Right tail of the two-time distribution of the KPZ fixed point

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Content

Introduction

Totally asymmetric simple exclusion process (TASEP)

- Assume there are N particles in the system.
- Denote their locations at time t by $X(t) = (x_1(t), \cdots, x_N(t)) \in \mathbb{Z}^N$ where $x_N(t) < \cdots < x_1(t)$ for all t.
- There is at most one particle per site.
- Each particle jumps independently to the right with rate 1, provided its neighboring site is empty.

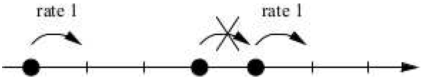

Figure: TASEP dynamics in continuous time.

Introduction

- Consider its height function $h(x,t)$ at location x and time t.
- Step (wedge) initial condition: $h(x,0) = |x|$

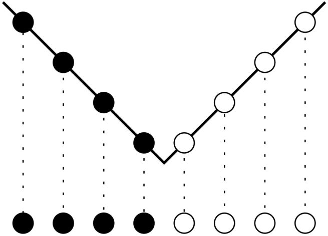

Figure: Height configuration for the step initial condition

Introduction

- (Johansson'00) For any $\xi \in (-1, 1)$,

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{h(\xi t, t) - c_1(\xi)t}{-c_2(\xi)t^{1/3}} \leq s\right) = F_2(s), \tag{1.1}$$

  where $F_2(s)$ is the GUE Tracy-Widom distribution.
- A broad class of random growth models share the scaling limits $t : t^{2/3} : t^{1/3}$ for the time, spatial correlation length and fluctuation order, called Kardar-Parisi-Zhang (KPZ) universality class.

Introduction

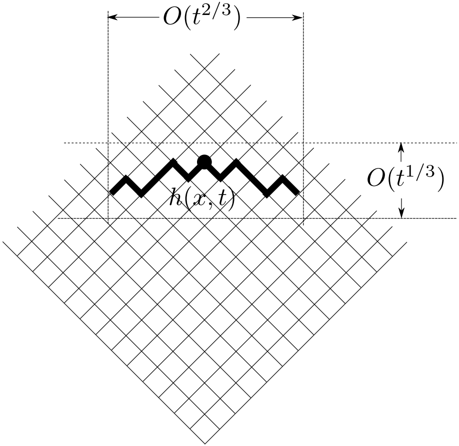

Figure: Fluctuations live on the $t^{1/3}$ scale are correlated spatially in the $t^{2/3}$ scale.

KPZ fixed point

- The scaled limiting space-time field is believed to be universal and independent of the models, but only depends on the initial condition:

$$\lim_{T\to\infty} \frac{h\left(c_1 x T^{2/3}, c_2 \tau T\right) - c_3 \tau T}{c_4 T^{1/3}} = H(x, \tau). \tag{2.1}$$

  Here c_1, c_2, c_3 and c_4 are model-dependent constants.
- $H(x, \tau)$ is called the KPZ fixed point, characterized by Matetski, Quastel and Remenik [MQR17] as a Markov process with explicit transition probabilities.

KPZ fixed point

- Along the spatial direction, the multi-point distribution was obtained by Prähofer-Spohn [PS02],Johansson [Joh03] and Borodin-Prähofer-Ferrari-Sasamoto [BPFS07], etc. for standard initial conditions and Matetski-Quastel-Remenik [MQR17] for general initinal condition.
- For the step initial condition $H(x, 0) = |x|$,

$$\mathbb{P}\left(\bigcap_{\ell=1}^{m}\left\{\frac{H\left(2x_\ell T^{2/3}, 2T\right) - T}{-T^{1/3}} \leq h_\ell\right\}\right) = \mathbb{P}\left(\bigcap_{\ell=1}^{m}\{\mathcal{A}_2(x_\ell) \leq h_\ell\}\right),$$

where $\mathcal{A}_2$ is the $Airy_2$ process shifted by a parabola.
- Along the time direction, the multi-point distribution was obtained recently by Johansson-Raham [JR19] and Liu [Liu19] independently.

