SELF-STANDARDIZED CENTRAL LIMIT THEOREMS FOR TRIMMED SUBORDINATORS

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ABSTRACT

We prove under general regularity conditions that a trimmed subordinator satisfies a self-standardized central limit theorem [SSCLT]. Our basic tools are a classic representation for subordinators and a distributional approximation result of Zaitsev (1987).

The main results in this talk have recently appeared in D. M. Mason, Self-standardized central limit theorems for trimmed Lévy processes, *Journal of Theoretical Probability.*

A MOTIVATING TRIMMED SUM CLT

Let $X_1, X_2 \ldots,$ be i.i.d. nonnegative nondegenerate random variables and for each $n \geq 1$ let $X_n^{(1)} \geq X_n^{(2)} \geq \cdots \geq X_n^{(n)}$ denote their order statistics.

A special case of results of S. Csörgő, Haeusler and Mason (1988) characterizes when for a sequence $\{r_n\}_{n\geq 1}$ of positive integers $1 \leq r_n \leq n$ satisfying $r_n \to \infty$ and $r_n/n \to 0$ as $n \to \infty$ there exist norming and centering constants $B_{r_n} > 0$ and A_{r_n} such that

$$\frac{\sum_{i=1}^{n} X_i - X_n^{(1)} - \cdots - X_n^{(r_n)} - A_{r_n}}{B_{r_n}} \overset{\mathrm{D}}{\to} Z \,, \tag{1}$$

where Z is a standard normal random variable. We shall soon see that an analogous CLT holds for trimmed subordinators.

Let V_t, $t \geq 0$, be a subordinator with Lévy measure Λ on $(0, \infty)$ and drift 0. This means that V_t is a stationary independent increment process with nonnegative jumps satisfying $V_0 = 0$ having Laplace transform

$$\mathbf{E} \exp\left(-\theta V_t\right) = \exp\left(-t\Phi\left(\theta\right)\right), \; \theta \geq 0, \tag{2}$$

where

$$\Phi\left(\theta\right) = \int_0^\infty \left(1 - \exp\left(-\theta v\right)\right) \Lambda\left(\mathrm{d}v\right). \tag{3}$$

SOME BASIC NOTATION

Put $\overline{\Lambda}(x) = \Lambda((x,\infty))$, and for $u > 0$ let

$$\varphi(u) = \sup\{x : \overline{\Lambda}(x) > u\}, \tag{4}$$

where $\sup\varnothing := 0$. We shall assume that $\overline{\Lambda}(0+) = \infty$.

Note that for V_t to be a subordinator its Lévy measure must satisfy

$$\int_0^1 x\Lambda(\mathrm{d}x) < \infty, \tag{5}$$

equivalently, for all $y > 0$

$$\int_y^\infty \varphi(x)\,\mathrm{d}x < \infty. \tag{6}$$

For future use set for any $y > 0$

$$\mu(y) := \int_y^\infty \varphi(x)\,\mathrm{d}x \text{ and } \sigma^2(y) := \int_y^\infty \varphi^2(x)\,\mathrm{d}x. \tag{7}$$

A REPRESENTATION FOR SUBORDINATORS

Let $\omega_1, \omega_2, \ldots$ be i.i.d. exponential random variables with mean 1. Put for $n \geq 1$, the partial sums,

$$\Gamma_n = \omega_1 + \ldots + \omega_n. \tag{8}$$

V_t has the distributional representation

$$V_t \stackrel{\mathrm{D}}{=} \sum_{i=1}^{\infty} \varphi(\Gamma_i / t). \tag{9}$$

Denote for $t > 0$ the ordered jump sequence $m_t^{(1)} \geq m_t^{(2)} \geq \dots$ of V_t on the interval $[0, t]$. It turns out for any given $t > 0$

$$\left(m_t^{(k)}\right)_{k\geq 1} \stackrel{\mathrm{D}}{=} \left(\varphi\left(\frac{\Gamma_k}{t}\right)\right)_{k\geq 1}. \tag{10}$$

