Analysis of systems of stochastic delay differential equations and applications

Nhu N. Nguyen

University of Connecticut

(joint work with Dang Nguyen (University of Alabama) & George Yin (University of Connecticut))

Frontier Probability Days 2021, Las Vegas, Nevada

Dec. 3-5, 2021

Outline

Introduction

- Since the delays or past dependence are unavoidable in natural phenomena and dynamical systems, the framework of stochastic functional differential equations is more realistic, more effective, and more general for the population dynamics in real life than a stochastic differential equation counterpart.

Introduction

- Since the delays or past dependence are unavoidable in natural phenomena and dynamical systems, the framework of stochastic functional differential equations is more realistic, more effective, and more general for the population dynamics in real life than a stochastic differential equation counterpart.
- For example, in population dynamics, some delay mechanisms studied in the literature include age structure, feeding times, replenishment or regeneration time for resources, see e.g. Cushing (2013).

Introduction

- Since the delays or past dependence are unavoidable in natural phenomena and dynamical systems, the framework of stochastic functional differential equations is more realistic, more effective, and more general for the population dynamics in real life than a stochastic differential equation counterpart.
- For example, in population dynamics, some delay mechanisms studied in the literature include age structure, feeding times, replenishment or regeneration time for resources, see e.g. Cushing (2013).
- The work on Kolmogorov stochastic differential equations with delay is relatively scarce. Results seem limited.

General Stochastic Functional Kolmogorov Systems

- $\mathscr{C}$: set of $\mathbb{R}^n$-valued continuous functions on $[-r,0]$.
  $\mathscr{C}_+ := \{\varphi = (\varphi_1,\dots,\varphi_n) \in \mathscr{C} : \varphi_i(s) \geq 0 \ \forall s \in [-r,0], i = 1,\dots,n\}$
  $\partial\mathscr{C}_+ := \{\varphi = (\varphi_1,\dots,\varphi_n) \in \mathscr{C} : \|\varphi_i\| = 0 \text{ for some } i = 1,\dots,n\}$
  $\mathscr{C}_+^\circ := \{\varphi \in \mathscr{C}_+ : \varphi_i(s) > 0, \forall s \in [-r,0], i = 1,\dots,n\} \neq \mathscr{C}_+ \setminus \partial\mathscr{C}_+$
- Consider the stochastic delay Kolmogorov system

$$\begin{cases} dX_i(t) = X_i(t)f_i(\mathbf{X}_t)dt + X_i(t)g_i(\mathbf{X}_t)dE_i(t), \quad i = 1,\dots,n, \\ \mathbf{X}_0 = \phi \in \mathscr{C}_+, \end{cases} \tag{2.1}$$

  and denoted by $\mathbf{X}^\phi(t)$ its solution.
  Here $\mathbf{X}_t := \{\mathbf{X}(t+s) : s \in [-r,0]\}$.

Standing Assumption

Assumption 2.1

The coefficients of (2.1) satisfy

(1) $\mathrm{diag}(g_1(\varphi),\dots,g_n(\varphi))\Gamma^\top\Gamma\mathrm{diag}(g_1(\varphi),\dots,g_n(\varphi)) = (g_i(\varphi)g_j(\varphi)\sigma_{ij})_{n\times n}$ is a positive definite matrix for any $\varphi\in\mathscr{C}_+$.

(2) $f_i(\cdot), g_i(\cdot):\mathscr{C}_+\to\mathbb{R}$ are Lipschitz continuous in each bounded set of $\mathscr{C}_+$ for any $i=1,\dots,n$.

(3) There exist $\mathbf{c}=(c_1,\dots,c_n)\in\mathbb{R}^n$, $c_i>0,\forall i$ and $\gamma_b,\gamma_0>0$, $A_0>0$, $A_1>A_2>0$, $M>0$, a continuous function $h:\mathbb{R}^n\to\mathbb{R}_+$ and a probability measure μ concentrated on $[-r,0]$ such that for any $\varphi\in\mathscr{C}_+$

$$\frac{\sum_{i=1}^n c_i x_i f_i(\varphi)}{1+\mathbf{c}^\top\mathbf{x}} - \frac{1}{2}\frac{\sum_{i,j=1}^n \sigma_{ij}c_ic_jx_ix_jg_i(\varphi)g_j(\varphi)}{(1+\mathbf{c}^\top\mathbf{x})^2} + \gamma_b\sum_{i=1}^n\Big(|f_i(\varphi)|+g_i^2(\varphi)\Big)$$
$$\le A_0\mathbf{1}_{\{|\mathbf{x}|<M\}} - \gamma_0 - A_1h(\mathbf{x}) + A_2\int_{-r}^0 h(\varphi(s))\mu(ds), \tag{2.2}$$

where $\mathbf{x}:=\varphi(0)$.

Assumption 2.2

One of following assumptions holds:

(a) There is a constant $\widetilde{K}$ such that for any $\varphi \in \mathscr{C}_+$, $\mathbf{x} = \varphi(0)$

$$\sum_{i=1}^{n} |f_i(\varphi)| + \sum_{i=1}^{n} g_i^2(\varphi) \le \widetilde{K}\left[h(\mathbf{x}) + \int_{-r}^{0} h(\varphi(s))\mu(ds)\right]. \tag{2.3}$$

(b) There exist $b_1, b_2 > 0$, a function $h_1 : \mathbb{R}^n \to [1, \infty]$, and a probability measure μ_1 on $[-r, 0]$ such that for any $\varphi \in \mathscr{C}_+$, $\mathbf{x} = \varphi(0)$

$$b_1 h_1(\mathbf{x}) \le \sum_{i=1}^{n} |f_i(\varphi)| + \sum_{i=1}^{n} g_i^2(\varphi) \le b_2\left[h_1(\mathbf{x}) + \int_{-r}^{0} h_1(\varphi(s))\mu_1(ds)\right]. \tag{2.4}$$

Some Definitions

Definition 2.1

The process $\mathbf{X}$ *is strongly stochastically persistent if for any* $\varepsilon > 0$*, there exists* $\delta > 1$ *such that for any* $\phi \in \mathscr{C}_+^\circ$

$$\liminf_{t\to\infty} \mathbb{P}_\phi \{\delta \le |X_i(t)|\} \ge 1 - \varepsilon \text{ for all } i = 1, \ldots, n. \tag{2.5}$$

Definition 2.2

For $\phi \in \mathscr{C}_+^\circ$*, we say the population* X_i *goes extinct with probability* $p_\phi > 0$ *if*

$$\mathbb{P}_\phi \left\{ \lim_{t\to\infty} X_i(t) = 0 \right\} = p_\phi .$$

The subscript ϕ is the initial value.

