A Stock Market Model
Based on CAPM and Market Size

Andrey Sarantsev

University of Nevada, Reno

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Returns

A stock costs 10$

Grew to 12$, paid 50 cents dividend

Gives 12.5$ wealth

Price returns: $\ln(12/10)$

Total returns: $\ln(12.5/10)$

Equity Premium

$R(t)$ = risk-free returns

Usually short-term Treasury bills: no default risk

Equity premium = Total returns - risk-free returns

Compensation for taking risk

Stock Size

Market Capitalization = Stock Price × Number of Stocks

$S(t)$ = market cap at time t

$Q(t) = \ln(S(t+1)/S(t))$ = price returns

Benchmark

The entire market: all stocks in proportion to their size

A popular benchmark: Standard & Poor 500, Russell 3000, NASDAQ

Used to gauge the overall movement of the market

Can invest in a benchmark just by buying an exchange-traded fund

Capital Asset Pricing Model

Take a stock or a portfolio

Correlation of its equity premium $P(t)$ with that of the benchmark $P_0(t)$

Only systematic risk matters: This correlation

The rest of risk can be diversified away by taking enough stocks

$$P(t) = \beta P_0(t) + \varepsilon(t), \quad \mathbb{E}[\varepsilon(t)] = 0.$$

Beta β is the main characteristics: market exposure

(Littner, Sharpe, 1960s).

Corrections to CAPM

(Banz, 1981; Fama, French, 1993; Fama, French, 2015)

CAPM does not strictly hold, although useful

Excess return α:

$$P(t) = \alpha + \beta P_0(t) + \varepsilon(t)$$

where $\mathbb{E}[\varepsilon(t)] = 0$ and $\mathbb{E}\varepsilon^2(t)] = \sigma^2$.

Market Size Dependence

Make α, β, σ dependent on market size $S(t)$

Or, more precisely, upon relative market size $\ln(S(t)/S_0(t))$

Here, $S_0(t)$ is the benchmark size

System of Equations for Market Caps

Replace equity premium with price returns $\ln(S(t+1)/S(t))$

Want to ignore dividends and risk-free returns

Have a closed system of time series for market capitalizations

In continuous time, replace $\ln(S(t+1)/S(t))$ with $\mathrm{d}\ln S(t)$

Stochastic Differential Equation

$$\mathrm{d}\ln S(t) = \alpha(C(t))\,\mathrm{d}t + \beta(C(t))\,\mathrm{d}\ln S_0(t) + \sigma(C(t))\,\mathrm{d}W(t)$$

where W is a standard Brownian motion

relative market size $C(t) = \ln(S_0(t)/S(t))$

$\ln S_0$ is another Brownian motion, independent of W

This is a closed system for 2 processes: S_0 and S

System of Stochastic Equations

$$\mathrm{d}\ln S_i(t) = \alpha(C_i(t))\,\mathrm{d}t + \beta(C_i(t))\,\mathrm{d}\ln S_0(t) + \sigma(C_i(t))\,\mathrm{d}W_i(t)$$

$(W_1, \ldots, W_n)$ is an N-dimensional Brownian motion

$C_i(t) = \ln(S_0(t)/S_i(t))$ are relative market sizes

Long-Term Stability

$C = (C_1, \ldots, C_n)$ is a Markov process, driven by an SDE

$\mathbb{P}(C(t) \in A \mid C(0) = x) = P^t(x, A)$: transition kernel

We say a probability measure π on $\mathbb{R}^N$ is a stationary distribution if $C(0) \sim \pi$ implies $C(t) \sim \pi$ for $t \geq 0$

We say the process is ergodic if the stationary distribution is unique, and for every initial condition $C(0) = x$, as $t \to \infty$,

$$\sup_{A \subseteq \mathbb{R}^N} |P^t(x, A) - \pi(A)| \to 0.$$

Main Stability Results

Assume g is the drift (growth rate) of $\ln S_0$

$$\underline{\lim}_{c \to +\infty} [\alpha(c) + g(\beta(c) - 1)] > 0, \quad \overline{\lim}_{c \to -\infty} [\alpha(c) + g(\beta(c) - 1)] < 0.$$

Then the process C is ergodic

Example: $\alpha(c) = \alpha_0 c$ and $\beta(c) = 1 + \beta_0 c$.

Financial Data

Kenneth French's Dartmouth College Data Library

Monthly returns, equally-weighted portfolios

Market deciles: Top = benchmark, ignore bottom two

Years: July 1926 – June 2020

Stocks: Center for Research in Securities Prices (USA)

Data Analysis

Suggests the linear functions

$$\alpha(c) = 0.0045c, \quad \beta(c) = 1 + 0.0069c, \quad \sigma(c) = 0.052c$$

Thus the system is stable

References

A Stock Market Model Based on CAPM and Market Size (2021)

With Brandon Flores (Mathematics undergrad UNR) and

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