Frontier Probability days

Pete Rigas, December 4

Recent arXiv posting: adaptation of 2019 arguments for crossing probabilities

Mathematics > Probability

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Renormalization of crossing probabilities in the dilute Potts model

Pete Rigas

A recent paper due to Duminil-Copin and Tassion from 2019 introduces a novel argument for obtaining estimates on horizontal crossing probabilities of the random cluster model, in which a range of four possible behaviors is established. To apply the novel renormalization of crossing probabilities that the authors propose can be studied in other models of interest that are not self-dual, we collect results to formulate vertical and horizontal strip, and renormalization, inequalities for the dilute Potts model, whose measure is obtained from the high temperature expansion of the loop $O(n)$ measure supported over the hexagonal lattice in the presence of two external fields. The dilute Potts model was originally introduced in 1991 by Nienhuis and is another model that enjoys the RSW box crossing property in the Continuous Critical phase, which is one of the four possible behaviors that the model is shown to enjoy. Through a combination of the Spatial Markov Property (SMP) and Comparison between Boundary Conditions (CBC) of the high-temperature spin measure, four phases of the dilute Potts model can be analyzed, exhibiting a class of boundary conditions upon which the probability of obtaining a horizontal crossing is significantly dependent. The exponential factor that is inserted into the Loop $O(n)$ model to quantify properties of the high-temperature phase is proportional to the summation over all spins, and the number of monochromatically colored triangles over a finite volume, which is in exact correspondence with the parameter of a Boltzmann weight introduced in Nienhuis' 1991 paper detailing extensions of the q-state Potts model. Asymptotically, in the infinite volume limit we obtain strip and renormalization inequalities that provide conditions on the RSW constants $1-c$ and c.

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Preliminaries

- Classical Russo-Seymour-Welsh (RSW) arguments have historically been applied to Bernoulli percolation, which is a self dual model computed in Kesten's seminal 1980 paper
- It is of research interest to determine whether RSW results can still be obtained for other models in statistical mechanics that are not self-dual

Presentation overview

- Description of renormalization of crossing probabilities arguments from the random cluster model from 2019,
- Applications of the argument to the dilute Potts model originally introduced by Nienhuis in 1991,

- Main goal: Classify four regimes of behavior of the dilute Potts model, from analyzing estimates initially introduced for the random cluster model

Our prize

Russo-Seymour-Welsh (RSW) result

1 Support of the random-cluster measure

2 Horizontal crossing event

$$c_\rho \leq \phi^{\xi}_{[-n,(\rho+1)n]\times[-n,2n]}[\mathcal{H}_{[0,\rho n]\times[0,n]}] \leq 1 - c_\rho.$$

3 Boundary condition independence

4 Crossing probabilities degenerate to neither 0 nor 1

Random cluster model

1 Ratio of probabilities of obtaining open to closed edges in random subgraph over the square lattice

2 Number of clusters q

Measure on random subgraphs of the square lattice

$$\phi_G^{\xi}[\omega] = \frac{1}{Z_G^{\xi}} \left(\frac{p}{1-p}\right)^{|\omega|} q^{k_{\xi}(\omega)},$$

Definition 1 ([14, Theorem 2 & Corollary 3]): The *strip density* corresponding to the measure across a rectangle $\mathcal{R}$ of aspect ratio $[0, \alpha n] \times [-n, 2n]$ with free boundary conditions is of the form,

Strip densities for wired and free boundary conditions

$$p_n = \limsup_{\alpha \to \infty} \left(\phi^0_{[0,\alpha n]\times[-n,2n]} \left[\mathcal{H}_{[0,\alpha n]\times[0,n]} \right] \right)^{\frac{1}{\alpha}},$$

Horizontal crossing density taken under free boundary conditions

where $\mathcal{H}$ denotes the event that $\mathcal{R}$ is crossed horizontally, whereas for the measure supported over $\mathcal{R}$ with wired boundary conditions, the crossing density is of the form,

$$q_n = \limsup_{\alpha \to \infty} \left(\phi^1_{[0,\alpha n]\times[-n,2n]} \left[\mathcal{V}^c_{[0,\alpha n]\times[0,n]} \right] \right)^{\frac{1}{\alpha}},$$

Vertical crossing density taken under wired boundary conditions

where $\mathcal{V}^c$ denotes the complement of a vertical crossing across $\mathcal{R}$.

