Orthogonal polynomial duality functions for multi-species SEP($2j$) and ASEP(q, j)

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Multi-species SEP($2j$) and ASEP(q, j)

Asymmetric simple exclusion process (ASEP) was introduced by Spitzer '70 and by MacDonald–Gibbs–Pipkin '68.
Two-species ASEP was introduced by Liggett '76.
Later, Carinci-Giardina-Redig-Sasamoto '15 generalized ASEP to ASEP(q, j) which allows up to $2j \in \mathbb{N}$ particles at each site.

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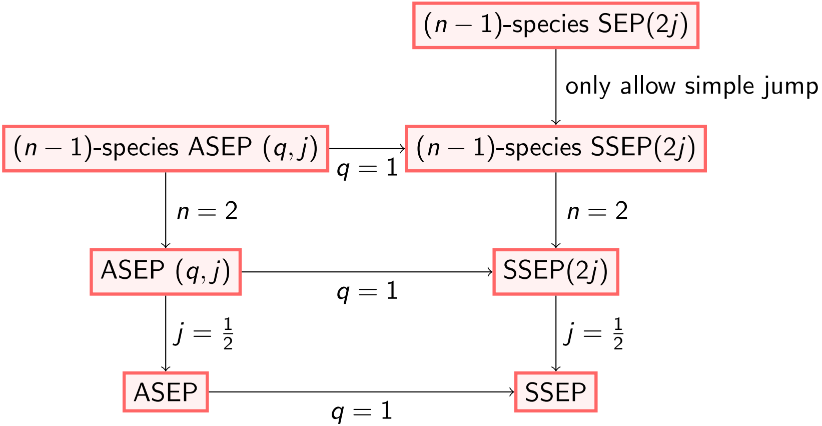
We consider the multi-species ASEP(q, j) with closed boundaries on a finite lattice $G = \{1, \cdots, L\}$, introduced by Kuan '17[4].

Relations



$(n-1)$-species SEP$(2j)$
only allow simple jump
$(n-1)$-species ASEP (q, j)
$q = 1$
$(n-1)$-species SSEP$(2j)$
$n = 2$
$n = 2$
ASEP (q, j)
$q = 1$
SSEP$(2j)$
$j = \frac{1}{2}$
$j = \frac{1}{2}$
ASEP
$q = 1$
SSEP

Notations

The **state space** of particle configurations consists of variables $\xi = (\xi_i^x : 1 \leq i \leq n, x \in G)$, where $G = \{1, \cdots, L\}$, ξ_i^x denotes the number of particles of species i at site x, and

$$\xi^x = (\xi_1^x, \cdots, \xi_n^x) \qquad \text{such that } \sum_{i=1}^{n} \xi_i^x = 2j$$

for any $x \in G$.
We think of ξ_n^x as the number of holes.

Multi-species SEP($2j$)

Definition

The generator of $(n-1)$-species SEP($2j$) is given by

$$\mathcal{L}f(\xi) = \sum_{x<y\in G} \mathcal{L}_{x,y} f(\xi)$$

$$\mathcal{L}_{x,y} f(\xi) = \sum_{1\le k<l\le n} \xi_l^x \xi_k^y [f(\xi_{l,k}^{x,y}) - f(\xi)] + \xi_l^y \xi_k^x [f(\xi_{l,k}^{y,x}) - f(\xi)], \tag{1}$$

where $\xi_{l,k}^{x,y}$ denotes the particle configuration obtained by switching a particle of the l^{th} species at position x with a particle of the k^{th} species at position y.

[default]

Multi-species ASEP(q,j)

Definition

The generator of $(n-1)$-species ASEP(q,j) is given by:

$$\mathcal{L}f(\xi) = \sum_{x=1}^{L-1} \mathcal{L}_{x,x+1} f(\xi),$$

$$\mathcal{L}_{x,x+1} f(\xi) = \sum_{1 \le k < l \le n} \alpha(\xi)[f(\xi_{k,l}^{x,x+1}) - f(\xi)] + \beta(\xi)[f(\xi_{l,k}^{x,x+1}) - f(\xi)],$$

where $\xi_{l,k}^{x,x+1}$ means switching a particle of the l^{th} species at position x with a particle of the k^{th} species at position $x+1$, and the jump rates are

