On the local magnetization in the Sherrington-Kirkpatrick model

(Joint work with W.-K. Chen)

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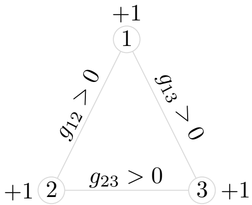
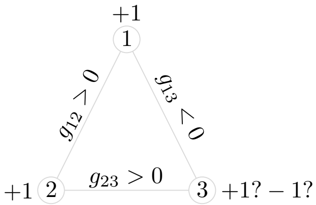
Dean's problem

- Assign N students in two dorms, labeled $+1$ and -1.
  $\sigma_i \in \{+1, -1\}$: assignment of student i for $i = 1, 2, \ldots, N$
- g_{ij}: friendship between student i and student j
  $g_{ij} > 0$: "i and j like each other"
  $g_{ij} < 0$: "i and j hate each other"
- The **best** assignment $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_N)$ maximizes the "happiness function"

$$H(\boldsymbol{\sigma}) = \sum_{\substack{i,j=1 \\ i<j}}^{N} g_{ij}\sigma_i\sigma_j$$





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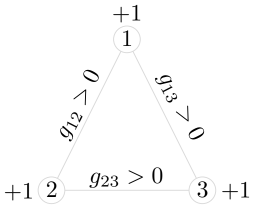

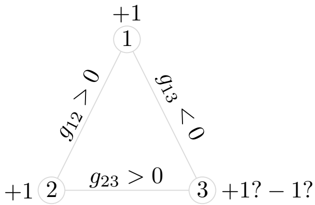

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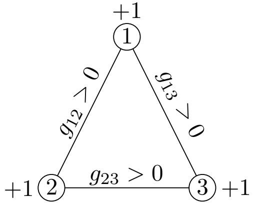

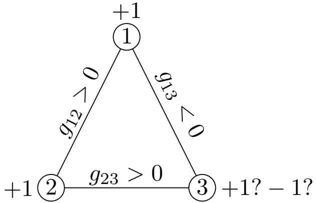

The Sherrington-Kirkpatrick Model [SK'72]

- $\boldsymbol{G} = (g_{ij})_{i,j=1}^N$: symmetric matrix with $g_{ii} = 0$ and $g_{ij,i<j} \overset{iid}{\sim} N(0,1)$.
  Fix an (inverse) temperature parameter $\beta > 0$ and external field $h > 0$.

- For each configuration $\boldsymbol{\sigma} \in \Sigma_N := \{+1, -1\}^N$,
  the Hamiltonian ("energy")

$$H_N(\boldsymbol{\sigma}) = H_{N,\beta,h}(\boldsymbol{\sigma}) := \frac{\beta}{\sqrt{N}} \sum_{\substack{i,j=1 \\ i<j}}^N g_{ij}\sigma_i\sigma_j + h\sum_{i=1}^N \sigma_i$$

- Gibbs measure G_N: $G_N(\boldsymbol{\sigma}) = \dfrac{e^{H_N(\boldsymbol{\sigma})}}{Z_N}$, where $Z_N := \sum_{\boldsymbol{\sigma}\in\Sigma_N} e^{H_N(\boldsymbol{\sigma})}$.

- "Best" configuration $\boldsymbol{\sigma}^*$? $\boldsymbol{\sigma}^* \in \Sigma_N$ that maximizes $H_N(\boldsymbol{\sigma})$

- "Average" configuration $\langle\boldsymbol{\sigma}\rangle = \langle\boldsymbol{\sigma}\rangle_{N,\beta,h} := (\langle\sigma_1\rangle_{N,\beta,h}, \ldots, \langle\sigma_N\rangle_{N,\beta,h})$?

