Moments, intermittency, and growth indices for nonlinear stochastic PDE's with rough initial conditions

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Abstract

In this thesis, we study several stochastic partial differential equations (SPDE's) in the spatial domain $\mathbb{R}$, driven by multiplicative space-time white noise. We are interested in how rough and unbounded initial data affect the random field solution and the asymptotic properties of this solution.

We first study the nonlinear stochastic heat equation. A central special case is the parabolic Anderson model. The initial condition is taken to be a measure on $\mathbb{R}$, such as the Dirac delta function, but this measure may also have non-compact support and even be non-tempered (for instance with exponentially growing tails). Existence and uniqueness is proved without appealing to Gronwall's lemma, by keeping tight control over moments in the Picard iteration scheme. Upper and lower bounds on all p-th moments ($p \ge 2$) are obtained. These bounds become equalities for the parabolic Anderson model when $p = 2$. We determine the growth indices introduced by Conus and Khoshnevisan [19] and, despite the irregular initial conditions, we establish Hölder continuity of the solution for $t > 0$.

In order to study a wider class of SPDE's, we consider a more general problem, consisting in a stochastic integral equation of space-time convolution type. We give a set of assumptions which guarantee that the stochastic integral equation in question has a unique random field solution, with moment formulas and sample path continuity properties. As a first application, we show how certain properties of an extra potential term in the nonlinear stochastic heat equation influence the admissible initial data. As a second application, we investigate the nonlinear stochastic wave equation on $\mathbb{R}_+ \times \mathbb{R}$. All the properties obtained for the stochastic heat equation – moment formulas, growth indices, Hölder continuity, etc. – are also obtained for the stochastic wave equation.

Keywords: nonlinear stochastic heat equation, nonlinear stochastic wave equation, parabolic Anderson model, hyperbolic Anderson model, rough initial data, Hölder continuity, Lyapunov exponents, growth indices.

Résumé

Dans cette thèse, nous étudions plusieurs équations aux dérivées partielles stochastiques (EDPS) définies sur le domaine spatial $\mathbb{R}$, perturbées par un bruit multiplicatif et blanc en espace-temps. Nous nous intéressons à la façon dont des données initiales irrégulières et non-bornées affectent la solution du champ aléatoire et les propriétés asymptotiques de cette solution.

Nous étudions d'abord l'équation de la chaleur stochastique non-linéaire. Un cas particulier central est le modèle parabolique d'Anderson. La condition initiale est alors une mesure sur $\mathbb{R}$, comme par exemple la fonction delta de Dirac, mais cette mesure pourrait également avoir un support non-compact et même ne pas être tempérée (par exemple avec des queues en croissance exponentielle). L'existence et l'unicité sont établies sans utiliser le lemme de Gronwall, en gardant un contrôle serré des moments dans le schéma itératif de Picard. Des bornes supérieures et inférieures sur tous les moments d'ordre p ($p \geq 2$) sont obtenues. Ces bornes deviennent des égalités pour le modèle parabolique d'Anderson lorsque $p = 2$. Nous déterminons les indices de croissance introduites par Conus et Khoshnevisan [19] et, malgré l'irrégularité de conditions initiales, nous établissons la continuité de Hölder de la solution pour $t > 0$.

Afin d'étudier une catégorie plus large d'EDPS, nous considérons un problème plus général, consistant en une équation intégrale stochastique de type convolution en espace-temps. Nous donnons une famille d'hypothèses qui garantissant que l'équation intégrale stochastique en question aura une solution unique de type champ aléatoire, avec des formules pour les moments et des propriétés de continuité de la trajectoire. Comme première application, nous montrons comment certaines propriétés d'un terme potentiel supplémentaire dans l'équation de la chaleur stochastique non-linéaire modifie l'ensemble des données initiales admissibles. Comme seconde application, nous étudions l'équation des ondes stochastique non-linéaire sur $\mathbb{R}_+ \times \mathbb{R}$. Toutes les propriétés obtenues pour l'équation de la chaleur stochastique – formules pour les moment, les indices de croissance, la continuité de Hölder, etc. – sont également obtenues pour l'équation des ondes stochastique.

Mots-clés: équation de la chaleur stochastique non-linéaire, équation des ondes stochastique non-linéaire, modèle parabolique d'Anderson, modèle hyperbolique d'Anderson, données initiales irrégulières, continuité de Hölder, exposants de Lyapunov, indices de croissance.

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Time flies quickly especially with the teaching and supervising of students. In the beginning of a semester, we are busy preparing the course materials and guiding students for their projects. Then a dozen weeks slip by in a fixed pattern. At the end of the semester, it is a time for harvests. We receive many papers to grade and semester project reports to read. Finally, we have a month or two of peaceful summer/winter break, when we can concentrate more on our own research. As the old Chinese sage – Confucius – said, *"the processes of teaching and learning stimulate one another."* I did learn a lot from teaching and, especially, from supervising my students. I would like to thank them – *Sarah Grandjean, Hélène Ruffleux, Denis Garcia, David Spiess, David Krief, Alexandre Villard, Louis Larmonier*, and *François Pagano* (in chronological order) – for their hard and excellent work.

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I know it’s a rare privilege,
but if one can really tackle something
in adult life that means that much to you,
then it’s more rewarding than
anything I can imagine.
— Andrew Wiles

To my mother and my late father

Contents

1 Introduction

In this thesis, we study the following nonlinear stochastic partial differential equation

$$\mathscr{L}u(t,x) = \rho\left(u(t,x)\right)\dot{W}(t,x)\,, \qquad t\in\mathbb{R}_+^*,\ x\in\mathbb{R}\,, \tag{1.0.1}$$

subject to certain initial conditions, where $\mathscr{L}$ is a partial differential operator, $\mathbb{R}_+^* =]0,\infty[$, the function $\rho:\mathbb{R}\mapsto\mathbb{R}$ is Lipschitz continuous, and $\dot{W}$ is space-time white noise. We work in Walsh's framework; see [42] and [68] for an introduction. More generally, the problem (1.0.1) is formulated as a stochastic integral equation

$$u(t,x) = J_0(t,x) + \iint_{\mathbb{R}_+\times\mathbb{R}} G\left(t-s,x-y\right)\rho\left(u\left(s,y\right)\right)W(\mathrm{d}s\mathrm{d}y)\,, \tag{1.0.2}$$

where the kernel function $G(t,x)$ is usually, but not necessarily, the fundamental solution corresponding to the partial differential operator $\mathscr{L}$, and $J_0(t,x)$ is usually, but not necessarily, the solution to the homogeneous equation,

$$\mathscr{L}u(t,x) = 0\,, \qquad t>0,\ x\in\mathbb{R}\,,$$

subject to certain initial conditions. We use the convention that $G(t,x)\equiv 0$ for $t<0$.

According to the theory introduced by Dalang in [23], a minimal condition that needs to be examined first is whether the linear case – the case where $\rho(u)\equiv 1$ – admits a random field solution. This solution, if it exists, will be a Gaussian random field. Define, for $t\in\mathbb{R}_+$, and $x,y\in\mathbb{R}$,

$$\Theta(t,x,y) := \iint_{[0,t]\times\mathbb{R}} G(t-s,x-z)G\left(t-s,y-z\right)\mathrm{d}s\mathrm{d}z\,. \tag{1.0.3}$$

Clearly, $2\Theta(t,x,y)\le\Theta(t,x,x)+\Theta\left(t,y,y\right)$. The condition, called *Dalang's condition* in [18], is

$$\Theta(t,x,x) < +\infty\,, \qquad \text{for all } (t,x)\in\mathbb{R}_+\times\mathbb{R}\,. \tag{1.0.4}$$

1.1 Stochastic Heat Equation

We will first study the stochastic heat equation in Chapter 2. In this case,

$$\mathscr{L} = \frac{\partial}{\partial t} - \frac{\nu}{2}\frac{\partial^2}{\partial x^2}\,,$$

where $\nu > 0$ and the heat kernel function is

$$G_\nu(t,x) := \frac{1}{\sqrt{2\pi\nu t}} \exp\left\{-\frac{x^2}{2\nu t}\right\}, \quad \text{for all } (t,x) \in \mathbb{R}_+^* \times \mathbb{R}. \tag{1.1.1}$$

Clearly, Dalang's condition (1.0.4) holds in this case: for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$,

$$\Theta_\nu(t,x,x) = \iint_{[0,t]\times\mathbb{R}} G_\nu^2\left(t-s, x-y\right) \mathrm{d}s\mathrm{d}y = \frac{\sqrt{t}}{\sqrt{\pi\nu}} < +\infty. \tag{1.1.2}$$

For reference purpose, we write this equation as follows:

$$\begin{cases} \left(\dfrac{\partial}{\partial t} - \dfrac{\nu}{2}\dfrac{\partial^2}{\partial x^2}\right) u(t,x) = \rho(u(t,x))\,\dot{W}(t,x), & x \in \mathbb{R},\ t \in \mathbb{R}_+^*, \\ u(0,\cdot) = \mu(\cdot)\,, \end{cases} \tag{1.1.3}$$

where μ is the initial data. This problem has been intensively studied during last two decades by many authors: See [2, 3, 5, 12, 17, 19, 18, 30, 37] for the intermittency problem, [28, 29] for probabilistic potential theory, [62, 63] for regularity of the solution, and some other properties in [47, 48, 58, 65]. In particular, the special case $\rho(u) = \lambda u$ is called *the parabolic Anderson model* [12]. Our work focuses on (1.1.3) with general deterministic initial data μ, and we study how the initial data affects the solution.