Set $V_t^{(0)} := V_t$ and for any integer $k \geq 1$ consider the trimmed subordinator

$$V_t^{(k)} := V_t - m_t^{(1)} - \cdots - m_t^{(k)}, \tag{11}$$

which on account of (10) says for any integer $k \geq 0$

$$V_t^{(k)} \stackrel{\mathrm{D}}{=} \sum_{i=k+1}^{\infty} \varphi\left(\frac{\Gamma_i}{t}\right) =: \widetilde{V}_t^{(k)}. \tag{12}$$

OUR GOAL

We shall prove under regularity conditions that given a sequence of positive integers $\{k_n\}_{n\geq 1}$ converging to infinity and a sequence of positive constants $\{t_n\}_{n\geq 1}$ that the following SSCLT holds for $\widetilde{V}_{t_n}^{(k_n)}$

$$\frac{\widetilde{V}_{t_n}^{(k_n)} - t_n\mu\left(\Gamma_{k_n}/t_n\right)}{\sqrt{t_n}\sigma\left(\Gamma_{k_n}/t_n\right)} \xrightarrow{\mathrm{D}} Z. \tag{13}$$

As a special case we get the SSCLT of Ipsen, Maller and Resnick [IMR] (2020), who consider the case when $t_n = t$ is fixed and $k_n = n$.

Theorem 1 *Assume that* $\overline{\Lambda}(0+) = \infty$. *For any sequence of positive integers* $\{k_n\}_{n\geq 1}$ *converging to infinity and sequence of positive constants* $\{t_n\}_{n\geq 1}$ *satisfying*

$$\frac{\sqrt{t_n}\sigma\left(\Gamma_{k_n}/t_n\right)}{\varphi\left(\Gamma_{k_n}/t_n\right)} \xrightarrow{\mathrm{P}} \infty, \text{ as } n \to \infty, \tag{14}$$

we have uniformly in x, *as* $n \to \infty$

$$\left| P\left\{ \frac{\widetilde{V}_{t_n}^{(k_n)} - t_n\mu\left(\Gamma_{k_n}/t_n\right)}{\sqrt{t_n}\sigma\left(\Gamma_{k_n}/t_n\right)} \leq x | \Gamma_{k_n} \right\} - P\left\{Z \leq x\right\} \right| \xrightarrow{\mathrm{P}} 0, \tag{15}$$

which implies as $n \to \infty$

$$\frac{\widetilde{V}_{t_n}^{(k_n)} - t_n\mu\left(\Gamma_{k_n}/t_n\right)}{\sqrt{t_n}\sigma\left(\Gamma_{k_n}/t_n\right)} \xrightarrow{\mathrm{D}} Z. \tag{16}$$

There always exist $k_n \to \infty$ and $t_n \to \infty$ such that (14) holds. For example for any $k_n \to \infty$, let $t_n = \rho n$ for some $\rho > 0$. Since $\Gamma_{k_n}/k_n \overset{\mathrm{P}}{\to} 1$ and $\Gamma_{k_n}/t_n \overset{\mathrm{P}}{\to} 1/\rho$, which implies that

$$P\left\{\frac{\sqrt{t_n}\sigma\left(\Gamma_{k_n}/t_n\right)}{\varphi\left(\Gamma_{k_n}/t_n\right)} > \frac{\sqrt{\rho k_n}\sigma\left(2/\rho\right)}{\varphi\left(1/\left(2\rho\right)\right)}\right\} \to 1 \tag{17}$$

and thus (14) holds.

EXAMPLE 2

Assume the Feller class at zero condition

$$\limsup_{x\downarrow 0} \frac{x^2\overline{\Lambda}(x)}{\int_0^x u^2\Lambda(\mathrm{d}u)} < \infty. \tag{18}$$

Since $\overline{\Lambda}(\varphi(y)-) \geq y$, (18) says that

$$\limsup_{y\to\infty} \frac{\varphi^2(y)\,y}{\int_0^{\varphi(y)} u^2\Lambda(\mathrm{d}u)} \leq \limsup_{y\to\infty} \frac{\varphi^2(y)\,\overline{\Lambda}(\varphi(y)-)}{\int_0^{\varphi(y)-} u^2\Lambda(\mathrm{d}u)} < \infty, \tag{19}$$

which implies that

$$\liminf_{y\to\infty} \int_y^\infty \varphi^2(x)\,\mathrm{d}x \Big/ \left(y\varphi^2(y)\right) := \tau > 0. \tag{20}$$