KPZ fixed point

In [Liu19], assume the parameters $0 < \tau_1 < \cdots < \tau_m$. With the step initial condition $H(x,0) = |x|$, we have

$$\lim_{T\to\infty} \mathbb{P}\left(\bigcap_{\ell=1}^{m}\left\{\frac{H\left(2x_\ell T^{2/3}, 2\tau_\ell T\right) - \tau_\ell T}{-T^{1/3}} \le h_\ell\right\}\right) = F_{\text{step}} ,$$

where the function F_{step} is given by

$$F_{\text{step}} = \oint \cdots \oint \left[\prod_{\ell=1}^{m-1} \frac{1}{1-z_\ell}\right] \mathrm{D}_{\text{step}}\,(z_1, \cdots, z_{m-1}) \frac{\mathrm{d}z_1}{2\pi \mathrm{i} z_1} \cdots \frac{\mathrm{d}z_{m-1}}{2\pi \mathrm{i} z_{m-1}}.$$

Here D_{step} is defined in terms of a Fredholm determinant.

KPZ fixed point

For $m = 2$, we have

$$F_{step} = \oint_{|\mathrm{z}|<1} \frac{\mathrm{dz}}{2\pi \mathrm{i}(1-\mathrm{z})\mathrm{z}} \sum_{n_1,n_2\geq 0} \frac{1}{(n_1!)^2(n_2!)^2} \mathrm{T}_{n_1,n_2}.$$

Here

$$\mathrm{T}_{n_1,n_2} := \prod_{i_2=1}^{n_2} \left[\frac{1}{1-\mathrm{z}} \int_{\Gamma_{\mathrm{L,in}}} \frac{\mathrm{d}\xi_{i_2}^{(2)}}{2\pi \mathrm{i}} - \frac{\mathrm{z}}{1-\mathrm{z}} \int_{\Gamma_{\mathrm{L,out}}} \frac{\mathrm{d}\xi_{i_2}^{(2)}}{2\pi \mathrm{i}} \right]$$

$$\cdot \left[\frac{1}{1-\mathrm{z}} \int_{\Gamma_{\mathrm{R,in}}} \frac{\mathrm{d}\eta_{i_2}^{(2)}}{2\pi \mathrm{i}} - \frac{\mathrm{z}}{1-\mathrm{z}} \int_{\Gamma_{\mathrm{R,out}}} \frac{\mathrm{d}\eta_{i_2}^{(2)}}{2\pi \mathrm{i}} \right]$$

$$\cdot \prod_{i_1=1}^{n_1} \int_{\Gamma_{\mathrm{L}}} \frac{\mathrm{d}\xi_{i_1}^{(1)}}{2\pi \mathrm{i}} \int_{\Gamma_{\mathrm{R}}} \frac{\mathrm{d}\eta_{i_1}^{(1)}}{2\pi \mathrm{i}} (1-\mathrm{z})^{n_1} \left(1 - \frac{1}{\mathrm{z}}\right)^{n_2} \cdot \prod_{\ell=1}^{2} \mathrm{f}_\ell(\boldsymbol{\xi}^{(\ell)}) \mathrm{f}_\ell(\boldsymbol{\eta}^{(\ell)})$$

$$\cdot \prod_{\ell=1}^{2} \frac{\Delta(\boldsymbol{\xi}^{(\ell)})^2 \Delta(\boldsymbol{\eta}^{(\ell)})^2}{\Delta(\boldsymbol{\xi}^{(\ell)};\boldsymbol{\eta}^{(\ell)})^2} \cdot \frac{\Delta(\boldsymbol{\xi}^{(1)};\boldsymbol{\eta}^{(2)})\Delta(\boldsymbol{\eta}^{(1)};\boldsymbol{\xi}^{(2)})}{\Delta(\boldsymbol{\xi}^{(1)};\boldsymbol{\xi}^{(2)})\Delta(\boldsymbol{\eta}^{(1)};\boldsymbol{\eta}^{(2)})}.$$

KPZ fixed point

- Here, we use notations

$$\Delta(W) := \prod_{1\le i<j\le n} (w_j - w_i), \quad \Delta\left(W; W'\right) := \prod_{i=1}^{n} \prod_{i'=1}^{n'} \left(w_i - w'_{i'}\right),$$

$$f(W) = \prod_{i=1}^{n} f(w)$$

for any two vectors $W = (w_1, \cdots, w_n) \in \mathbb{C}^n$ and $W' = \left(w'_1, \cdots, w'_{n'}\right) \in \mathbb{C}^{n'}$.

