Mean field spin glasses under weak external fields

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Spin glass models

Consider a configuration $\boldsymbol{\sigma} \in \Sigma_N := \{-1, +1\}^N$,

- Hamiltonian: $H_N(\boldsymbol{\sigma})$, Gaussian random fields on Σ_N such that

$$\mathrm{E}\, H_N(\boldsymbol{\sigma}) H_N(\boldsymbol{\tau}) = N\beta^2 \xi\left(\frac{1}{N}\boldsymbol{\sigma}\cdot\boldsymbol{\tau}\right), \text{ where } \beta > 0 \text{ inverse temperature}$$

- Examples:
  - Sherrington-Kirkpatrick Model:

$$H_N(\boldsymbol{\sigma}) = \frac{\beta}{\sqrt{N}} \sum_{i,j} J_{ij}\sigma_i\sigma_j \text{ with } J_{ij} \sim \mathcal{N}(0,1) \iff \xi(x) = x^2,$$

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- Free energy: $F_N(\beta, h) := \frac{1}{N}\mathrm{E}_J \log Z_N(\beta, h)$, where

$$Z_N(\beta, h) := \sum_{\boldsymbol{\sigma}\in\Sigma_N} \exp\left(H_N(\boldsymbol{\sigma}) + h\sum_i \sigma_i\right).$$

  $h \geq 0$ is the external field.

- Limiting free energy is given by the solution of Parisi's variational formula for all β.

Fluctuation problems

Take SK model as example:

- Fluctuations of free energy is know as Gaussian when β small.
  - For $h = 0$, fluctuation order is $O(1)$. [Aizenman-Lebowitz-Ruelle87, Comets-Neveu95, Bovier-Kurkova-Lowe02, etc.]
  - For $h \neq 0$, fluctuation order is $O(N)$. [Guerra et.al, Tindel05].
- For β large (low temperature),
  - For $h = 0$, some debate about the fluctuation order: $O(1)$ or $O(N^{\varepsilon})$? Best known bound: at least $O(1)$ by [Chatterjee19]. largely open!
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Related results on spherical SK models:

- For $h \neq 0$, Gaussian at all temperature. [Chen-Dey-Panchenko17]
- For $h = 0$, Gaussian at high temperature, GOE Tracy-Widom in low temperature. [Baik et.al.]

A Fluctuation Question

What happens when h changes from 0 to a nonzero constant?

- How the order of fluctuation changes from $O(1)$ to $O(N)$?
- What happens to the limiting distribution?

Take $h = h_N = \lambda N^{-\alpha}$, $\alpha \in (0, \infty)$ with some $\lambda > 0$.

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Challenges:

- Lack of nice tools and loss of symmetry and smoothness compared to spherical models.
- Prove the results up to the critical temperature is very hard when the external field is present.

Our Results

Theorem (Dey-W. 2021+)

For $\beta < \beta_0$, $h_N = \lambda N^{-\alpha}$,

- *super-critical:* $\alpha \in [0, 1/4)$

$$\sqrt{N h_N^{-4}} \cdot (F_N(\beta, h_N) - \mu_1) \to \mathcal{N}(0, \sigma_1^2) \text{ as } N \to \infty.$$

- *critical:* $\alpha = 1/4$,

$$N(F_N(\beta, h_N) - \mu_2) \to \mathcal{N}(0, \sigma_2^2) \text{ as } N \to \infty.$$

- *sub-critical:* $\alpha \in (1/4, \infty]$,

$$N(F_N(\beta, h_N) - \mu_3) \to \mathcal{N}(0, \sigma_3^2) \text{ as } N \to \infty.$$

Our Results

- Remarks
  - Same fluctuation order in critical and sub-critical regime.
  - The explicit forms of μ_i, σ_i^2 in each regime are known.
  - β_0 can be pushed up to critical temperature β_c in sub-critical and critical regime.

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$$\log Z_N(\beta, h_N) \approx \text{independent sum} + \text{loop cluster} + \text{path cluster}.$$

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  - In the sub-critical regime, the partition function nearly has no difference with the zero external field case.
  - In the critical regime, the log-partition function is composed of 2 independent random objects (corresponding to different clusters (path (new structure) + loop) in cluster expansion).
  - In the super-critical regime, the partition function has exponential growth, and can be controlled via overlap at small enough β.

Summary of Approaches

Sub-critical	Critical	Super-critical
Direct proof Cluster expansion Quadratic coupling	Quadratic coupling Cluster expansion	Interpolation Scheme
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- Cluster expansion works for general disorders, not just Gaussian. The new path cluster analysis should also be applied to many other spin glass models, diluted model, perceptron, etc.
- Other methods depend on the Gaussian disorder, ans also can be generalized to pure even p-spin model, multi-species SK model, spherical models.
- In the Cluster expansion approach, we finally apply multi-variate Stein's method to different clusters (loop+path) to prove the CLT. The proof gives a more transparent picture of the combinatorial structures than [ALR87].

Cluster Expansion proof sketch

Step 1: The decomposition,

$$Z_N(\beta,h) = \bar{Z}_N(\beta,h)\cdot\hat{Z}_N(\beta,h),$$

with

$$\bar{Z}_N(\beta,h) := \prod_{e\in\mathcal{E}_N}\cosh\left(\frac{\beta J_e}{\sqrt{N}}\right),\quad \hat{Z}_N(\beta,h) := \sum_{\Gamma\subseteq\mathcal{E}_N}(\tanh h)^{|\partial\Gamma|}\prod_{e\in\Gamma}\tanh\left(\frac{\beta J_e}{\sqrt{N}}\right).$$

Step 2: Finite graphs dominate in (sub)-critical regime,

$$\hat{Z}_N(\beta,h)\approx\hat{Z}_{N,m}(\beta,h) := \sum_{\Gamma\subseteq\mathcal{E}_N,|\Gamma|\le m}(\tanh h)^{|\partial\Gamma|}\prod_{e\in\Gamma}\tanh\left(\frac{\beta J_e}{\sqrt{N}}\right).$$

Step 3: Combinatorial analysis:

$$\hat{Z}_{N,m}(\beta,h)\approx\prod_{\gamma:|\gamma|\le m'}\left(1+\prod_{e\in\gamma}\tanh\left(\frac{\beta J_e}{\sqrt{N}}\right)\right)\prod_{p:|p|\le m''}\left(1+\tanh^2 h\cdot\prod_{e\in p}\tanh\left(\frac{\beta J_e}{\sqrt{N}}\right)\right).$$

Related results in spherical model

Similar results have been established in spherical SK model.

- Random matrix approach: Contour integral representation of the partition function $Z_N(\beta, h)$. Steepest descent methods to study the asymptotics.[Baik-Woodfin-Le Doussal-Wu21],[Landon-Sosoe20].
- Geometric side: Kac-Rice formula to study the complexity, there is a topological trivialization regime. [Belius-Cerny-Nakajima-Schmidt21].
- Our results match the results by [Baik-Woodfin-Le Doussal-Wu21] when $\beta < \beta_c$!

Thank you