Main Ideas of Our Approach

- If all i.p.m on $\partial\mathscr{C}_+^\circ$ are repellers, the system must be persistent.
- If there is one subspace in $\partial\mathscr{C}_+^\circ$ on which all i.p.m are attractors, then the extinction happens.

- In the deterministic setting one usually characterizes the asymptotics of the system by first looking at the equilibrium points (or rest points). The stability of an equilibrium is quantified by the Lyapunov exponents of the linearized system
- We want to do something similar in the stochastic setting. We look at the behaviors of the systems near the boundary to determine whether or not the system is persistent.
-

$$\frac{\ln X_i(t)}{t} = \frac{\ln X_i(0)}{t} + \frac{1}{t}\int_0^t g_i(\mathbf{X}_s)dE_i(s) + \frac{1}{t}\int_0^t \left[f_i(\mathbf{X}_s) - \frac{g_i^2(\mathbf{X}_s)\sigma_{ii}}{2} \right] ds \tag{2.6}$$

- If $\mathbf{X}_t$ is close to the support of an ergodic measure μ supported on $\partial\mathscr{C}_+$ for a long time, then

$$\frac{1}{t}\int_0^t \left[f_i(\mathbf{X}_s) - \frac{g_i^2(\mathbf{X}_s)\sigma_{ii}}{2}\right] ds$$

can be approximated by the average with respect to μ

$$\lambda_i(\mu) = \int_{\partial C_+} \left(f_i(\phi) - \frac{g_i^2(\phi)\sigma_{ii}}{2}\right)\mu(d\phi),\ i = 1,\ldots,n$$

- As $t \to \infty$ the term

$$\frac{\ln X_i(0)}{t} + \frac{1}{t}\int_0^t g_i(\mathbf{X}_s)dE_i(s)$$

is negligible. This implies that

$$\lambda_i(\mu) = \int_{\partial C_+} \left(f_i(\phi) - \frac{g_i^2(\phi)\sigma_{ii}}{2}\right)\mu(d\phi),\ i = 1,\ldots,n$$

are the Lyapunov exponents of μ.
It can also be seen that $\lambda_i(\mu)$ gives the long-term growth rate of $X_i(t)$ if $\mathbf{X}_t$ is close to the support of μ.

Some Notations

- For a subset I of $\{1, \ldots, n\}$, denote $I^c := \{1, \ldots, n\} \setminus I$,

Some Notations

- For a subset I of $\{1,\ldots,n\}$, denote $I^c := \{1,\ldots,n\} \setminus I$,
- $\mathscr{C}_+^I := \left\{\varphi \in \mathscr{C}_+ : \left\|\varphi_j\right\| = 0 \text{ if } j \in I^c\right\},$

Some Notations

- For a subset I of $\{1,\dots,n\}$, denote $I^c := \{1,\dots,n\} \setminus I$,
- $\mathscr{C}_+^I := \left\{\varphi \in \mathscr{C}_+ : \left\|\varphi_j\right\| = 0 \text{ if } j \in I^c\right\}$,
- $\mathscr{C}_+^{I,\circ} := \{\varphi \in \mathscr{C}_+ : \|\varphi_i\| = 0 \text{ if } i \in I^c \text{ and } \varphi_i(s) > 0 \, \forall\, s \in [-r,0], i \in I\}$

$$\partial \mathscr{C}_+^I := \left\{\varphi = (\varphi_1,\dots,\varphi_n) \in \mathscr{C} : \|\varphi_i\| = 0 \text{ if } i \in I^c \cup J, J \subset I, J \neq \emptyset\right\}.$$

- Note that for the case $I = \emptyset$, $\mathscr{C}_+^I = \mathscr{C}_+^{I,\circ} = \{\mathbf{0}\}$.

Some Notations

- For a subset I of $\{1,\dots,n\}$, denote $I^c := \{1,\dots,n\} \setminus I$,
- $\mathscr{C}_+^I := \left\{\varphi \in \mathscr{C}_+ : \left\|\varphi_j\right\| = 0 \text{ if } j \in I^c\right\}$,
- $\mathscr{C}_+^{I,\circ} := \{\varphi \in \mathscr{C}_+ : \|\varphi_i\| = 0 \text{ if } i \in I^c \text{ and } \varphi_i(s) > 0 \, \forall s \in [-r,0], i \in I\}$

$$\partial\mathscr{C}_+^I := \left\{\varphi = (\varphi_1,\dots,\varphi_n) \in \mathscr{C} : \|\varphi_i\| = 0 \text{ if } i \in I^c \cup J, J \subset I, J \neq \emptyset\right\}.$$

- Note that for the case $I = \emptyset$, $\mathscr{C}_+^I = \mathscr{C}_+^{I,\circ} = \{\mathbf{0}\}$.
- Denote by $\mathscr{M}^I, \mathscr{M}^{I,\circ}, \partial\mathscr{M}^I$ the sets of ergodic measures on $\mathscr{C}_+^I, \mathscr{C}_+^{I,\circ}$ and $\partial\mathscr{C}_+^I$ respectively. $\mathscr{M}$ is the sets of ergodic measure on $\partial\mathscr{C}_+$.

Some Notations

- For a subset I of $\{1,\ldots,n\}$, denote $I^c := \{1,\ldots,n\} \setminus I$,
- $\mathscr{C}_+^I := \left\{\varphi \in \mathscr{C}_+ : \left\|\varphi_j\right\| = 0 \text{ if } j \in I^c\right\},$
- $\mathscr{C}_+^{I,\circ} := \{\varphi \in \mathscr{C}_+ : \|\varphi_i\| = 0 \text{ if } i \in I^c \text{ and } \varphi_i(s) > 0 \,\forall\, s \in [-r,0], i \in I\}$

$$\partial\mathscr{C}_+^I := \left\{\varphi = (\varphi_1,\ldots,\varphi_n) \in \mathscr{C} : \|\varphi_i\| = 0 \text{ if } i \in I^c \cup J, J \subset I, J \neq \emptyset\right\}.$$

- Note that for the case $I = \emptyset$, $\mathscr{C}_+^I = \mathscr{C}_+^{I,\circ} = \{\mathbf{0}\}$.
- Denote by $\mathscr{M}^I, \mathscr{M}^{I,\circ}, \partial\mathscr{M}^I$ the sets of ergodic measures on $\mathscr{C}_+^I, \mathscr{C}_+^{I,\circ}$ and $\partial\mathscr{C}_+^I$ respectively. $\mathscr{M}$ is the sets of ergodic measure on $\partial\mathscr{C}_+$.
- If $\pi \in \mathscr{M}^{I,\circ}$ then $\lambda_i(\pi) = 0, i \in I$. We call these Lyapunov exponents internal Lyapunov exponent. The other are called external ones.

