Nonlocal Dispersal Equations with Almost Periodic Dependence

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Introduction

Main equation

In this talk, we discuss the principal spectral theory of the following linear nonlocal dispersal equation,

$$\partial_t u = \int_D \kappa(y-x)u(t,y)dy + a(t,x)u, \quad x \in \bar{D}, \tag{1}$$

where

- **(H1)** $\kappa(\cdot) \in C^1(\mathbb{R}^N, [0,\infty))$, $\kappa(0) > 0$, $\int_{\mathbb{R}^N} \kappa(x)dx = 1$, and $\exists\ \mu, M > 0$ s.t. $\kappa(x) \leq e^{-\mu|x|}$ and $|\nabla\kappa| \leq e^{-\mu|x|}$ for $|x| \geq M$.
- **(H2)** $a(t,x)$ is uniformly continuous in $(t,x) \in \times\bar{D}$, and is almost periodic in t uniformly with respect to $x \in \bar{D}$.
- **(H2)$'$** $a(t,x)$ is limiting almost periodic in t with respect to x and is also limiting almost periodic in x when $D = \mathbb{R}^n$.

Nonlocal dispersal equations are used to model the dynamics of populations having a long range dispersal strategy. This model of spatial spread is obtained by replacing the Laplacian in the usual reaction-diffusion equation $\partial_t u = \Delta u + ug(t,x,u)\ x \in \bar{D} \subset \mathbb{R}^n$, with an integral operator,

$$\partial_t u = \int_D \kappa(y-x)[u(t,y) - u(t,x)]dy + ug(t,x,u),\ x \in \bar{D}, \quad (2)$$

- $u(t,x)$ - population density function at time t and location x.
- $\kappa(y-x)$ - probability of a specie jumping from location y to x
- $\int_D \kappa(y-x)u(t,y)dy$ - total population arriving at position x from all other places $y \in D$
- $\int_D \kappa(y-x)u(t,x)dy$ – total population leaving location x.
- $ug(t,x,u)$ - reaction term (proliferation, death rate ...).

When $D = \mathbb{R}^n$, (2) reads as

$$\partial_t u = \int_D \kappa(y-x)u(t,y)dy - u(t,x) + ug(t,x,u),\ x \in \mathbb{R}^n,$$

which can be written as

$$\partial_t u = \int_D \kappa(y-x)u(t,y)dy + uf(t,x,u), \quad x \in \mathbb{R}^n \qquad (3)$$

with $f(t,x,u) = -1 + g(t,x,u)$

Dirichlet-type boundary condition on bounded $D \subset \mathbb{R}^n$:

$u(t,x) = 0$ for $x \in \bar{D}^c = \mathbb{R}^n \setminus \bar{D} \Longrightarrow$

$\int_{\mathbb{R}^n} \kappa(y-x)[u(t,y)-u(t,x)]dy = \int_D \kappa(y-x)u(t,y)dy - u(t,x)$

$\Longrightarrow$ (2) becomes

$$\partial_t u = \int_D \kappa(y-x)u(t,y)dy - u(t,x) + ug(t,x,u), \ \ x \in \bar{D},$$

Neumann-type boundary condition on bounded domain D:

$\int_{\mathbb{R}^n\setminus D} \kappa(y-x)u(t,y)dy = \int_{\mathbb{R}^n\setminus D} \kappa(y-x)u(t,x)dy \Longrightarrow$

$\int_{R^n} \kappa(y-x)[u(t,y)-u(t,x)]dy = \int_D \kappa(y-x)[u(t,y)-u(t,x)]dy$

$\Longrightarrow$ (2) becomes

$$\partial_t u = \int_D \kappa(y-x)[u(t,y)-u(t,x)]dy + ug(t,x,u), \ x \in \bar{D}.$$

Introduction

motivations

Solutions of

$$\begin{cases} u_t = \Delta u + ug(t,x,u), & x \in D \\ u = 0, & x \in \partial D, \end{cases}$$

and

$$\begin{cases} u_t = \Delta u + ug(t,x,u), & x \in D \\ \frac{\partial u}{\partial n} = 0, & x \in \partial D, \end{cases}$$

can be approximated by solutions of

$$\partial_t u = \int_D \kappa(y-x)u(t,y)dy - u(t,x) + ug(t,x,u),\ x \in \bar{D},$$

and

$$\partial_t u = \int_D \kappa(y-x)[u(t,y) - u(t,x)]dy + ug(t,x,u),\ x \in \bar{D},$$

respectively.