KPZ fixed point

- the vectors $\boldsymbol{\xi}^{(\ell)} = \left(\xi_1^{(\ell)}, \cdots, \xi_{i_\ell}^{(\ell)}\right)$ and $\boldsymbol{\eta}^{(\ell)} = \left(\eta_1^{(\ell)}, \cdots, \eta_{i_\ell}^{(\ell)}\right)$ for $\ell \in \{1, 2\}$ and the functions f_ℓ are defined by

$$\mathrm{f}_\ell(\zeta) = \begin{cases} \exp\left(-\frac{\tau_\ell}{3}\zeta^3 + x_\ell\zeta^2 + \mathrm{h}_\ell\zeta\right), & \mathrm{Re}\zeta < 0, \\ \exp\left(\frac{\tau_\ell}{3}\zeta^3 - x_\ell\zeta^2 - \mathrm{h}_\ell\zeta\right), & \mathrm{Re}\zeta > 0, \end{cases} \tag{2.2}$$

- The three contours in the left half plane from left to right are $\Gamma_{\mathrm{L,in}}, \Gamma_{\mathrm{L}}$ and $\Gamma_{\mathrm{L,\,out}}$ respectively, the three contours in the right half plane from left to right are $\Gamma_{\mathrm{R,\,out}}, \Gamma_{\mathrm{R}}$ and $\Gamma_{\mathrm{R,\,in}}$ respectively.

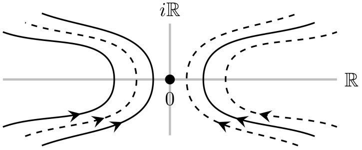

Figure: Illustration of the contours:

Main Results

We want to investigate the right tail behavior of the two-time distribution

$$\mathbb{P}\left(H(x_1,\tau_1) \geq \mathrm{h}_1, H(x_1+x_2,\tau_1+\tau_2) \geq \mathrm{h}_1+\mathrm{h}_2\right)$$

when $\mathrm{h}_1 \to \infty$.

Theorem (Nissim-Z'21+)

Suppose x_1, x_2, τ_1, τ_2 are all fixed and τ_1, τ_2 are positive. Also assume that h_2 is fixed. Then

$$\begin{aligned}&\mathbb{P}\left(H(x_1+x_2,\tau_1+\tau_2) \geq \mathrm{h}_1+\mathrm{h}_2 \mid H(x_1,\tau_1) \geq \mathrm{h}_1\right)\\ &= \mathbb{P}\left(H(x_2,\tau_2) \geq \mathrm{h}_2\right) - 2\tau_1^{1/2}\mathrm{h}_1^{-1/2}\frac{\mathrm{d}}{\mathrm{dh}_2}\mathbb{P}\left(H(x_2,\tau_2) \geq \mathrm{h}_2\right)\\ &\quad + 2\tau_1\mathrm{h}_1^{-1}\frac{\mathrm{d}^2}{\mathrm{dh}_2^2}\mathbb{P}\left(H(x_2,\tau_2) \geq \mathrm{h}_2\right) + \mathcal{O}(\mathrm{h}_1^{-3/2})\end{aligned} \tag{3.1}$$

as $\mathrm{h}_1 \to \infty$.

Main Results

Proof strategy:

- By a residue calculus, we obtain

$$\mathbb{P}\left(\mathrm{H}(x_1,\tau_1) \geq \mathrm{h}_1, \mathrm{H}(x_1+x_2,\tau_1+\tau_2) \geq \mathrm{h}_1+\mathrm{h}_2\right)$$
$$= \oint_{|\mathrm{z}|>1} \frac{\mathrm{dz}}{2\pi \mathrm{i} \mathrm{z}(1-\mathrm{z})} \sum_{n_1,n_2\geq 1} \frac{1}{(n_1!)^2(n_2!)^2} \mathrm{T}_{n_1,n_2}.$$

- The main contribution comes from T_{1,n_2}.
- Only one term survives if we evaluate $\oint_{|z|>1} \frac{\mathrm{dz}}{2\pi \mathrm{i} \mathrm{z}(1-\mathrm{z})} \mathrm{T}_{1,n_2}$.
- Evaluate its asymptotic expansion of in terms of h_1.

Thank you!