Some Notations

- For a subset I of $\{1,\dots,n\}$, denote $I^c := \{1,\dots,n\}\setminus I$,
- $\mathscr{C}_+^I := \left\{\varphi \in \mathscr{C}_+ : \left\|\varphi_j\right\| = 0 \text{ if } j \in I^c\right\}$,
- $\mathscr{C}_+^{I,\circ} := \{\varphi \in \mathscr{C}_+ : \|\varphi_i\| = 0 \text{ if } i \in I^c \text{ and } \varphi_i(s) > 0 \,\forall\, s \in [-r,0], i \in I\}$

$$\partial\mathscr{C}_+^I := \left\{\varphi = (\varphi_1,\dots,\varphi_n) \in \mathscr{C} : \|\varphi_i\| = 0 \text{ if } i \in I^c \cup J, J \subset I, J \neq \emptyset\right\}.$$

- Note that for the case $I = \emptyset$, $\mathscr{C}_+^I = \mathscr{C}_+^{I,\circ} = \{\mathbf{0}\}$.
- Denote by $\mathscr{M}^I, \mathscr{M}^{I,\circ}, \partial\mathscr{M}^I$ the sets of ergodic measures on $\mathscr{C}_+^I, \mathscr{C}_+^{I,\circ}$ and $\partial\mathscr{C}_+^I$ respectively. $\mathscr{M}$ is the sets of ergodic measure on $\partial\mathscr{C}_+$.
- If $\pi \in \mathscr{M}^{I,\circ}$ then $\lambda_i(\pi) = 0, i \in I$. We call these Lyapunov exponents internal Lyapunov exponent. The other are called external ones.
- Biologically, $\lambda_j(\pi), j \in I^c$ is the invasion rate of species j when its density is rare to a subsystem $\{X_i, i \in I\}$ evolving according to π.

The following conditions will imply persistence cannot happen.

Assumption 2.3

There exists a subset $I \subset \{1, \ldots, n\}$ such that

$$\max_{i \in I_\pi^c,\, \pi \in \mathscr{M}^{I,\circ}} \{\lambda_i(\pi)\} < 0. \tag{2.7}$$

If $I \neq \emptyset$, we assume further that

$$\max_{i \in I} \{\lambda_i(\nu)\} > 0, \tag{2.8}$$

for any $\nu \in \mathrm{Conv}(\partial \mathscr{M}^I)$.

Assumption 2.4

The inverse of the matrix $(x_i x_j \sigma_{ij} g_i(\varphi) g_j(\varphi))_{n \times n}$ is uniformly bounded in $D_{\varepsilon,R}$ for each $\varepsilon, R > 0$, where

$$D_{\varepsilon,R} := \{\varphi \in \mathscr{C}_+ : \|\varphi\| \leq R, x_i \geq \varepsilon \ \forall i; \mathbf{x} := \varphi(0)\}.$$

Theorem 2.1

Assume Assumptions 2.1, 2.2, 2.3, *and* 2.4 *hold. For any* $p < p_0$ *with* p_0 *being a sufficiently small constant, and any initial value* $\phi \in \mathscr{C}_+^\circ$, *we have*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \mathbb{E}_\phi \bigwedge_{i=1}^{n} \|X_{i,t}\|^p \, dt = 0, \tag{2.9}$$

where $\bigwedge_{i=1}^n x_i := \min_{i=1,\dots,n}\{x_i\}$ *and* $\mathbf{X}_t =: (X_{1,t}, \dots, X_{n,t})$.

With additional technical conditions, we can determine which species go extinct, which persist. First, we define the random normalized occupation measures

$$\widetilde{\Pi}_t(\cdot) := \frac{1}{t}\int_0^t \mathbf{1}_{\{\mathbf{X}_s\in\cdot\}} ds,\ t > 0. \tag{2.10}$$

Moreover, for any initial condition $\phi \in \mathscr{C}_+$, denote the weak*-limit set of the family $\left\{\widetilde{\Pi}_t(\cdot), t \geq 1\right\}$ by $\mathscr{U} = \mathscr{U}(\omega)$.

A technical condition, a bit stronger than Assumption 2.2, to ensure the diffusion does not fluctuate too widely.

Assumption 2.5

Assume one of the following conditions hold.

- Assumption 2.2(a) holds and there exist constants $p_2 > 0$ and $B_1 > B_2 > 0, B_0 > 0$, $B_3 > 0$ such that for any $\varphi \in \mathscr{C}_+, \mathbf{x} := \varphi(0)$

$$(1+\mathbf{c}^\top\mathbf{x})^{p_2}\left(\frac{\sum_{i=1}^n c_i x_i f_i(\varphi)}{1+\mathbf{c}^\top\mathbf{x}} - \frac{1}{2}\frac{\sum_{i,j=1}^n \sigma_{ij} c_i c_j x_i x_j g_i(\varphi) g_j(\varphi)}{(1+\mathbf{c}^\top\mathbf{x})^2}\right) \le B_0 - B_1(1+\mathbf{c}^\top\mathbf{x})^{p_2} h(\mathbf{x}) + B_2\int_{-r}^0 (1+\mathbf{c}^\top\varphi(s))^{p_2} h(\varphi(s))\mu(ds), \tag{2.11}$$

  and

$$(1+\mathbf{c}^\top\mathbf{x})^{2p_2}\sum_{i=1}^n g_i^2(\varphi) \le B_3(1+\mathbf{c}^\top\mathbf{x})^{p_2} h(\mathbf{x}) + B_3\int_{-r}^0 (1+\mathbf{c}^\top\varphi(s))^{p_2} h(\varphi(s))\mu(\tag{2.12}$$

- Assumption 2.2(b) is satisfied, and (2.11) and (2.12) hold with h, μ replaced by h_1, μ_1.