Observing that

$$\frac{t_n \sigma^2\left(\Gamma_{k_n}/t_n\right)}{\varphi^2\left(\Gamma_{k_n}/t_n\right)} = \Gamma_{k_n} \frac{\int_{\Gamma_{k_n}/t_n}^{\infty} \varphi^2(x)\,\mathrm{d}x}{\left(\Gamma_{k_n}/t_n \varphi^2\left(\Gamma_{k_n}/t_n\right)\right)}, \qquad (21)$$

we see that (14) holds, whenever $\Gamma_{k_n} \xrightarrow{\mathrm{P}} \infty$ and $\Gamma_{k_n}/t_n \xrightarrow{\mathrm{P}} \infty$. We note in passing that (18) is satisfied whenever $\overline{\Lambda}$ is regularly varying at zero with index $-\alpha$, $0 < \alpha < 2$.

IMR SELF-STANDARDIZED CLT

Ipsen, Maller and Resnick [IMR] (2020) have shown whenever there exist constants a_n and b_n such that for a nondegenerate random variable Δ

$$\frac{m_1^{(n)} - b_n}{a_n} \stackrel{\mathrm{D}}{=} \frac{\varphi\left(\Gamma_n\right) - b_n}{a_n} \stackrel{\mathrm{D}}{\to} \Delta \tag{22}$$

then for all $t > 0$ the following self-standardized trimmed CLT holds

$$\frac{\widetilde{V}_t^{(n)} - t\mu\left(\Gamma_n/t\right)}{\sqrt{t}\sigma\left(\Gamma_n/t\right)} \stackrel{\mathrm{D}}{\to} Z. \tag{23}$$

After some analysis one can show that Theorem 1 implies the IMR (2020) CLT.

With added regularity one can use non random norming and centering.

Corollary 1 *Assume that* V_t, $t \geq 0$, *is a subordinator with drift* 0, *whose Lévy tail function* $\overline{\Lambda}$ *is regularly varying at zero with index* $-\alpha$, *where* $0 < \alpha < 1$. *For any sequence of positive integers* $\{k_n\}_{n\geq 1}$ *converging to infinity and sequence of positive constants* $\{t_n\}_{n\geq 1}$ *such that* $k_n/t_n \to \infty$ *we have as* $n \to \infty$,

$$\frac{V_{t_n}^{(k_n)} - t_n\mu\left(k_n/t_n\right)}{\sqrt{t_n}\sigma\left(k_n/t_n\right)} \xrightarrow{\mathrm{D}} \sqrt{\frac{2}{\alpha}}Z. \tag{24}$$

The analog of Corollary 1 for a sequence of i.i.d. positive random variables $\xi_1, \xi_2 \ldots$ in the domain of attraction of a stable law of index $0 < \alpha < 2$ says that as $n \to \infty$,

$$\frac{\sum_{i=r_n+1}^{n} \xi_n^{(i)} - nc\left(r_n/n\right)}{\sqrt{n}a\left(r_n/n\right)} \xrightarrow{\mathrm{D}} \sqrt{\frac{2}{2-\alpha}}Z, \tag{25}$$

where for each $n \geq 2$, $\xi_n^{(1)} \geq \cdots \geq \xi_n^{(n)}$ denote the order statistics of $\xi_1, \ldots, \xi_n$, $\{r_n\}_{n\geq 1}$ is a sequence of positive integers $1 \leq r_n \leq n$ satisfying $r_n \to \infty$ and $r_n/n \to 0$ as $n \to \infty$, and $c\left(r_n/n\right)$ and $a\left(r_n/n\right)$ are appropriate centering and norming constants. For details refer to S. Csörgő, Horváth and Mason (1986). The proof of Corollary 1 borrows ideas from the proof of their Theorem 1.

