Mixing and stabilization in the abelian sandpile model

Robert Hough

SUNY Stony Brook

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Outline

1 Overview

2 Abelian sandpiles

Random walk on a group

Figure: Persi Diaconis

Random walk on a group

- G a locally compact (finite) group
- $\mathscr{P}(G)$ the set of Borel probability measures on G
- For $\mu, \nu \in \mathscr{P}(G)$, $f \in C_c(G)$,

$$\langle f, \mu * \nu \rangle = \int_G \int_G f(xy) d\mu(x) d\nu(y)$$

- Consider, for $\mu \in \mathscr{P}(G)$, the large N behavior of μ^{*N} as a weak-* limit in one of several function spaces, e.g. $L^\infty(G)$, Lipschitz functions, Sobolev spaces, etc., and also the growth of $\mathrm{supp}(\mu^{*N})$
- We seek quantitative statements, e.g. a rate of convergence.

Example: riffle shuffling

Let $N > 1$ and consider the following random walk on the symmetric group $\mathfrak{S}_N$ (Gilbert-Shannon-Reeds)

- μ is the distribution on $\mathfrak{S}_N$ given by
  - Choose $1 \leq n \leq N$ according to the binomial distribution $\mathrm{P}(n) = \frac{\binom{N}{n}}{2^N}$
  - Conditioned on the value of n, the measure is uniform over all permutations which preserve the relative order of the first n and last $N - n$ cards
- Convergence to uniform is observed after $\frac{3}{2} \log_2 N + O(1)$ steps in the total variation (L^1) metric [1], [2].

Outline

1. Overview
2. Abelian sandpiles

Sandpiles on the square lattice

(a) Daniel Jerison

(b) Lionel Levine

The abelian sandpile model

- The abelian sandpile model (ASM) is a statistical physics model proposed by Bak, Tang, and Wiesenfeld in the 80s.
- The model exhibits self-organized criticality, and is a model for phenomena like earthquakes
- Given a graph $G = (V, E)$ a non-negative number of chips, or sand grains, is allocated to each node. If a node has at least as many chips as its degree, it is unstable and can *topple* passing one chip to each neighboring node
- If the graph has sink vertices, chips landing on these vertices disappear from the model.
- Well studied, more than 10000 hits on Google Scholar.

Sandpiles on the square lattice

- A *sandpile* on the square lattice $\mathbb{Z}^2$ is a sand allocation

$$\sigma : \mathbb{Z}^2 \to \mathbb{Z}_{\geq 0}.$$

- If $\sigma(x) \geq 4$ the pile at x can *topple*, passing one grain of sand to each of its neighbors. This toppling procedure is known to be abelian.
- An allocation is *stable* if $\sigma \leq 3$, and *unstable* otherwise.

Sandpiles on the square lattice

- We consider parallel toppling dynamics in which time progresses in discrete steps, and at each time step, every site that can topple topples once.
- A configuration σ is said to be *stabilizable* if under parallel toppling, each vertex topples finitely many times. The stabilized sandpile is written σ^∞.

Sandpiles on the square lattice

Theorem (H., Jerison, Levine, '17)

Let $(\sigma_x)_{x\in\mathbb{Z}^2}$, distributed i.i.d. on $\mathbb{Z}_{\geq 0}$, be a sandpile configuration. If $(\sigma_x)_{x\in\mathbb{Z}^2}$ is stabilizable almost surely then

$$\mathrm{E}[\sigma_0] \leq 3 - \epsilon$$

$$\epsilon \gg \min\left(1, \int\int |z_1 - z_2|^{\frac{2}{3}} d\sigma_0(z_1) d\sigma_0(z_2)\right).$$

Graph Laplacian and Green's function

- Let Δ denote the graph Laplacian on $\mathbb{Z}^2$,

$$\Delta f(x) = 4f(x) - \sum_{y:\|y-x\|_1=1} f(y)$$

- Denote $G(x)$ the Green's function, which satisfies $\Delta G = \delta_0$.

A function harmonic modulo 1

- Write $D_1 f(x) = f(x + (1,0)) - f(x)$.
- The argument uses that $\xi = D_1^3 G$ satisfies
  1. $\xi \in L^1(\mathbb{Z}^2)$
  2. ξ is 'harmonic modulo 1', that is, $\Delta\xi \equiv 0 \bmod 1$
  3. For $A > 1$,
  $$\sum_{x : |\xi_x| < \frac{1}{A}} |\xi_x|^2 \asymp A^{-\frac{4}{3}}.$$

Several lemmas

Lemma (Fey, Meester, Redig)

Let $(\sigma_x)_{x\in\mathbb{Z}^2} \in \mathbb{Z}_{\geq 0}^{\mathbb{Z}^2}$ *be an i.i.d. sandpile which stabilizes a.s.. Then*

$$\mathrm{E}[\sigma_0] = \mathrm{E}[\sigma_0^\infty].$$

Lemma

Let $(\sigma_x)_{x\in\mathbb{Z}^2} \in \mathbb{Z}_{\geq 0}^{\mathbb{Z}^2}$ *be an i.i.d. sandpile which stabilizes a.s.. Then*

$$\langle \xi, \sigma \rangle \equiv \langle \xi, \sigma^\infty \rangle \bmod 1, \qquad a.s..$$

Proof.