Assumption 2.6

Let S be the family of subsets I satisfying the Assumption 2.3. We assume either that $S^c := 2^{\{1,\ldots,n\}} \setminus S$ is empty, where $2^{\{1,\ldots,n\}}$ denotes the family of all subsets of $\{1,\ldots,n\}$ or that

$$\max_{i=1,\ldots,n}\{\lambda_i(\nu)\} > 0 \text{ for any } \nu \in \mathrm{Conv}(\cup_{J\notin S}\mathscr{M}^{J,\circ}).$$

Theorem 2.2

Suppose that Assumptions 2.1, 2.3, 2.4, 2.5, and 2.6 are satisfied. Then for any $\phi \in \mathscr{C}_+^\circ$

$$\sum_{I\in S} P_\phi^I = 1, \quad P_\phi^I > 0, \tag{2.13}$$

where for $\phi \in \mathscr{C}_+^\circ$,

$$P_\phi^I := \mathbb{P}_\phi\left\{\mathscr{U}(\omega) \subset \mathrm{Conv}(\mathscr{M}^{I,\circ}) \ \&\ \lim_{t\to\infty}\frac{\ln X_i(t)}{t} \subset \left\{\lambda_i(\pi) : \pi \in \mathrm{Conv}(\mathscr{M}^{I,\circ})\right\}\right.,$$

Persistence

Assumption 2.7

For any $\pi \in \mathrm{Conv}(\mathscr{M})$, we have

$$\max_{i=1,\dots,n} \{\lambda_i(\pi)\} > 0,$$

where

$$\lambda_i(\pi) := \int_{\partial \mathscr{C}_+} \left(f_i(\varphi) - \frac{\sigma_{ii} g_i^2(\varphi)}{2} \right) \pi(d\varphi). \tag{2.14}$$

Theorem 2.3

Assume that Assumptions 2.1, 2.2, *and* 2.7 *hold. For any* $\varepsilon > 0$, *there exists a positive number* $R_* = R_*(\varepsilon)$ *such that for all* $\phi \in \mathscr{C}_+^\circ$,

$$\liminf_{t\to\infty} \mathbb{P}_\phi \left\{ R_*^{-1} \leq |X_i(t)| \leq R_* \right\} \geq 1 - \varepsilon \text{ for all } i = 1, \dots, n.$$

Assumption 2.8

The following conditions hold:

(i) There are some constants $D_0, d_0 > 0$ such that for any $\varphi^{(1)}, \varphi^{(2)} \in \mathscr{C}_+^\circ$, $i \in \{1, \dots, n\}$,

$$\left|f_i(\varphi^{(1)}) - f_i(\varphi^{(2)})\right| \le D_0 \left|\mathbf{x}^{(1)} - \mathbf{x}^{(2)}\right| \left|1 + \mathbf{x}^{(1)} + \mathbf{x}^{(2)}\right|^{d_0} + D_0 \int_{-r}^{0} \left|\varphi^{(1)}(s) - \varphi^{(2)}(s)\right| \left|1 + \varphi^{(1)}(s) + \varphi^{(2)}(s)\right|^{d_0} \mu(ds),$$

where $\mathbf{x}^{(1)} := \varphi^{(1)}(0), \mathbf{x}^{(2)} := \varphi^{(2)}(0)$.

(ii) The conditions in (i) above holds with $f_i(\cdot)$ replaced by $g_i(\cdot)$ and $g_i^2(\cdot)$.

(iii) The inverse of matrix $(g_i(\varphi)g_j(\varphi)\sigma_{ij})_{n\times n}$ is uniformly bounded in $\mathscr{C}_+^\circ$.

Theorem 2.4

Under Assumptions 2.1, 2.2, 2.7, and 2.8, system (2.1) has a unique invariant measure concentrated on $\mathscr{C}_+^\circ$.

Functional Itô formula

- We need a "bona fide" operator in the function space $\mathscr{C}$. We use the function Itô's formula initiated by Dupire to do that.
- In a recent insightful work, Dupire (2009) proposed a method to extend the Itô formula to a functional setting using a pathwise functional derivative that quantifies the sensitivity of a functional variation in the endpoint of a path.

  This work encouraged subsequent development (for example by R. Cont and D.-A. Fournié).

Let $\mathbb{D}$ be the space of cadlag functions $f : [-r, 0] \mapsto \mathbb{R}^n$. For $\phi \in \mathbb{D}$, we define horizontal (time) and vertical (space) perturbations for $h \geq 0$ and $y \in \mathbb{R}^n$ as

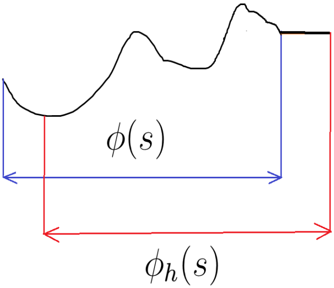
Let $\mathbb{D}$ be the space of cadlag functions $f : [-r,0] \mapsto \mathbb{R}^n$. For $\phi \in \mathbb{D}$, we define horizontal (time) and vertical (space) perturbations for $h \geq 0$ and $y \in \mathbb{R}^n$ as

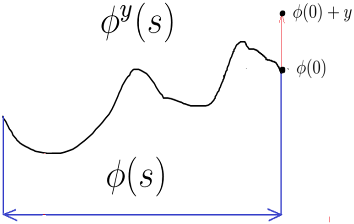
$$\phi_h(s) = \begin{cases} \phi(s+h) \text{ if } s \in [-r,-h], \\ \phi(0) \text{ if } s \in [-h,0], \end{cases}$$

and

$$\phi^y(s) = \begin{cases} \phi(s) \text{ if } s \in [-r,0), \\ \phi(0)+y, \end{cases}$$

respectively.





Let $V : \mathbb{D} \times \mathbb{N} \mapsto \mathbb{R}$. The horizontal derivative at (ϕ, i) and vertical partial derivative of V are defined as

Let $V : \mathbb{D} \times \mathbb{N} \mapsto \mathbb{R}$. The horizontal derivative at (ϕ, i) and vertical partial derivative of V are defined as

$$V_t(\phi, i) = \lim_{h \to 0} \frac{V(\phi_h, i) - V(\phi)}{h} \tag{3.1}$$

Let $V : \mathbb{D} \times \mathbb{N} \mapsto \mathbb{R}$. The horizontal derivative at (ϕ, i) and vertical partial derivative of V are defined as

$$V_t(\phi, i) = \lim_{h \to 0} \frac{V(\phi_h, i) - V(\phi)}{h} \tag{3.1}$$

$$\partial_i V(\phi, i) = \lim_{h \to 0} \frac{V(\phi^{he_i}, i) - V(\phi)}{h} \tag{3.2}$$

if these limits exist. In (3.2), e_i is the standard unit vector in $\mathbb{R}^n$ whose i-th component is 1 and other components are 0.

Let **F** be the family of function $V(\cdot) : \mathbb{D} \mapsto \mathbb{R}$ satisfying that

Let **F** be the family of function $V(\cdot) : \mathbb{D} \mapsto \mathbb{R}$ satisfying that

- V is continuous, that is, for any $\varepsilon > 0$, $(\phi) \in \mathbb{D} \times \mathbb{N}$, there is a $\delta > 0$ such that $|V(\phi) - V(\phi')| < \varepsilon$ as long as $\|\phi - \phi'\| < \delta$.

