesssim\eta^{\beta}$ allows to eliminate the feedback and simultaneously preserve the ergodic properties of fast dynamics (u_2 negligible as $\epsilon\to 0$).

Large Deviations: Tightness and limiting behavior

- Define a family of random occupation measures $\{P^\epsilon\,; \epsilon > 0\}$ on $\mathscr{B}([0,T] \times \mathcal{U}_1 \times \mathcal{U}_2 \times \mathcal{Y})$ by

$$P^\epsilon(A_1 \times A_2 \times A_3 \times A_4) := \int_{A_1} \mathbb{1}_{A_2}\big(\underbrace{K_H^{-1} u_1(s)}_{\in L^2}\big)\mathbb{1}_{A_3}\big(\underbrace{\dot{u}_2(s)}_{\in L^2}\big)\mathbb{1}_{A_4}\big(Y_s^{\epsilon,\eta,u}\big)ds,$$

- Family $\{(X^{\epsilon,\eta,u}, P^\epsilon); \epsilon < \epsilon_0, u \in \mathcal{A}_b\}$ is tight. Hence, up to subsequences, $(X^{\epsilon,\eta,u}, P^\epsilon) \to (X^0, P)$ in distribution in $C([0,T];\mathcal{X})$ $\times \mathscr{P}([0,T] \times \mathcal{U}_1 \times \mathcal{U}_2 \times \mathcal{Y})$, endowed with the product of the uniform and weak convergence topologies.

- How can we characterize the limiting pairs (X^0, P)?

Large Deviations: Tightness and limiting behavior

Definition (Viable pairs)

Fix $x_0 \in \mathcal{X}$. A pair (ψ, P)will be called viable (notation: $\mathscr{V}_{x_0}$) if (i) P has finite second moments (ii) $\forall h \in C_b([0,T] \times \mathcal{U}_1 \times \mathcal{U}_2 \times \mathcal{Y})$:

$$\int_{[0,T]\times\mathcal{U}_1\times\mathcal{U}_2\times\mathcal{Y}} h dP = \int_0^T \int_{\mathcal{Y}} \int_{\mathcal{U}_1\times\mathcal{U}_2} h(s,u,v,y) d\Theta(u,v|y,s) d\mu(y) ds,$$

where $\Theta(\cdot|\cdot)$ is a stochastic kernel on μ is the unique invariant measure of Y.

$$(iii) \quad \psi(t) = x_0 + \int_{[0,t]\times\mathcal{U}_1\times\mathcal{U}_2\times\mathcal{Y}} \nabla\Psi(y_2)\tau(y_2) u_2 dP(s,u_1,u_2,y_2)$$

$$+ \frac{c_H}{\Gamma(H-1/2)} \iint_{([0,t]\times\mathcal{U}_1\times\mathcal{U}_2\times\mathcal{Y})^2} \mathbb{1}_{[0,s]}(r) s^{H-\frac{1}{2}} r^{\frac{1}{2}-H} (s-r)^{H-\frac{3}{2}} \sigma(\psi_s, y_2) v_1 dP^{\otimes 2}(r,v_1,v_2,y_1,s,u_1,u_2,y_2).$$

Theorem (Gailus, G.)

Every limit point of the family $\{(X^{\epsilon,\eta,u}, P^{\epsilon})\}_{\epsilon,u}$ is a viable pair with probability 1.

Laplace Pinciple Upper Bound

An application of Fatou's lemma yields the LP upper bound:

$$\limsup_{\epsilon\to 0} \epsilon \log \mathbb{E}\left[e^{-h(X^{\epsilon,\eta})/\epsilon}\right] \leq - \inf_{\phi\in C([0,T];\mathcal{X})} \left[S_{x_0}(\phi) + h(\phi)\right]$$

with a variational-form rate function:

$$S_{x_0}(\phi) = \inf_{P:(\phi,P)\in\mathscr{V}_{x_0}} \frac{1}{2}\int_{[0,T]\times\mathcal{U}_1\times\mathcal{U}_2\times\mathcal{Y}} \left[|u_1|^2 + |u_2|^2\right] dP(t,u_1,u_2,y)$$