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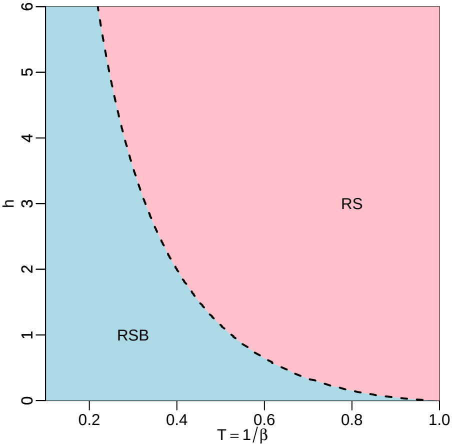
Phase Transition: high and low temperature regimes



- Overlap: for $\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2 \in \Sigma_N$

$$R(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2) := \frac{1}{N} \sum_{i=1}^{N} \sigma_i^1 \sigma_i^2$$

- "High-temperature" Regime $\mathcal{D}$

$$\left\{ (\beta, h) : \mathbb{E}\langle |R(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2) - q|^2 \rangle \xrightarrow{N\to\infty} 0 \right\}$$

where $q = q_{\beta,h}$ solves

$$q_{\beta,h} = \mathbb{E} \tanh^2(\beta \sqrt{q_{\beta,h}} Z + h).$$

- "AT-line" condition $\mathcal{A}$ [AT78]

$$\left\{ (\beta, h) : \beta^2 \mathbb{E} \cosh^{-4}(\beta \sqrt{q} Z + h) < 1 \right\}$$

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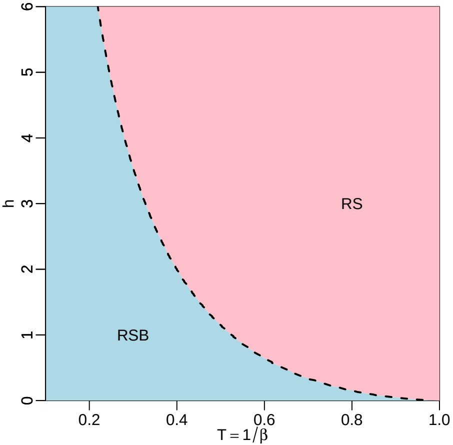

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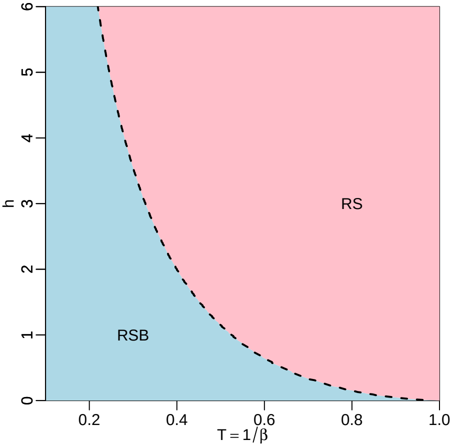

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The Cavity Equations [MPV'87]

For sufficiently high temperature (e.g., $\beta < 1/2$),

$$\langle \sigma_1 \rangle \approx \tanh \left(\frac{\beta}{\sqrt{N}} \sum_{j=2}^{N} g_{1j} \langle \sigma_j \rangle_{N-1,\beta',h} + h \right)$$

To see this, compute $\langle \sigma_1 \rangle$ directly

$$\langle \sigma_1 \rangle = \frac{\sum_{\boldsymbol{\sigma} \in \Sigma_N} \sigma_i e^{H_N(\boldsymbol{\sigma})}}{\sum_{\tilde{\boldsymbol{\sigma}} \in \Sigma_N} e^{H_N(\tilde{\boldsymbol{\sigma}})}} = \frac{\sum_{\sigma_2 \ldots, \sigma_N} e^{H_N\big((+1,\sigma_2,\ldots,\sigma_N)\big)} - e^{H_N\big((-1,\sigma_2,\ldots,\sigma_N)\big)}}{\sum_{\tilde{\sigma}_2 \ldots, \tilde{\sigma}_N} e^{H_N\big((+1,\tilde{\sigma}_2,\ldots,\tilde{\sigma}_N)\big)} + e^{H_N\big((-1,\tilde{\sigma}_2,\ldots,\tilde{\sigma}_N)\big)}}$$

where, with $\beta' = \frac{\beta\sqrt{N-1}}{\sqrt{N}}$

$$H_N\big((\pm 1, \sigma_2, \ldots, \sigma_N)\big) = \pm \left[\frac{\beta}{\sqrt{N}} \sum_{j=2}^{N} g_{1j}\sigma_j + h \right] + \overbrace{\frac{\beta}{\sqrt{N}} \sum_{\substack{i,j=2 \\ i<j}}^{N} g_{ij}\sigma_i\sigma_j + h \sum_{i=2}^{N} \sigma_i}^{=H_{N-1,\beta',h}\big((\sigma_2,\ldots,\sigma_N)\big)} .$$

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Recall: $\cosh(x) = \frac{e^x + e^{-x}}{2}$, $\sinh(x) = \frac{e^x - e^{-x}}{2}$, $\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$.