For the existence of random field solutions to (1.1.3), the case where the initial data μ is a bounded and measurable function is covered by the classical theory of Walsh [68]. When μ is a positive Borel measure on $\mathbb{R}$ such that

$$\sup_{t\in[0,T]} \sup_{x\in\mathbb{R}} \sqrt{t}\left(\mu * G_\nu(t,\circ)\right)(x) < \infty, \quad \text{for all } T > 0, \tag{1.1.4}$$

where $*$ denotes convolution in the spatial variable, Bertini and Cancrini [3] gave an ad-hoc definition for the Anderson model via a smoothing of the space-time white noise and a Feynman-Kac type formula. Their analysis depends heavily on properties of the local times of Brownian bridges. Recently, Conus and Khoshnevisan [18] constructed a weak solution defined through certain norms on random fields. The initial data has to verify certain technical conditions, which include the Dirac delta function in some of their cases. In particular, the solution is defined for almost all (t,x), but not at specific (t,x). More recently, Conus, Joseph, Khoshnevisan and Shiu [17] also studied random field solutions. In particular, they require the initial data to be a finite measure of compact support. We improve the existence result by working under a much weaker condition on initial data, namely, μ can be any signed Borel measure over $\mathbb{R}$ such that

$$\left(|\mu| * G_\nu(t,\cdot)\right)(x) < +\infty\,, \quad \text{for all } t>0 \text{ and } x\in\mathbb{R}\,, \tag{1.1.5}$$

where, from the Jordan decomposition, $\mu = \mu_+ - \mu_-$ where $\mu_\pm$ are two non-negative Borel measures with disjoint support and $|\mu| := \mu_+ + \mu_-$. On the one hand, the condition (1.1.5) allows the measure-valued initial data, for example, the Dirac delta function. Proposition 2.2.9 below shows that initial data cannot be extended beyond measures to other Schwartz distributions, even with compact support. On the other hand, the condition (1.1.5) permits certain exponential growth at infinity. For instance, if $\mu(\mathrm{d}x) = f(x)\mathrm{d}x$, then $f(x) = \exp\left(a|x|^p\right)$, $a>0$, $p\in\,]0,2[$, (i.e., exponential growth at $\pm\infty$), will satisfy this condition. Note that the case where the initial data is a continuous function with the linear exponential growth (i.e.,$p=1$) has been considered by many authors; see [48, 58, 65] and the references therein. Note that the set of μ satisfying (1.1.5) is the set of locally finite Borel measures such that for all $a>0$, $\int_{\mathbb{R}} e^{-ax^2}|\mu|(\mathrm{d}x) < +\infty$.

Moreover, we obtain estimates for the moments $\mathbb{E}(|u(t,x)|^p)$ with both t and x fixed for all even integers $p\ge 2$. In particular, for the parabolic Anderson model, we give an explicit formula for the second moment of the solution. When the initial data is either the Lebesgue measure or the Dirac delta function, we give explicit formulas for the two-point correlation functions (see (2.2.17) and (2.2.20) below), which can be compared to the integral form in Bertini and Cancrini's paper [3, Corollaries 2.4 and 2.5] (see also Remark 2.2.4 below).

Recently, Borodin and Corwin [5] also obtained the moment formulas for the parabolic Anderson model in the case where the initial data is the Dirac delta function. When $p=2$, we give the same explicit formula. For $p>2$, their p-th moments are represented by a multiple contour integral. Our methods are very different from theirs: They use the arguments of approximating the continuous system by a discrete one. Our formulas allow more general initial data than the Dirac delta function, and are useful for proving other properties like sample path regularity and growth indices.

Our proof of existence is based on the standard Picard iteration scheme. The main difference from the conventional situation is that instead of applying Gronwall's lemma to bound the second moment from above, we show that the sequence of the second moments in the Picard iteration converges to an explicit formula (in the case of the parabolic Anderson model).

After establishing the existence of random field solutions, we study whether the solution exhibits intermittency properties. More precisely, define the *upper and lower Lyapunov exponents* for constant initial data (the Lebesgue measure) as follows

$$\overline{\lambda}_p(x) := \limsup_{t\to+\infty} \frac{\log \mathbb{E}\left[|u(t,x)|^p\right]}{t}\,, \qquad \underline{\lambda}_p(x) := \liminf_{t\to+\infty} \frac{\log \mathbb{E}\left[|u(t,x)|^p\right]}{t}\,. \tag{1.1.6}$$

When the initial data is constant, these two exponents do not depend on x. In this case, following Bertini and Cancrini [3], we say that the solution is *intermittent* if $\lambda_n := \underline{\lambda}_n = \overline{\lambda}_n$

and the strict inequalities

$$\lambda_1 < \frac{\lambda_2}{2} < \cdots < \frac{\lambda_n}{n} < \cdots \tag{1.1.7}$$

are satisfied. Carmona and Molchanov gave the following definition [12, Definition III.1.1, on p. 55]:

Definition 1.1.1 (Intermittency)**.** Let p be the smallest integer for which $\lambda_p > 0$. When $p < \infty$, we say that the solution $u(t,x)$ shows *(asymptotic) intermittency of order* p and *full intermittency* when $p = 2$.

They showed that full intermittency implies the intermittency defined by (1.1.7) (see [12, III.1.2, on p. 55]). This mathematical definition of intermittency is related to the property that the solutions develop high peaks on some small "islands". The parabolic Anderson model has been well studied: see [12, 20] for a discrete approximation and [3, 37, 30] for the continuous version. Further discussion can be found in [70].

When the initial data are not homogeneous, in particular, when they have certain exponential decrease at infinity, Conus and Khoshnevisan [19] defined the following *lower and upper exponential growth indices*:

$$\underline{\lambda}(p) := \sup\left\{\alpha > 0 : \limsup_{t\to\infty} \frac{1}{t} \sup_{|x|\ge\alpha t} \log \mathbb{E}\left(|u(t,x)|^p\right) > 0\right\}, \tag{1.1.8}$$

$$\overline{\lambda}(p) := \inf\left\{\alpha > 0 : \limsup_{t\to\infty} \frac{1}{t} \sup_{|x|\ge\alpha t} \log \mathbb{E}\left(|u(t,x)|^p\right) < 0\right\}, \tag{1.1.9}$$

and proved that if the initial data μ is a non-negative, lower semicontinuous function with compact support of positive measure, then for the Anderson model ($\rho(u) = \lambda u$),

$$\frac{\lambda^2}{2\pi} \le \underline{\lambda}(2) \le \overline{\lambda}(2) \le \frac{\lambda^2}{2}.$$

We improve this result by showing that $\underline{\lambda}(2) = \overline{\lambda}(2) = \lambda^2/2$, and extend this to more general measure-valued initial data. This is possible mainly thanks to our explicit formula for the second moment.

We now discuss the regularity of the random field solution. Denote by $C_{\beta_1,\beta_2}(D)$ the set of trajectories that are β_1-Hölder continuous in time and β_2-Hölder continuous in space on the domain $D \subseteq \mathbb{R}_+ \times \mathbb{R}$, and let

$$C_{\beta_1-,\beta_2-}(D) := \bigcap_{\alpha_1 \in \,]0,\beta_1[} \; \bigcap_{\alpha_2 \in \,]0,\beta_2[} C_{\alpha_1,\alpha_2}(D).$$

In Walsh's notes [68, Corollary 3.4, p. 318], a slightly different equation was studied and the Hölder exponents given (for both space and time) are $1/4 - \epsilon$. Bertini and Cancrini [3] stated in their paper that the random field solution for the parabolic Anderson model with initial data satisfying (1.1.4) belongs to $C_{\frac{1}{4}-,\frac{1}{2}-}(\mathbb{R}_+^* \times \mathbb{R})$. In [58, 65], the authors showed that if the initial data is a continuous function with certain exponentially growing tails, then

$$u \in C_{\frac{1}{4}-,\frac{1}{2}-}(\mathbb{R}_+ \times \mathbb{R}), \quad \text{a.s.} \tag{1.1.10}$$

Sanz-Solé and Sarrà [63] considered the stochastic heat equation over $\mathbb{R}^d$ with spatially homogeneous colored noise which is white in time. Let $\tilde{\mu}$ be the spectral measure satisfying

$$\int_{\mathbb{R}^d} \frac{\tilde{\mu}(d\xi)}{\left(1+|\xi|^2\right)^\eta} < +\infty, \quad \text{for some } \eta \in]0,1[. \tag{1.1.11}$$

They proved that if the initial data is a bounded ρ-Hölder continuous function for some $\rho \in]0,1[$, then the solution is in

$$u \in C_{\frac{1}{2}(\rho\wedge(1-\eta))-,\rho\wedge(1-\eta)-}\left(\mathbb{R}_+^* \times \mathbb{R}\right), \quad \text{a.s.},$$

where $a \wedge b := \min(a,b)$. For the case of space-time white noise on $\mathbb{R}_+ \times \mathbb{R}$, the spectral measure $\tilde{\mu}$ is the Lebesgue measure and hence η in (1.1.11) (with $d = 1$) can be $1/2-\epsilon$ for any $\epsilon > 0$. Their result ([62, Theorem 4.3]) reduces to

$$u \in C_{\left(\frac{1}{4}\wedge\frac{\rho}{2}\right)-,\left(\frac{1}{2}\wedge\rho\right)-}\left(\mathbb{R}_+^* \times \mathbb{R}\right), \quad \text{a.s.}$$

More recently, Conus *et al* proved in their paper [17, Lemma 9.3] that the random field solution is Hölder continuous in x with exponent $1/2-\epsilon$ (for initial data that is a finite measure). They did not give the regularity estimate over the time variable. In their papers [28, 29], Dalang, Khoshnevisan and Nualart considered a system of heat equations with vanishing initial conditions subject to space-time white noise, and proved that the solution is jointly Hölder continuous with exponents $1/4-$ in time and $1/2-$ in space. We extend the $C_{\frac{1}{4}-,\frac{1}{2}-}\left(\mathbb{R}_+^* \times \mathbb{R}\right)$-Hölder continuity result to measure-valued initial data satisfying (1.1.5). We show that in general, the result in (1.1.10) should exclude the time line $t = 0$.