Let **F** be the family of function $V(\cdot) : \mathbb{D} \mapsto \mathbb{R}$ satisfying that

- V is continuous, that is, for any $\varepsilon > 0$, $(\phi) \in \mathbb{D} \times \mathbb{N}$, there is a $\delta > 0$ such that $|V(\phi) - V(\phi')| < \varepsilon$ as long as $\|\phi - \phi'\| < \delta$.
- The functions V_t, $V_x = (\partial_k V)$, and $V_{xx} = (\partial_{kl} V)$ exist and are continuous.

Let **F** be the family of function $V(\cdot) : \mathbb{D} \mapsto \mathbb{R}$ satisfying that

- V is continuous, that is, for any $\varepsilon > 0$, $(\phi) \in \mathbb{D} \times \mathbb{N}$, there is a $\delta > 0$ such that $|V(\phi) - V(\phi')| < \varepsilon$ as long as $\|\phi - \phi'\| < \delta$.
- The functions V_t, $V_x = (\partial_k V)$, and $V_{xx} = (\partial_{kl} V)$ exist and are continuous.
- V, V_t, $V_x = (\partial_k V)$ and $V_{xx} = (\partial_{kl} V)$ are bounded in each $B_R := \{(\phi, i) : \|\phi\| \leq R, i \leq R\}$, $R > 0$.

Let **F** be the family of function $V(\cdot) : \mathbb{D} \mapsto \mathbb{R}$ satisfying that

- V is continuous, that is, for any $\varepsilon > 0$, $(\phi) \in \mathbb{D} \times \mathbb{N}$, there is a $\delta > 0$ such that $|V(\phi) - V(\phi')| < \varepsilon$ as long as $\|\phi - \phi'\| < \delta$.
- The functions V_t, $V_x = (\partial_k V)$, and $V_{xx} = (\partial_{kl} V)$ exist and are continuous.
- V, V_t, $V_x = (\partial_k V)$ and $V_{xx} = (\partial_{kl} V)$ are bounded in each $B_R := \{(\phi, i) : \|\phi\| \leq R, i \leq R\}$, $R > 0$.

Let $V(\cdot) \in \mathbf{F}$, we define the following operator for a SFDE with drift and diffusion coefficients b, σ respectively

$$\begin{aligned} \mathscr{L} V(\phi) &= V_t(\phi) + V_x(\phi) b(\phi) + \frac{1}{2} \operatorname{tr}\Big(V_{xx}(\phi) A(\phi) \Big) \\ &= V_t(\phi) + \sum_{k=1}^{n} b_k(\phi) V_k(\phi) + \frac{1}{2} \sum_{k,l=1}^{n} a_{kl}(\phi) V_{kl}(\phi) \end{aligned} \tag{3.3}$$

where $A = (a_{kl}) = \sigma\sigma^\top$.

for any bounded stopping time $\tau_1 \le \tau_2$, we have the functional Itô formula:

$$\mathbb{E}V(X_{\tau_2}, \alpha(\tau_2)) = \mathbb{E}V(X_{\tau_1}, \alpha(\tau_1)) + \mathbb{E}\int_{\tau_1}^{\tau_2} \mathscr{L}V(X_s, \alpha(s))ds \qquad (3.4)$$

if the expectations involved exist.

Consider functionals of the form

$$\mathscr{V}(\phi, i) = f_1(\phi(0), i) + \int_{-r}^{0} g(t, i) f_2(\phi(t), i) dt.$$

where $f_2(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{N} \mapsto \mathbb{R}$ is a continuous function and $f_1(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{N} \mapsto \mathbb{R}$ is a function that is twice continuously differentiable in the first variable and $g(\cdot,\cdot) : \mathbb{R}_+ \times \mathbb{N} \mapsto \mathbb{R}$ be a continuously differentiable function in the first variable. Then at $(\phi) \in \mathscr{C} \times \mathbb{N}$ we have

$$\mathscr{V}_t(\phi) = g(0) f_2(\phi(0)) - g(-r) f_2(\phi(-r)) - \int_{-r}^{0} f_2(\phi(t)) dg(t),$$

$$\partial_k \mathscr{V}(\phi) = \frac{\partial f_1}{\partial x_k}(\phi(0)), \qquad \partial_{kl} \mathscr{V}(\phi) = \frac{\partial^2 f_1}{\partial x_k \partial x_l}(\phi(0)).$$

If

$$\mathscr{V}_2(\phi) = \int_{-r}^{0} g_2(s)\mu(ds) \int_{s}^{0} g_1(u) f_2(\phi(u)) du$$

where $f_2(\cdot,\cdot) : \mathbb{R}^n \times \mathscr{M} \mapsto \mathbb{R}$ is a function that is twice continuously differentiable in the first variable and g_1 be a continuously differentiable function in the first variable and g_2 is continuous. Then

$$\begin{aligned} \mathscr{V}_{2t}(\phi) = g_1(0,i) f_2(\phi(0)) \int_{-r}^{0} g_2(s)\mu(ds) \\ - \int_{-r}^{0} g_1(s) g_2(s) f_2(\phi(s))\mu(ds) \\ - \int_{-r}^{0} g_2(s)\mu(ds) \int_{s}^{0} f_2(\phi(t)) dg_1(t), \end{aligned} \tag{3.5}$$

$$\partial_k \mathscr{V}_2(\phi) = 0, \qquad \partial_{kl} \mathscr{V}(\phi) = 0.$$

$$V_\rho(\varphi) := \left(1 + \mathbf{c}^\top \mathbf{x}\right) \prod_{i=1}^{n} x_i^{\rho_i} \exp\left\{A_2 \int_{-r}^{0} \mu(ds) \int_{s}^{0} e^{\gamma(u-s)} h(\varphi(u))\, du\right\}.$$

Then, we have

$$\begin{aligned}
\mathscr{L} V_\rho^{p_0}(\varphi) \leq & p_0 V_\rho^{p_0}(\varphi) \Bigg[A_0 \mathbf{1}_{\{|\mathbf{x}|<M\}} - \gamma_0 - Ah(\mathbf{x}) \\
& - A_2 \gamma \int_{-r}^{0} \mu(ds) \int_{s}^{0} e^{\gamma(u-s)} h(\varphi(u))\, du \qquad (3.6) \\
& - \frac{\gamma_b}{2} \sum_{i=1}^{n} \left(|f_i(\varphi)| + g_i^2(\varphi) \right) \Bigg],
\end{aligned}$$

