Numerical approximations for stochastic differential equations driven by fractional Brownian motion

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Based on a joint work with Y. Hu and D. Nualart.

Outline

Presentation Outline

1. Introduction

2. Crank-Nicolson method

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- In classical stochastic analysis, the SDE is solved when x is a semi-martingale.

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- If R has finite 2D ρ-variation for $\rho \in [1,2)$, then x gives raise to a β-rough path $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3)$ provided $\beta < \frac{1}{2\rho}$, and the SDE is well-defined. (Friz-Victoir '11).

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Let $H \in (0, 1)$. A fBm $B = (B_t, t \geq 0)$ with Hurst parameter H is:

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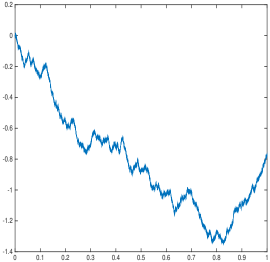
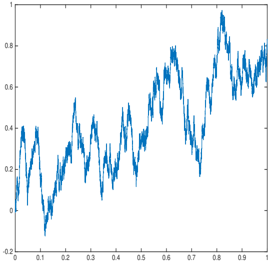
Applications: finance, statistical mechanics, hydrodynamics, etc.

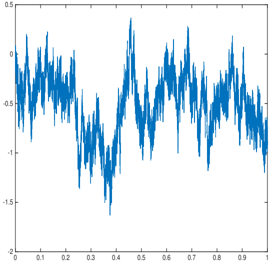
Simulation of fBms



$H = 0.7$



$H = 0.5$



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- The classical Euler scheme is defined as:

$$\delta y^n_{t_k t_{k+1}} = V(y^n_{t_k})\delta B_{t_k t_{k+1}},$$

where $\delta B_{t_k t_{k+1}} := B_{t_{k+1}} - B_{t_k}$ and $\delta y^n_{t_k t_{k+1}} := y^n_{t_{k+1}} - y^n_{t_k}$.

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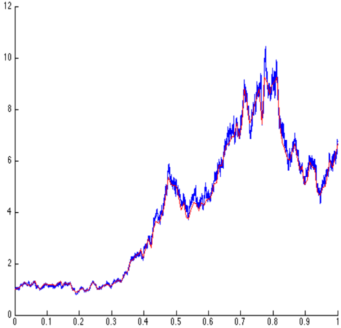

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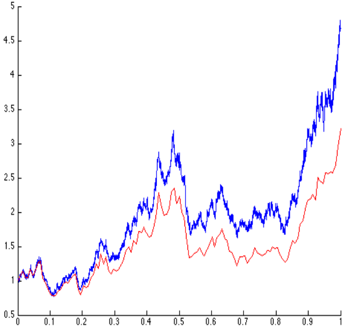

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Blue curve: Solution path   Red curve: Euler scheme

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- The modified Euler scheme of the SDE is defined as [Hu-L.-Nualart '16; L.-Tindel '20]:

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- When $3/4 > H > 1/3$ the strong convergence rates of the Euler scheme is $n^{1/2-2H}$. (L-Tindel '18)

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$$\vartheta_n \quad = \quad \begin{cases} n^{H+\frac{1}{2}} & \text{when} \quad (D1) \text{ holds,} \\ n^{2H} & \text{when} \quad (D2) \text{ holds,} \\ n^{2H-\frac{1}{2}} & \text{otherwise.} \end{cases}$$

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- While the Crank-Nicolson scheme is still convergent, the convergence rate is "throttled" due to the interactions among the driving processes of the equation.

Theorem (Hu-L.-Nualart '21)

- *Suppose that* $V \in C_b^3$.

- *Then for any* $p \geq 1$ *there exists a constant* $K = K_p$ *independent of* n *such that the following strong convergence result holds true for all* $n \in \mathbb{N}$*:*

$$\sup_{t\in[0,T]} \left(\mathbb{E}|X_t - X_t^n|^p\right)^{1/p} \leq K/\vartheta_n, \tag{2.2}$$

where ϑ_n *is defined as*

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- While the Crank-Nicolson scheme is still convergent, the convergence rate is "throttled" due to the interactions among the driving processes of the equation.

- In the degenerate cases these interactions disappear and the convergence of $X - X^n$ is taken over by some higher order terms.

Theorem (Hu-L.-Nualart '21)

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- *When $m > 1$, we have the convergence*

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$$dU_t = \sum_{j=0}^{m} \partial V_j(X_t) U_t dB_t^j + T^{2H-\frac{1}{2}} \sqrt{\frac{\kappa}{2}} \sum_{1 \le j' < j \le m} \phi_{jj'}(X_t) dW_t^{j'j}$$