The difficulties for the proof of the Hölder continuity of the random field solution lie in the fact that for the initial data satisfying (1.1.5), the p-th moment $\mathbb{E}[|u(t,x)|^p]$ is neither bounded for $x \in \mathbb{R}$, nor for $t \in [0,T]$. Standard techniques, which isolate the effects of initial data by the $L^p(\Omega)$-boundedness of the solution, fail in our case. Instead, the initial data play an active role in our proof. Note that Fourier transforms are not applicable here because μ need not be a tempered measure.

1.2 Stochastic Integral Equation of Space-time Convolution Type

In Chapter 3, we will consider the following stochastic integral equation,

$$u(t,x) = J_0(t,x) + I(t,x), \quad (t,x) \in \mathbb{R}_+^* \times \mathbb{R}^d, \tag{1.2.1}$$

where $d \geq 1$ and

$$I(t,x) := \iint_{\mathbb{R}_+ \times \mathbb{R}^d} G\left(t-s, x-y\right)\theta\left(s,y\right)\rho\left(u\left(s,y\right)\right) W\left(ds, dy\right) .$$

As before, $\dot{W}$ is the space-time white noise and ρ is a Lipschitz continuous function. Let $\{\Omega, \mathscr{F}, \{\mathscr{F}_t : t \geq 0\}, P\}$ be a filtered probability space, which will be specified latter. Both functions $J_0(t,x)$ and $\theta(t,x)$ are real-valued deterministic Borel measurable functions.

The main motivation is the case where $G(t,x)$ is the fundamental solution for a partial differential operator $\mathscr{L}$, and the study of the stochastic partial differential equation

$$\mathscr{L} u(t,x) = \rho(u(t,x))\theta(t,x)\dot{W}(t,x) , \quad x \in \mathbb{R}^d,\ t \in \mathbb{R}_+^* ,$$

which is a slight variation of (1.0.1). Note that in the literature of the stochastic differential equations, for example [39, 40], the function in front of the driving noise is sometimes called *dispersion matrix (or function)*. In general, the dispersion function is not necessarily time homogeneous. In our case, one can think that the dispersion function in question is not space-time homogeneous, but it has the following factorized form

$$\rho_*\left(t,x,u(t,x)\right) := \theta(t,x)\rho\left(u(t,x)\right) .$$

In [31], Dalang *et al* work under a similar framework. More precisely, they considered the stochastic integral equation (1.2.1) with $\theta(t,x) \equiv 1$ and $\rho(u) = u$, where the driving noise is spatially homogeneous and white in time. They proved existence and uniqueness of a random field solution and then obtained Feynman-Kac-type formulas for all p-moments of the random field solution. Their requirements on $J_0(t,x)$ are as follows (see [31, Proposition 4.1]): for all $T > 0$,

(1) $\left(J_0(t,x),\ (t,x) \in [0,T] \times \mathbb{R}^d\right)$ is a predictable process;

(2) $\sup_{t \leq T,\, x \in \mathbb{R}^d} \mathbb{E}\left[J_0^2(t,x)\right] < +\infty$.

The condition (2) is slightly restrictive. Let us consider the stochastic heat equation (1.1.3). If the initial data is the Dirac delta function, then $J_0(t,x) = G_\nu(t,x)$ and the supremum of $J_0^2(t,0)$ over $t \in [0,T]$ is infinite. If the initial data is $\mu(dx) = x^2 dx$, then the supremum of $J_0^2(t,x)$ over $x \in \mathbb{R}$ is infinite. We will consider weaker conditions on $J_0(t,x)$ in our settings: Assume that for all compact sets $K \subseteq \mathbb{R}_+^* \times \mathbb{R}^d$ and $v \in \mathbb{R}$,

$$\sup_{(t,x) \in K} \int_0^t ds \int_{\mathbb{R}^d} \left(v^2 + J_0^2\left(s,y\right)\right) \theta^2\left(s,y\right) G^2\left(t-s, x-y\right) dy < +\infty .$$

Under some additional assumptions on the kernel function $G(t,x)$ and the function $\theta(t,x)$, we prove the existence and uniqueness of the random field solution. Though we do not give exact formulas for p-th moments $\mathbb{E}\left(|u(t,x)|^p\right)$ of the solution, we obtain good estimates on them, which are exact formulas when $p = 2$ and $\rho(u)^2 = \lambda^2(\varsigma^2 + u^2)$ for some constants $\lambda > 0$ and $\varsigma \in \mathbb{R}$. These estimates are convenient to study sample path regularity and certain asymptotic properties of the solution.

As a first application, we show that under certain conditions on the potential function $\theta(t,x)$, one can include some distribution-valued initial data for the one-dimensional stochastic heat equation such that the system still admits a random field solution. More precisely, consider the following equation

$$\begin{cases} \left(\frac{\partial}{\partial t} - \frac{\nu}{2}\frac{\partial^2}{\partial x^2}\right) u(t,x) = \theta(t,x)\,\rho(u(t,x))\,\dot{W}(t,x), & x \in \mathbb{R},\ t \in \mathbb{R}_+^*\,, \\ u(0,\cdot) = \mu(\cdot)\,, \end{cases} \tag{1.2.2}$$

which is the same as (1.1.3) except an extra potential function $\theta(t,x)$. The characterization of the balance between the admissible initial data and certain properties of $\theta(t,x)$ is stated in Theorem 3.2.17. For simplicity, we assume $|\theta(t,x)|$ is uniformly bounded. Here we only highlight this balance by some examples: If $\theta(t,x) \equiv 1$, then the initial data cannot go beyond measures; If $\theta(t,x) = t^r \wedge 1$ for some $r > 0$, then the initial data can be $\delta_0^{(k)}$ for all integer $k \in [0, r+1/4[$, where $\delta_0^{(k)}$ is the k-th distributional derivative of the Dirac delta function δ_0; If $\theta(t,x) = \exp(-1/t)$, then any Schwartz (or tempered) distribution can serve as the initial data.

Chapter 4 is an application of Chapter 3 to the stochastic wave equation in the setting: $d = 1$ and $\theta(t,x) \equiv 1$, which we now discuss.

1.3 Stochastic Wave Equation

The stochastic wave equation, like the stochastic heat equation, has been widely studied: See for example [8, 10, 11, 54, 68] for some early work, [22, 68] for an introduction, [30, 31] for the intermittency problems, [16, 23, 27, 45, 56, 57] for the stochastic wave equation in the spatial domain $\mathbb{R}^d$, $d > 1$, [33, 62] for regularity of the solution, [6, 7] for the stochastic wave equation with values in Riemannian manifolds, [14, 52, 53] for wave equations with polynomial nonlinearities, and [46, 49, 59] for smoothness of the law. In Chapter 4, we will study a simple case: the nonlinear stochastic wave equation in spatial domain 1. In this case,

$$\mathscr{L} = \frac{\partial^2}{\partial t^2} - \kappa^2 \frac{\partial^2}{\partial x^2}\,,$$

where $\kappa > 0$ is the speed of wave propagation, and the wave kernel function is

$$G_\kappa(t,x) = \frac{1}{2} H(t) 1_{[-\kappa t, \kappa t]}(x), \quad \kappa > 0, \tag{1.3.1}$$

where $H(t)$ is the Heaviside function, i.e., $H(t) = 1$ if $t > 0$ and 0 otherwise. Clearly, Dalang's condition (1.0.4) holds in this case: for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$,

$$\Theta_\kappa(t,x,x) = \iint_{[0,t]\times\mathbb{R}} G_\kappa^2\left(t-s, x-y\right) \mathrm{d}s\mathrm{d}y = \frac{\kappa t^2}{2} < +\infty\,. \tag{1.3.2}$$

More precisely, we will study the following equation

$$\begin{cases} \left(\frac{\partial^2}{\partial t^2} - \kappa^2 \frac{\partial^2}{\partial x^2}\right) u(t,x) = \rho(u(t,x))\,\dot{W}(t,x), & x \in \mathbb{R},\ t \in \mathbb{R}_+^*, \\ u(0,\cdot) = g(\cdot)\,, \quad \frac{\partial u}{\partial t}(0,\cdot) = \mu(\cdot)\,, \end{cases} \tag{1.3.3}$$

where $g(\cdot)$ and μ are the (deterministic) initial position and initial velocity, respectively. The linear case, $\rho(u) = \lambda u$ with $\lambda \neq 0$, is called *the hyperbolic Anderson model* [30].

The general aim of this study is to understand how irregular (possibly unbounded) initial data affects the random field solutions to (1.3.3). Here are our assumptions on the initial data:

(1) The initial position g is a Borel measurable, locally square integrable function, which is denoted by $g \in L^2_{loc}(\mathbb{R})$;

(2) The initial velocity μ is a locally finite Borel measure, which is denoted by $\mu \in \mathcal{M}(\mathbb{R})$.