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The TAP equation

[TAP'77] At sufficiently high temperature (e.g., $\beta < 1/2$), local magnetizations asymptotically satisfy a system of consistency equations

$$\langle \sigma_1 \rangle \approx \tanh\left(\frac{\beta}{\sqrt{N}} \sum_{j=1}^{N} g_{1j} \langle \sigma_j \rangle + h - \beta^2 (1 - \|\langle \boldsymbol{\sigma} \rangle\|^2) \langle \sigma_1 \rangle \right).$$

where

- $\beta^2 (1 - \|\langle \boldsymbol{\sigma} \rangle\|^2) \langle \sigma_1 \rangle$ is called the "Onsager term".
- $\boldsymbol{x} = (x_1, \ldots, x_N)$, $\|\boldsymbol{x}\|^2 := \frac{1}{N} \sum_{i=1}^{N} x_i^2$.

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Bolthausen's Iteration [B'14]

Initialization.

Set $\boldsymbol{m}^{[0]} = (0, \dots, 0) \in \mathbb{R}^N$

$\boldsymbol{m}^{[1]} = (\sqrt{q}, \dots, \sqrt{q}) \in \mathbb{R}^N$

Iteration. For $k = 1, 2, 3, \dots$

$$\boldsymbol{m}^{[k+1]} = \tanh\left(\frac{\beta}{\sqrt{N}}\boldsymbol{G}\boldsymbol{m}^{[k]} + h - \beta^2(1 - \|\boldsymbol{m}^{[k]}\|^2)\boldsymbol{m}^{[k-1]}\right) \quad (*)$$

If (β, h) satisfies the AT-line condition, then

$$\lim_{k,k'\to\infty} \lim_{N\to\infty} \mathbb{E}\|\boldsymbol{m}^{[k]} - \boldsymbol{m}^{[k']}\| = 0,$$

i.e., the iteration converges.

- Bolthausen's iteration is a special case of the *Approximate Message Passing* (AMP) algorithm, where the iteration ("state evolution") is defined as

$$u_i^{[k+1]} = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} g_{ij} f_k(u_j^{[k]}) - \left(\frac{1}{N} \sum_{j=1}^{k} f_k'(u_j^{[k]}) \right) f_{k-1}(u_i^{[k-1]}), \ \ i = 1, 2, \ldots, N.$$

  Setting $\boldsymbol{m}^{[k]} = f_k(\boldsymbol{u}^{[k]})$ with $f_k(x) = \tanh(\beta x + h)$ $(k \geq 2)$ recovers $(*)$.

- AMP iteration enjoys the following Weak Law of Large Numbers.

  Fix $k \geq 0$ and for any bounded Lipschitz function $\psi : \mathbb{R}^{k+1} \to \mathbb{R}$

$$\frac{1}{N} \sum_{i=1}^{N} \psi(u_i^{[k]}, u_i^{[k-1]}, \ldots, u_i^{[0]}) \xrightarrow{P} \mathbb{E}\psi(U_k, \ldots, U_0)$$

  where $(U_k, \ldots, U_1)$ is jointly centered Gaussian independent of U_0, which follows the limiting empirical distribution of $\boldsymbol{u}^{[0]}$.

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Main Result: convergence to the local magnetization

Theorem [Chen & T.'21]. Assuming locally uniform concentration of the overlap, i.e., $\beta, h > 0$ satisfy that for some $\delta > 0$,

$$\lim_{N\to 0} \sup_{\beta-\delta\le\beta'\le\beta} \mathbb{E}\,\langle |R(\boldsymbol{\sigma}^1,\boldsymbol{\sigma}^2) - q_{\beta',h}|^2\rangle_{N,\beta',h} = 0,$$

then

$$\lim_{k\to\infty}\lim_{N\to\infty} \mathbb{E}\|\langle\boldsymbol{\sigma}\rangle - \boldsymbol{m}^{[k]}\|^2 = 0$$

where $\boldsymbol{m}^{[k]} \in \mathbb{R}^N$ is the output of the k-th iteration in Bolthausen's iteration.