Theorem 3.1

For any initial condition $\phi \in \mathscr{C}_+$*, there exists a unique global solution of* (2.1)*. It remains in* $\mathscr{C}_+$ *(resp.* $\mathscr{C}_+^\circ$*), provided* $\phi \in \mathscr{C}_+$ *(resp.,* $\phi \in \mathscr{C}_+^\circ$*). Moreover, for sufficiently small* p_0, ρ*, we have*

$$\mathbb{E}_\phi V_\rho^{p_0}(\mathbf{X}_t) \le V_\rho^{p_0}(\phi) e^{A_0 p_0 t}. \tag{3.7}$$

In addition, if $\rho_i \ge 0, \forall i$*, then*

$$\mathbb{E}_\phi V_\rho^{p_0}(\mathbf{X}_t) \le V_\rho^{p_0}(\phi) e^{-\gamma_0 p_0 t} + \overline{M}_{p_0,\rho}, \tag{3.8}$$

where

$$\overline{M}_{p_0,\rho} := \frac{A_0}{\gamma_0} \sup_{\varphi \in \mathscr{C}_{V,M}} V_\rho^{p_0}(\varphi) < \infty \text{ provided } \rho_i \ge 0\ \forall i,$$

and

$$\mathscr{C}_{V,M} = \{\varphi \in \mathscr{C}_+ : A_2 \gamma \int_{-r}^0 \mu(ds) \int_s^0 e^{\gamma(u-s)} h(\varphi(u))\, du \le A_0 \text{ and } |\mathbf{x}| \le M\}.$$

Lemma 3.2

Under Assumption 2.2(b)*, there is a constant, still denoted by* H_1 *(for simplicity of notation) such that*

$$\int_r^T \mathbb{E}_\phi \left(\left(1 + \sum_{i=1}^n c_i X_i(t)\right)^{p_0} h_1(\mathbf{X}(t)) + \int_{-r}^0 \left(1 + \sum_{i=1}^n c_i X_i(t+s)\right)^{p_0} h_1(\mathbf{X}(t+s)) \right.$$

$$\leq H_1 \left(T + V_{\mathbf{0}}^{p_0}(\phi)\right), \ \forall T \geq r \tag{3.9}$$

by the min-max principle that Assumption 2.7 is equivalent to the existence of $\rho^* = (\rho_1^*, \ldots, \rho_n^*)$ with $\rho_i^* > 0$ such that

$$\inf_{\pi \in \mathscr{M}} \left\{ \sum_{i=1}^{n} \rho_i^* \lambda_i(\pi) \right\} > 0. \tag{3.10}$$

Rescaling ρ^* if needed, the pair $\ln V_{-\rho^*}(\mathbf{X}_t)$ and $\mathscr{L} \ln V_{-\rho^*}(\mathbf{X}_t)$ play the same role as the pair (V, H) in Benaim (2018).
With the same idea and some additional handling of cases when $X(t)$ is close to the boundary (i.e. at least one component is small) but $X(s), s < t$ is not, we can obtain the proof for persistence.

Existence and uniqueness of invariant probability measure

- For stochastic functional differential equations, the associated Markov semigroups are often not strong Feller, even in some simple cases, see Scheutzow (2005).

Existence and uniqueness of invariant probability measure

- For stochastic functional differential equations, the associated Markov semigroups are often not strong Feller, even in some simple cases, see Scheutzow (2005).
- We generalize an elegant method initially termed *asymptotic coupling* (in Hairer et.al. (2011) and later referred to as generalized coupling (see e.g. A. Kulik, M. Scheutzow(2018)) since the method can be used in a nonasymptotic manner.

Existence and uniqueness of invariant probability measure

- For stochastic functional differential equations, the associated Markov semigroups are often not strong Feller, even in some simple cases, see Scheutzow (2005).
- We generalize an elegant method initially termed *asymptotic coupling* (in Hairer et.al. (2011) and later referred to as generalized coupling (see e.g. A. Kulik, M. Scheutzow(2018)) since the method can be used in a nonasymptotic manner.
- The basic ideas is that, to "asymptotically" comple two processes X, Y, we construct $\widetilde{Y}$ such that the law of Y and $\widetilde{Y}$ are close (in some metric) and X and $\widetilde{Y}$ are asymptotically close to each other in a probability space.

Existence and uniqueness of invariant probability measure

- However, when apply to SFDE, the conditions needed in oder to successfully couple are often restrictive, such as the one-sided Lispchitz condition, etc.

Existence and uniqueness of invariant probability measure

- However, when apply to SFDE, the conditions needed in oder to successfully couple are often restrictive, such as the one-sided Lispchitz condition, etc.
- To relax the conditions, we need better estimates for the sample paths.
- We show that the sample path cannot exponentially grow. Then we put a weight $e^{-\varepsilon t}$ and make new coupling.

2D Systems

- Consider the two dimensional system

$$\begin{cases} dX(t) = X(t)f_1(X_t, Y_t)dt + X(t)g_1(X_t, Y_t)dE_1(t) \\ dY(t) = Y(t)f_2(X_t, Y_t)dt + Y(t)g_2(X_t, Y_t)dE_2(t) \end{cases} \quad (4.1)$$

2D Systems

- Consider the two dimensional system

$$\begin{cases} dX(t) = X(t)f_1(X_t, Y_t)dt + X(t)g_1(X_t, Y_t)dE_1(t) \\ dY(t) = Y(t)f_2(X_t, Y_t)dt + Y(t)g_2(X_t, Y_t)dE_2(t) \end{cases} \tag{4.1}$$

- Let δ be the Dirac measure concentrated at the origin (0-valued function). Its Lyapunov exponents given by

$$\lambda_i(\delta) = f_i(\mathbf{0}) - \frac{1}{2}g_i^2(\mathbf{0})\sigma_{ii}.$$

2D Systems

- Consider the two dimensional system

$$\begin{cases} dX(t) = X(t)f_1(X_t, Y_t)dt + X(t)g_1(X_t, Y_t)dE_1(t) \\ dY(t) = Y(t)f_2(X_t, Y_t)dt + Y(t)g_2(X_t, Y_t)dE_2(t) \end{cases} \tag{4.1}$$

- Let δ be the Dirac measure concentrated at the origin (0-valued function). Its Lyapunov exponents given by

$$\lambda_i(\delta) = f_i(\mathbf{0}) - \frac{1}{2} g_i^2(\mathbf{0})\sigma_{ii}.$$

- If $\lambda_1(\delta) < 0$, there is no i.p.m on $\mathscr{C}^\circ_{1+}$. (*the set of $\mathbb{R}^2_+$-valued functions whose second component is 0)*.
  If $\lambda_1(\delta) > 0$, there is an i.p.m μ_1 on $\mathscr{C}^\circ_{1+}$.
  Similarly, If $\lambda_2(\delta) < 0$, there is no i.p.m on $\mathscr{C}^\circ_{2+}$. If $\lambda_2(\delta) > 0$, there is an i.p.m μ_2 on $\mathscr{C}^\circ_{2+}$.