The weak solution to the homogeneous equation

$$\begin{cases} \left(\frac{\partial^2}{\partial t^2} - \kappa^2 \frac{\partial^2}{\partial x^2}\right) u(t,x) = 0\,, & x \in \mathbb{R},\ t \in \mathbb{R}_+^*, \\ u(0,\cdot) = g(\cdot)\,, \quad \frac{\partial u}{\partial t}(0,\cdot) = \mu(\cdot)\,, \end{cases} \tag{1.3.4}$$

is

$$J_0(t,x) := \frac{1}{2}\big(g(x+\kappa t) + g(x-\kappa t)\big) + (\mu * G_\kappa(t,\cdot))(x)\,, \tag{1.3.5}$$

where $*$ is the convolution in the space variable. We formally rewrite the stochastic partial differential equation (1.3.3) in the integral form (mild form)

$$u(t,x) = J_0(t,x) + \iint_{[0,t]\times\mathbb{R}} G_\kappa\left(t-s, x-y\right) \rho\left(u\left(s,y\right)\right) W\left(\mathrm{d}s, \mathrm{d}y\right), \tag{1.3.6}$$

and denote the stochastic integral part by $I(t,x)$ as in Section 1.1.

Orsingher studied the linear case $\rho(u) \equiv 1$ with vanishing initial data ($\mu = 0$ and $g = 0$) in [54]: Two-point correlation functions and the upcrosing rate were derived. This case is briefly covered in Walsh's notes [68, Chapter 3, p. 308–311] for existence of a solution. Carmona and Nualart [11] considered this problem in a slightly more general setting:

$$\left(\frac{\partial^2}{\partial t^2} - \kappa^2 \frac{\partial^2}{\partial x^2}\right) u(t,x) = \rho(u(t,x))\,\dot{W}(t,x) + b(u(t,x))\,, \tag{1.3.7}$$

where both ρ and b are Lipschitz continuous. In order to show that the solution has a density and the density is smooth, they first proved the existence and uniqueness of the solution. Their requirement (see [11, Proposition II.3]) on the initial data for the corresponding integral equation (1.3.6) is

$$\int_0^t \mathrm{d}s \int_{\mathbb{R}} J_0^2\left(s,y\right) G_\kappa^2\left(t-s,x-y\right) \mathrm{d}y < +\infty\,, \quad \text{for all } (t,x) \in \mathbb{R}_+ \times \mathbb{R}\,. \tag{1.3.8}$$

In particular, regarding the (deterministic) initial position g and the initial velocity μ, they showed in [11, Proposition II.4] that if g is a continuous function and μ is a measure with a continuous density function, then there is a solution to (1.3.7) with initial condition (g,μ). As for the stochastic integral, they used the notion of stochastic integral in the plane introduced by Cairoli and Walsh [8]. The random field solution to the stochastic wave equation in the higher dimension spatial domain $\mathbb{R}^d$ (driven by spatially homogeneous noise) has been studied in [27] for $d=2$, [23] for $d=3$, and [16] for $d>3$. Peszat and Zabczyk studied the function-valued solution in [56] and [57]. See [32] for a comparison of these two methods. We prove the existence results for the case where $d=1$ using Walsh's integral [68] and different estimates on the p-th moments. In our case, the initial position g can be any locally square integrable function, and the initial velocity μ can be any locally finite Borel measure. We establish the existence of random field $I(t,x)$ and its sample path Hölder continuity (see below) such that the solution to (1.3.3) (or (1.3.6)) is $u(t,x) = J_0(t,x) + I(t,x)$.

Moreover, we obtain estimates for the higher moments $\mathbb{E}(|u(t,x)|^p)$ for all $p \geq 2$ with both t and x fixed. In particular, for the hyperbolic Anderson model, we give an explicit formula for the second moment of the solution. When both initial position and initial velocity are the Lebesgue measure, or when the initial position vanishes and the initial velocity is the Dirac delta function, we give explicit formulas for the two-point correlation functions (see Corollaries 4.2.2 and 4.2.3 below).

We remark that Brzeźniak and Ondreját [6] studied a nonlinear stochastic wave equation in spatial dimension one, with values in a Riemannian manifold, driven by a spatially homogeneous Gaussian noise with a finite spectral measure on $\mathbb{R}$ that also has a finite second moment. See also their recent work in [7].

As for the sample path regularity of the random field solutions, Carmona and Nualart showed that if the initial position is constant and the initial velocity vanishes, then the solution is in $C_{1/2-,1/2-}(\mathbb{R}_+ \times \mathbb{R})$ a.s.; see [11, p. 484 – p. 485]. Another reference is [62, Theorem 4.1] where Sanz-Solé and Sarrà proved that the solution with vanishing initial conditions is in $C_{1/2-,1/2-}(\mathbb{R}_+ \times \mathbb{R})$ a.s. This reference also covers the cases where the spatial domain is either $\mathbb{R}^2$ or $\mathbb{R}^3$. For the case where the spatial domain is $\mathbb{R}^3$, this problem has been studied in full detail in [33]. See also [22] for a presentation of the main ideas of [33]. Instead of vanishing or constant initial data, we study this equation with rough initial data. In particular, we show that if $g \in L_{loc}^{2p}(\mathbb{R})$ with $p \geq 1$ and μ is any locally finite Borel measure on $\mathbb{R}$, then the random field part $I(t,x)$ is almost surely Hölder continuous:

$$I \in C_{\frac{1}{2p'}-,\frac{1}{2p'}-}\left(\mathbb{R}_+^* \times \mathbb{R}\right)\,, \quad \text{a.s.}\,, \quad \frac{1}{p} + \frac{1}{p'} = 1. \tag{1.3.9}$$

As a consequence of (1.3.9), if g is a bounded Borel measurable function ($p = +\infty$), then

$$I \in C_{\frac{1}{2}-,\frac{1}{2}-}\left(\mathbb{R}_+^* \times \mathbb{R}\right), \quad \text{a.s.}$$

Clearly, $1/(2p') \le 1/2$. The estimates in (1.3.9) are optimal in certain sense: The singularity of the initial position propagates along the characteristic lines in such a way that the random field part $I(t,x)$ of the solution is less regular there; see Remark 4.2.7 for more details.

After establishing the existence of random field solutions, we study whether the solution exhibits intermittency properties. When the initial data are spatially homogeneous, so is the solution $u(t,x)$, and then the Lyapunov exponents are independent of the spatial variable x. In [30], Dalang and Mueller showed that in this case, for the wave equation in spatial domain $\mathbb{R}^3$ with spatially homogeneous colored noise, the Lyapunov exponents $\overline{\lambda}_p$ and $\underline{\lambda}_p$ are both bounded by some constant times $p^{4/3}$, from above and below respectively. They considered the linear case – the hyperbolic Anderson model – using a Feynman-Kac-type formula developed in [31]. It turns out that for the nonlinear one-dimensional stochastic wave equation driven by space-time white noise, the upper Lyapunov exponents $\overline{\lambda}_p$ are bounded by constant times $p^{3/2}$; see Theorem 4.2.8 below. The different exponents, 4/3 versus 3/2, reflect the distinct natures of the driving noises.

When the initial data are not spatially constant, in particular, when they have certain exponential decrease at infinity, the exponential growth indices proposed by Conus and Khoshnevisan (see (1.1.8) and (1.1.9)) give a way to describe the location of high peaks of the solution. They proved in [19, Theorem 5.1] that if g and μ are bounded and lower semicontinuous functions with a certain decrease at infinity such that $g > 0$ on a set of positive measure and $\mu \ge 0$, then

$$0 < \underline{\lambda}(p) \le \overline{\lambda}(p) < +\infty, \quad \text{for all } p \in [2,\infty[\,. \tag{1.3.10}$$

If, in addition, both g and μ have compact support, then

$$\underline{\lambda}(p) = \overline{\lambda}(p) = \kappa, \quad \text{for all } p \in [2,\infty[\,.$$

We improve their results by allowing more general initial data and giving non-trivial lower and upper bounds in (1.3.10) when initial data have certain exponential decrease at infinity. See Theorem 4.2.11 for more details.

1.4 Some Notation

Throughout this thesis, the function $\rho : \mathbb{R} \mapsto \mathbb{R}$ is Lipschitz continuous with Lipschitz constant $\mathrm{Lip}_\rho > 0$, i.e.,

$$\left|\rho(x) - \rho(y)\right| \le \mathrm{Lip}_\rho\, |x-y|, \quad \text{for all } x, y \in \mathbb{R}.$$

We need some growth conditions on ρ: Assume that for some constants $L_\rho > 0$ and $\overline{\varsigma} \ge 0$,

$$|\rho(x)|^2 \le L_\rho^2 \left(\overline{\varsigma}^2 + x^2\right), \quad \text{for all } x \in \mathbb{R}\,. \tag{1.4.1}$$

When we want to bound the second moment from below, we will assume that for some constants $l_\rho > 0$ and $\underline{\varsigma} \ge 0$,

$$|\rho(x)|^2 \ge l_\rho^2 \left(\underline{\varsigma}^2 + x^2\right), \qquad \text{for all } x \in \mathbb{R}\,. \tag{1.4.2}$$

We shall also specially consider the linear case (the *Anderson model*): $\rho(u) = \lambda u$ with $\lambda \ne 0$, which is a special case of the following quasi-linear growth condition:

$$|\rho(x)|^2 = \lambda^2 \left(\varsigma^2 + x^2\right), \qquad \text{for all } x \in \mathbb{R}\,, \tag{1.4.3}$$

for some $\varsigma \ge 0$.

