Concentration inequalities for ultra log-concave distributions

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(joint work with Arnaud Marsiglietti & James Melbourne)

FRONTIER PROBABILITY DAYS - 2021

1. Introduction
2. Problem & Motivation
3. Results
4. Remarks
5. Proof Techniques

Definition:
A random variable X on $\mathbb{Z}$ is said to be log-concave if its probability mass function p satisfies,

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A random variable X taking values in $\{0, 1, 2, ...\}$ is said to be ultra log-concave (ULC) if its probability mass function p satisfies,

$$p^2(n) \geq \frac{n+1}{n}\, p(n+1)\, p(n-1) \text{ \textit{for all} } n \geq 1\,.$$

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- The number of independent k-subsets of a claw-free finite graph. (**Hamidoune '90, Chudnowsky & Seymour '07**)

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What does $D(t)$ look like?

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where $h(x) = 2\,\frac{(1+x)\log(1+x) - x}{x^2}$ defined on $[-1, \infty)$.

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$\{V_i(K) : i = 0, 1, 2..., n\}$ is ultra log-concave (**McMullen '91**).

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Since the sum of independent ultra log-concave random variables is ultra log-concave, the theorem applies to $X = \sum_{i=1}^{n} X_i$, with X_i's independent ultra log-concave.

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$$Var[Z_k] \leq n\,.$$

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- For the variance bound, expand the inequality in lemma, and then, take the limit $t \to 0$.

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Consider the following set.

$\mathcal{P}_h^{\gamma}([M, N]) = \{\mathbb{P}_X \in \mathcal{P}([M, N])$: X log-concave w.r.t γ, $\mathbb{E}[h(X)] \geq 0\}$.

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If $\mathbb{P}_X \in \text{Conv}(\mathcal{P}_h^{\gamma}([M,N]))$ is an extreme point, then its proba. mass function f w.r.t γ satisfies,

$$f(n) = Cp^n q(n)\, 1_{[k,l]} \ , \qquad \textbf{(}\star\textbf{)}$$

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Let $\Phi : \mathcal{P}_h^\gamma([M,N]) \to \mathbb{R}$ be convex. Then,

$$\sup_{\mathbb{P}_X \in \mathcal{P}_h^\gamma([M,N])} \Phi(\mathbb{P}_X) \leq \sup_{\mathbb{P}_X \in \mathcal{A}_h^\gamma([M,N])} \Phi(\mathbb{P}_X)\,,$$

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where $\mathcal{A}_h^\gamma([M,N]) = \mathcal{P}_h^\gamma([M,N]) \cap \{\mathbb{P}_X$: X with PMF as in $(\star)\}$

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where $\frac{1}{C} = \sum_{n=k}^{l} \frac{p^n}{n!}$.

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- f is convex.

Thank you! Any questions?