(assuming that $\inf\varnothing = \infty$). Rate function in ordinary control formulation: Let $u_i(t,y) = \int z_i d\Theta(z_1,z_2|y,t)$,

$$\mathscr{A}^o_{\phi,x_0} := \Bigg\{ (u_1,u_2) : \int_0^T \int_{\mathcal{Y}} \left[|u_1(t,y)|^2 + |u_2(t,y)|^2\right] d\mu(y)dt < \infty,$$

$$\phi(t) = x_0 + \int_0^t \left[\bar{c}(\phi(s)) + \bar{\sigma}(\phi(s))\dot{K}_H\bar{u}_1(s) + \overline{\nabla\Psi\tau u_2}(s)\right] ds \; t\in[0,T] \Bigg\},$$

where $\dot{K}_H = \frac{d}{dt}\circ K_H$. Then

$$S_{x_0}(\phi) = \inf_{(u_1,u_2)\in\mathscr{A}^o_{\phi,x_0}} \frac{1}{2}\int_0^T\int_{\mathcal{Y}} \left[|u_1(t,y)|^2 + |u_2(t,y)|^2\right] d\mu(y)dt.$$

Laplace Pinciple Lower Bound

- For each ϕ, the state-dependent coefficient $\bar{\sigma}$ induces a bounded multiplication operator $\bar{\Sigma}(\phi) : L^2[0,T] \to L^2[0,T] : \Sigma(\phi)[v](t) := \bar{\sigma}(\phi_t)v(t)$
- The operator $\dot{K}_H L^2[0,T] \to L^2[0,T]$ is bounded (studied by Pipiras-Taqqu [PT01]).
- Letting $Q_H(\phi) : L^2[0,T] \oplus L^2([0,T]\times\mathcal{Y}, dt\otimes d\mu) \to L^2[0,T]$

$$Q_H(\phi)[\bar{u}_1, u_2](t) := \bar{\Sigma}(\phi)\dot{K}_H[\bar{u}_1](t) + \overline{\nabla\Psi\tau u_2}(t)$$

we have

$$S_{x_0}(\phi) \geq \frac{1}{2} \inf_{(\bar{u}_1,u_2)\in Q_H^{-1}(\phi)[\dot{\phi}-\bar{c}(\phi)]} \|(\bar{u}_1, u_2)\|^2_{L^2[0,T]\oplus L^2([0,T]\times\mathcal{Y},dt\otimes d\mu)}$$

- Optimization problem on RHS explicitly solvable as long as $Q_H(\phi)Q_H^*(\phi) : L^2[0,T] \to L^2[0,T]$ is boundedly invertible (not true if $\nabla\Psi = 0$). Optimal controls available in closed form and used to prove the LP lower bound:

$$\limsup_{\epsilon\to 0} -\epsilon \log \mathbb{E}\left[e^{-h(X^{\epsilon,\eta})/\epsilon}\right] \leq \inf_{\phi\in C([0,T];\mathcal{X})} \left[S_{x_0}(\phi) + h(\phi)\right].$$

Large Deviation Principle

Theorem (Gailus, G.)

Let $T > 0, (x_0, y_0) \in \mathbb{R}^d$. *Assume that* $\sqrt{\eta}/\sqrt{\epsilon} \to 0$ *and one of the following holds:*

(H1) $\sigma = \sigma(x, y), H \in (\frac{3}{4}, 1)$ *and* $\exists \beta \in (1 - H, \frac{1}{4})$ *s.t.* $\sqrt{\epsilon}/\eta^{\beta} \to 0$

(H2) $\sigma = \sigma(x), H \in (\frac{1}{2}, 1)$.

Moreover, assume that

$$\bar{Q}_H(\phi) := [\bar{\Sigma}(\phi)\dot{K}_H][\bar{\Sigma}(\phi)\dot{K}_H]^* + \overline{[\nabla \Psi_T][\nabla \Psi_T]^T} \in \mathscr{L}(L^2[0, T])$$

has a bounded inverse. Then the family $\{X^{\epsilon,\eta}; \epsilon > 0\}$ *satisfies an LDP with good rate function* $S_{x_0} : C([0, T]; \mathcal{X}) \to [0, \infty]$ *given by*

$$S_{x_0}(\phi) = \inf_{P:(\phi,P)\in \mathscr{V}_{x_0}} \frac{1}{2} \int_{[0,T]\times \mathcal{U}_1 \times \mathcal{U}_2 \times \mathcal{Y}} \left[|u_1|^2 + |u_2|^2\right] dP(t, u_1, u_2, y)$$
$$= \frac{1}{2} \int_0^T \langle \dot{\phi}(t) - \bar{c}(\phi(t)), \bar{Q}_H^{-1}(\phi)[\dot{\phi} - \bar{c}(\phi)](t) \rangle dt$$

if $\dot{\phi} - \bar{c}(\phi) \in L^2[0, T], \phi(0) = x_0$ *and* $S_{x_0} = \infty$ *otherwise.*