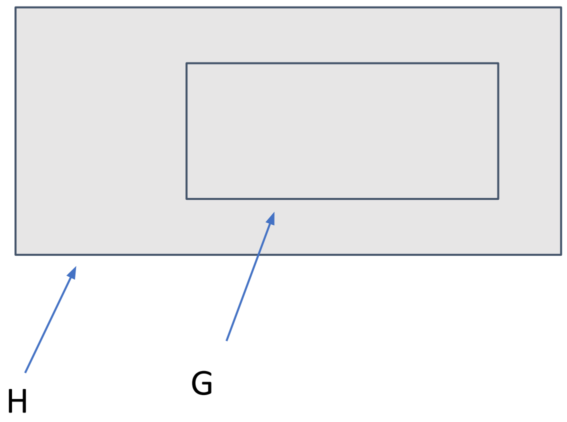
Properties of the random cluster measure

A, B are increasing events

Ingredients

$$\phi_G^\xi[\mathcal{A}\cap\mathcal{B}] \ge \phi_G^\xi[\mathcal{A}]\phi_G^\xi[\mathcal{B}]. \quad \text{(FKG)}$$

$$\phi_G^\xi[\cdot_{|F}|\omega_e = \xi_e \forall e \in E \setminus F] = \phi_H^{\xi\cup\xi'}, \quad \text{(SMP)}$$



$$\phi_G^\xi[\mathcal{A}] \le \phi_G^\zeta[\mathcal{A}]. \quad \text{(CBC)}$$

Special case of CBC property with a maximum of k new clusters wired on the boundary of finite volume

$$\phi_G^\xi[\mathcal{A}] \le q^{\max\{k_\xi(\omega)-k_\zeta(\omega):\omega\}-\min\{k_\xi(\omega)-k_\zeta(\omega):\omega\}}\phi_G^\zeta[\mathcal{A}].$$

Random cluster strip & renormalization inequalities

Objective: 'Renormalize' horizontal and vertical crossing probabilities on smaller scales with crossings on larger scales

Strip inequality

Lemma 1 ([14, Lemma 12]) There exists a constant $C > 0$ such that for every integer $\lambda \geq 2$, and for every $n \in 3\mathbf{N}$,

Horizontal crossing probability

$$p_{3n} \geq \frac{1}{\lambda^C} q_n^{3+\frac{3}{\lambda}},$$

Vertical crossing probability

while a similar inequality holds between horizontal and the complement of vertical crossing probabilities of the complement $\mathcal{V}^c$ across $\mathcal{R}$, which takes the form,

$$q_{3n} \geq \frac{1}{\lambda^C} p_n^{3+\frac{3}{\lambda}}.$$

Renormalization inequality

Lemma 2 ([14, Lemma 15]) There exists $C > 0$ such that for every integer $\lambda \geq 2$ and for every $n \in 3\mathbf{N}$,

Horizontal crossing probability

$$p_{3n} \leq \lambda^C p_n^{3-\frac{9}{\lambda}} \quad \& \quad q_{3n} \leq \lambda^C q_n^{3-\frac{9}{\lambda}}.$$

Vertical crossing probability

Introducing the Loop O(n) model

Probability measure on loops

The Gibbs measure on a random configuration σ in the loop $O(n)$ model is of the form,

Boundary conditions

Support of the measure

$$\mathbf{P}^{\xi}_{\Lambda,x,n}(\sigma) = \frac{x^{e(\sigma)} n^{l(\sigma)}}{Z^{\xi}_{\Lambda,x,n}}, \qquad \text{(Loop measure)}$$

where $\sigma(e)$ denotes the number of edges, $\sigma(l)$ the number of loops, $\Lambda \subset \mathbf{H}$, $\xi \in \{0, 1, 0/1\}$ and $Z^{\xi}_{\Lambda,e,n}$