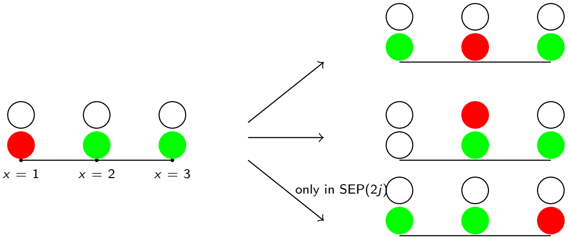
$$\alpha(\xi) = q^{-1+2(\xi_1^x + \cdots + \xi_{k-1}^x) + 2(\xi_{l+1}^{x+1} + \cdots + \xi_n^{x+1})} \{\xi_k^x\}_{q^2} \{\xi_l^{x+1}\}_{q^2},$$

$$\beta(\xi) = q^{1+2(\xi_1^x + \cdots + \xi_{l-1}^x) + 2(\xi_{l+1}^{x+1} + \cdots + \xi_n^{x+1})} \{\xi_l^x\}_{q^2} \{\xi_k^{x+1}\}_{q^2}, \ \{n\}_{q^2} = \frac{1-q^{2n}}{1-q^2}.$$

[default]

Example

Example: Suppose $L = 3$, $n = 3$, $2j = 2$,



Markov Duality

Definition

Two Markov processes $\mathfrak{s}_t$ and $\hat{\mathfrak{s}}_t$ on state spaces $\mathfrak{S}$ and $\hat{\mathfrak{S}}$ are dual with respect to duality function $D(\cdot,\cdot)$ on $\mathfrak{S}\times\hat{\mathfrak{S}}$ if

$$E_{\mathfrak{s}}[D(\mathfrak{s}_t,\hat{\mathfrak{s}})] = E_{\hat{\mathfrak{s}}}[D(\mathfrak{s},\hat{\mathfrak{s}}_t)] \text{ for all } \mathfrak{s}\in\mathfrak{S},\ \hat{\mathfrak{s}}\in\hat{\mathfrak{S}},\ \text{and } t>0, \tag{2}$$

where $E_{\mathfrak{s}}$ denotes $\mathfrak{s}_0=\mathfrak{s}$ and $E_{\hat{\mathfrak{s}}}$ denotes $\hat{\mathfrak{s}}_0=\hat{\mathfrak{s}}$. If $\hat{\mathfrak{s}}_t$ is a copy of $\mathfrak{s}_t$, we say that the process $\mathfrak{s}_t$ is self-dual.

It is equivalent to define duality at the level of process generators. We say that generator L_1 is dual to L_2 with respect to duality function $D(\cdot,\cdot)$ if

$$[L_1 D(\cdot,\hat{\mathfrak{s}})](\mathfrak{s}) = [L_2 D(\mathfrak{s},\cdot)](\hat{\mathfrak{s}}). \tag{3}$$

If $L_1=L_2$, we have self-duality.
For discrete state space, it is equlivant to $L_1 D = D L_2^T$.

[default]

Known Results

Theorem (Carinci, Franceschini,Groenevelt '21[1])

The ASEP(q,j) is self dual with orthogonal functions:

$$D_\alpha(\xi,\eta) = \prod_{x=1}^{L} K_{\eta^x}\left(q^{-2\xi^x}, p_\alpha^x(\xi,\eta), 2j, q^2\right), \quad \alpha \in (0, q^{-1+2j(2L+1)}) \quad (4)$$

where

$$p_\alpha^x(\xi,\eta) = \alpha^{-1} q^{-2\left(\sum_{y=1}^{x-1}\xi^y - \sum_{y=x+1}^{L}\eta^y\right)+2j(2x-1)-1},$$

and the q-Krawtchouk polynomials are given by

$$K_n(q^x; p, c; q) = {}_2\phi_1(q^{-x}, q^{-n}, q^{-c}; q, pq^{n+1}). \quad (5)$$

The authors used the method of scalar products and also the method of symmetries.

Known Results

Theorem (Z. 21' [5])

The multi-species SEP(2j) is self dual with respect to duality functions

$$\prod_{x \in G} K(\xi^x, \eta^x, \kappa, 2j), \tag{6}$$

where $K(\xi, \eta, \kappa, 2j)$ is the multivariate Krawtchouk polynomial.

The author used the method introduced by Groenevelt '17 [3]. Namely, unitary intertwiners between different $*$-representations of the special linear Lie algebra $\mathfrak{sl}_n$ yield the duality function.

Orthogonal polynomial duality functions for multi-species ASEP(q, j)

Theorem (Franceschini-Kuan-Z, '2? [2])

The multi-species ASEP(q, j) is self dual with respect to products of multivariate q-Krawtchouk polynomials, which are orthogonal.

