Stochastic elliptic-parabolic system arising in flow in porous media

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Frontier Probability Days

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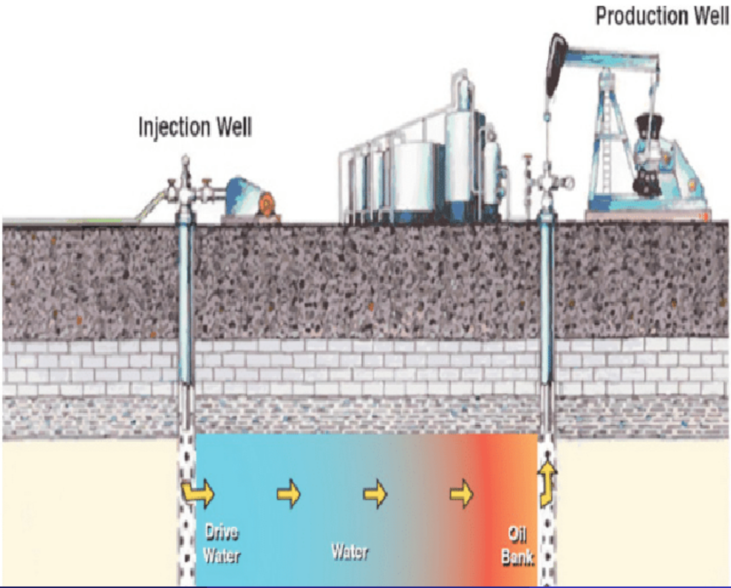
Motivation



Deterministic model

We are interested in modeling the miscible displacement flow model, in porous medium, of one incompressible fluid by another

$$\begin{cases} \phi\partial_t c - \nabla\cdot(D\,\nabla c - c\,\boldsymbol{v}) + (q^P c)(x,t) = (q^I \hat{c}) & \text{in } \mathcal{U}\times[0,T], \\ \nabla\cdot\boldsymbol{v}(x,t) = q^I - q^P & \text{in } \mathcal{U}\times[0,T], \\ c(x,0) = c_0(x) & \text{on } \mathcal{U}, \end{cases} \quad (1)$$

where

- c - Concentration
- $\boldsymbol{v}$ - Darcy velocity
- ϕ - Porosity
- D - Diffusion coefficient
- q^I - Sum of injected well source
- q^P - Sum of produced well sink
- $\hat{c}$ - Concentration at source
- c_0 - Initial condition

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Coupled through the Darcy velocity

$$\boldsymbol{v} = -\frac{k}{\mu(c)}\nabla p$$

k - Permeability
μ - Viscosity
p - Fluid pressure

The coupled system, describe the behavior of: the total fluid pressure of the mixture p, the Darcy velocity v of the fluid mixture, computed with respect to the concentration c of one of the components in the mixture.

- Peaceman and Rachford 1962 : Numerical approximation of the solution / compared to the laboratory data.
- Sammon in 1986 : Theoretically study: Modeled the injection q^I and production q^p well source/sink terms as a modified Dirac delta-function.
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Stochastic model

The model reads

$$\begin{cases} dc(t) - \nabla\cdot(D(x)\nabla c(t) - c(t)\,v(t))dt + q(t)\,c(t)dt = (f(t) + dW(t))\,dt \\ \nabla\cdot v(t) = h(t), \\ v(t) = -\kappa(c(t))\nabla p(t), \end{cases}$$

where $W(t)$ is an $H^1(\mathcal{U})$-valued Wiener process defined on a complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$:

$$\begin{cases} W(x,t,\omega) \in L^2(\Omega, C(0,T; H^1(\mathcal{U}))), \\ D(x)\nabla W(x,t,\omega)\cdot\vec{n} = 0, \text{ for } x\in\partial\mathcal{U} \text{ and for a.e. } (t,\omega)\in[0,T]\times\Omega. \end{cases}$$

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where

$$\kappa(c(x,t,\omega)) := \frac{k(x)}{\mu(c(x,t,\omega))}, \quad v(x,t,\omega) = -\kappa(c(x,t,\omega))\nabla p(x,t,\omega),$$
$$f(x,t) := (q^I\hat{c})(x,t), \quad h(x,t) := q^I(x,t) - q^P(x,t),$$
$$q(x,t) := q^P(x,t),$$

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The stochastic process $(p(t), v(t), c(t))$

$$\begin{aligned} (p, v, c) : \mathcal{U} \times [0, T] \times \Omega &\longrightarrow \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}, \\ (x,t,\omega) &\longmapsto (p(x,t,\omega), v(x,t,\omega), c(x,t,\omega)), \end{aligned}$$

that satisfies the no-flow boundary conditions,

$$D\nabla c(x,t) \cdot \vec{n} = 0, \quad (x,t) \in \partial\mathcal{U} \times [0,T], \tag{2}$$

$$v(x,t) \cdot \vec{n} = 0, \quad (x,t) \in \partial\mathcal{U} \times [0,T], \tag{3}$$

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Compatibility condition:

$$\int_{\mathcal{U}} h(t,x)dx := \int_{\mathcal{U}} q^I(x,t) - q^P(x,t)dx = 0, \quad \forall t \in [0,T].$$

We normalize the pressure p by an average condition,

$$\int_{\mathcal{U}} p(x,t)dx = 0, \quad t \in [0,T].$$

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Existence of weak solution:

- Pathwise argument: for fixed $\omega \in \Omega$, we solve for $(p(t), v(t), c(t))$ the system.
- Measurability of $(p(t), v(t), c(t))$.