High-temperature probability measure on spins

First external field

Second external field

$$\mu^{\tau}_{G,x,n}(\sigma') = \frac{n^{k(\sigma')} x^{e(\sigma')} \exp\left(hr(\sigma') + \frac{h'}{2} r'(\sigma') \right)}{Z^{\tau}_{G,x,n}}, \qquad \text{(Spin Measure)}$$

where $\tau \in \{-1,+1\}^{\mathbf{T}}$, $\Sigma(G,\tau)$ is the set of spin configurations coinciding with σ' outside of G, $r(\sigma') = \sum_{u \in G} \sigma'_u$ is the summation of spins inside G, $r'(\sigma') = \sum_{\{u,v,w\} \in G} \sigma'_u \mathbf{1}_{\sigma'_u = \sigma'_v = \sigma'_w}$ is the difference

Correspondence between the high-temperature measure and Nienhuis' dilute Potts model from 1991

$$\mu^{\mathcal{T}}_{G,x,n}(\sigma') = \frac{n^{k(\sigma')} x^{e(\sigma')} \exp\left(hr(\sigma') + \frac{h'}{2} r'(\sigma') \right)}{Z^{\mathcal{T}}_{G,x,n}} , \qquad \text{(Spin Measure)}$$

Correspondence

1. Dual monochromatically colored triangles over **H*=T**
2. Summation of all spins over finite volume
3. Number of edges and loops in the sampled configuration

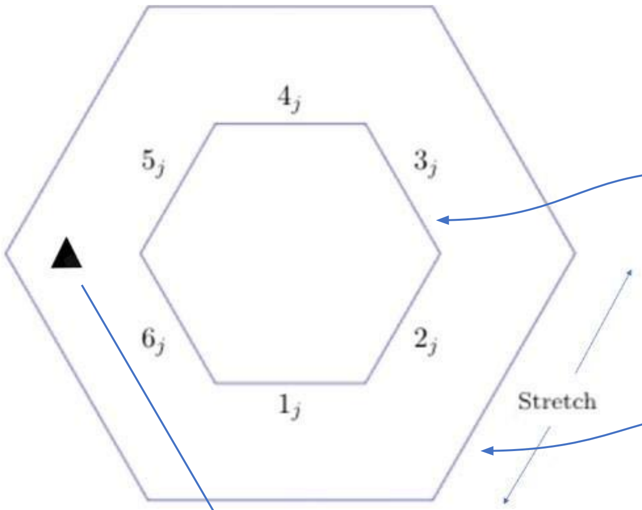
$$\mathscr{W} \equiv \prod_{i \sim j} \left(1 - t_i t_j + t_i t_j \delta_{s_j, s_k} \right) \exp\left(K_1 \sum_i t_i + K_2 \sum_{i \sim j} t_i t_j + K_3 \sum_{i \sim j \sim k} t_i t_j t_k \right) ,$$

spins from the q-state Potts model

Positive parameters

where the quantities in the nearest-neighbor product above include the occupation number t_j, which is either equal to 0 or 1, corresponding to whether a triangle is colored white and black, respectively.

dilute Potts CBC & SMP



Inner domain

Outer domain

S CBC

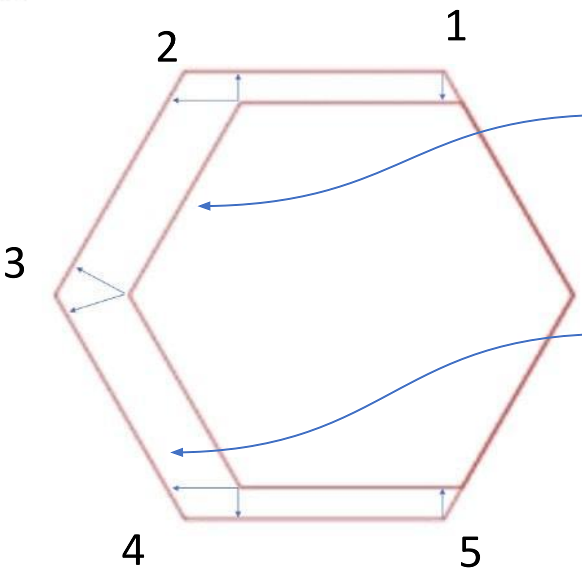
$$\mu^{\tau}_{G,n,x,h,h'}[A] \le \mu^{\tau'}_{G,n,x,h,h'}[A]$$