- Let $u^n(x)$ denote the number of times that site x topples in the first n rounds of toppling, and let $\sigma^n = \sigma - \Delta u^n$ be the sandpile after n topplings.
- We have $u^n \leq n$ and $\sigma^n(x) \leq \max(\sigma^{n-1}(x), 7)$.
- $\langle \xi, \sigma \rangle$ converges absolutely a.s. by the weak law of large numbers.
- For each n, a.s. $\langle \xi, \sigma \rangle - \langle \xi, \sigma^n \rangle = -\langle \xi, \Delta u^n \rangle = -\langle \Delta \xi, u^n \rangle \in \mathbb{Z}$.
- Thus a.s. $\langle \xi, \sigma \rangle - \langle \xi, \sigma^\infty \rangle \in \mathbb{Z}$, by dominated convergence.

□

Proof sketch of theorem

- The previous lemma implies that the equality

$$\chi(\xi,\sigma) = \mathsf{E}\left[e^{-2\pi i\langle \xi,\sigma\rangle}\right] = \mathsf{E}\left[e^{-2\pi i\langle \xi,\sigma^\infty\rangle}\right] = \chi(\xi,\sigma^\infty).$$

- By independence,

$$|\mathrm{LHS}| = \prod_{x\in\mathbb{Z}^2}\left|\mathsf{E}\left[e^{-2\pi i \xi_x \sigma_0}\right]\right|$$

while $\sigma^\infty \leq 3$ implies

$$|\,\mathsf{RHS}\,| = 1 - O\left(3 - \mathsf{E}[\sigma_0]\right).$$

- The theorem follows on comparing the LHS and RHS, we omit the details.

Torus sandpiles

Consider sandpile dynamics on the torus $\mathbb{T}_m = (\mathbb{Z}/m\mathbb{Z})^2$, given as follows.

- The point $(0, 0)$ is designated 'sink' and is special, in that any grain of sand which falls on the sink is lost from the model.
- Each non-sink point on the torus has a sand allocation indicated by

$$\sigma : \mathbb{T}_m \setminus \{(0, 0)\} \to \mathbb{Z}_{\geq 0}.$$

- A move in the model consists of dropping a grain of sand on a uniformly chosen vertex v, then performing topplings until the configuration is stable.

Torus sandpiles

- Those states $\mathscr{S}_m$ for which $\sigma \leq 3$ are *stable*, while those states $\mathscr{R}_m$ which may be reached from the maximal state $\sigma \equiv 3$ are recurrent.
- The stationary distribution of the model is the uniform distribution on recurrent states.

Torus sandpiles

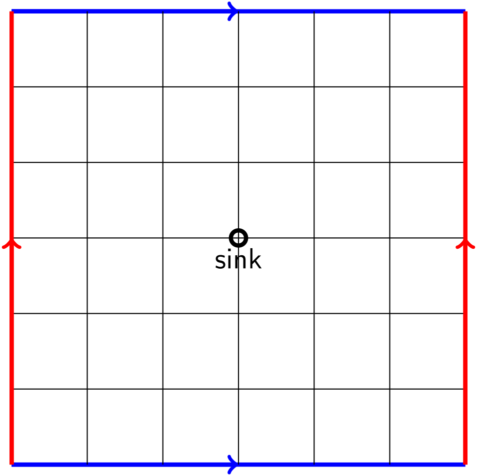

Figure: The square lattice configuration with periodic boundary condition and a single sink.

Torus sandpiles

Theorem (H., Jerison, Levine, 2017)

There is a constant $c_0 > 0$ *and* $t_m^{\text{mix}} = c_0 m^2 \log m$ *such that the following holds. For each fixed* $\epsilon > 0$,

$$\lim_{m\to\infty} \min_{\sigma\in\mathscr{S}_m} \left\| P^{\lceil (1-\epsilon)t^{\text{mix}}\rceil}\delta_\sigma - \mathbb{U}_{\mathscr{R}} \right\|_{\text{TV}(\mathscr{S}_m)} = 1, \tag{1}$$

$$\lim_{m\to\infty} \max_{\sigma\in\mathscr{S}_m} \left\| P^{\lfloor (1+\epsilon)t^{\text{mix}}\rfloor}\delta_\sigma - \mathbb{U}_{\mathscr{R}} \right\|_{\text{TV}(\mathscr{S}_m)} = 0.$$

We say that sandpile dynamics on the torus exhibits a cut-off phenomenon with mixing time asymptotic to $c_0 m^2 \log m$.

Open boundary condition

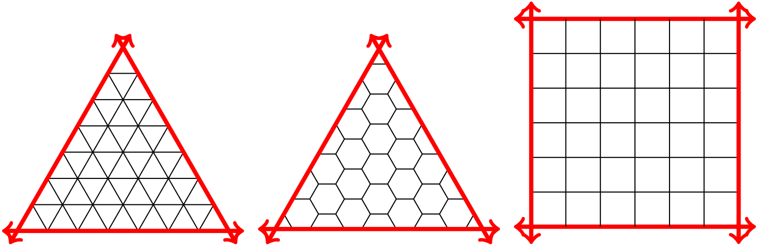

Figure: The triangular, hex and square lattice configurations with open boundary condition.