Assumptions

Assume the following assumptions, hold for $\mathbb{P}$-a.e. $\omega \in \Omega$,

- $c_0 \in L^2(\mathcal{U})$.
- $h, q \in L^\infty([0,T]; L^2(\mathcal{U}))$.
- $h(t,x) + 2q(t,x) \geq 0$, for a.e, $x \in \mathcal{U}$ and $\forall t \in [0,T]$.
- $f \in L^2(0,T; L^2(\mathcal{U}))$.
- $D \in L^\infty(\mathcal{U})$ and there exists $D^*, D_* > 0$ such that $D_* \leq D(x) \leq D^*$ for a.e., $x \in \mathcal{U}$.
- $\kappa \in \mathcal{C}(\mathbb{R}) \cap L^\infty(\mathbb{R})$, with $0 < \kappa_* \leq \kappa(\xi) \leq \kappa^*$ for a.e $\xi \in \mathbb{R}$.
- $W(x,t,\omega) \in L^2(\Omega, C(0,T; H^1(\mathcal{U})))$,
- $D(x)\nabla W(x,t,\omega) \cdot \vec{n} = 0,$ for $x \in \partial\mathcal{U}$ and for a.e. $(t,\omega) \in [0,T] \times \Omega$.

Weak solution

A stochastic process $(p(t), v(t), c(t))$ is called a weak solution to the system on $[0, T]$, if for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$p(\cdot,\omega) \in L^\infty(0,T;H^1(\mathcal{U})), \quad \boldsymbol{v}(\cdot,\omega) \in L^\infty(0,T;L^2(\mathcal{U})^2),$$
$$c(\cdot,\omega) \in C(0,T;L^2(\mathcal{U})) \cap L^2(0,T;H^1(\mathcal{U})),$$

and for a.e. $0 \le t \le T$,

$$< c, \psi > + \int_o^T \underbrace{< D\nabla c, \nabla\psi > - < c\,\boldsymbol{v}, \nabla\psi > + < qc, \psi >}_{:=\Lambda(c(t),\boldsymbol{v}(t),\psi)} dt$$

$$=< c_0, \psi > + \int_0^T < f, \psi > dt + < W(t), \psi >, \quad \forall \psi \in H^1(\mathcal{U}),$$

$$- < \boldsymbol{v}(t), \nabla\phi > =< h(t), \phi >, \quad \forall \phi \in H^1(\mathcal{U}),$$
$$\boldsymbol{v}(x,t) = -\kappa(c(x,t))\nabla p(x,t).$$

Theorem

Under the assumptions: There exists a stochastic process (p, v, c) *solution to system and for* $\mathbb{P}$*-a.e.* $\omega \in \Omega$,

$$\|p\|_{L^\infty(0,T;H^1(\mathcal{U}))} \leq (\kappa_* b_0^2)^{-1} \|h\|_{L^\infty(0,T;L^2(\mathcal{U}))} \tag{2}$$

$$\|c\|^2_{L^\infty(0,T;L^2(\mathcal{U}))} \leq \beta e^T + \|W\|^2_{L^\infty(0,T;L^2(\mathcal{U}))} \tag{3}$$

$$\|c\|^2_{L^2(0,T;H^1(\mathcal{U}))} \leq (D_* b_0^2)^{-1} (\beta + 1) e^T + \|W\|^2_{L^2(0,T;H^1(\mathcal{U}))}, \tag{4}$$

where

$$\beta := \|c_0\|_0^2 + \|f\|^2_{L^2(0,T;L^2(\mathcal{U}))} + \frac{3}{D_*} \|W\|^2_{L^2(0,T;H^1(\mathcal{U}))}$$
$$\left(\frac{1}{{b_0}^2} \|q\|^2_{L^\infty(0,T;L^2(\mathcal{U}))} + (D^*)^2 + (\kappa^*)^2 \|h\|^2_{L^\infty(0,T;L^2(\mathcal{U}))} \right).$$

Sketch of the proof

- In order to construct a weak solution, we define for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$\begin{cases} \alpha(x,t) & := c(x,t) - W(x,t), \\ \alpha_0(x) & := c(x,0) - W(x,0) = c_0(x). \end{cases}$$

- $\alpha(t)$ satisfies the boundary condition

$$D(x)\nabla\alpha(x,t)\cdot\vec{n} = 0, \quad \text{for } x \in \partial\mathcal{U} \text{ and for a.e. } t \in [0,T].$$

- We re-write the stochastic system, for a.e. $0 \le t \le T$,

$$< \partial_t\alpha(t), \psi > + \Lambda(\alpha(t), \boldsymbol{v}(t), \psi) = -\Lambda(W(t), \boldsymbol{v}(t), \psi) + < f(t), \psi >, \quad \forall\psi \in H^1(\mathcal{U}).$$

$$- < \boldsymbol{v}(t), \nabla\phi > = < h(t), \phi >, \quad \forall\phi \in H^1(\mathcal{U}).$$

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- For given $\tilde{\alpha} \in L^2(0,T;L^2(\mathcal{U}))$, we prove the existence of unique $p_{\tilde{\alpha}}$ and $v_{\tilde{\alpha}}$ solution to the elliptic equation.
- For given $c_0 \in L^2(\mathcal{U}), \tilde{\alpha} \in L^2(0,T;L^2(\mathcal{U}))$, and $(p_{\tilde{\alpha}}, v_{\tilde{\alpha}})$, we construct a unique weak solution $\alpha(t)$ to the parabolic equation.
- Define:

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  where $\alpha(t)$ is the unique solution of parabolic equation.
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- Since $c(t) = \tilde{\alpha}(t) + W(t,x)$, then $c(t)$ is a weak solution of the parabolic equation
- We prove the measurability of $(p(t), v(t), c(t))$.

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