Difference in number of edges

Difference in number of connected components

$$\mu^{\tau}_{\mathbf{H}}[\mathcal{A}'] \le n^{k_{\tau'}(\sigma)-k_{\tau}(\sigma)}\, x^{e_{\tau'}(\sigma)-e_{\tau}(\sigma)} \exp\Big(h\big(r_{\tau'}(\sigma)-r_{\tau}(\sigma)\big) + \frac{h'}{2}\big(r_{\tau'}(\sigma')-r_{\tau}(\sigma')\big) \Big)\, \mu^{\tau'}_{\mathbf{H}}[\mathcal{A}'] ,$$

S SMP



Inner domain **I**

Outer domain **O**

$$\mu^{\tau}_{\mathcal{I}}[\;\cdot\,|_{\mathcal{I}} \mid \sigma'_{\mathcal{I}} \equiv \sigma'_{\mathcal{O}} \;\text{ for } \mathcal{I}\cap\mathcal{O}\,] = \mu^{\tau\cup\tau'}_{\mathcal{O}}$$

FKG

A, B are increasing events

$$\mu^{\tau}_{\mathbf{H}}[\,A\cap B\,] \ge \mu^{\tau}_{\mathbf{H}}[\,A\,]\;\mu^{\tau}_{\mathbf{H}}[\,B\,]$$

Lower bound the probability of the intersection of horizontal or vertical crossing events

dilute Potts model quadrichotomy

Theorem 2^* (*hexagonal crossing probabilities*): For the dilute regime $x \leq \frac{1}{\sqrt{n}}$, aspect ratio n of a regular hexagon $H \subset \mathcal{S}_T$, $c > 0$, and horizontal crossing $\mathcal{H}$ across H, estimates on crossing probabilities with free, wired or mixed boundary conditions satisfy the following criterion in the following **4** possible behaviors.

Four phases of interest

- *Subcritical*: For every $n \geq 1$, under wired boundary conditions, $\mu^{1}_{G,x,n}[\mathcal{H}] \leq \exp(-cn)$,
- *Supercritical*: For every $n \geq 1$, under free boundary conditions, $\mu^{0}_{G,x,n}[\mathcal{H}] \geq 1 - \exp(-cn)$,
- *Continuous Critical* (*Russo-Seymour-Welsh property*): For every $n \geq 1$, independent of boundary conditions τ, $c \leq \mu^{\tau}_{G,x,n}[\mathcal{H}] \leq 1 - c$,
- *Discontinuous Critical*: For every $n \geq 1$, $\mu^{1}_{G,x,n}[\mathcal{H}] \geq 1 - \exp(-cn)$ for free boundary conditions, while $\mu^{0}_{G,x,n}[\mathcal{H}] \leq \exp(-cn)$ for wired boundary conditions.

dilute Potts horizontal & vertical strip densities

Definition 1* (*dilute Potts horizontal and vertical strip densities*): For $n \geq 1$, $x \leq \frac{1}{\sqrt{n}}$, $nx^2 \leq \exp(-|h'|)$, and (n, x, h, h'), with external fields h, h', the strip density for horizontal crossings across $\mathcal{H}_j$ under the Spin measure with free boundary conditions is,

- Free boundary conditions. no edges wired together
- Horizontal crossing event

$$p_n^{\mu} = \text{lim sup}_{\rho \to \infty} \bigg(\mu^{0}_{[0\,,\,\rho n] \times_H [0,\,\lambda\text{Stretch}]} \bigg[\mathcal{H}_{[0\,,\,\rho n] \times_H [0\,,\,\lambda\text{Stretch}\,]} \bigg] \bigg)^{\frac{1}{\rho}} ,$$

while for vertical crossings across $\mathcal{H} + j$, under the Spin measure with wired boundary conditions, is,

$$q_n^{\mu} = \text{lim sup}_{\rho \to \infty} \bigg(\mu^{1}_{[0\,,\,\rho n] \times_H [0,\,\lambda\text{Stretch}]} \bigg[\mathcal{V}^{c}_{[0\,,\,\rho n] \times_H [0\,,\,\lambda\text{Stretch}]} \bigg] \bigg)^{\frac{1}{\rho}} .$$