SPECIAL CASE OF A RESULT OF ZAITSEV

Fact (Zaitsev (1987)) *Let Y be an infinitely divisible mean 0 and variance 1 random variable with Lévy measure Λ and Z be a standard normal random variable. Assume that the support of Λ is contained in a closed ball with center 0 of radius τ. Then for universal positive constants C_1 and C_2 for any $\lambda > 0$*

$$\pi\left(Y, Z; \lambda\right) \leq C_1 \exp\left(-\frac{\lambda}{C_2 \tau}\right), \tag{26}$$

where for $\lambda > 0$

$$\pi\left(Y, Z; \lambda\right) := \sup_{B \in \mathcal{B}} \max\left\{F, G\right\}, \tag{27}$$

where

$$F = P\left\{Y \in B\right\} - P\left\{Z \in B^{\lambda}\right\} \tag{28}$$

and

$$G = P\left\{Z \in B\right\} - P\left\{Y \in B^{\lambda}\right\} \tag{29}$$

with $B^{\lambda} = \{y \in \mathbb{R}\colon \inf_{x \in B} |x - y| < \lambda\}$ for $B \in \mathcal{B}$, the Borel sets of $\mathbb{R}$

IN PARTICULAR

A particular, the Zaitsev Fact says that for all $x \in \mathbb{R}$ and $\lambda > 0$,

$$P\{Z \leq x - \lambda\} - C_1 \exp\left(-\frac{\lambda}{C_2 \tau}\right) \leq P\{Y \leq x\} \tag{30}$$

$$\leq P\{Z \leq x + \lambda\} + C_1 \exp\left(-\frac{\lambda}{C_2 \tau}\right). \tag{31}$$

SKETCH OF PROOF OF THEOREM 1

For each $t > 0$ and $y > 0$ consider the random variable

$$T(t, y) = \sum_{i=1}^{\infty} \varphi\left(\frac{y}{t} + \frac{\Gamma_i'}{t}\right), \tag{32}$$

with Γ_i', $i \geq 1$, $\stackrel{\mathrm{D}}{=} \Gamma_i$, $i \geq 1$, which has Laplace transform

$$\Upsilon_{t,y}(\theta) := E \exp(-\theta T(t, y)) = \exp(-t\Phi_{t,y}(\theta)), \tag{33}$$

where $\Phi_{t,y}(\theta)$ is the Laplace exponent,

$$\Phi_{t,y}(\theta) = \int_0^{\infty} t\left(1 - \exp\left(-\theta\varphi\left(\frac{y}{t} + u\right)\right)\right) \mathrm{d}u. \tag{34}$$

LEVY MEASURE OF $T\left(t,y\right)$

Introducing the Lévy measure $\Lambda_{y/t}$ defined on $(0,\infty)$ by

$$\overline{\Lambda}_{y/t}(u) = \begin{cases} \overline{\Lambda}(u) - \frac{y}{t}, \text{ for } 0 < u < \varphi\left(\frac{y}{t}\right) \\ 0, \text{ for } u \geq \varphi\left(\frac{y}{t}\right) \end{cases}, \tag{35}$$

we see that

$$\Phi_{t,y}\left(\theta\right) = \int_0^\infty t\left(1 - \exp\left(-\theta v\right)\right)\Lambda_{y/t}\left(\mathrm{d}v\right). \tag{36}$$

Clearly $T\left(t,y\right)$ is an infinitely divisible random variable and the support of $\Lambda_{y/t}$ is contained in $[0,\varphi(y/t)]$.

STANDARDIZED VERSION OF $T(t, y)$

One finds that

$$ET(t, y) = t \int_{y/t}^{\infty} \varphi(u)\,\mathrm{d}u =: t\mu\left(\frac{y}{t}\right) \qquad (37)$$

and

$$VarT(t, y) = t \int_{y/t}^{\infty} \varphi^2(u)\,\mathrm{d}u =: t\sigma^2\left(\frac{y}{t}\right). \qquad (38)$$

For each $t > 0$ and $y > 0$ consider the standardized version of $T(t, y)$

$$S(t, y) = \frac{T(t, y) - t\mu\left(\frac{y}{t}\right)}{\sqrt{t}\sigma\left(\frac{y}{t}\right)}. \qquad (39)$$

By definition $ES(t, y) = 0$ and $VarS(t, y) = 1$.