LDP: Remarks on conditions and rate function

- On the invertibility of $\bar{Q}_H(\phi) := [\bar{\Sigma}(\phi)\dot{K}_H][\bar{\Sigma}(\phi)\dot{K}_H]^* + \overline{[\nabla\Psi\tau][\nabla\Psi\tau]^T}$
Example: $\mathcal{X} = \mathcal{Y} = \mathbb{R}, f(y) = -ay, \tau(y) = \sqrt{2a}, a > 0$ (fast process is an OU) and $b(y) = \lambda y, \lambda \neq 0$. Then $\mu \sim \mathcal{N}(0,1)$ and $a\Psi''(y) - ay\Psi'(y) = -\lambda y$ is explicitly solved by $\Psi(y) = \lambda y/a$. Thus $\Psi'(y) = \frac{\lambda}{a}$, $(\Psi'\tau)(\Psi'\tau)^T = \frac{2\lambda^2}{a}$,

$$\|\bar{Q}_H(\phi)h\|^2_{L^2} = \int_0^T \left|\dot{K}_H^*\bar{\Sigma}^*(\phi)[h](t)\right|^2 dt + \int_0^T \int_{\mathcal{Y}} \left|(\nabla\Psi\tau)^T h(t)\right|^2 d\mu(y)dt$$
$$\geq \int_{\mathcal{Y}} (\Psi'\tau)^2 d\mu(y) \int_0^T h^2(t)dt = \frac{2\lambda^2}{a}\|h\|^2_{L^2} \text{ so condition holds.}$$

- What if $\nabla\Psi = 0$? Then we have a variational rate function:

$$S_{x_0}(\phi) = \inf_{\{u_1 \in \mathcal{H}_H : \dot{\phi} - \bar{c}(\phi) = \bar{\sigma}(\phi)\dot{u}_1, \phi(0) = x_0\}} \frac{1}{2}\|u_1\|^2_{\mathcal{H}_H}.$$

LDP: Remarks on conditions and rate function

- In the case $\nabla\Psi = 0$ we can obtain an explicit rate function by assuming e.g. that $\bar{\sigma}(x)$ is invertible and $\sup_x |\bar{\sigma}^{-1}(x)| \leq C$. Then:

$$S_{x_0}(\phi) = \frac{1}{c_H\Gamma(\frac{3}{2}-H)}\int_0^T \Bigg| t^{1/2-H}\bar{\sigma}(\phi(t))^{-1}\big(\dot{\phi}(t)-\bar{c}(\phi(t))\big) + \left(H-\frac{1}{2}\right)\times$$
$$t^{2H-1}\int_0^t \frac{t^{\frac{1}{2}-H}\bar{\sigma}(\phi(t))^{-1}\big(\dot{\phi}(t)-\bar{c}(\phi(t))\big) - s^{\frac{1}{2}-H}\bar{\sigma}(\phi(s))^{-1}\big(\dot{\phi}(s)-\bar{c}(\phi(s))\big)}{(t-s)^{H+\frac{1}{2}}}ds\Bigg|^2 dt$$

for $\phi \in C[0,T]$ s.t. $\phi(0) = x_0$ and $\dot{\phi} - \bar{c}(\phi) \in K_H(L^2[0,T])$ and $S_{x_0} = \infty$ otherwise (defined for smoother paths than $H = 1/2$.)

Thank you!