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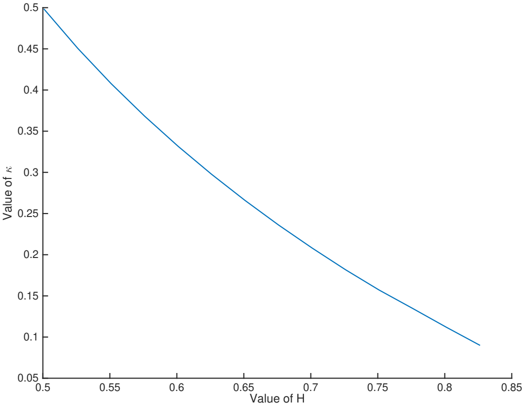
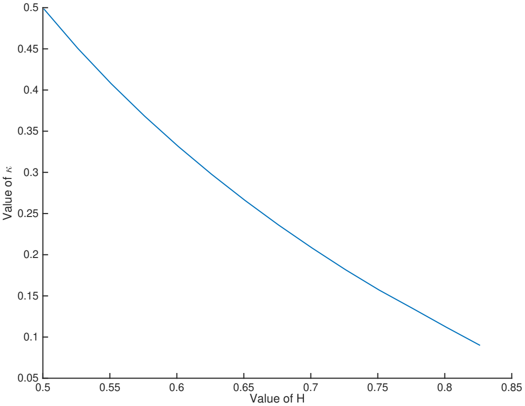

0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
Value of κ
0.5
0.55
0.6
0.65
0.7
0.75
0.8
0.85
Value of H



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Theorem (Cont'd)

When (D1) holds, the normalized error process $n^{-1/2-H}(X - X^n)$ *converges in distribution to the solution of the linear SDE on* $[0, T]$*:* $U_0 = 0$ *and*

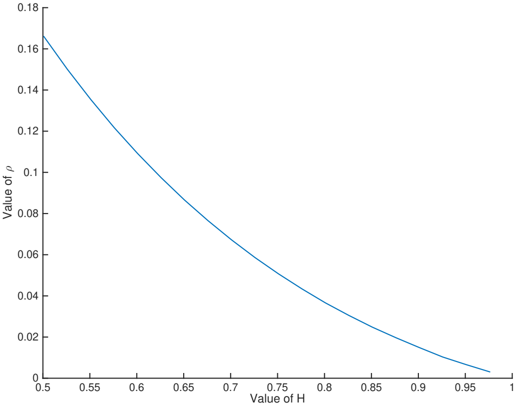
$$dU_t \quad = \quad \partial b(X_t)U_t dt + \sum_{j=1}^{m} \partial V_j(X_t)U_t dB_t^j + T^{H+\frac{1}{2}}\sqrt{\frac{\varrho}{2}}\sum_{j=1}^{m}\phi_{j0}(X_t)dW_t^j.$$

Theorem (Cont'd)

When (D1) holds, the normalized error process $n^{-1/2-H}(X - X^n)$ converges in distribution to the solution of the linear SDE on $[0, T]$: $U_0 = 0$ and

$$dU_t = \partial b(X_t)U_t dt + \sum_{j=1}^{m} \partial V_j(X_t)U_t dB_t^j + T^{H+\frac{1}{2}}\sqrt{\frac{\varrho}{2}}\sum_{j=1}^{m} \phi_{j0}(X_t)dW_t^j.$$

- *W is a one-dimensional standard Brownian motion independent of B and ρ is some constant depending on H.*


0.18
0.16
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0.12
0.1
0.08
0.06
0.04
0.02
0
Value of ρ
0.5
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0.6
0.65
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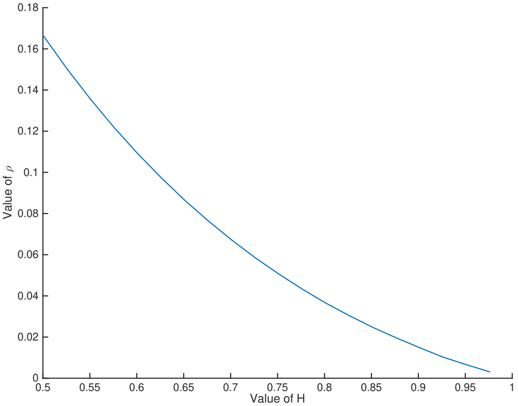
0.75
0.8
0.85
0.9
0.95
1
Value of H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- Suppose that (D2) holds, and let U be the solution of the linear equation:

$$dU_t = \sum_{j=0}^{m} \partial V_j(X_t) U_t dB_t^j - \frac{T^{2H}}{4} \sum_{j=1}^{m} \int_0^t \psi_{jjj}(s) dB_s^j + \sum_{j=0}^{m} \int_0^t \varphi_j(s) dB_s^j,$$

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- We denote $\psi_{jj'j''}(t) = \sum_{i,i'=1}^{d} (V_{j'}^i V_{j''}^{i'} \partial_{i'} \partial_i V_j)(X_t)$ and

$$\varphi_j(t) = \frac{1}{2} T^{2H} \sum_{j' \notin \{0,j\}} \Big(\frac{1-2H}{4H+2} \psi_{jj'j'}(t) - \frac{1}{2H+1} \psi_{j'j'j}(t) \Big), \quad t \in [0, T].$$

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- When $m = 1$ and $b \equiv 0$ the above linear SDE is reduced to

$$dU_t = \partial V(X_t) U_t dB_t - \frac{T^{2H}}{4} \sum_{i,i'=1}^{d} (V^i V^{i'} \partial_i \partial_{i'} V)(X_t) dB_t.$$

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- The asymptotic error distribution results are obtained with the help of the fourth moment theorem and fractional calculus.

Thank you very much for your attention!