

Disordered Monomer-Dimer Model on Cylinder Graphs

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- We will denote the collection of all matchings by \mathcal{M} .

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- Statistics of interest are typical number of edges or typical number of unpaired vertices, denoted U .

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- The recurrence can be used to prove the absence of phase transition, as well as exact computation of the partition function in special cases (line graph, complete graph, regular trees).

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- The partition function and free energy $\log Z_n$ can have non trivial limiting behavior.
- For statistics like U , we have environmental and ensemble contributions to fluctuations, need to distinguish.

What is Already Known?

- Alberici, Contucci and Mignione (2014) analyse the monomer dimer model on the complete graph with i.i.d random vertex weights, and establish an exact solution for the limiting free energy

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- Alberici and Contucci also analyse the monomer dimer model on locally tree like graphs, such as the Erdős-Renyi graph, establish exact solution.

Definition (Cylinder Graph)

Let $H = (V_H, E_H)$ be a fixed graph with $|V_H| = h$ and G_n be the line graph on n vertices with vertex set $[n]$. A cylinder graph \mathcal{G}_n is given by the graph Cartesian product

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- We will work with i.i.d families $\{\nu\}_{v \in V}$ and $\{\omega_e\}_{e \in E}$, though not necessarily with same distribution.
- We will require $\mathbb{E}|\nu|^{2+\epsilon} + \mathbb{E}|\omega|^{2+\epsilon} < \infty$ for some $\epsilon > 0$.

- Let $\mathcal{G}_{[k:l]}$ denote the principal subgraph of \mathcal{G}_n generated by the vertices with G_n components in the interval $[k, l]$.
- Let $Z_{[k:l]}$ denote the partition function of the monomer-dimer model on $\mathcal{G}_{[k:l]}$.
- Let $U_{[k,l]}$ denote the number of unpaired vertices of a matching \mathfrak{m} on \mathcal{G}_n contained in the section $\mathcal{G}_{[k:l]}$.

Main Result (1)

Theorem (Dey, K. 2021)

Assume that $E(|\nu_\nu|^{2+\varepsilon} + |\omega_e|^{2+\varepsilon})$ is finite for some $\varepsilon > 0$. We have $f \in \mathbb{R}$ and $\sigma_F > 0$ depending only on the distributions of ν and ω such that

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and

$$n^{-1/2} \cdot (\log Z_n - \mathbb{E} \log Z_n) \xrightarrow{(d)} \mathbb{N}(0, \sigma_F^2) \text{ as } n \rightarrow \infty.$$

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Theorem (Dey, K. 2021)

Recall the definition of $U_{[k,l]}$. Let $t \in [0, 1]$ and let

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Then as $n \rightarrow \infty$, $\exists u > 0$ and $\sigma > 0$

$$\left(\frac{\theta_n(t) - ntu}{\sqrt{n}} \right)_{t \in [0,1]} \xrightarrow{(d)} (\sigma B_t)_{t \in [0,1]}$$

in probability, in the sense of finite dimensional distributions, B_t is standard Brownian Motion.

Key Ideas for Proof

- This situation is essentially one dimensional, Hielmann and Lieb recursion can be recast into a form allowing application of a subadditive theorem.

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- This situation is essentially one dimensional, Hielmann and Lieb recursion can be recast into a form allowing application of a subadditive theorem.
- The new form of the recursion also enables us to write the free energy and the number of unpaired vertices as a sum of i.i.d random variables, with an error term.
- We can show that the error vanishes in all relevant limits.