Large Deviations: The weak convergence method

Using a variational representation on abstract Wiener spaces (X.Zhang [Zha09], see also Budhiraja-Song [BS20]) we express the Laplace functional as follows:

$$-\epsilon \log \mathbb{E} e^{-h(X^{\epsilon,\eta})/\epsilon} = \inf_{u=(u_1,u_2)\in\mathcal{A}_b} \mathbb{E}\left\{ h\left(X^{\epsilon,\eta,u}\right) + \frac{1}{2}\left(\|u_1\|^2_{\mathcal{H}_H} + \|u_2\|^2_{\mathcal{H}_{1/2}} \right) \right\}, \tag{2}$$

$\mathcal{H}_H \oplus \mathcal{H}_{1/2}$: Cameron-Martin space of mixed noise, $\mathcal{H}_H := K_H[L^2([0,T])]$,

$$K_H(f)(t) = c_H I^1_{0^+}\left(\psi \cdot I^{H-\frac{1}{2}}_{0^+}(\psi^{-1} \cdot f)\right)(t), \quad \text{"}(H+1/2)\text{−integral"}$$

$\psi(t) = t^{H-\frac{1}{2}}, c_H^2 = \frac{2H\Gamma(\frac{3}{2}-H)\Gamma(H+\frac{1}{2})}{\Gamma(2-2H)}$, $I^r_{0^+}f(t) = \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1} f(s)ds$, inner product $\langle f, g\rangle_{\mathcal{H}_H} := \langle K_H^{-1} f, K_H^{-1} g\rangle_{L^2}$, $\mathcal{A}_b$: a.s. bounded stochastic controls, adapted to common filtration, $X^{\epsilon,\eta,u}$ corresponds to the controlled system:

$$\begin{cases} dX_t^{\epsilon,\eta,u} = \frac{\sqrt{\epsilon}}{\sqrt{\eta}} b(Y_t^{\epsilon,\eta,u})dt + c(X_t^{\epsilon,\eta,u}, Y_t^{\epsilon,\eta,u})dt + \sigma(X_t^{\epsilon,\eta,u}, Y_t^{\epsilon,\eta,u})du_1(t) \\ \qquad + \sqrt{\epsilon}\sigma(X_t^{\epsilon,\eta,u}, Y_t^{\epsilon,\eta,u})dB_t^H \\ dY_t^{\epsilon,\eta,u} = \frac{1}{\eta} f(Y_t^{\epsilon,\eta,u})dt + \frac{1}{\sqrt{\epsilon\eta}}\tau(Y_t^{\epsilon,\eta,u})du_2(t) + \frac{1}{\sqrt{\eta}}\tau(Y_t^{\epsilon,\eta,u})dW_t \end{cases}$$

- Finding $\lim_{\epsilon\to 0}$ (2) $\Longrightarrow$ LP

Stochastic integral, function spaces and solutions

$$\begin{cases} dX_t^{\epsilon,\eta,u} = \frac{\sqrt{\epsilon}}{\sqrt{\eta}} b(Y_t^{\epsilon,\eta,u})dt + c(X_t^{\epsilon,\eta,u}, Y_t^{\epsilon,\eta,u})dt + \sigma(X_t^{\epsilon,\eta,u}, Y_t^{\epsilon,\eta,u})\dot{u}_1(t)dt \\ \qquad + \sqrt{\epsilon}\sigma(X_t^{\epsilon,\eta,u}, Y_t^{\epsilon,\eta,u})dB_t^H \\ dY_t^{\epsilon,\eta,u} = \frac{1}{\eta} f(Y_t^{\epsilon,\eta,u})dt + \frac{1}{\sqrt{\epsilon\eta}}\tau(Y_t^{\epsilon,\eta,u})\dot{u}_2(t)dt + \frac{1}{\sqrt{\eta}}\tau(Y_t^{\epsilon,\eta,u})dW_t \end{cases}$$

- Interpretation of solutions: fix ϵ, η, a realization of the mixed noise, deterministic initial conditions and a Hölder continuous version of the non-feedback fast process $Y^{\epsilon,\eta,u}(\omega)$.
- Assume b, c globally Lipschitz, linear growth in x, y, σ bounded, globally Lipschitz with a locally θ-Hölder derivative in x, $\theta > \frac{1}{H} - 1$. Let $\alpha \in (1 - H, \frac{1}{2} \wedge \frac{\theta}{1+\theta})$ and $W_+^{\alpha,\infty}([0, T]; \mathcal{X})$ be the Banach space of measurable paths such that

$$\|X\|_{\alpha,\infty} := \sup_{t\in[0,T]} \left[|X_t| + \int_0^t \frac{|X_t - X_r|}{(t-r)^{\alpha+1}} dr \right] < \infty.$$

- The Young SDE for $X^{\epsilon,\eta,u}$ has a unique pathwise solution $X^{\epsilon,\eta,u} \in L^0(\Omega; W_+^{\alpha,\infty}([0, T]; \mathcal{X}))$ and $X^{\epsilon,\eta,u}(\omega) \in C^{1-\alpha}([0, T]; \mathcal{X})$.