Remark 1.4.1. The Lipschitz continuity of ρ implies the linear growth of the form (1.4.1) for some $\overline{\varsigma} > 0$ and $L_\rho > 0$. In fact, by the Lipschitz continuity of ρ, we have that $\left|\rho(x) - \rho(0)\right| \le \operatorname{Lip}_\rho |x|$. Hence, $|\rho(x)| \le |\rho(0)| + \operatorname{Lip}_\rho |x|$ and so $|\rho(x)|^2 \le 2|\rho(0)|^2 + 2\operatorname{Lip}_\rho^2 |x|^2$. Therefore, we can always choose $L_\rho = \sqrt{2}\operatorname{Lip}_\rho$ and $\overline{\varsigma} = \frac{|\rho(0)|}{\operatorname{Lip}_\rho}$, but there are cases where (1.4.1) may be satisfied with L_ρ much smaller than $\sqrt{2}\operatorname{Lip}_\rho$.

We will also use the constant $a_{p,\overline{\varsigma}}$ defined as follows:

$$a_{p,\overline{\varsigma}} := \begin{cases} 2^{(p-1)/p} & \text{if } \overline{\varsigma} \ne 0,\ p > 2, \\ \sqrt{2} & \text{if } \overline{\varsigma} = 0,\ p > 2, \\ 1 & \text{if } p = 2\,. \end{cases} \tag{1.4.4}$$

2 The One-Dimensional Nonlinear Stochastic Heat Equation

2.1 Introduction

In this chapter, we will study the stochastic heat equation

$$\begin{cases} \left(\dfrac{\partial}{\partial t} - \dfrac{\nu}{2}\dfrac{\partial^2}{\partial x^2}\right) u(t,x) = \rho(u(t,x))\,\dot{W}(t,x), & x \in \mathbb{R},\ t \in \mathbb{R}_+^*, \\ u(0,\cdot) = \mu(\cdot)\,, \end{cases} \tag{2.1.1}$$

where $\dot{W}$ is space-time white noise, $\rho(u)$ is globally Lipschitz, μ is the initial data, and $\mathbb{R}_+^* =]0,\infty[$. Our main contributions in this chapter are as follows:

(1) A random field solution to (2.1.1) exists for any measure-valued initial condition which satisfies (1.1.5), and the solution is almost surely $C_{1/4-,1/2-}(\mathbb{R}_+^* \times \mathbb{R})$-Hölder continuous.

(2) We obtain sharp estimates for the moments of the solution with both t and x fixed. For the parabolic Anderson model, we get an explicit formula for the second moment.

(3) We get sharper lower bounds for the exponential growth indices, which then answers the first open problem given by Conus and Khoshnevisan [19].

The main results and some examples are presented in Section 2.2. Theorem 2.2.2 states the first main result about the existence, uniqueness, moment estimates and two-point correlations of the random field solution. Before proving Theorem 2.2.2, we first prepare some results in Section 2.3. The complete proofs are in Section 2.4. The second main result –Theorem 2.2.10– is about the exponential growth indices. It is proved in Section 2.5. We give some discussions in Section 2.7. Finally, in Section 2.6, we prove the third main result: space-time Hölder continuity of the random field solution.

2.2 Main Results

Denote the solution to the homogeneous equation

$$\begin{cases} \left(\dfrac{\partial}{\partial t} - \dfrac{\nu}{2}\dfrac{\partial^2}{\partial x^2}\right) u(t,x) = 0, \quad x \in \mathbb{R},\ t \in \mathbb{R}_+^*, \\ u(0,\cdot) = \mu(\cdot)\,, \end{cases} \tag{2.2.1}$$

by

$$J_0(t,x) := \left(\mu * G_\nu(t,\cdot)\right)(x) = \int_{\mathbb{R}} G_\nu\left(t, x-y\right)\mu(\mathrm{d}y)\,, \quad (t,x) \in \mathbb{R}_+^* \times \mathbb{R}\,.$$

Note that $J_0(t,x)$ is well defined by the hypothesis (1.1.5). It solves (2.2.1) for $t > 0$ and $\lim_{t\to 0_+} J_0(t,x) = \mu$ in the sense of distributions (see Lemma 2.6.15 below). We formally rewrite the stochastic partial differential equation (2.1.1) in the integral form (mild form):

$$u(t,x) = J_0(t,x) + I(t,x) \tag{2.2.2}$$

where

$$I(t,x) := \iint_{[0,t]\times\mathbb{R}} G_\nu\left(t-s, x-y\right)\rho\left(u\left(s,y\right)\right) W\left(\mathrm{d}s,\mathrm{d}y\right)\,. \tag{2.2.3}$$

By convention, $I(0,x) = 0$. In Section 2.4, we prove that the above stochastic integral is well defined in the sense of Walsh [68, 21].

2.2.1 Notation and Conventions

We use the convention that $G_\nu(t,\cdot) \equiv 0$ if $t < 0$. Hence, the integral region in the stochastic integral in (2.2.2) can be written as $\mathbb{R}_+ \times \mathbb{R}$.

Define a kernel function

$$\mathscr{K}\left(t,x;\nu,\lambda\right) := G_{\frac{\nu}{2}}(t,x)\left(\frac{\lambda^2}{\sqrt{4\pi\nu t}} + \frac{\lambda^4}{2\nu}\, e^{\frac{\lambda^4 t}{4\nu}}\,\Phi\left(\lambda^2\sqrt{\frac{t}{2\nu}}\right)\right), \quad (t,x) \in \mathbb{R}_+^* \times \mathbb{R}\,, \tag{2.2.4}$$

where $\Phi(x)$ is the probability distribution function of the standard normal distribution:

$$\Phi(x) := \int_{-\infty}^{x} \frac{e^{-y^2/2}}{\sqrt{2\pi}}\mathrm{d}y\,.$$

We also use the error function $\mathrm{erf}(x) := \frac{2}{\sqrt{\pi}}\int_0^x e^{-y^2}\mathrm{d}y$ and its complement $\mathrm{erfc}(x) := 1 - \mathrm{erf}(x)$. Clearly,

$$\Phi(x) = \frac{1}{2}\left(1 + \mathrm{erf}\left(x/\sqrt{2}\right)\right), \quad \mathrm{erf}(x) = 2\Phi\left(\sqrt{2}\,x\right) - 1, \quad \mathrm{erfc}(x) = 2\left(1 - \Phi\left(\sqrt{2}\,x\right)\right)\,.$$

We use $\star$ to denote the simultaneous convolution in both space and time variables. Define another function

$$\mathscr{H}(t;\nu,\lambda) := (1\star\mathscr{K})(t,x) = 2e^{\frac{\lambda^4 t}{4\nu}}\Phi\left(\lambda^2\sqrt{\frac{t}{2\nu}}\right)-1\,, \tag{2.2.5}$$

where the second equality is due to (2.3.7) below. Clearly, $\mathscr{K}(t,x;\nu,\lambda)$ can be written as

$$\mathscr{K}(t,x;\nu,\lambda) = G_{\nu/2}(t,x)\left(\frac{\lambda^2}{\sqrt{4\pi\nu t}}+\frac{\lambda^4}{4\nu}\left[\mathscr{H}(t;\nu,\lambda)+1\right]\right).$$

We use the following conventions:

$$\mathscr{K}(t,x) := \mathscr{K}(t,x;\nu,\lambda)\,, \tag{2.2.6}$$
$$\overline{\mathscr{K}}(t,x) := \mathscr{K}\left(t,x;\nu,L_\rho\right)\,, \tag{2.2.7}$$
$$\underline{\mathscr{K}}(t,x) := \mathscr{K}\left(t,x;\nu,l_\rho\right)\,, \tag{2.2.8}$$
$$\widehat{\mathscr{K}}_p(t,x) := \mathscr{K}\left(t,x;\nu,a_{p,\overline{\varsigma}}\,z_p\,L_\rho\right)\,,\quad \text{for all } p>2\,, \tag{2.2.9}$$

where z_p (in particular, $z_2=1$) is the universal constant in the Burkholder-Davis-Gundy inequality (see Theorem 2.3.18 below) and $a_{p,\overline{\varsigma}}$ is a constant defined in Lemma 2.4.3 below (see (1.4.4)). We only need to keep in mind that $a_{p,\overline{\varsigma}}\le 2$. Note that the kernel function $\widehat{\mathscr{K}}_p(t,x)$ implicitly depends on $\overline{\varsigma}$ through $a_{p,\overline{\varsigma}}$ which will be clear from the context. If $p=2$, then $\widehat{\mathscr{K}}_2(t,x)=\overline{\mathscr{K}}(t,x)$.

Similarly $\overline{\mathscr{H}}(t)$, $\underline{\mathscr{H}}(t)$ and $\widehat{\mathscr{H}}_p(t)$ denote the kernel functions with λ in $\mathscr{H}(t)$ replaced by L_ρ, l_ρ and $a_{p,\overline{\varsigma}}z_p\,L_\rho$, respectively. Again $\widehat{\mathscr{H}}_p(t)$ depends on $\overline{\varsigma}$ implicitly which will be clear from the context.

Let us set up the filtered probability space. Let

$$W=\left\{W_t(A):\ A\in\mathscr{B}_b(\mathbb{R}),\, t\ge 0\right\}$$

be a space-time white noise defined on a probability space $(\Omega,\mathscr{F},P)$, where $\mathscr{B}_b(\mathbb{R})$ is the collection of Borel measurable sets with finite Lebesgue measure. Let $(\mathscr{F}_t, t\ge 0)$ be the standard filtration generated by this space-time white noise. More precisely, let

$$\mathscr{F}_t^0 := \sigma\left(W_s(A): 0\le s\le t, A\in\mathscr{B}_b(\mathbb{R})\right)\vee\mathscr{N},\quad t\ge 0$$

be the natural filtration augmented by the σ-field $\mathscr{N}$ generated by all P-null sets in $\mathscr{F}$. Define $\mathscr{F}_t := \mathscr{F}_{t+}^0 = \wedge_{s>t}\mathscr{F}_s^0$ for any $t\ge 0$. [1] In the following, we fix this filtered probability space $\{\Omega,\mathscr{F},\{\mathscr{F}_t: t\ge 0\},P\}$. We use $\|\cdot\|_p$ to denote the $L^p(\Omega)$-norm ($p\ge 1$). Denote $\lceil p\rceil_2 := 2\lceil p/2\rceil$, which is the smallest even integer greater than or equal to p.