[W. Shen and X. Xie *DCDS*, (2015)]

The dynamics of solutions of (2) heavily relies on the probability kernel $\kappa(y-x)$ and the nonlinear term $f(t,x,u)$. In general, the dispersal range δ can be incorporated into the model by using the modified dispersal kernel $\tilde{\kappa}(z)=\frac{1}{\delta^n}\kappa(\frac{z}{\delta})$ for some $\delta>0$.

Observe that $u(t,x)\equiv 0$ is a solution of equation (2), referred to as the *trivial solution.* Linearizing (2) at this trivial solution with $f(t,x,0)=a(t,x)$ yields the linear nonlocal dispersal equation (1). The principal spectral theory of (1) has its own interests and also plays an important role in studying the asymptotic dynamics of (2).

Principal spectral theory: The behaviour (growth/decay rate) of positive solutions of

$$\partial_t u = \int_D \kappa(y-x)u(t,y)dy + a(t,x)u, \quad x \in \bar{D}, \tag{1}$$

can be determined by the "principal eigenvalue" in a certain sense.

Questions:

- How to define "principal eigenvalue"?
- What about its existence?
- How does temporal and spatial variation of $a(t,x)$ influence the " principal eigenvalue"?

Introduction

Definition

Almost Periodic functions

Let $E \subset \mathbb{R}^n$ and $a \in C(\mathbb{R} \times E, \mathbb{R})$; $a(t,x)$ is almost periodic in t uniformly w.r.t $x \in E$ if it is uniformly continuous in $(t,x) \in \mathbb{R} \times E$ and for any $\epsilon > 0$, $\exists\ l_\epsilon > 0$ such that any interval of length l_ϵ contains at least one point of the set

$$T(\epsilon) = \{\tau \in \mathbb{R} \,|\, |a(t+\tau, x) - a(t,x)| \leq \epsilon \; \forall\, t \in \mathbb{R},\, x \in E\}.$$

Limiting almost periodic functions

$f \in C(\mathbb{R} \times E, \mathbb{R})$ is limiting almost periodic in t uniformly w.r.t $x \in E$ if there is a sequence $f_n(t,x)$ of functions periodic in t such that $\lim_{n\to\infty} f_n(t,x) = f(t,x)$ uniformly in $(t,x) \in \mathbb{R} \times E$.

Known Results

Principal eigenvalue theory in the time independent case: $a(t,x) \equiv a(x)$

When $a(t,x) = a(x)$, the associated eigenvalue equation of

.

$$\partial_t u = \int_D \kappa(y-x)u(t,y)dy + a(x)u, \quad x \in \bar{D}, \qquad (1)$$

reads as

$$\int_D \kappa(y-x)u(y)dy + a(x)u(x) = \lambda u(x), \quad x \in \bar{D}. \qquad (4)$$

Consider (11) on $X = C^b_{\mathrm{unif}}(\bar{D})$. Let

$$(\mathcal{L}(a)u)(x) = \int_D \kappa(y-x)u(y)dy + a(x)u(x), \quad x \in \bar{D}.$$

$\lambda_p \in \mathbb{R}$ is called the **principal eigenvalue of (1)** if it is an algebraically simple eigenvalue of $\mathcal{L}(a)$ with an eigenfunction $\phi \in X^{++}$ and for every $\lambda \in \sigma(\mathcal{L}(a))$, $\mathrm{Re}(\lambda) < \lambda_p$. (λ_p, ϕ) is called an **eigenpair.**

Known results

Spectral theory of nonlocal dispersal operators

- If (λ_p, v) is an eigenpair, then $u(t,x) = e^{\lambda_p t} v(x)$ solves

$$\partial_t u = \int_D \kappa(y-x)u(t,y)dy + a(x)u, \quad x \in \bar{D}$$

- If (λ_p, v) is an eigenpair, then for any $u_0 \in X^+$,

$$\limsup_{t\to\infty} \frac{\ln \|u(t,\cdot;u_0)\|_X}{t} \leq \lambda_p,$$

where $u(t,x;u_0)$ is the solution of (1) with $u(0,x;u_0) = u_0(x)$.