Large Deviations: Tightness and limiting behavior

- Define a family of random occupation measures $\{P^\epsilon\,;\epsilon > 0\}$ on $\mathscr{B}([0,T]\times\mathcal{U}_1\times\mathcal{U}_2\times\mathcal{Y})$ by

$$P^\epsilon(A_1\times A_2\times A_3\times A_4) := \int_{A_1} \mathbb{1}_{A_2}\Big(\underbrace{K_H^{-1}u_1(s)}_{\in L^2}\Big)\mathbb{1}_{A_3}\Big(\underbrace{\dot{u}_2(s)}_{\in L^2}\Big)\mathbb{1}_{A_4}\Big(Y_s^{\epsilon,\eta,u}\Big)ds,$$

- Family $\{(X^{\epsilon,\eta,u},P^\epsilon);\epsilon<\epsilon_0, u\in\mathcal{A}_b\}$ is tight. Hence, up to subsequences, $(X^{\epsilon,\eta,u},P^\epsilon)\to(X^0,P)$ in distribution in $C([0,T];\mathcal{X})$ $\times\mathscr{P}([0,T]\times\mathcal{U}_1\times\mathcal{U}_2\times\mathcal{Y})$, endowed with the product of the uniform and weak convergence topologies.
- How to characterize (X^0,P)? Key step:

$$\int_0^t \sigma(X_s^{\epsilon,\eta,u},Y_s^{\epsilon,\eta,u})\dot{u}_1(s)ds = \int_0^t \sigma(X_s^{\epsilon,\eta,u},Y_s^{\epsilon,\eta,u})\frac{d}{dt}K_H(K_H^{-1}u_1)(s)ds$$

$$= \frac{c_H}{\Gamma(H-\frac{1}{2})}\int_0^t\int_0^s s^{H-\frac{1}{2}}r^{\frac{1}{2}-H}(s-r)^{H-\frac{3}{2}}\sigma(X_s^{\epsilon,\eta,u},Y_s^{\epsilon,\eta,u})K_H^{-1}u_1(r)drds =$$

$$\frac{c_H}{\Gamma(H-\frac{1}{2})}\int_{[0,t]\times\mathcal{Y}}\int_{\times[0,s]\times\mathcal{U}_1} s^{H-\frac{1}{2}}r^{\frac{1}{2}-H}(s-r)^{H-\frac{3}{2}}\sigma(X_s^{\epsilon,\eta,u},y_2)v_1\,dP^\epsilon(r,v_1)dP^\epsilon(s,y_2)$$

On the proof of tightness

Let

$$\Delta_\alpha \left| Y^{\epsilon,\eta,u}_{t_1,t_2} \right| := \int_{t_1}^{t_2} \frac{\left| Y^{\epsilon,\eta,u}_{t_2} - Y^{\epsilon,\eta,u}_{s} \right|}{(t_2 - s)^{\alpha+1}} ds$$

Lemma

For $T > 0$ the following hold:
(i) Let $\alpha < \frac{1}{2}$ and $\theta \in (\alpha, 1/2)$. There exists $C > 0$ and $\epsilon_0 > 0$ such that

$$\sup_{\epsilon<\epsilon_0, u\in\mathcal{A}_b} \mathbb{E} \sup_{t_1 \neq t_2} \frac{1}{(t_2 - t_1)^{\frac{1}{2}-\alpha}} \int_{t_1}^{t_2} \Delta_\alpha \left| Y^{\epsilon,\eta,u}_{t_1,s} \right| ds \leq C\eta^{-\frac{\alpha}{\theta}}. \quad (3)$$