[default]

Definitions

Define the q-deformed integers by $[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$ and $[n]_q! = [1]_q \cdots [n]_q$.
The q-Pochhammer symbol for $a \in \mathbb{R}$ and $n \in \mathbb{N}$ is defined by
$(a; q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$.

[default]

Definitions

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Recall a reversible measure of $(n-1)$-species ASEP(q,j) given by Kuan '17 [4] is

$$\mu^n(\xi) = \prod_x \prod_{i=1}^{n} \frac{1}{[\xi_i^x]_q^!} q^{(\xi_i^x)^2/2} \prod_{y<x} \prod_{i=1}^{n-1} q^{-2\xi_{[1,i]}^x \xi_{i+1}^y}. \tag{7}$$

[default]

Definitions

Let $\theta = (\theta^1, \cdots, \theta^L)$, $\theta^x > 0$. Define

$$D_\alpha^\theta(\xi, \eta) = \prod_{x=1}^{L} K_{\eta^x}\left(q^{-2\xi^x}, p^x(\xi, \eta), \theta^x, q^2\right), \tag{8}$$

with

$$p^x(\xi, \eta) = \alpha^{-1} q^{-2\left(\sum_{y<x} \xi^y - \sum_{y>x} \eta^y\right) + 2\sum_{y<x} \theta_y - 1}, \tag{9}$$

and

$$\mu_\alpha^\theta(\xi) = \prod_{x=1}^{L} \alpha^{\xi^x} \binom{\theta_x}{\xi^x}_q q^{-\left(2\sum_{y<x} \theta_y + \theta_x\right)\xi^x}. \tag{10}$$

Example

Then, the 2-species ASEP(q, j) is self dual with respect to orthogonal functions

$$\mathcal{D}(\xi,\eta) = D_{\alpha_1}^{(\eta_1+\eta_2)}(\xi_1,\eta_1) \times D_{\alpha_2}^{(N-\xi_1)}(\xi_2,\eta_1+\eta_2-\xi_1) \times\sqrt{\frac{\mu_{\alpha_1}^{(\eta_1+\eta_2)}(\xi_1)\mu_{\alpha_1}^{(\eta_1+\eta_2)}(\eta_1)\mu_{\alpha_2}^{(N-\xi_1)}(\xi_2)\mu_{\alpha_2}^{(N-\xi_1)}(\eta_1+\eta_2-\xi_1)}{\mu^3(\xi)\mu^3(\eta)}}. \tag{11}$$

Proof Idea

- Kuan '17 [4] constructed $(n-1)$-species ASEP(q, j) using representations of $\mathcal{U}_q(\mathfrak{gl}_n)$.
  Namely, the generator $\mathcal{L}$ of multi-species ASEP(q, j) could be written as $\mathcal{L} = G^{-1}HG$, where G is a diagonal matrix and H is a self adjoint quantum Hamiltonian. In addition, G^2 is a reversible measure of $\mathcal{L}$, i.e. $G = \sqrt{\mu(\xi)}\delta_{\xi,\eta}$.

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- If S is a symmetry of H, i.e. $SH = HS$, then $G^{-1}SG^{-1}$ is a self duality function of $\mathcal{L}$. Moreover, if S is unitary, then $G^{-1}SG^{-1}$ is an orthogonal self duality function of $\mathcal{L}$.

[default]

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- If S is a symmetry of H, i.e. $SH = HS$, then $G^{-1}SG^{-1}$ is a self duality function of $\mathcal{L}$. Moreover, if S is unitary, then $G^{-1}SG^{-1}$ is an orthogonal self duality function of $\mathcal{L}$.
- Carinci-Franceschini-Groenevelt '21[1] found symmetries in $\mathcal{U}_q(\mathfrak{sl}_2)$ which yields the orthogonal duality functions for ASEP(q,j).