2D Systems

- Consider the two dimensional system

$$\begin{cases} dX(t) = X(t)f_1(X_t, Y_t)dt + X(t)g_1(X_t, Y_t)dE_1(t) \\ dY(t) = Y(t)f_2(X_t, Y_t)dt + Y(t)g_2(X_t, Y_t)dE_2(t) \end{cases} \tag{4.1}$$

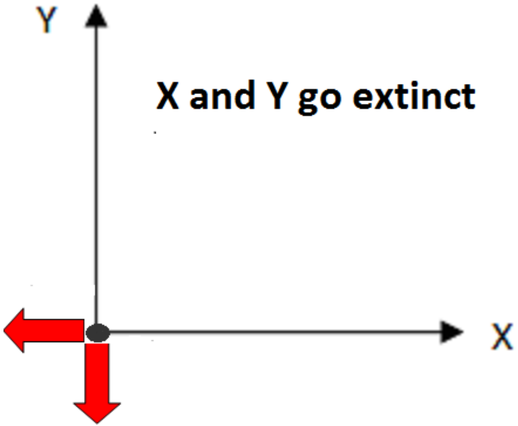
- Let δ be the Dirac measure concentrated at the origin (0-valued function). Its Lyapunov exponents given by

$$\lambda_i(\delta) = f_i(\mathbf{0}) - \frac{1}{2}g_i^2(\mathbf{0})\sigma_{ii}.$$

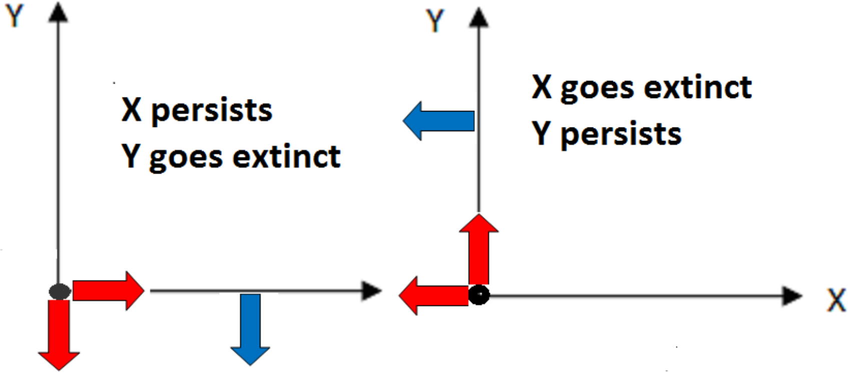
- If $\lambda_1(\delta) < 0$, there is no i.p.m on $\mathscr{C}^\circ_{1+}$. (*the set of* $\mathbb{R}^2_+$*-valued functions whose second component is 0)*.
  If $\lambda_1(\delta) > 0$, there is an i.p.m μ_1 on $\mathscr{C}^\circ_{1+}$.
  Similarly, If $\lambda_2(\delta) < 0$, there is no i.p.m on $\mathscr{C}^\circ_{2+}$. If $\lambda_2(\delta) > 0$, there is an i.p.m μ_2 on $\mathscr{C}^\circ_{2+}$.
- δ, μ_1, μ_2 are all possible ergodic measures on ∂C.

Classification

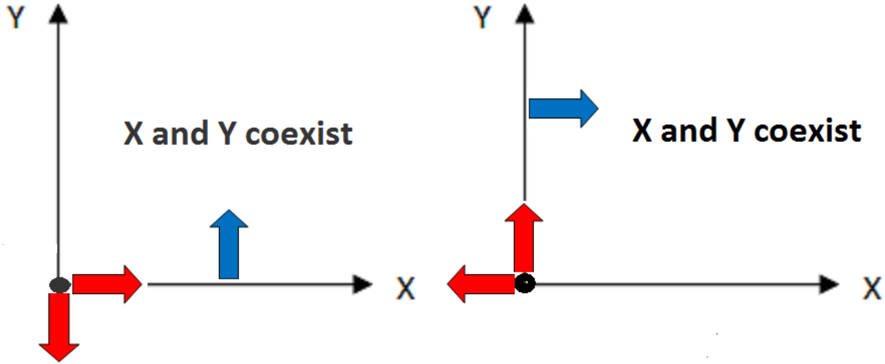
Note that We use the plane coordinates to represent 2 component functions



If $\lambda_i(\delta) < 0, i = 1,2$ then $X(t), Y(t)$ converge to 0 almost surely at the exponential rate $\lambda_1(\delta), \lambda_2(\delta)$ respectively.

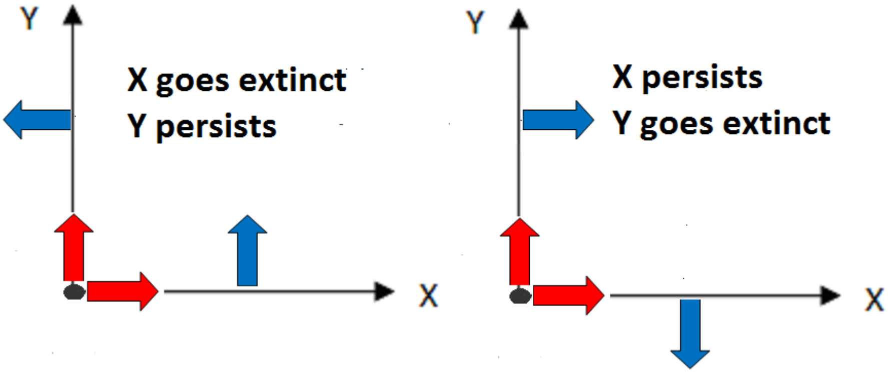


- $\lambda_1(\delta) > 0, \lambda_2(\delta) < 0$ and $\lambda_2(\mu_1) < 0$ then $Y(t)$ converges to 0 almost surely at the exponential rate $\lambda_2(\mu_1)$ (left)
- $\lambda_1(\delta) < 0, \lambda_2(\delta) > 0$ and $\lambda_1(\mu_2) < 0$ then $X(t)$ converges to 0 almost surely at the exponential rate $\lambda_1(\mu_2)$ (right)



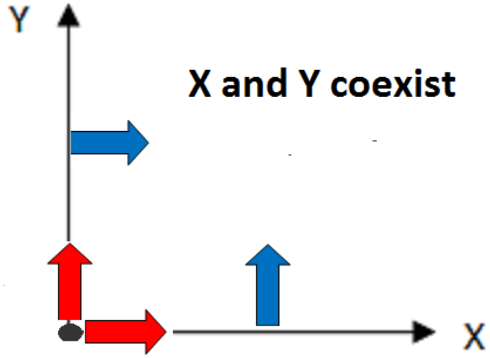
X and Y coexist if either of the following holds.

