Heat kernel estimates of non-local operators with multisingular critical killing

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History for local operators

For any stochatic process, the transition density(heat kernel) of it provides us with a lot of information. Thus it is always good to know the formula of the heat kernel.

Simplest case: Heat kernel of Brownian motion

A standard Brownian motion B_t in $\mathbb{R}^d$ has $\frac{1}{2}\Delta$ as its generator, the transition density(heat kernel) $p^B(t,x,y)$ has an explicit formula:

$$p^B(t,x,y) = \frac{1}{(2\pi t)^{d/2}} \exp\left(\frac{-|x-y|^2}{2t}\right)$$

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Heat kernel estimates: DeGiorgi-Nash-Moser-Aronson theory

Given a diffusion operator $\mathcal{L} = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right)$, where $a_{ij}(x)$ is an $n \times n$ matrix function which is uniformly elliptic and bounded, then there is a diffusion process $\{X_t\}_{t\geq 0}$ having $\mathcal{L}$ as its L^2 generator.

The celebrated DeGiorgi-Nash-Moser-Aronson theory says that the transition density (or heat kernel) $p(t,x,y)$ of X_t satisfies the following two-sided estimates: Given $T>0$, there exist $c_1, c_2, c_3, c_4 > 0$, such that

$$c_1 t^{-d/2} \exp\left(\frac{-c_2|x-y|^2}{t}\right) \leq p(t,x,y) \leq c_3 t^{-d/2} \exp\left(\frac{-c_4|x-y|^2}{t}\right),$$

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Beyond DeGiorgi-Nash-Moser-Aronson theory

Given a domain $D \subseteq \mathbb{R}^d$ and a stochastic process X_t on $\mathbb{R}^d$, we also want to know the heat kernel of X_t^D, where X_t^D is the killed process upon leaving D.

Heat kernel estimates for killed process

Since the boundary of the domain can be complicated, two-sided estimates for the transition density (equivalently, the Dirichlet heat kernels) of killed processes are harder to derive. The first related result was established after 21st century, by (Zhang, 2002), who showed that the heat kernel estimates of B_t^D has the following form: For any $T > 0$, there exists $c_1, c_2, c_3, c_4 > 0$, such that $\forall t \in (0, T), x, y \in D$, $(\delta_D(x) := dist(x, D))$

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Heat kernel estimates under Feynman-Kac transformation

Another type of stochastic processes are given by Feynman-Kac transformation. For simplicity, given an open set D, we consider the heat kernel of the operator $\mathcal{L}^V := -\Delta|_D + V$, where V is a function defined on D. For this type of generators, some highlighted results for the heat kernel estimates are

1. If V is in Kato class, the heat kernel estimates are the same as the case without V. For example, when $V \asymp |x|^{-\beta}$ where $\beta < 2$(we call this case subcritical).
2. If $D = \mathbb{R}^d \setminus \{0\}$, $V = c|x|^{-2}$ the heat kernel estimates has similar form as killed process but has more detailed relation with the constant c.(we call this case critical.)
3. If D is a bounded C^2 domain, $V = c\delta_D^{-2}$, where $\delta_D(x) := dist(x, D)$. Similar results are derived, but with complicated arguments to deal with the regularity of the boundary.

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Some open problems

- For the case $D = \mathbb{R}^d \backslash \{0\}$, $V = c|x|^{-2-\beta}$, where $\beta > 0$. If $c < 0$(creation), it is known that there is no fundamental solutions, i.e., the stochastic process associated with the operator blows up. What about the heat kernel estimates when $c > 0$(killing)? The heat kernel estimates should change radically.
- All of the above theories were established mainly by analytical methods. We believe we can give probabilistic arguments for killing case, i.e., V is a positive function.

History for non-local operators

Heat kernel estimates of stable processes

The starting point of heat kernel estimates for non-local operators is the estimates of transition probability of α-stable process.

Definition

A stochastic process X_t in $\mathbb{R}^d$ is called α-stable process with $\alpha \in (0,2)$ if the generator is given by $\Delta^{\alpha/2}$, where

$$\Delta^{\alpha/2} f(x) := \mathcal{A}(d,\alpha) p.v. \int_{\mathbb{R}^d} \frac{f(y) - f(x)}{|y-x|^{d+\alpha}} dy$$

for some constant $\mathcal{A}(d,\alpha) > 0$.

Heat kernel estimates of stable processes

We do not have explicit formula of the heat kernel of α-stable process except for special values of α, e.g. $\alpha = 1$. Instead, it is known that the heat kernel estimates are the following:

$$p(t,x,y) \asymp t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}, t > 0, x, y \in \mathbb{R}^d$$

The general theory of heat kernel estimates of non-local operators(non-local analogue of DeGiorgi-Nash-Moser-Aronson theory) is still undergoing active research.