Algebraic structures

The Drinfeld–Jimbo quantum group $\mathcal{U}_q(\mathfrak{gl}_n)$ is the Hopf algebra with generators

$$\{E_{i,i+1}, E_{i+1,i} : 1 \leq i < n\}, \{q^{E_{ii}} : 1 \leq i \leq n\}$$

and relations

$$q^{E_{ii}} q^{E_{jj}} = q^{E_{jj}} q^{E_{ii}} = q^{E_{ii}+E_{jj}}$$

$$[E_{i,i+1}, E_{i+1,i}] = \frac{q^{E_{ii}-E_{i+1,i+1}} - q^{E_{i+1,i+1}-E_{ii}}}{q-q^{-1}} \quad [E_{i,i+1} E_{j+1,j}] = 0, \quad i \neq j$$

$$q^{E_{ii}} E_{i,i+1} = qE_{i,i+1} q^{E_{ii}} \ \ q^{E_{ii}} E_{i-1,i} = q^{-1} E_{i-1,i} q^{E_{ii}} \ \ \left[q^{E_{ii}}, E_{j,j+1}\right] = 0, \ j \neq i, i-1$$

$$q^{E_{ii}} E_{i,i-1} = qE_{i,i-1} q^{E_{ii}} \ \ q^{E_{ii}} E_{i+1,i} = q^{-1} E_{i+1,i} q^{E_{ii}} \ \ \left[q^{E_{ii}}, E_{j,j-1}\right] = 0, \ j \neq i, i+1$$

$$E_{i,i+1}^2 E_{j,j+1} - \left(q + q^{-1}\right) E_{i,i+1} E_{j,j+1} E_{i,i+1} + E_{j,j+1} E_{i,i+1}^2 = 0, \quad i = j \pm 1$$

$$E_{i,i-1}^2 E_{j,j-1} - \left(q + q^{-1}\right) E_{i,i-1} E_{j,j-1} E_{i,i-1} + E_{j,j-1} E_{i,i-1}^2 = 0, \quad i = j \pm 1$$

$$[E_{i,i+1}, E_{j,j+1}] = 0 = [E_{i,i-1}, E_{j,j-1}], \quad i \neq j \pm 1$$

[default]

Algebraic structures

The coproduct $\Delta : \mathcal{U}_q(\mathfrak{gl}_n) \to \mathcal{U}_q(\mathfrak{gl}_n) \otimes \mathcal{U}_q(\mathfrak{gl}_n)$ is given by

$$\begin{aligned} \Delta(E_{i,i+1}) &= E_{i,i+1} \otimes q^{-A_i^0} + q^{A_i^0} \otimes E_{i,i+1}, \\ \Delta(E_{i+1,i}) &= E_{i+1,i} \otimes q^{-A_i^0} + q^{A_i^0} \otimes E_{i+1,i}, \\ \Delta(q^{E_{i,i}}) &= q^{E_{i,i}} \otimes q^{E_{i,i}}, \end{aligned} \tag{12}$$

where $A_i^0 = \frac{E_{i,i} - E_{i+1,i+1}}{2}$.
The coproduct is **asymmetric.**

[default]

Algebraic structures

The coproduct $\Delta : \mathcal{U}_q(\mathfrak{gl}_n) \to \mathcal{U}_q(\mathfrak{gl}_n) \otimes \mathcal{U}_q(\mathfrak{gl}_n)$ is given by

$$\begin{aligned} \Delta(E_{i,i+1}) &= E_{i,i+1} \otimes q^{-A_i^0} + q^{A_i^0} \otimes E_{i,i+1}, \\ \Delta(E_{i+1,i}) &= E_{i+1,i} \otimes q^{-A_i^0} + q^{A_i^0} \otimes E_{i+1,i}, \\ \Delta(q^{E_{i,i}}) &= q^{E_{i,i}} \otimes q^{E_{i,i}}, \end{aligned} \tag{12}$$

where $A_i^0 = \frac{E_{i,i} - E_{i+1,i+1}}{2}$.
The coproduct is **asymmetric.**
We define a representation of $\mathcal{U}_q(\mathfrak{gl}_n)$ by

$$\begin{aligned} E_{i,i+1}|\eta\rangle &= \sqrt{[\eta_{i+1}]_q[\eta_i + 1]_q}|\eta_{i,i+1}^{+1,-1}\rangle, \\ E_{i+1,i}|\eta\rangle &= \sqrt{[\eta_i]_q[\eta_{i+1} + 1]_q}|\eta_{i,i+1}^{-1,+1}\rangle, \\ E_{i,i}|\eta\rangle &= \eta_i|\eta\rangle. \end{aligned} \tag{13}$$

[default]

Symmetries

The q-exponential functions are given by

$$E_{q^2}(X) = (-X; q^2)_\infty = \sum_{n=0}^{\infty} q^{n(n-1)} \frac{X^n}{(q^2; q^2)_n},$$

$$e_{q^2}(X) = \frac{1}{(X; q^2)_\infty} = \sum_{n=0}^{\infty} \frac{X^n}{(q^2; q^2)_n}.$$