Let $\mathscr{M}(\mathbb{R})$ be the set of locally finite (signed) Borel measures over $\mathbb{R}$. Let $\mathscr{M}_H(\mathbb{R})$ be

[1] By [40, Proposition 7.7 on p. 90], the augmented filtration $\mathscr{F}_t^0$ is already right continuous. Indeed, we can just use this filtration.

the set of signed Borel measures over $\mathbb{R}$ satisfying (1.1.5). Define

$$\mathcal{M}_G^{\beta}(\mathbb{R}) := \left\{\mu \in \mathcal{M}(\mathbb{R}) : \int_{\mathbb{R}} e^{\beta|x|}|\mu|(\mathrm{d}x) < +\infty\right\}, \quad \beta \ge 0, \tag{2.2.10}$$

where $|\mu| = \mu_+ + \mu_-$ is the Jordan decomposition of a measure into two non-negative measures. We use subscript "+" to denote the subset of non-negative measures. For example, $\mathcal{M}_+(\mathbb{R})$ is the set of non-negative Borel measures over $\mathbb{R}$ and $\mathcal{M}_{G,+}^{\beta}(\mathbb{R}) = \mathcal{M}_G^{\beta}(\mathbb{R}) \cap \mathcal{M}_+(\mathbb{R})$.

A random field $Y = \left(Y(t,x) : (t,x) \in \mathbb{R}_+^* \times \mathbb{R}\right)$ is said to be $L^p(\Omega)$*-continuous*, $p \ge 2$, if for all $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}$,

$$\lim_{(t',x')\to(t,x)} \left|\left|Y(t,x) - Y\left(t',x'\right)\right|\right|_p = 0\,.$$

2.2.2 Existence, Uniqueness and Moments

We first give the definition of the random field solution as follows:

Definition 2.2.1. A process $u = \left(u(t,x) : (t,x) \in \mathbb{R}_+^* \times \mathbb{R}\right)$ is called a *random field solution* to (2.1.1) (or (2.2.2)) if

(1) u is adapted, i.e., for all $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}$, $u(t,x)$ is $\mathscr{F}_t$-measurable;

(2) u is jointly measurable with respect to $\mathscr{B}\left(\mathbb{R}_+^* \times \mathbb{R}\right) \times \mathscr{F}$;

(3) $\left(G_\nu^2 \star \left|\left|\rho(u)\right|\right|_2^2\right)(t,x) < +\infty$ for all $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}$, and the function $(t,x) \mapsto I(t,x)$ mapping from $\mathbb{R}_+^* \times \mathbb{R}$ into $L^2(\Omega)$ is continuous;

(4) u satisfies (2.1.1) (or (2.2.2)) a.s., for all $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}$.

The first main result is stated as follows.

Theorem 2.2.2 (Existence, uniqueness, and moments)**.** *Suppose that*

(i) the initial data μ is a signed Borel measure such that (1.1.5) *holds;*

(ii) the function ρ is Lipschitz continuous such that the linear growth condition (1.4.1) *holds.*

Then the stochastic integral equation (2.2.2) *has a random field solution* $u = \{u(t,x) : t > 0, x \in \mathbb{R}\}$ *(note that* $t > 0$*) in the sense of Definition 2.2.1. This solution has the following properties:*

(1) u is unique (in the sense of versions);

(2) $(t,x) \mapsto u(t,x)$ *is* $L^p(\Omega)$*-continuous for all integers* $p \ge 2$*;*

(3) For all even integers $p \ge 2$*, the* p*-th moment of the solution* $u(t,x)$ *satisfies the upper*

bounds

$$||u(t,x)||_p^2 \le \begin{cases} J_0^2(t,x) + \left(J_0^2 \star \overline{\mathscr{K}}\right)(t,x) + \overline{\varsigma}^2\, \overline{\mathscr{H}}(t), & \text{if } p=2,\\ \\ 2J_0^2(t,x) + \left(2J_0^2 \star \widehat{\mathscr{K}_p}\right)(t,x) + \overline{\varsigma}^2\, \widehat{\mathscr{H}_p}(t), & \text{if } p>2, \end{cases} \tag{2.2.11}$$

for all $t>0$, $x\in\mathbb{R}$, and the two-point correlation satisfies the upper bound

$$\begin{aligned} &\mathbb{E}\left[u(t,x)u\left(t,y\right)\right] \\ &\quad\le J_0(t,x)J_0\left(t,y\right) + L_\rho^2 \int_0^t \mathrm{d}s \int_{\mathbb{R}} \overline{f}(s,z) G_\nu(t-s,x-z) G_\nu\left(t-s,y-z\right)\mathrm{d}z \\ &\qquad + \frac{L_\rho^2\,\overline{\varsigma}^2}{\nu}|x-y|\left(\Phi\left(\frac{|x-y|}{\sqrt{2\nu t}}\right)-1\right) + 2\,L_\rho^2\,\overline{\varsigma}^2\, t\, G_{2\nu}\left(t,x-y\right), \end{aligned} \tag{2.2.12}$$

for all $t>0$, $x,y\in\mathbb{R}$, where $\overline{f}(s,z)$ denotes the right hand side of (2.2.11) *for $p=2$;*

(4) *If ρ satisfies* (1.4.2)*, then the second moment satisfies the lower bound*

$$||u(t,x)||_2^2 \ge J_0^2(t,x) + \left(J_0^2 \star \underline{\mathscr{K}}\right)(t,x) + \underline{\varsigma}^2\, \underline{\mathscr{H}}(t) \tag{2.2.13}$$

for all $t>0$, $x\in\mathbb{R}$, and the two-point correlation satisfies the lower bound

$$\begin{aligned} &\mathbb{E}\left[u(t,x)u\left(t,y\right)\right] \\ &\quad\ge J_0(t,x)J_0\left(t,y\right) + l_\rho^2 \int_0^t \mathrm{d}s \int_{\mathbb{R}} \underline{f}(s,z) G_\nu(t-s,x-z) G_\nu\left(t-s,y-z\right)\mathrm{d}z \\ &\qquad + \frac{l_\rho^2\,\underline{\varsigma}^2}{\nu}|x-y|\left(\Phi\left(\frac{|x-y|}{\sqrt{2\nu t}}\right)-1\right) + 2\,l_\rho^2\,\underline{\varsigma}^2\, t\, G_{2\nu}\left(t,x-y\right), \end{aligned} \tag{2.2.14}$$

for all $t>0$, $x,y\in\mathbb{R}$, where $\underline{f}(s,z)$ denotes the right hand side of (2.2.13)*;*

(5) *In particular, for the quasi-linear case $|\rho(u)|^2 = \lambda^2\left(\varsigma^2+u^2\right)$, the second moment has the explicit expression*

$$||u(t,x)||_2^2 = J_0^2(t,x) + \left(J_0^2 \star \mathscr{K}\right)(t,x) + \varsigma^2\, \mathscr{H}(t)\,, \tag{2.2.15}$$

for all $t>0$, $x\in\mathbb{R}$, and the two-point correlation is given by

$$\begin{aligned} &\mathbb{E}\left[u(t,x)u\left(t,y\right)\right] \\ &\quad= J_0(t,x)J_0\left(t,y\right) + \lambda^2 \int_0^t \mathrm{d}s \int_{\mathbb{R}} f(s,z) G_\nu(t-s,x-z) G_\nu\left(t-s,y-z\right)\mathrm{d}z \\ &\qquad + \frac{\lambda^2\,\varsigma^2}{\nu}|x-y|\left(\Phi\left(\frac{|x-y|}{\sqrt{2\nu t}}\right)-1\right) + 2\lambda^2\,\varsigma^2\, t\, G_{2\nu}\left(t,x-y\right), \end{aligned} \tag{2.2.16}$$

for all $t>0$, $x,y\in\mathbb{R}$, where $f(s,z)=||u(s,z)||_2^2$ is defined in (2.2.15).

This theorem is proved in several parts: The proofs of existence, uniqueness and moment estimates are presented in Section 2.4.2. The proofs of the two-point estimates are in Section 2.4.3. The following two corollaries 2.2.3 and 2.2.6 are proved in Section 2.4.4.