Our contribution

Heat kernel estimates of non-local operators with critical killing

We mainly focus on the heat kernel estimates of the non-local operator of the form $\mathcal{L}^V = -(\Delta|_D)^{\alpha/2} + V$, where D is an open set, V is a positive function defined on D and

$$-(\Delta|_D)^{\alpha/2} f(x) := -\mathcal{A}(d, -\alpha) p.v. \int_D \frac{f(y) - f(x)}{|y - x|^{d+\alpha}} dy$$

We have affirmative answers to the following two natural questions:

1. Do we have an analogue of heat kernel estimates for non-local operators with Feynman-Kac transform?
2. Do we have unique nature of non-local operators which is not shared by local operators?

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Main result I: Critical killing with single singularity

Suppose the open set D is $C^{1,\beta}$, i.e., the boundary of which is uniform $C^{1,\beta}$ with codimension $1 \le k \le d$ and $(\alpha - 1)_+ < \beta \le 1$. The killing function is given by $V(x) = \lambda \delta_D^{-\alpha}(x)$ with $\lambda > 0$. The heat kernel $q^D(t,x,y)$ of $\mathcal{L}^V = -\Delta|_D^{\alpha/2} + V$ has the following estimate: For any $T > 0$, there exists c_1, c_2, such that

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where λ and p are related through an increasing function $C(p,d,k,\alpha) = \lambda$.

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where λ and p are related through an increasing function $C(p,d,k,\alpha) = \lambda$.

Remark

- Compared to the previous result, we require less regularity of the boundary of the open set, by making use of the non-local nature of the operator. Roughly speaking, we require the boundary to be $C^{\alpha+}$.
- We have less requirement of the codimension of the boundary of the open set. For example, suppose $D = \mathbb{R}^4 \setminus S^2$, where S^2 is the unit 2-dimensional sphere in $\mathbb{R}^4$, then the heat kernel estimates of $-\Delta|_D^{\alpha/2} + \lambda\delta_D^{-\alpha}$ has the form

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- For the killed α-stable process upon leaving an open set D, the generator has the form $-\Delta|_D^{\alpha/2} + \int_{D^c} \frac{1}{|y-\cdot|^{d+\alpha}} dy$, with slightly more calculation, we can recover the heat kernel estimates of killed α-stable process(first established by Chen et al. in 2010) and we need less regularity of the open set D. This means we only need to consider non-local operators with critical killing, which will cover the case of killed process. This feature is unique for non-local operators, not shared by local operators!

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Main result II: Critical killing with several singularities

Suppose the open set D is given by $\mathbb{R}^d \backslash \{x_1, x_2\}$, where x_1, x_2 are two distinct points in $\mathbb{R}^d$. Suppose our killing function $V(x) := \lambda_1 |x - x_1|^{-\alpha} + \lambda_2 |x - x_2|^{-\alpha}$, where $\lambda_1, \lambda_2 > 0$. Then the heat kernel estimates of $\mathcal{L}^V$ has the following form: For any $T > 0$,

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where λ_i and p_i are related through the increasing function $C(p_i, d, d, \alpha) = \lambda_i, i = 1, 2.$

Remark

For more general open set D, we have similar type of heat kernel estimates if the killing function has multi-singular critical killing potential.

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Main result II: Critical killing with several singularities

Suppose the open set D is given by $\mathbb{R}^d \backslash \{x_1, x_2\}$, where x_1, x_2 are two distinct points in $\mathbb{R}^d$.Suppose our killing function $V(x) := \lambda_1 |x - x_1|^{-\alpha} + \lambda_2 |x - x_2|^{-\alpha}$, where $\lambda_1, \lambda_2 > 0$. Then the heat kernel estimates of $\mathcal{L}^V$ has the following form: For any $T > 0$,

$$p^D(t,x,y) \asymp_T (1 \wedge \frac{|x-x_1||y-x_1|}{t^{2/\alpha}})^{p_1} (1 \wedge \frac{|x-x_2||y-x_2|}{t^{2/\alpha}})^{p_2} (t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}})$$

$\forall t \in (0, T), x, y \in D.$

where λ_i and p_i are related through the increasing function $C(p_i, d, d, \alpha) = \lambda_i, i = 1, 2$.

Remark

For more general open set D, we have similar type of heat kernel estimates if the killing function has multi-singular critical killing potential.

Thank you!