Define operators S_i and $\hat{S}_i$ for $i = 1, 2 \cdots, n-1$.

$$S_i = e_{q^2}\left(-\sqrt{\alpha_i}(1-q^2)\Delta^{L-1}(q^{A_i^0} E_{i,i+1})\right), \tag{14}$$

$$\hat{S}_i = E_{q^2}\left(\sqrt{\alpha_i}(1-q^2)\Delta^{L-1}(q^{-(E_{i,i}+E_{i+1,i+1})} q^{-A_i^0} E_{i,i+1})\right). \tag{15}$$

[default]

Symmetries & Duality Functions

Fact: S_i and $\hat{S}_i$ are symmetries of the quantum Hamiltonian H.
Then $\hat{S}_{n-1}S^*_{n-1}\cdots\hat{S}_1S^*_1$ is a symmetry of H.

Symmetries & Duality Functions

Fact: S_i and $\hat{S}_i$ are symmetries of the quantum Hamiltonian H.
Then $\hat{S}_{n-1}S^*_{n-1}\cdots\hat{S}_1S^*_1$ is a symmetry of H.
Explicit formula:

$$< \xi|\hat{S}_iS^*_i|\eta >= D^{(\xi_i+\xi_{i+1})}_{\alpha_i}(\xi_i,\eta_i)q^{N(\eta_i)-N(\xi_i)}\sqrt{\mu^{(\xi_i+\xi_{i+1})}_{\alpha_i}(\xi_i)\mu^{(\xi_i+\xi_{i+1})}_{\alpha_i}(\eta_i)}\,(-1)^{N(\eta_i)}\delta_{\left\{\substack{\xi_j=\eta_j\\ j\neq i,i+1}\right\}},$$

where $N(\eta_i)=\sum_{x=1}^{L}\eta_i^x$, which is a constant.

[default]

Orthogonality

To show orthogonality, we normalize $\langle\xi|\hat{S}_i S_i^*|\eta\rangle$,

$$\langle\xi|F_i|\eta\rangle = (-1)^{N(\eta_i)} q^{\binom{N(\xi_i)}{2}-\binom{N(\eta_i)}{2}} \sqrt{\frac{\left(\alpha_i q^{1+2N(\xi_i)-2N(\eta_i+\eta_{i+1})}\right)_\infty}{\left(\alpha_i q^{1-2N(\eta_i)}\right)_\infty}} \langle\xi|\hat{S}_i S_i^*|\eta\rangle. \tag{16}$$

Then F_i are unitary in the standard inner product, i.e.

$$\langle\xi|F_i F_i^*|\eta\rangle = \delta_{\xi,\eta}. \tag{17}$$

Thus $(G^n)^{-1}F_{n-1}\cdots F_1(G^n)^{-1}$ is a self duality function, moreover, it satisfies orthogonal relation:

$$\langle\xi|(G^n)^{-1}F_{n-1}\cdots F_1(G^n)^{-1}(G^n)^2\left((G^n)^{-1}F_{n-1}\cdots F_1(G^n)^{-1}\right)^*|\eta\rangle = \frac{\delta_{\xi,\eta}}{\mu^n(\xi)}. \tag{18}$$

[default]

References I

[1] Gioia Carinci, Chiara Franceschini, and Wolter Groenevelt. *q-Orthogonal dualities for asymmetric particle systems*. 2021. arXiv: 2003.07837 [math.PR].

[2] Chiara Franceschini, Jeffrey Kuan, and Zhengye Zhou. "work in progress". In: ().

[3] Wolter Groenevelt. "Orthogonal Stochastic Duality Functions from Lie Algebra Representations". In: *Journal of Statistical Physics* 174.1 (Oct. 2018), pp. 97–119. ISSN: 1572-9613. DOI: 10.1007/s10955-018-2178-7. URL: http://dx.doi.org/10.1007/s10955-018-2178-7.

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References II

[4] Jeffrey Kuan. "A Multi-species ASEP **(q, j)** and **q**-TAZRP with Stochastic Duality". In: *International Mathematics Research Notices* 2018.17 (Mar. 2017), pp. 5378–5416. ISSN: 1073-7928. DOI: 10.1093/imrn/rnx034. URL: http://dx.doi.org/10.1093/imrn/rnx034.

[5] Zhengye Zhou. *Orthogonal polynomial Stochastic Duality functions for Multi-species SEP$(2j)$ and Multi-species IRW*. 2021. arXiv: 2110.07042 [math.PR].