Corollary 2.2.3 (Constant initial data)**.** *Suppose that* $|\rho(u)|^2 = \lambda^2(\varsigma^2+u^2)$ *and* μ *is the Lebesgue measure. Then for all* $t>0$ *and* $x, y \in \mathbb{R}$,

$$\begin{aligned}&\mathbb{E}\left[u(t,x)u\left(t,y\right)\right]\\ &\quad= 1+(1+\varsigma^2)\left(\exp\left(\frac{\lambda^4 t-2\lambda^2|x-y|}{4\nu}\right)\operatorname{erfc}\left(\frac{|x-y|-\lambda^2 t}{2\sqrt{\nu t}}\right)-\operatorname{erfc}\left(\frac{|x-y|}{2\sqrt{\nu t}}\right)\right). \qquad (2.2.17)\end{aligned}$$

In particular, when $y=x$, *we have*

$$\mathbb{E}\left[|u(t,x)|^2\right] = 1+(1+\varsigma^2)\mathcal{H}(t)\,. \qquad (2.2.18)$$

Remark 2.2.4. If $\rho(u)=u$ (i.e., $\lambda=1$ and $\varsigma=0$), then the second moment formula (2.2.18) recovers, in the case $n=2$, the moment formulas of Bertini and Cancrini [3, Theorem 2.6]:

$$\mathbb{E}\left[|u(t,x)|^n\right] = 2\exp\left\{\frac{n(n^2-1)}{4!\,\nu}t\right\}\Phi\left(\sqrt{\frac{n(n^2-1)}{12\nu}t}\right).$$

As for the two-point correlation function, Bertini and Cancrini [3, Corollary 2.4] gave the following integral form:

$$\mathbb{E}\left[u(t,x)u\left(t,y\right)\right] = \int_0^t \mathrm{d}s\frac{|x-y|}{\sqrt{\pi\nu s^3}}\exp\left\{-\frac{(x-y)^2}{4\nu s}+\frac{t-s}{4\nu}\right\}\Phi\left(\sqrt{\frac{t-s}{2\nu}}\right). \qquad (2.2.19)$$

This integral can be evaluated explicitly and equals

$$= \exp\left(\frac{t-2|x-y|}{4\nu}\right)\operatorname{erfc}\left(\frac{|x-y|-t}{\sqrt{4\nu t}}\right),$$

so their result differs from ours. The difference is a term $\operatorname{erf}\left(\frac{|x-y|}{\sqrt{4\nu t}}\right)$. By letting $x=y$ in the two-point correlation function, both results do give the correct second moment (the difference term is zero for $x=y$). However, for $x\neq y$, this is not the case. For instance, as t tends to zero, the correlation function should have a limit equal to one, while (2.2.19) has limit zero. The argument in [3] should be modified as follows (we use the notation in their paper): (4.6) on p. 1398 should be

$$\mathbb{E}_0^{\beta,1}\left[\exp\left(\frac{L_t^{\xi}(\beta)}{\sqrt{2\nu}}\right)\right] = \int_0^t P_\xi(\mathrm{d}s)\mathbb{E}_0^{\beta}\left[\exp\left(\frac{L_{t-s}(\beta)}{\sqrt{2\nu}}\right)\right]+P(T_\xi\geq t)\,.$$

The extra term is the last term, which is

$$P(T_\xi\geq t) = \int_t^\infty \frac{|\xi|}{\sqrt{2\pi s^3}}\exp\left(-\frac{\xi^2}{2s}\right)\mathrm{d}s = \operatorname{erf}\left(\frac{|\xi|}{\sqrt{2t}}\right) = \operatorname{erf}\left(\frac{|x-x'|}{\sqrt{4\nu t}}\right).$$

With this term, (2.2.17) is recovered.

Example 2.2.5 (Higher moments for constant initial data)**.** Suppose that $\mu(\mathrm{d}x) = \mathrm{d}x$. Clearly, $J_0(t,x) \equiv 1$. By the above bound (2.2.11), we have

$$\mathbb{E}[|u(t,x)|^p] \le 2^{p-1} + 2^{p/2-1}\left(2+\overline{\varsigma}^2\right)^{p/2} \exp\left\{\frac{a_{p,\overline{\varsigma}}^4\, z_p^4\, p\, L_\rho^4\, t}{8\nu}\right\}\left|\Phi\left(a_{p,\overline{\varsigma}}^2\, L_\rho^2\, z_p^2\sqrt{\frac{t}{2\nu}}\right)\right|^{p/2}.$$

We can replace z_p by $2\sqrt{p}$ thanks to Theorem 2.3.18 below, and $a_{p,\overline{\varsigma}}$ by 2. Then the upper Lyapunov exponent of order p defined in (1.1.6) is bounded by

$$\overline{\lambda}_p \le \frac{2^5\, p^3\, L_\rho^4}{\nu}.$$

If $\overline{\varsigma} = 0$, we can replace $a_{p,\overline{\varsigma}}$ by $\sqrt{2}$ instead of 2, which gives a slightly better bound $\overline{\lambda}_p \le 2^3 p^3 L_\rho^4/\nu$. In particular, for the parabolic Anderson model $\rho(u) = \lambda u$, we have

$$\overline{\lambda}_p \le 2^3 p^3 \lambda^4/\nu\,,$$

which is consistent with Bertini and Cancrini's formulas $\lambda_p = \frac{\lambda^4}{4!\nu}p(p^2-1)$ (see [3, (2.40)]).

Corollary 2.2.6 (Dirac delta initial data)**.** *Suppose that $|\rho(u)|^2 = \lambda^2(\varsigma^2+u^2)$ and μ is the Dirac delta measure with a unit mass at zero. Then for all $t>0$ and $x, y \in \mathbb{R}$,*

$$\begin{aligned}\mathbb{E}\left[u(t,x)u\left(t,y\right)\right] &= G_\nu(t,x)G_\nu\left(t,y\right) - \varsigma^2\operatorname{erfc}\left(\frac{|x-y|}{2\sqrt{\nu t}}\right)\\ &+\left(\frac{\lambda^2}{4\nu}G_{\nu/2}\left(t,\frac{x+y}{2}\right)+\varsigma^2\right)\exp\left(\frac{\lambda^4 t - 2\lambda^2|x-y|}{4\nu}\right)\operatorname{erfc}\left(\frac{|x-y|-\lambda^2 t}{2\sqrt{\nu t}}\right). \end{aligned} \tag{2.2.20}$$

In addition, when $y = x$, we have

$$\mathbb{E}\left[|u(t,x)|^2\right] = \frac{1}{\lambda^2}\mathcal{K}(t,x) + \varsigma^2\mathcal{H}(t)\,. \tag{2.2.21}$$

Remark 2.2.7. If $\rho(u) = u$ (i.e., $\lambda = 1$ and $\varsigma = 0$), then the second moment formula (2.2.21) coincides with the result by Bertini and Cancrini [3, (2.27)] (see also [5, 2]):

$$\mathbb{E}\left[|u(t,x)|^2\right] = \frac{1}{2\pi\nu t}e^{-\frac{x^2}{\nu t}}\left[1+\sqrt{\frac{\pi t}{\nu}}e^{\frac{t}{4\nu}}\Phi\left(\sqrt{\frac{t}{2\nu}}\right)\right],$$

which equals $\mathcal{K}(t,x;\nu,1)$. As for the two-point correlation function, Bertini and Cancrini [3, Corollary 2.5] gave the following integral form:

$$\begin{aligned}\mathbb{E}\left[u(t,x)u\left(t,y\right)\right] &= \frac{1}{2\pi\nu t}\exp\left\{-\frac{x^2+y^2}{2\nu t}\right\}\int_0^1 \frac{|x-y|}{\sqrt{4\pi\nu t}}\frac{1}{\sqrt{s^3(1-s)}}\\ &\exp\left\{-\frac{(x-y)^2}{4\nu t}\frac{1-s}{s}\right\}\left(1+\sqrt{\frac{\pi t(1-s)}{\nu}}\exp\left\{\frac{t}{2\nu}\frac{1-s}{2}\right\}\Phi\left(\sqrt{\frac{t(1-s)}{2\nu}}\right)\right)\mathrm{d}s\,. \end{aligned} \tag{2.2.22}$$

This integral can be evaluated explicitly (see Lemma 2.4.9 below), and is equal to

$$= G_\nu(t,x)G_\nu\left(t,y\right) + \frac{1}{4\nu} G_{\frac{\nu}{2}}\left(t, \frac{x+y}{2}\right) \exp\left(\frac{t-2|x-y|}{4\nu}\right) \operatorname{erfc}\left(\frac{|x-y|-t}{\sqrt{4\nu t}}\right).$$

This coincides with our result (2.2.20) for $\varsigma = 0$ and $\lambda = 1$.

Example 2.2.8 (Higher moments for delta initial data)**.** Suppose that $\mu = \delta_0$ and $\overline{\varsigma} = 0$. Let $p \geq 2$ be an even integer. Clearly, $J_0(t,x) \equiv G_\nu(t,x)$. Then, by (2.2.11), we have that

$$\begin{aligned}
\mathbb{E}\left[|u(t,x)|^p\right] &\leq 2^{p-1} G_\nu^p(t,x) + 2^{(p-2)/2} \left|\left(2G_\nu^2 \star \widehat{\mathcal{K}}_p\right)(t,x)\right|^{p/2} \\
&\leq 2^{p-1} G_\nu^p(t,x) + 2^{(p-2)/2} L_\rho^{-p} z_p^{-p} \left|\widehat{\mathcal{K}}_p(t,x)\right|^{p/2} \\
&= 2^{p-1} G_\nu^p(t,x) + 2^{p-1} G_{\nu/2}^{p/2}(t,x) \left(\frac{1}{\sqrt{4\pi\nu t}} + \frac{z_p^2 L_\rho^2}{\nu} e^{\frac{z_p^4 L_\rho^4 t}{\nu}} \Phi\left(z_p^2 L_\rho^2 \sqrt{\frac{2t}{\nu}}\right)\right)^{p/2}
\end{aligned}$$

where the second inequality is due to (2.3.3) below. Hence, for all $x \in \mathbb{R}$, the upper Lyapunov exponent (1.1.6) of order p is bounded by

$$\overline{\lambda}_p \leq \frac{L_\rho^4 z_p^4 p}{2\nu} \leq \frac{2^3 p^3 L_\rho^4}{\nu},$$

where the last inequality is due to the fact that $z_p \leq 2\sqrt{p}$ for all $p \geq 2$. Note that this upper bound is identical to the case of the constant initial data. We can also bound the exponential growth indices explicitly in this case:

$$\lim_{t\to+\infty} \frac{1}{t} \sup_{|x|>\alpha t} \log \mathbb{E}\left[|u(t,x)|^p\right] \leq -\frac{\alpha^2 p}{2\nu} + \frac{L_\rho^4 p z_p^4}{2\nu}, \quad \text{for all } \alpha \geq 0.$$

Hence, the upper growth indices of order p is bounded by $\overline{\lambda}(p) \leq z_p^2 L_\rho^2$. Similarly, one can derive that $\underline{\lambda}(2) \geq l_\rho^2/2$. Finally, since $\underline{\lambda}(2) \leq \underline{\lambda}(p)$ for all $p \geq 2$, we have that, for all even integers $p \geq 2$,

$$\frac{l_\rho^2}{2} \leq \underline{\lambda}(p) \leq \overline{\lambda}(p) \leq z_p^2 L_\rho^2.$$

Similar bounds are obtained for more general initial data: see Theorem 2.2.10 below.