Renormalization of crossing probabilities statement

Lemma 1* (7, *hexagonal strip density inequalities*): In the **Non**(*Subcritical*) and **Non**(*Supcritical*) regimes, for every integer $\lambda \geq 2$, and every $n \in \text{Stretch}\,\mathbf{N}$, there exists a positive constant C satisfying,

$$p_{\text{Stretch } n} \geq \frac{1}{\lambda^C}\left(q_{\text{Stretch } n}\right)^{\text{Stretch}+\frac{\text{Stretch}}{\lambda}},$$

while a similar upper bound for vertical crossings is of the form,

$$q_{\text{Stretch } n} \geq \frac{1}{\lambda^C}\left(p_{\text{Stretch } n}\right)^{\text{Stretch}+\frac{\text{Stretch}}{\lambda}}.$$

Lemma 2* (9, *hexagonal renormalization inequalities*): In the **Non**(*Subcritical*) and **Non**(*Supcritical*) regimes, for every integer $\lambda \geq 2$, and every $n \in \text{Stretch}\,\mathbf{N}$, there exists a positive constant C satisfying,

$$p_{\text{Stretch } n} \geq \lambda^C\left(p_{\text{Stretch } n}\right)^{\text{Stretch}-\frac{n\,\text{Stretch}}{\lambda}} \;\&\; q_{\text{Stretch } n} \geq \lambda^C\left(q_{\text{Stretch } n}\right)^{\text{Stretch}-\frac{n\,\text{Stretch}}{\lambda}},$$

Similar ingredients for constructing auxiliary symmetric domains from random cluster inequalities

Exponent to which the crossing probability is raised is dependent upon some integer and the dilation parameter Stretch

RSW results proof sketch

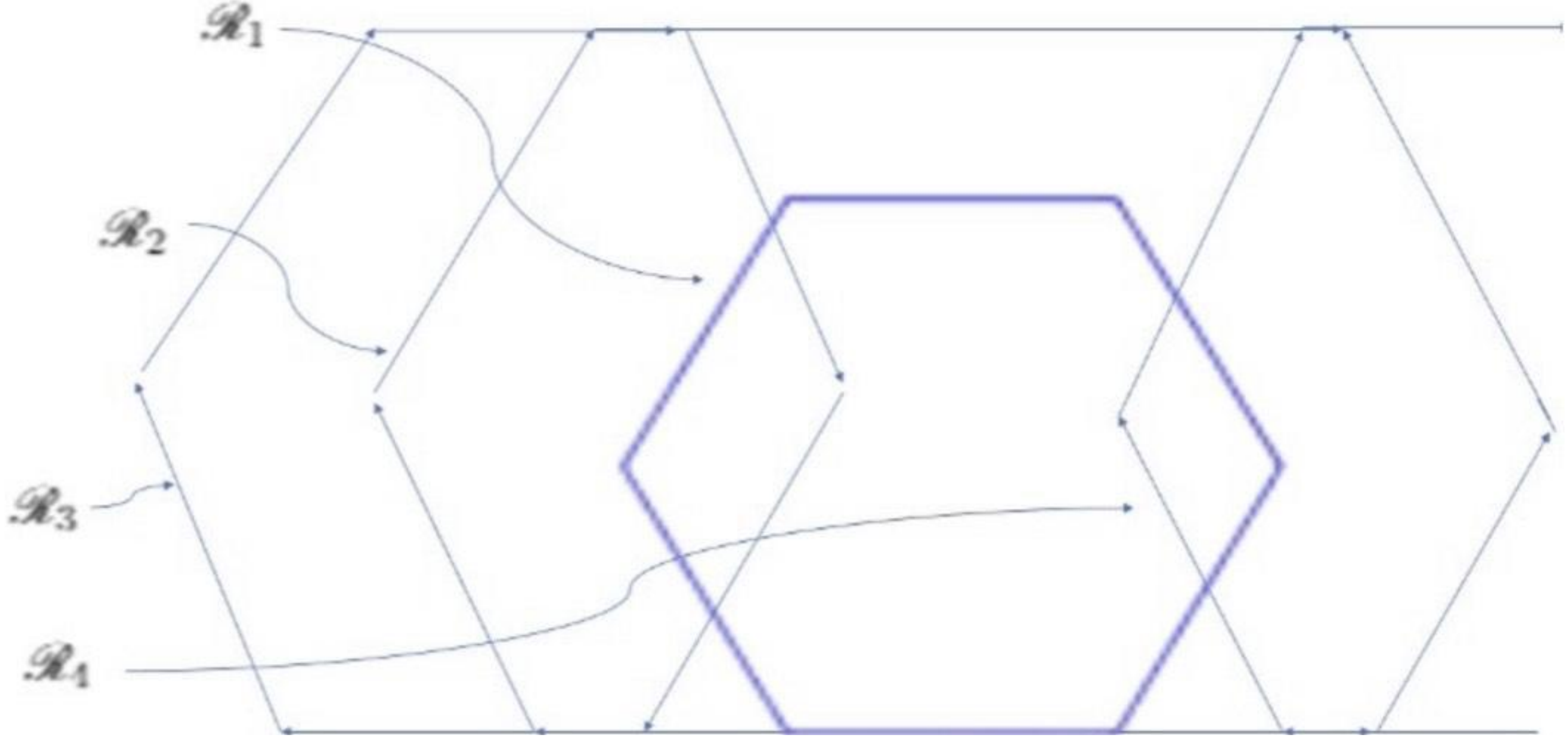
Recall the structure of classical RSW results for the random cluster model,