The random variable $S(t, y)$ is an infinitely divisible random variable with mean zero, variance one whose Lévy measure is contained in

$$\left[0, \frac{\varphi(y/t)}{\sqrt{t}\sigma\left(\frac{y}{t}\right)}\right]. \tag{40}$$

This allows us to use the Zaitsev fact.

Applying the Zaitsev Fact to the infinitely divisible random variable $S(t,y)$ we get for any $t>0$, $y>0$ and $\lambda>0$ and for universal positive constants C_1 and C_2

$$\pi\left(S(t,y), Z;\lambda\right) \le C_1 \exp\left(-\frac{\lambda\sqrt{t}\sigma\left(\frac{y}{t}\right)}{C_2\varphi(y/t)}\right). \tag{41}$$

Since $\varphi(z) \to 0$, as $z\to\infty$, this implies that whenever $\{t_n\}_{n\ge 1}$ is a sequence of positive constants and Y_{k_n} is a sequence of random variables such that each Y_{k_n} is independent of Γ_i', $i\ge 1$, and as $n\to\infty$

$$\frac{\sqrt{t_n}\sigma\left(Y_{k_n}/t_n\right)}{\varphi\left(Y_{k_n}/t_n\right)} \stackrel{\mathrm{P}}{\to} \infty, \tag{42}$$

then uniformly in x as $n\to\infty$

$$\left|P\left\{S\left(t_n, Y_{k_n}\right) \le x | Y_{k_n}\right\} - P\left\{Z\le x\right\}\right| \stackrel{\mathrm{P}}{\to} 0, \tag{43}$$

and thus as $n\to\infty$

$$\left|P\left\{S\left(t_n, Y_{k_n}\right) \le x\right\} - P\left\{Z\le x\right\}\right| \to 0. \tag{44}$$

PARTICULAR CASE

In particular by choosing $Y_{k_n} = \Gamma_{k_n}$ and independent of $\{\Gamma'_i\}_{i\geq 1} \overset{\mathrm{D}}{=} \{\Gamma_i\}_{i\geq 1}$, we get

$$\frac{\widetilde{V}_{t_n}^{(k_n)} - t_n \mu\left(\frac{\Gamma_{k_n}}{t_n}\right)}{\sqrt{t_n}\sigma\left(\frac{\Gamma_{k_n}}{t_n}\right)} \overset{\mathrm{D}}{=} \frac{\sum_{i=1}^{\infty} \varphi((Y_{k_n} + \Gamma'_i)/t_n) - t_n \mu\left(\frac{Y_{k_n}}{t_n}\right)}{\sqrt{t_n}\sigma\left(\frac{Y_{k_n}}{t_n}\right)} \tag{45}$$

$$= \frac{T(t_n, Y_{k_n}) - t_n \mu\left(\frac{Y_{k_n}}{t_n}\right)}{\sqrt{t_n}\sigma\left(\frac{Y_{k_n}}{t_n}\right)} = S(t_n, Y_{k_n}). \tag{46}$$

Keeping (14) in mind, (15) and (16) follow from (43) and (44), respectively.

Let $(X_t)_{t\geq 0}$, be a Lévy process with a nontrivial Lévy measure Λ. For any $t > 0$ denote the ordered positive jump sequence

$$m_t^{(1)} \geq m_t^{(2)} \geq \dots \tag{47}$$

of X_t on the interval $[0, t]$ and let

$$n_t^{(1)} \leq n_t^{(2)} \leq \dots \tag{48}$$

denote the corresponding ordered negative jump sequence of X_t. Consider for any positive integers k and l, the trimmed subordinator

$$X_t^{(k,l)} := X_t - m_t^{(1)} - \dots - m_t^{(k)} - n_t^{(1)} - \dots - n_t^{(l)}. \tag{49}$$

Thank you for the invitation to speak here!