Convergence of Densities of Spatial Averages for Stochastic Heat Equation via Malliavin-Stein Approach

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Stochastic Heat Equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + \sigma(u)\dot{W}, \qquad x \in \mathbb{R},\ t > 0,$$

- $\dot{W}$ space-time white noise, $\mathrm{E}\left[\dot{W}(t,x)\dot{W}(s,y)\right] = \delta_0(t-s)\delta_0(x-y)$
- $\sigma : \mathbb{R} \to \mathbb{R}$, $\sigma(1) \neq 0$, deterministic, Lipschitz
- $u(0,x) = u_0(x) = 1$

Theorem ([Walsh, 1986])

There exists a unique mild solution measurable and adapted random field $u = \{u(t,x) : (t,x) \in \mathbb{R}_+ \times \mathbb{R}\}$ *such that for all* $T > 0$ *and* $p \geq 2$

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathrm{E}\left[|u(t,x)|^p\right] = C_{T,p}$$

and for all $t > 0$ *and* $x \in \mathbb{R}$

$$u(t,x) = 1 + \int_{[0,t]\times\mathbb{R}} p_{t-s}(x-y)\sigma(u(s,y))W(ds,dy).$$

Spatial Averages

- Fix $t > 0$ and $\{u(t,x)\}_{x\in\mathbb{R}}$ has localization property [Conus et al., 2013].
- Consider

$$F_{R,t} := \frac{1}{\sigma_{R,t}} \left(\int_{-R}^{R} u(t,x)dx - 2R \right)$$

where

$$\sigma_{R,t}^2 := \mathrm{Var}\left(\int_{-R}^{R} u(t,x)dx \right) \sim R,$$

Theorem ([Huang et al., 2020])

$$d_{TV}(F_{R,t}, N) \leq \frac{C_t}{\sqrt{R}}.$$

- How does one obtain such quantitative normal approximations?
- **Malliavin-Stein approach** Let $F = \delta(v) \in \mathbb{D}^{1,2}$, N standard normal random variable.

$$d_{\mathrm{TV}}(F, N) \leq 2\sqrt{\operatorname{Var}\langle DF, v\rangle_{\mathfrak{H}}}$$

- In the aforementioned model:

$$F_{R,t} = \delta(v) = \int_{[0,t]\times\mathbb{R}} \left(\frac{1}{\sigma_{R,t}} \int_{-R}^{R} p_{t-s}(x-y)dx\sigma(u(s,y)) \right) W(ds, dy)$$

- What about convergence in densities?

Theorem ([Caballero et al., 1998, Hu et al., 2014])

Assume that

- $v \in \mathbb{D}^{1,6}(\Omega; \mathfrak{H}),\ F = \delta(v) \in \mathbb{D}^{2,6}$
- $\mathrm{E}\left[\left| \langle DF, v \rangle_{\mathfrak{H}} \right|^{-4} \right] < \infty$

Then,

$$\sup_{x \in \mathbb{R}} |f_F(x) - \phi(x)| \leq C \sqrt{\mathrm{Var} \langle DF, v \rangle_{\mathfrak{H}}} + \sqrt{\mathrm{E}\left[\left| \langle D \langle DF, v \rangle_{\mathfrak{H}}, v \rangle_{\mathfrak{H}} \right|^2 \right]}.$$

Need regularity and nondegeneracy conditions.

Theorem (1)

Assume,

- **H1**: $\sigma : \mathbb{R} \to \mathbb{R} \in C^2$ *with* σ' *bounded and* $|\sigma''(x)| \leq C(1 + |x|^m)$, *for some* $m > 0$;
- **H2**: *For some* $q > 10$, $\mathrm{E}\left[|\sigma(u(t,0))|^{-q}\right] < \infty$.

Then,

$$\sup_{x \in \mathbb{R}} |f_{F_{R,t}}(x) - \phi(x)| \leq \frac{C_t}{\sqrt{R}}.$$

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Remark

H2 *holds if* σ *is bounded away from zero or if* $|\sigma(x)| \leq \Lambda |x|$.

[Chen et al., 2016]

Highlights of the proof

For $t \in [0,T]$ and $r < s < t$

- $$\|D_{s,y}u(t,x)\|_p \le C_{T,p}p_{t-s}(x-y)$$

- Under **H2**: there exists $R_0 > 0$ such that

$$\sup_{R \ge R_0} \mathrm{E}\left[|\langle DF_{R,t}, v_{R,t}\rangle_{\mathfrak{H}}|^{-p} \right] < \infty.$$

- Recall

$$\sup_{x \in \mathbb{R}} |f_F(x) - \phi(x)| \le C \sqrt{\mathrm{Var}\langle DF, v\rangle_{\mathfrak{H}}} + \sqrt{\mathrm{E}\left[|\langle D\langle DF, v\rangle_{\mathfrak{H}}, v\rangle_{\mathfrak{H}}|^2 \right]}.$$