$$c_\rho \leq \phi^{\xi}_{[-n,(\rho+1)n]\times[-n,2n]}[\mathcal{H}_{[0,\rho n]\times[0,n]}] \leq 1 - c_\rho.$$

Goal: Obtain RSW constants dependent upon horizontal and vertical strip densities

Proof sketch

1 Configure the following 4 finite volumes as shown



RSW results proof sketch

2 Obtain part of the RSW constant from bounding the horizontal crossing event below with the horizontal strip density

$$\left(\mu^{\mathrm{Mixed}}_{\mathscr{R}_2}[\,\mathcal{H}_{\mathscr{R}_1}\,]\right)^{\alpha} \geq \mu^{0}_{[0\,,\,\rho n]\times_H[0,\,\lambda\mathrm{Stretch}]}\left[\,\mathcal{H}_{[0\,,\,\rho n]\times_H[\,0,\,\lambda\mathrm{Stretch}]}\,\right],$$

3 Resulting lower bound from the horizontal strip density

$$\mu^{\mathrm{Mixed}}_{\mathscr{R}_2}[\,\mathcal{H}_{\mathscr{R}_1}\,] \geq p_n^{\alpha\rho},$$

4 Repeat the arguments for a different horizontal crossing event, conditionally on the crossing event considered in 2-3, for the vertical strip density lower bound

$$\mu^{\mathrm{Mixed}}_{\mathscr{R}_2}\left[\, \mathcal{H}_{\mathscr{R}_3} \cap \mathcal{H}_{\mathscr{R}_4} \,\middle|\, \mathcal{H}_{\mathscr{R}_1} \right] \geq q^{\gamma}_{\frac{n}{3}}$$

Multiply inequalities for horizontal and vertical crossings

$$\mu^{\mathrm{Mixed}}_{\mathscr{R}_2}\left[\, \mathcal{H}_{\mathscr{R}_1} \,\middle|\, \mathcal{H}_{\mathscr{R}_3} \cap \mathcal{H}_{\mathscr{R}_4} \right] \geq \mu^{\mathrm{Mixed}}_{\mathscr{R}_2}\left[\, \mathcal{H}_{\mathscr{R}_1} \cap \mathcal{H}_{\mathscr{R}_3} \cap \mathcal{H}_{\mathscr{R}_4} \right] \geq p^{\alpha\rho}_n q^{\gamma}_{\frac{n}{3}} \; .$$

5 Synthesize results to obtain RSW results with constants c and 1-c, per classical results

$$p^{\alpha\rho}_n q^{\gamma}_{\frac{n}{3}} \leq \mu^{\xi}_{\mathscr{R}_2}[\, \mathcal{H}_{\mathscr{R}_1} \,] \leq 1 - p^{\alpha\rho}_n q^{\gamma}_{\frac{n}{3}} \; ,$$

Acknowledgments

Thank you to Philippe Sosoe for comments.

Future work

RSW results for the six-vertex model from crossings of the height function

Thank you for listening, are there any questions from the audience?