Open boundary condition

H.-Son study the corresponding mixing behavior on graphs built from periodic tilings in arbitrary dimension, and with periodic or open boundary condition.

Theorem (H.-Son, '21)

The abelian sandpile model on periodic tiling graphs exhibits a cut-off phenomenon for arbitrary tiling geometries and in arbitrary dimension, with either open or periodic boundary condition.

Open boundary condition

Theorem (H.-Son, '21)

If the periodic tiling satisfies a reflection condition, the asymptotic mixing time with torus and open boundary condition are equal.

Open boundary condition

Theorem (H.-Son, '21)

The asymptotic mixing time of the D4 lattice with open boundary is slower than for periodic boundary. The asymptotic mixing times are equal for the cubic lattices $\mathbb{Z}^d$ in all sufficiently large dimensions.

Ideas in the argument

- A simple coupon collector type argument shows that, started from an arbitrary state, a recurrent state is reached in $O(m^2\sqrt{\log m})$ steps with probability $1 + o(1)$.
- We thus reduce to considering the dynamics started from a recurrent state. These have the structure of an abelian group, isomorphic to $\mathscr{G} = \mathbb{Z}^{\mathbb{T}_m \setminus \{(0,0)\}} / \overline{\Delta}\mathbb{Z}^{\mathbb{T}_m \setminus \{(0,0)\}}$, where $\overline{\Delta}$ is the reduced graph Laplacian, found by omitting the row and column corresponding to the sink.
- In this identification, the dynamics are given by convolution with the measure μ which is uniform on the standard basis vectors of $\mathbb{Z}^{\mathbb{T}_m \setminus \{(0,0)\}}$ and 0.

Ideas in the argument

- Denote the dual group $\hat{\mathscr{G}} = \overline{\Delta}^{-1}\mathbb{Z}^{\mathbb{T}_m \setminus \{(0,0)\}} / \mathbb{Z}^{\mathbb{T}_m \setminus \{(0,0)\}}$
- The spectrum is given by the Fourier coefficients

$$\hat{\mu}(\xi) = \mathrm{E}_{x \in \mathbb{T}_m}\left[e^{2\pi i \xi_x}\right] : \qquad \xi \in \hat{\mathscr{G}}.$$

- Identify $\xi \in \hat{\mathscr{G}}$ with prevector $v = \overline{\Delta}\xi \in \mathscr{G}$. Choose representative v with $\|v\|_\infty \leq 3$.

Ideas in the argument

- The Fourier coefficients of high frequencies for which $\|v\|_1 > m^{2-\theta}$ are bounded by using that $\overline{\Delta}$ is bounded $L^2 \to L^2$.
- The remaining frequencies have v which are sparse. An agglomeration scheme is performed to decompose v into clustered components.
- The local nature of $\overline{\Delta}$ is used to show that cancellation in $\hat{\mu}(\xi)$ is essentially additive from separated clusters. This is the most technical part of the argument, since the inverse map, given by convolution with the Green's function, only satisfies a decay condition on derivatives.
- Strong additivity at small frequencies is used to demonstrate the cut-off phenomenon via second moment methods.

Torus sandpiles

We also evaluate the spectral gap as follows.

Theorem (H., Jerison, Levine, 2017)

Let $m \geq 1$. Restricted to recurrent states, the spectral gap of sandpile dynamics on $\mathbb{T}_m$ is given by $\mathrm{gap}_m = \frac{\gamma + o(1)}{m^2}$ where

$$\gamma = \inf\left\{\sum_{x \in \mathbb{Z}^2} (1 - \cos(2\pi \xi_x)) : \xi \in \mathbb{R}^{\mathbb{Z}^2}, \xi \not\equiv 0 \bmod 1, \Delta \xi \equiv 0 \bmod 1\right\}.$$

Torus sandpiles

The value of γ (and also c_0) is determined as follows.

- Let $\xi \in (-1/2, 1/2]^{\mathbb{Z}^2}$ and write $\Delta\xi = v \in \mathbb{Z}^{\mathbb{Z}^2}$.
- Given a subset $S \subset \mathbb{Z}^2$, define $N(S) = \{x \in \mathbb{Z}^2 : d(x, S) \leq 1\}$ it's distance-1 enlargement.
- Define $P(S; v)$ to be the program:

$$\begin{aligned} \text{minimize: } & \sum_{n \in N} (1 - \cos(2\pi x_n)) \\ \text{subject to: } & (x_n)_{n \in N} \subset \left[0, \frac{1}{2}\right)^N, \\ & \forall s \in S,\ 4x_s + \sum_{t: \|t-s\|_1 = 1} x_t \geq |v_s|. \end{aligned}$$

Thus $\sum_{x \in \mathbb{Z}^2} 1 - \cos(2\pi\xi_x) \geq P(S; v)$.

In practice this search is of a reasonable size.

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