Splitting the Graph

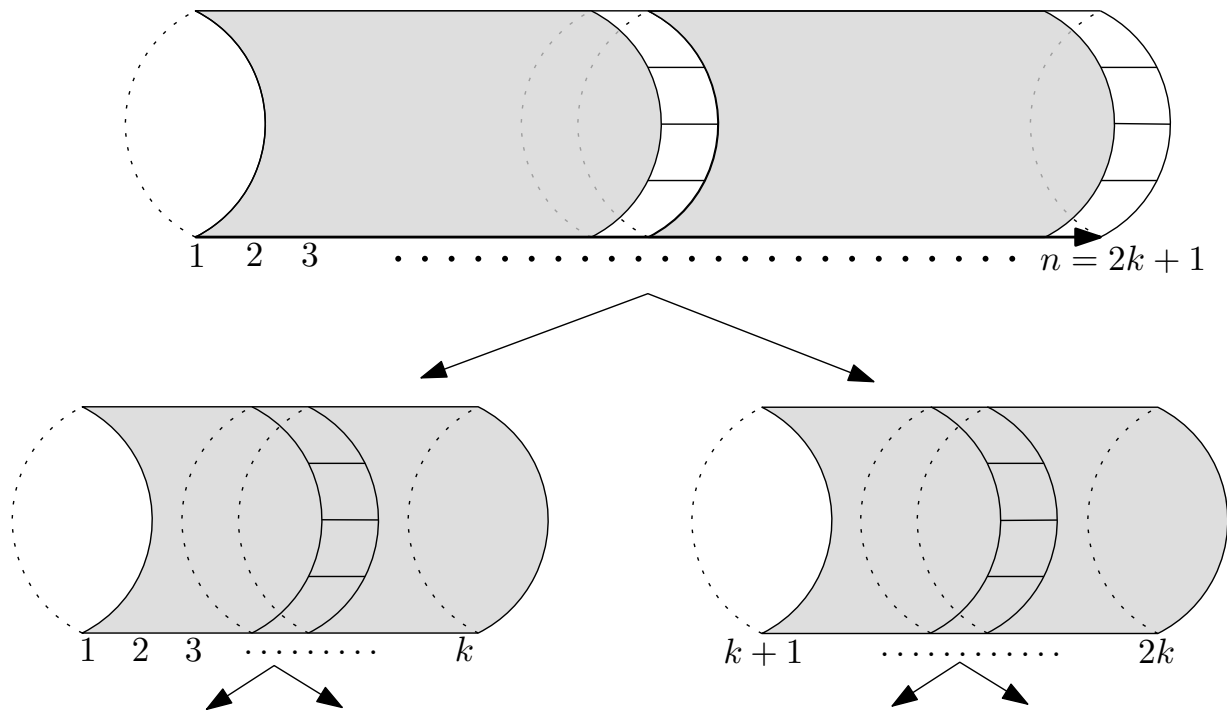


Figure 1: First step of the subdivision

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$$R_{n,k} = \log \left(\sum_{A \subseteq \mathcal{E}} \prod_{i: e_{k,i} \in A} e^{\omega_{k,i} - \nu_{k,i} - \nu_{k+1,i}} \cdot \frac{Z_{[1:k]}^A}{Z_{[1:k]}} \cdot \frac{Z_{[k+1:n]}^A}{Z_{[k+1:n]}} \right)$$

Error Control for Free Energy

- With moment condition as assumed, there is a constant depending only on distributions of ω and ν such that

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Theorem (Hammersley 1962)

Let a_n and b_n be sequences such that $a_{n+m} \leq a_n + a_m + b_{n+m}$. A sufficient condition for a_n/n to converge to limit $\ell < \infty$ is

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- Directly applicable to $\mathbb{E} \log Z_n$ and $\text{Var} \log Z_n$.
- Central Limit Theorem follows from the Lyapunov condition.

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- For the same error decomposition as the free energy, we need to control $\partial_x R_{n,k}$
- In particular, need to control $\partial_x Z_{(\cdot)}^A / Z_{(\cdot)}$, which is equivalent to bounding $\partial_x \log Z_{(\cdot)}^A - \partial_x \log Z_{(\cdot)}$.

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$$\lambda_i \leq \lambda_i^v \leq \lambda_{i+1}.$$

Interlacing Hierarchy

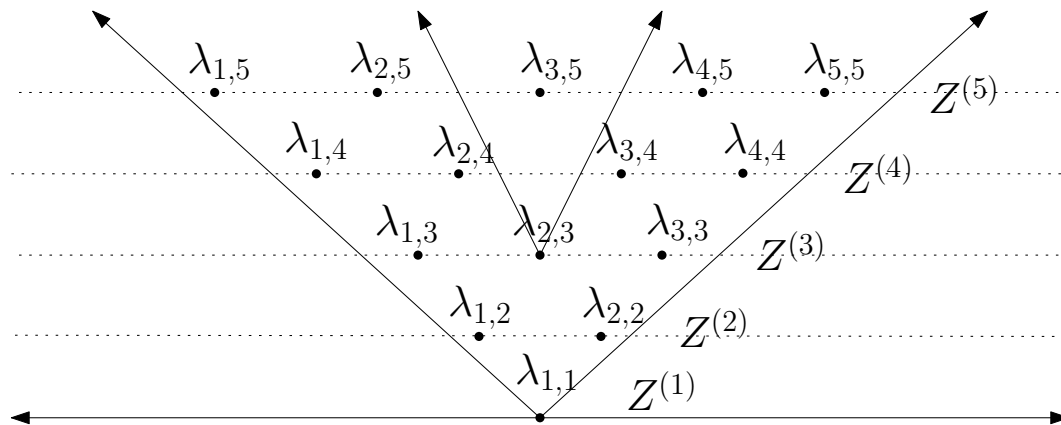


Figure 2: Interlacing shown for the first 5 levels

- The average number of unpaired vertices may be expressed as

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- This can be finitely iterated to yield a constant order bound for $\partial_x R_{n,k}$

Thanks!