(ii) Let $H \in (3/4, 1)$ and $\alpha \in (1 - H, 1/4)$. Furthermore, assume that there exists $\beta \in (2(1 - H), 1/2)$ such that $\sqrt{\epsilon}/\eta^{\beta} \to 0$ as $\epsilon \to 0$. Then

$$\sqrt{\epsilon}\, \mathbb{E} \sup_{t_2\in[0,T]} \int_0^{t_2} (t_2 - t_1)^{-\alpha-1} \int_{t_1}^{t_2} \Delta_\alpha |Y^{\epsilon,\eta,u}_{t_1,s}| ds dt_1 \leq C \frac{\sqrt{\epsilon}}{\eta^{\beta}} \longrightarrow 0\,,\ \epsilon \to 0. \quad (4)$$

Proof of LP upper bound

$$
\begin{aligned}
&\limsup_{\epsilon\to 0} \epsilon \log \mathbb{E}\left[e^{-h(X^\epsilon)/\epsilon}\right] \\
&\le \limsup_{\epsilon\to 0} -\left(\mathbb{E}\left[\frac{1}{2}\int_0^T \left|K_H^{-1}u_1^\epsilon(t)\right|^2 dt + \frac{1}{2}\int_0^T \left|\dot{u}_2^\epsilon(t)\right|^2 dt + h\left(X^{\epsilon,\eta,u^\epsilon}\right)\right] - \epsilon\right) \\
&= -\liminf_{\epsilon\to 0} \mathbb{E}\left[\frac{1}{2}\int_{[0,T]\times\mathcal{U}_1\times\mathcal{U}_2\times\mathcal{Y}} \left[|u_1|^2 + |u_2|^2\right] dP^\epsilon(t,u_1,u_2,y) + h\left(X^{\epsilon,\eta,u^\epsilon}\right)\right] \\
&\le -\mathbb{E}\left[\frac{1}{2}\int_{[0,T]\times\mathcal{U}_1\times\mathcal{U}_2\times\mathcal{Y}} \left[|u_1|^2 + |u_2|^2\right] dP(t,u_1,u_2,y) + h(\phi)\right] \\
&\le -\mathbb{E}\left[\inf_{P\in\mathscr{V}_{\Lambda,x_0,\phi}} \frac{1}{2}\int_{[0,T]\times\mathcal{U}_1\times\mathcal{U}_2\times\mathcal{Y}} \left[|u_1|^2 + |u_2|^2\right] dP(t,u_1,u_2,y) + h(\phi)\right] \\
&\le -\inf_{\phi\in C([0,T];\mathcal{X})}\left[\inf_{P\in\mathscr{V}_{\Lambda,x_0,\phi}} \frac{1}{2}\int_{[0,T]\times\mathcal{U}_1\times\mathcal{U}_2\times\mathcal{Y}} \left[|u_1|^2 + |u_2|^2\right] dP(t,u_1,u_2,y) + h(\phi)\right] \\
&= -\inf_{\phi\in C([0,T];\mathcal{X})}\left[S_{x_0}(\phi) + h(\phi)\right].
\end{aligned}
$$

Proof of LP lower bound

For any $\delta > 0$ and approximate minimizer sequence $(u^*_{1,\epsilon}, u^*_{2,\epsilon})$ in feedback form we have

$$\limsup_{\epsilon\to 0} -\epsilon \log \mathbb{E}\left[e^{-h(X^\epsilon)/\epsilon}\right]$$

$$\leq \limsup_{\epsilon\to 0}\left(\mathbb{E}\left[\frac{1}{2}\int_0^T \left|u^*_{1,\epsilon}(t, Y^\eta_t)\right|^2 dt + \frac{1}{2}\int_0^T \left|u^*_{2,\epsilon}(t, Y^\eta_t)\right|^2 dt + h\left(X^{\epsilon,\eta,u^*_\epsilon}\right)\right]\right)$$

$$= \frac{1}{2}\left(\int_0^T \int_{\mathcal{Y}} \left[|u^*_1(t,y)|^2 + |u^*_2(t,y)|^2\right] dt d\mu(y)\right) + h(\phi)$$

$$= \inf_{(u_1,u_2)\in\mathscr{A}^o_{x_0,\phi}} \frac{1}{2}\left(\int_0^T \int_{\mathcal{Y}} \left[|u_1(t,y)|^2 + |u_2(t,y)|^2\right] dt d\mu(y)\right) + h(\phi)$$

$$= S_{x_0}(\phi) + h(\phi) < \inf_{\psi\in C([0,T];\mathcal{X})}\left[S_{x_0}(\psi) + h(\psi) + \delta\right] + \delta.$$

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