- Unlike the random dispersal operators, even when $a(t,x) \equiv a(x)$, (4), may not have an eigenpair when $a(x)$ is not constant [W. Shen & A. Zhang (JDE - 2010)].

Known Results

Criteria for Existence of Principal eigenvalue in the case $a(t,x) \equiv a(x)$

We may abbreviate principal eigenvalue to PEVAL. The following are known:

- If there is some $x_0 \in \operatorname{Int}(D)$ satisfying $a(x_0) = \max_{x \in D} a(x)$ and the partial derivatives of $a(x)$ up to order $N-1$ at x_0 are zero, then (1) admits a PEVAL.
- If $\max_{x \in \bar{D}} a(x) - \min_{x \in \bar{D}} a(x) < \inf_{x \in D} \int_D \kappa(y-x)dy$ then (1) admits a PEVAL.
- If $\tilde{\kappa}(z) = \frac{1}{\delta^n}\kappa(\frac{z}{\delta})$ for some $\delta > 0$ and $\tilde{\kappa}(\cdot)$ with $\tilde{\kappa}(z) \geq 0$, $\operatorname{supp}(\tilde{\kappa}) = B_1(0) := \{z \in \mathbb{R}^n | \|z\| < 1\}$, $\int_{\mathbb{R}^n} \tilde{\kappa}(z)dz = 1$, and $\tilde{\kappa}(\cdot)$ being symmetric with respect to 0; then (1) admits a PEVAL provided that $0 < \delta \ll 1$.

[W. Shen and A. Zhang JDE (2010)]

Known Results

Existence of Principal eigenvalue in the case $a(t+T,x) \equiv a(t,x)$

When $a(t+T,x) = a(t,x)$, the associated eigenvalue equation of

.

$$\partial_t u = \int_D \kappa(y-x)u(t,y)dy + a(t,x)u, \quad x \in \bar{D}, \tag{1}$$

reads as

$$\begin{cases} -u_t + \int_D \kappa(y-x)u(t,y)dy + a(t,x)u(t,x) = \lambda u(t,x), \; x \in \bar{D} \\ u(t+T,x) = u(t,x), \; x \in \bar{D}. \end{cases}$$

Similar criteria for existence of PEVAL as in the time independent case have been established.

[N. Rawal and W. Shen, *JDDE*, (2012)]

Main Results

Principal eigenvalue in the almost periodic case

In general, the associated eigenvalue equation of

$$\partial_t u = \int_D \kappa(y-x)u(t,y)dy + a(t,x)u, \quad x \in \bar{D}, \qquad (1)$$

reads as

$$\begin{cases} \mathcal{L}(a)u(t,x) = \lambda u(t,x), & x \in \bar{D} \\ u(t,x) \text{ almost periodic in t} \end{cases}$$

where $\mathcal{L}(a) : Dom(\mathcal{L}) \subset \mathcal{X} := C^b_{unif}(\mathbb{R} \times \bar{D}) \to C^b_{unif}(\mathbb{R} \times \bar{D})$,

$$(\mathcal{L}(a)u)(t,x) = \; -u_t + \int_D \kappa(y-x)u(t,y)dy + a(t,x)u(t,x).$$

Main Results

Principal eigenvalue in the almost periodic case

It is very difficult to study the eigenvalue problem:

$$\begin{cases} \mathcal{L}(a)u(t,x) = \lambda u(t,x) \\ u(t,x) \ \text{ almost periodic} \end{cases}$$

We will introduce generalized principal eigenvalue, top Lyapunov exponents, and principal dynamical spectrum point to characterize the largest growth rate of the solutions of a linear evolution equation.

