Equivalence of Control Distances

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(Based on joint works with F. Baudoin, Q. Feng, X. Geng and S. Tindel.)

Consider

$$X_t = x + \sum_{\alpha=1}^{d} \int_0^t V_\alpha(X_s) dB_s^\alpha.$$

Here,

- B is a d-dimensional fBm with Hurst parameter $H > \frac{1}{4}$.
- $V_\alpha : \mathbb{R}^n \to \mathbb{R}^n$ are C_b^∞-vector fields on $\mathbb{R}^n$ that satisfy the uniform hypoelliptic condition.
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The *Control Distance* associated to the system is

$$d_H^2(x, y) = \inf_{h \in \mathcal{H}, \Phi(h)_1 = y} \left\{ \|h\|_{\mathcal{H}}^2 \right\}.$$

- When $H = \frac{1}{2}$: sub-Riemannian distance induced by V_α's. (Carathéodory 1909, Chow 1939, Rashovsky 1938)
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Main Result

Theorem (Geng-O-Tindel, '21)

Locally when $y \approx x$, we have

$$d_H(x, y) \approx d_{\frac{1}{2}}(x, y).$$

Note that $d_{\frac{1}{2}}$ is the sub-Riemannian distance induced by V_α's.

A special case

The *signature* of B (up to order N) is defined by

$$S_N(B)_t = \sum_{k=0}^{N} \int_{0<t_1<\cdots<t_k<t} dB_{t_1} \otimes \cdots \otimes dB_{t_k}, \quad t \in [0,1].$$

- An element in $T^N(\mathbb{R}^d) = \mathbb{R} \oplus \mathbb{R}^d \oplus (\mathbb{R}^d)^{\otimes 2} \oplus \cdots \oplus (\mathbb{R}^d)^{\otimes N}$.
- More precisely, $S_N(B)_t \in G^N(\mathbb{R}^d) \subset T^N(\mathbb{R}^d)$.
- It satisfies a canonical equation

$$dS_N(B)_t = S_N(B)_t \otimes dB_t,$$

with initial point $e = (1, 0, ..., 0)$, the group identity of $G^N(\mathbb{R}^d)$.

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Theorem (Baudoin-Feng-O, '20)

For all $g \in G^N(\mathbb{R}^d)$, we have

$$d_H(e,g) \approx d_{\frac{1}{2}}(e,g) = \|g\|_{\mathrm{CC}}.$$

In what follows, we use notation $\|g\|_H = d_H(e,g)$.

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Key Observation

For any $\lambda > 0$, there is dilation operator δ_λ on $G^N(\mathbb{R}^d)$ defined by, for any $g = (1, g_1, ..., g_N) \in G^N(\mathbb{R}^d)$

$$\delta_\lambda g = (1, \lambda g_1, ..., \lambda^N g_N).$$

We have

$$||\delta_\lambda g||_H = \lambda ||g||_H$$

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$$\|\delta_\lambda g\|_{\mathrm{CC}} = \lambda \|g\|_{\mathrm{CC}}.$$

Back to general case:

$$X_t = x + \sum_{\alpha=1}^{d} \int_0^t V_\alpha(X_s)dB_s^\alpha.$$

Locally, we can write

$$X_t = x + \sum_{k=1}^{N} \sum_{i_1,\dots,i_k=1}^{d} V_{i_1,\dots,i_k}(x) \int_{0<t_1<\cdots<t_k<t} dB_{t_1}^{i_1} \otimes \cdots \otimes dB_{t_k}^{i_k}$$

$$+R(t,x,B).$$

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Thank you!