- $\lambda_1(\delta) > 0, \lambda_2(\delta) < 0$ and $\lambda_2(\mu_1) > 0$ (left)
- $\lambda_1(\delta) < 0, \lambda_2(\delta) > 0$ and $\lambda_1(\mu_2) < 0$ (right)

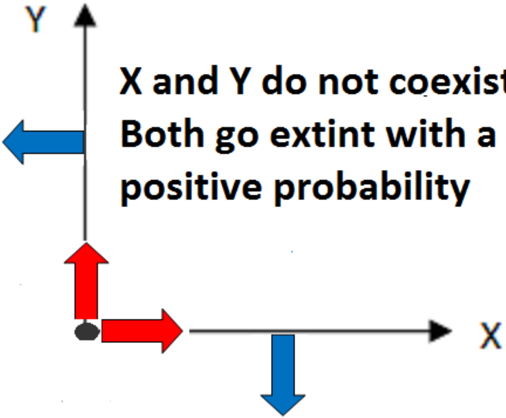


Suppose $\lambda_1(\delta) > 0, \lambda_2(\delta) > 0$.

- If $\lambda_2(\mu_1) < 0$ and $\lambda_1(\mu_2) > 0$ then $Y(t)$ converges to 0 almost surely at the exponential rate $\lambda_2(\mu_1)$ (right)
- If $\lambda_1(\mu_2) < 0$ and $\lambda_2(\mu_1) > 0$ then $X(t)$ converges to 0 almost surely at the exponential rate $\lambda_1(\mu_2)$ (left)



X and Y coexist if $\lambda_1(\delta) > 0, \lambda_2(\delta) > 0$, $\lambda_2(\mu_1) > 0$ and $\lambda_1(\mu_2) > 0$.



If $\lambda_1(\delta) > 0, \lambda_2(\delta) > 0$, $\lambda_2(\mu_1) < 0$ and $\lambda_1(\mu_2) < 0$, then $p_i^{x,y} > 0, i = 1, 2$ and $p_1^{x,y} + p_2^{x,y} = 1$ where

$$p_1^{x,y} = \mathbb{P}_{x,y}\left\{\lim_{t\to\infty} \frac{\ln X(t)}{t} = \lambda_1(\mu_2)\right\},$$

$$p_2^{x,y} = \mathbb{P}_{x,y}\left\{\lim_{t\to\infty} \frac{\ln Y(t)}{t} = \lambda_2(\mu_1)\right\}.$$

Stochastic delay Lotka-Volterra competitive model

$$
\begin{aligned}
dX_1(t) = X_1(t)\Big(a_1 - b_{11}X_1(t) - b_{12}X_2(t) - \widehat{b}_{11}X_1(t-r) \\
- \widehat{b}_{12}X_2(t-r)\Big)dt + X_1(t)dE_1(t), \\
dX_2(t) = X_2(t)\Big(a_2 - b_{21}X_1(t) - b_{22}X_2(t) - \widehat{b}_{21}X_1(t-r) \\
- \widehat{b}_{22}X_2(t-r)\Big)dt + X_2(t)dE_2(t).
\end{aligned}
\tag{4.2}
$$

$X_i(t)$ is the size of the species i at time t; $a_i > 0$ represents the growth rate of the species i; $b_{ii} > 0$ is the intra-specific competition of the i^{th} species; $b_{ij} \geq 0$, $(i \neq j)$ stands for the inter-specific competition; $\widehat{b}_{ij} > -b_{ii}$ $(i, j = 1, 2)$ (i.e. $\widehat{b}_{ij}$ can be negative); r is the delay time;

Stochastic delay Lotka-Volterra competitive model

- $\lambda_i(\delta^*) = a_i - \frac{\sigma_{ii}}{2}, i = 1, 2.$
- If $\lambda_i > 0$ there exists $\mu_i \in \mathscr{C}^\circ_{i+}$.
- Since $\lim_{t\to\infty} \frac{1}{t}\int_0^t X_i(s)ds = \lim_{t\to\infty} \frac{1}{t}\int_0^t X_i(s-\tau)ds$, we have $\int_{\mathscr{C}^\circ_{i+}} \varphi_i(-r)\pi_i(d\varphi) = \int_{\mathscr{C}^\circ_{i+}} \varphi_i(0)\mu_i(d\varphi)$
- From the equation $\lambda_i(\mu_i) = 0$, we have

$$\lambda_2(\mu_1) = a_2 - \frac{\sigma_{22}}{2} - \left(a_1 - \frac{\sigma_{11}}{2}\right) \cdot \frac{b_{21} + \widehat{b}_{21}}{b_{11} + \widehat{b}_{11}},$$

and

$$\lambda_1(\mu_2) = a_1 - \frac{\sigma_{11}}{2} - \left(a_2 - \frac{\sigma_{22}}{2}\right) \cdot \frac{b_{12} + \widehat{b}_{12}}{b_{22} + \widehat{b}_{22}}.$$

Stochastic delay Lotka-Volterra predator-prey model

We can apply the same method to a Lotka-Volterra predator-prey system with one prey and two competing predators as follows

$$\begin{cases} dX_1(t) = X_1(t)\Big\{a_1 - b_{11}X_1(t) - b_{12}X_2(t) - b_{13}X_3(t) \\ \qquad -\widehat{b}_{11}X_1(t-r) - \widehat{b}_{12}X_2(t-r) - \widehat{b}_{13}X_3(t-r)\Big\}dt + X_1(t)dE_1(t) \\ dX_2(t) = X_2(t)\Big\{-a_2 + b_{21}X_1(t) - b_{22}X_2(t) - b_{23}X_3(t) \\ \qquad -\widehat{b}_{21}X_1(t-r) - \widehat{b}_{22}X_2(t-r) - \widehat{b}_{23}X_3(t-r)\Big\}dt + X_2(t)dE_2(t) \\ dX_3(t) = X_3(t)\Big\{-a_3 + b_{31}X_1(t) - b_{32}X_2(t) - b_{33}X_3(t) \\ \qquad -\widehat{b}_{31}X_1(t-r) - \widehat{b}_{32}X_2(t-r) - \widehat{b}_{33}X_3(t-r)\Big\}dt + X_3(t)dE_3(t) \end{cases} \tag{4.3}$$

Thank you