We also study the relations between these concepts.

Consider

$$\partial_t u = \int_D \kappa(y-x)u(t,y)dy + a(t,x)u, \quad x \in \bar{D}, \qquad (1)$$

- $\lambda_{PE}(a) = \sup\{\lambda \in \mathbb{R} \,|\, \exists\, \phi \in \mathcal{X},\ \inf_{t\in\mathbb{R}} \phi(t,x) \geq\not\equiv 0,\ (\mathcal{L}(a)\phi)(t,x) \geq \lambda\phi(t,x) \text{for } a.e.\ t \in \mathbb{R},\ \text{all } x \in \bar{D}\}$
- $\lambda'_{PE}(a) = \inf\{\lambda \in \mathbb{R} \,|\, \exists\, \phi \in \mathcal{X},\ \inf_{t\in\mathbb{R}, x\in\bar{D}} \phi(t,x) > 0,\ (\mathcal{L}(a)\phi)(t,x) \leq \lambda\phi(t,x) \text{ for } a.e.\ t \in \mathbb{R},\ \text{all } x \in \bar{D}\}$

$\lambda_{PE}(a)$ and $\lambda'_{PE}(a)$ are called generalized PEVALs of (1)

Remark: If the PEVAL λ_p exists, then $\lambda_{PE}(a) = \lambda'_{PE}(a) = \lambda_p$.

Consider

$$\partial_t u = \int_D \kappa(y-x)u(t,y)dy + a(t,x)u, \quad x \in \bar{D}, \qquad (1)$$

Let $\Phi(t,s;a)u_0 = u(t,\cdot;s,u_0)$ denote the solution operator of (1) on $X = C^b_{\rm unif}(\bar{D})$ (here $u(t,x;s,u_0)$ is the solution of (1) with $u(s,x;s,u_0) = u_0(x)$). Define

$$\lambda_{PL}(a) = \limsup_{t-s\to\infty} \frac{\ln \|\Phi(t,s;a)\|}{t-s}, \quad \lambda'_{PL}(a) = \liminf_{t-s\to\infty} \frac{\ln \|\Phi(t,s;a)\|}{t-s}.$$

$\lambda_{PL}(a)$ and $\lambda'_{PL}(a)$ are called the top Lyapunov exponents of (1).

Main Results

Relation between $\lambda'_{PL}(a)$ and $\lambda_{PL}(a)$

Theorem 1. [M. A. Onyido, W. Shen JDE (2021)]

- For any $u_0 \in X$ with $\inf_{x\in D} u_0(x) > 0$,

$$\lambda'_{PL}(a) = \lambda_{PL}(a) = \lim_{t-s\to\infty} \frac{\ln \|\Phi(t,s;a)u_0\|}{t-s}$$
$$= \lim_{t-s\to\infty} \frac{\ln \|\Phi(t,s;a)\|}{t-s}.$$

Theorem 2. [M. A. Onyido, W. Shen JDE (2021)]

(a) $\lambda'_{PE}(a) = \lambda_{PL}(a)$.

(b) $\lambda_{PE}(a) \leq \lambda_{PL}(a)$. If $a(t,x)$ satisfies (H2)$'$, then $\lambda_{PE}(a) = \lambda_{PL}(a)$.

(c) If $a(t,x) \equiv a(t)$, then $\lambda_{PE}(a) = \lambda'_{PE}(a) = \lambda_{PL}(a) = \hat{a} + \lambda_{PL}(0)$, where $\hat{a}(x) = \lim\limits_{T\to\infty} \frac{1}{T}\int_0^T a(t,x)dt$

Remark: Theorems 1 and 2(a) shows that we always have $\lambda_{PD}(a) = \lambda_{PL}(a) = \lambda'_{PE}(a)$. However, in Theorem 2(b) we only have an inequality.