Iterating the Cavity Equation: set $f(x) = \tanh(\beta x + h)$

$$\langle\sigma_1\rangle \approx f\left(\frac{1}{\sqrt{N}}\sum_{j=2}^N g_{1j}\underbrace{\langle\sigma_j\rangle_{[N]\setminus\{1\},\beta',h}}\right)$$

$$\approx f\left(\frac{1}{\sqrt{N}}\sum_{j=2}^N g_{1j} f\left(\frac{1}{\sqrt{N}}\sum_{k=2,k\neq j}^N g_{jk}\underbrace{\langle\sigma_k\rangle_{[N]\setminus\{1,j\}}}\right)\right)$$

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Main Result: convergence to the local magnetization

Theorem [Chen & T.'21]. Assuming locally uniform concentration of the overlap, i.e., $\beta, h > 0$ satisfy that for some $\delta > 0$,

$$\lim_{N\to 0} \sup_{\beta-\delta\le\beta'\le\beta} \mathbb{E}\,\langle |R(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2) - q_{\beta',h}|^2\rangle_{N,\beta',h} = 0,$$

then

$$\lim_{k\to\infty}\lim_{N\to\infty} \mathbb{E}\|\langle\boldsymbol{\sigma}\rangle - \boldsymbol{m}^{[k]}\|^2 = 0$$

where $\boldsymbol{m}^{[k]} \in \mathbb{R}^N$ is the output of the k-th iteration in Bolthausen's iteration.

Iterating the Cavity Equation: set $f(x) = \tanh(\beta x + h)$

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$$\begin{aligned}
\langle\sigma_1\rangle &\approx f\left(\frac{1}{\sqrt{N}}\sum_{j=2}^N g_{1j}\underbrace{\langle\sigma_j\rangle_{[N]\setminus\{1\},\beta',h}}\right)\\
&\approx f\left(\frac{1}{\sqrt{N}}\sum_{j=2}^N g_{1j} f\left(\frac{1}{\sqrt{N}}\sum_{k=2,k\neq j}^N g_{jk}\underbrace{\langle\sigma_k\rangle_{[N]\setminus\{1,j\}}}\right)\right)\\
&\approx f\left(\frac{1}{\sqrt{N}}\sum_{j=2}^N g_{1j} f\left(\frac{1}{\sqrt{N}}\sum_{k=2,k\neq j}^N g_{jk} f\left(\frac{1}{\sqrt{N}}\sum_{l=2,l\neq k,j}^N g_{kl}\langle\sigma_k\rangle_{[N]\setminus\{1,j\}}\right)\right)\right) \approx \cdots
\end{aligned}$$

Proof outline: the Cavity iteration

For any vector $\boldsymbol{v} \in \mathbb{R}^N$, define $\boldsymbol{v}_S \in \mathbb{R}^{[N]\setminus S}$ where $S \subset [N]$ by

$$v_{S,i} = v_i, \quad i \in [N] \setminus S.$$

In particular, $\boldsymbol{v}_\emptyset = \boldsymbol{v}$.

Initialization.

Set $\boldsymbol{v}^{[0]} = (0, \ldots, 0) \in \mathbb{R}^N$

$\boldsymbol{v}^{[1]} = (\sqrt{q}, \ldots, \sqrt{q}) \in \mathbb{R}^N$

Iteration. For $k = 1, 2, 3, \ldots$

$$v_{S,i}^{[k+1]} = f\left(\frac{1}{\sqrt{N}} \sum_{j=1, j \notin S \cup \{i\}}^{N} g_{ij} v_{S \cup \{i\}, j}^{[k]}\right)$$

- The cavity iteration is asymptotically equivalent to Bolthausen's TAP iteration, i.e., $$\boldsymbol{v}^{[k]} \approx \boldsymbol{m}^{[k]}.$$
- The cavity iteration converges to the local magnetization $\langle \boldsymbol{\sigma} \rangle$.

Thank You!