Highlights of the proof continued

- [Chen et al., 2020a] If $\sigma(x) = x$ then

$$\|D_{r,z} D_{s,y} u(t,x)\|_p \leq C_{T,p} p_{t-s}(x-y) p_{s-r}(y-z).$$

- Under **H1**:

$$\left\|D_{r,z} D_{s,y} u(t,x)\right\|_p \leq C_{T,p} \Phi_{r,z,s,y}(t,x)$$

where

$$\Phi_{r,z,s,y}(t,x) := p_{t-s}(x-y) \cdot \left(p_{s-r}(y-z) + \frac{p_{t-r}(z-y) + p_{t-r}(z-x) + \mathbf{1}_{\{|y-x|>|z-y|\}}}{(s-r)^{1/4}} \right).$$

Parabolic Anderson Model

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + u\dot{W}, \qquad x \in \mathbb{R},\ t > 0, \tag{0.1}$$

- $u(0,x) = u_0(x) = \delta_0$
- $\dot{W}$ is space-time white noise

Theorem ([Chen and Dalang, 2015])

There exists a unique measurable and adapted random field solution $u = \{u(t,x)\}_{(t,x)\in\mathbb{R}_+\times\mathbb{R}}$ *such that for all* $T > 0$ *and* $p \geq 2$

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathrm{E}\left[|u(t,x)|^p\right] = C_{T,p}\,, \tag{0.2}$$

and for all $t \geq 0$ *and* $x \in \mathbb{R}$

$$u(t,x) = p_t(x) + \int_{[0,t]\times\mathbb{R}} p_{t-s}(x-y)u(s,y)W(ds,dy).$$

Spatial Averages

- Fix $t > 0$. The process $x \mapsto U(t,x) := u(t,x)/p_t(x)$ is stationary. [Amir et al., 2011]
- Consider

$$G_{R,t} := \frac{1}{\Sigma_{R,t}} \left(\int_{-R}^{R} U(t,x)dx - 2R \right)$$

where

$$\Sigma_{R,t}^2 := \text{Var} \left(\int_{-R}^{R} U(t,x)dx \right) \sim R \log R.$$

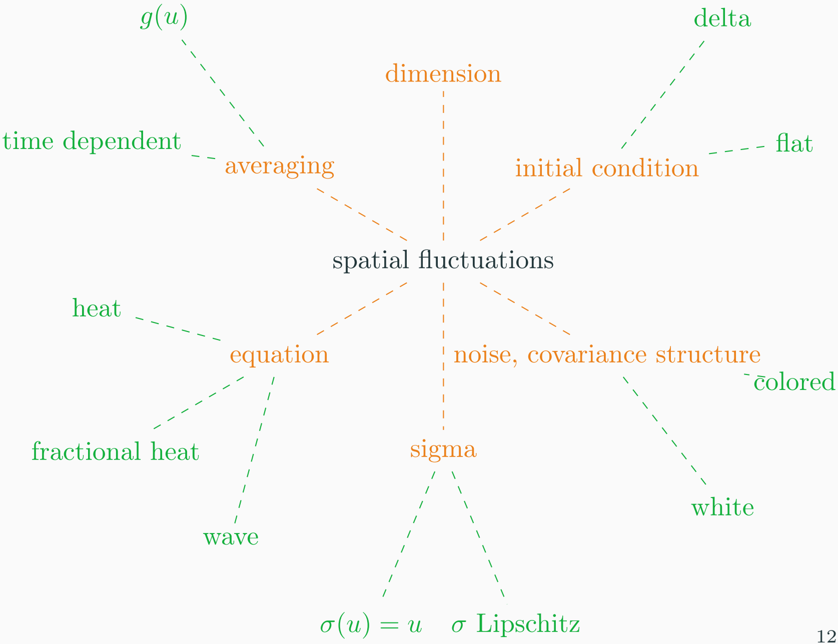
Theorem ([Chen et al., 2020b])

$$d_{TV}(G_{R,t}, N) \leq \frac{C_t \sqrt{\log R}}{\sqrt{R}}.$$

Theorem (2)

Fix $\gamma > \frac{19}{2}$*. Then, there exists an* $R_0 \geq 1$ *such that for all* $R \geq R_0$

$$\sup_{x\in\mathbb{R}} |f_{G_{R,t}}(x) - \phi(x)| \leq \frac{C_t(\log R)^\gamma}{\sqrt{R}}.$$



$g(u)$
delta
dimension
time dependent
flat
averaging
initial condition
spatial fluctuations
heat
equation
noise, covariance structure
colored
fractional heat
sigma
white
wave
$\sigma(u) = u$
σ Lipschitz

Current Project

- Convergence in densities for spatial averages of the solution to stochastic heat equation in general space dimension, noise white in time colored in space with Riesz kernel $|x-y|^{-\beta}$

Thank you for your attention!

☺

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