Main Results

Characterization of $\lambda_{PE}(a)$ and $\lambda'_{PE}(a)$ in the time independent case

Theorem 3(a). [M. A. Onyido, W. Shen JDE (2021)]

(a) If $a(t,x) \equiv a(x)$ then $\lambda_{PE}(a) = \lambda'_{PE}(a)$

(b) If $a(t+T,x) \equiv a(t,x)$, then $\lambda_{PE}(a) = \lambda'_{PE}(a)$

Theorem 4. [M. A. Onyido, W. Shen JDE (2021)]

(1) $\lambda_{PE}(a) \geq \sup_{x\in D} \hat{a}(x)$. If $a(t,x)$ satisfies $(H2)'$, then $\lambda_{PE}(a) \geq \lambda_{PE}(\hat{a}) \geq \sup_{x\in D} \hat{a}(x)$.

(2) If D is bounded, $a(t,x) \equiv a(x)$, and $\kappa(\cdot)$ is symmetric, then

$$\lambda_{PE}(a) \geq \bar{a} + \frac{1}{|D|}\int_D \int_D \kappa(y-x)dydx.$$

(3) If $D = \mathbb{R}^N$, $a(t,x) \equiv a(x)$ is almost periodic in x, and $\kappa(\cdot)$ is symmetric, then

$$\lambda_{PE}(a) \geq \bar{a} + 1,$$

where $\bar{a} = \frac{1}{|D|}\int_D \hat{a}(x)dx$ when D is bounded, and $\bar{a} = \lim_{q_1,\cdots,q_N\to\infty} \frac{1}{q_1\cdots q_N}\int_0^{q_N}\cdots\int_0^{q_1} \hat{a}(x_1,\cdots,x_N)dx_1\cdots dx_N$ when $D = \mathbb{R}^N$.

Main Results

Effects of time and space variations on $\lambda_{PL}(a)$

Theorem 4. [M. A. Onyido, W. Shen JDE (2021)]

(1) If D is bounded or $D = \mathbb{R}^N$ and a satisfies $(H2)'$, then $\lambda_{PL}(a) \geq \lambda_{PL}(\hat{a}) \geq \sup_{x \in D} \hat{a}(x)$.

(2) If D is bounded and $\kappa(\cdot)$ is symmetric, then $\lambda_{PL}(a) \geq \bar{a} + \frac{1}{|D|} \int_D \int_D \kappa(y-x)dydx$.

(3) If $D = \mathbb{R}^N$, $a(t,x)$ is almost periodic in x uniformly with respect to $t \in \mathbb{R}$, and $\kappa(\cdot)$ is symmetric, then $\lambda_{PL}(a) \geq \bar{a} + 1$, where $\bar{a} = \frac{1}{|D|} \int_D \hat{a}(x)dx$ when D is bounded, and $\bar{a} = \lim_{q_1, \cdots, q_N \to \infty} \frac{1}{q_1 \cdots q_N} \int_0^{q_N} \cdots \int_0^{q_1} \hat{a}(x_1, \cdots, x_N)dx_1 \cdots dx_N$ when $D = \mathbb{R}^N$.

Spectral theory of nonlocal dispersal operators

Remarks

- The generalized PEVAL for nonlocal dispersal operators was introduced and studied in the time independent case in
  [J. Coville JDE (2010)]
  [H. Berestycki, J. Coville, H. Vo, JFA (2016)]
- For the time periodic case, the generalized PEVAL was studied in
  [Z. Shen, H-H Vo, JDE (2019)]
  [Y-H Su, W-T Li, Y. Lou, F-Y Yang JDE (2020)]

They obtained some criteria for equality in those cases.
Almost periodic case: it remains open whether $\lambda_{PE}(a) = \lambda'_{PE}(a)$.
If $\lambda_{PE}(a) = \lambda'_{PE}(a)$, under what condition there is a positive function $\phi(t,x)$, such that for all $t \in \mathbb{R}, x \in D$,

$$-\phi_t + \int_D \kappa(y-x)\phi(t,y)dy + a(t,x)\phi(t,x) = \lambda_{PE}(a)\phi(t,x).$$

THANK YOU