This following proposition, which is proved in Section 2.4.5, shows that initial data cannot be extended beyond measures.

Proposition 2.2.9. *Suppose that the initial data is $\mu = \delta_0'$, the derivative of the Dirac delta measure at zero. Let $\rho(u) = \lambda u$ ($\lambda \neq 0$). Then* (2.2.2) *does not have a random field solution.*

2.2.3 Exponential Growth Indices

As an application of the above second moment formula, we partially answer the first open problem proposed by Conus and Khoshnevisan in [19]: the limits over t in the definitions of these two indices do exist when $n = 2$ and the lower and upper growth indices of order 2 (see (1.1.8) and (1.1.9)) coincide.

Before stating the main result, we first give some explanation concerning the exponential growth indices defined in (1.1.8) and (1.1.9). When the initial data is localized, for example, when it has compact support, we expect that the position of high peaks of the solution will exhibit a certain wave propagation phenomenon. As shown in Figure 2.1, when α is sufficiently large, it is likely that there is no high peaks outside of the space-time cone — the shaded region. Hence, the limit over t should be negative. The largest α such that this limit remains negative is then defined to be the upper growth index $\overline{\lambda}(p)$. On the other hand, when α is very small, say $\alpha = 0$, then there must be some high peaks in the shaded region so that the limit becomes positive. Hence, the smallest α such that this limit is positive is defined to be the lower growth index $\underline{\lambda}(p)$.

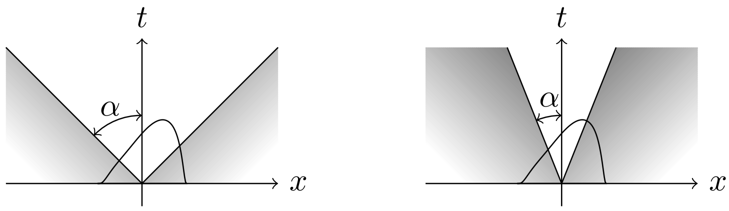


Figure 2.1 – Illustration of the exponential growth indices. The initial data, depicted by the curve, is localized around the origin.

Theorem 2.2.10 (Exponential growth indices)**.** *The following bounds hold:*

(1) If $|\rho(u)|^2 \le L_\rho^2\left(\overline{\varsigma}^2 + u^2\right)$ with $\overline{\varsigma} = 0$ (which implies $\underline{\varsigma} = \varsigma = 0$) and the initial data $\mu \in \mathcal{M}_G^\beta(\mathbb{R})$ for some $\beta > 0$, then for all $p \ge 2$,

$$\bar{\lambda}(p) \le \begin{cases} \dfrac{\beta\nu}{2} + \dfrac{z_{\lceil p\rceil_2}^4 L_\rho^4}{2\nu\beta}, & \text{if} \quad 0 \le \beta < \dfrac{z_{\lceil p\rceil_2}^2 L_\rho^2}{\nu}, \\ z_{\lceil p\rceil_2}^2 L_\rho^2, & \text{if} \quad \beta \ge \dfrac{z_{\lceil p\rceil_2}^2 L_\rho^2}{\nu}, \end{cases}$$

where z_m, $m \in \mathbb{N}$, $m \ge 2$, are the universal constants in the Burkholder-Davis-Gundy inequality (see Theorem 2.3.18 below). In addition, for $p = 2$,

$$\bar{\lambda}(2) \le \begin{cases} \dfrac{\beta\nu}{2} + \dfrac{L_\rho^4}{8\nu\beta}, & \text{if} \quad 0 \le \beta < \dfrac{L_\rho^2}{2\nu}, \\ \dfrac{1}{2} L_\rho^2, & \text{if} \quad \beta \ge \dfrac{L_\rho^2}{2\nu}. \end{cases} \tag{2.2.23}$$

(2) If $|\rho(u)|^2 \ge l_\rho^2\left(\underline{\varsigma}^2 + u^2\right)$ with $\underline{\varsigma} = 0$, then

$$\underline{\lambda}(p) \ge \frac{l_\rho^2}{2}, \qquad \text{for all } \mu \in \mathcal{M}_+(\mathbb{R}),\ \mu \ne 0 \text{ and all } p \ge 2;$$

if $\underline{\varsigma} \neq 0$, *then*

$$\underline{\lambda}(p) = \overline{\lambda}(p) = +\infty, \qquad \textit{for all } \mu \in \mathcal{M}_+(\mathbb{R}) \textit{ and } p \geq 2;$$

(3) *In particular, for the quasi-linear case* $|\rho(u)|^2 = \lambda^2\left(\varsigma^2 + u^2\right)$ *with* $\lambda \neq 0$*, if* $\varsigma = 0$ *and* $\beta \geq \frac{\lambda^2}{2\nu}$*, then*

$$\underline{\lambda}(2) = \bar{\lambda}(2) = \lambda^2/2, \qquad \textit{for all } \mu \in \mathcal{M}_{G,+}^{\beta}(\mathbb{R}),\ \mu \neq 0;$$

if $\varsigma \neq 0$*, then*

$$\underline{\lambda}(p) = \overline{\lambda}(p) = +\infty, \qquad \textit{for all } \mu \in \mathcal{M}_+(\mathbb{R}) \textit{ and } p \geq 2\,.$$

The lower bounds of this theorem are proved in Section 2.5.1; the upper bounds in Section 2.5.2.

This theorem generalizes the results by Conus and Khoshnevisan [19] in several regards: (i) more general initial data are allowed; (ii) both non trivial upper bound and lower bounds are given (compare with [19, Theorem 1.1]) for the Laplace operator case; (iii) for the parabolic Anderson model, the exact transition is proved (see Theorem 1.3 and the first open problem in [19]) for $n = 2$ and the Laplace operator case; (iv) our discussions above cover the case $\rho(0) \neq 0$.

Example 2.2.11 (Delta initial data)**.** Suppose that $\overline{\varsigma} = \underline{\varsigma} = 0$. Clearly, $\delta_0 \in \mathcal{M}_{G,+}^{\beta}(\mathbb{R})$ for all $\beta \geq 0$. Hence, the above theorem implies that for all even integers $k \geq 2$,

$$\frac{l_\rho^2}{2} \leq \underline{\lambda}(k) \leq \overline{\lambda}(k) \leq z_k^2 L_\rho^2\,.$$

This recovers the previous calculation in Example 2.2.8.

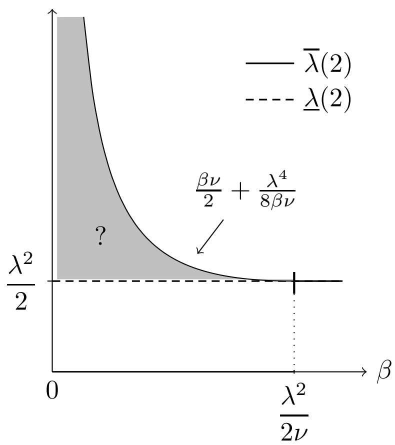


Figure 2.2 – Exponential growth indices of order two for the Anderson model $\rho(u) = \lambda u$ with initial data $\mu \in \mathcal{M}_{G,+}^{\beta}(\mathbb{R})$. When $\beta \geq \frac{\lambda^2}{2\nu}$, Theorem 2.2.10 says that there is an exact phase transition, namely, $\underline{\lambda}(2) = \overline{\lambda}(2)$. But it is not clear whether this is the case for small β.

Proposition 2.2.12. *Consider the parabolic Anderson model $\rho(u) = \lambda u$, $\lambda \neq 0$, with the initial data $\mu(\mathrm{d}x) = e^{-\beta|x|}\mathrm{d}x$ ($\beta > 0$). Then we have*

$$\underline{\lambda}(2) = \overline{\lambda}(2) = \begin{cases} \dfrac{\beta\nu}{2} + \dfrac{\lambda^4}{8\beta\nu} & \text{if } 0 < \beta \leq \dfrac{\lambda^2}{2\nu}, \\[2ex] \dfrac{\lambda^2}{2} & \text{if } \beta \geq \dfrac{\lambda^2}{2\nu}. \end{cases}$$

This proposition improves on Theorem 2.2.10, for the particular initial condition $\mu(\mathrm{d}x) = e^{\beta|x|}\mathrm{d}x$ when $0 < \beta < \frac{\lambda^2}{2\nu}$ (See Figure 2.2). This improvement is possible because $J_0(t,x)$ has an explicit form in this case. This proposition shows that for all $\beta \in]0,+\infty]$, the exact phase transition occurs, and hence our upper bounds (2.2.23) in Theorem 2.2.10 for the upper growth index $\overline{\lambda}(2)$ are sharp. See Section 2.5.3 for the proof.

2.2.4 Sample Path Regularity

Theorem 2.2.13. *Suppose that ρ is Lipschitz continuous. Then the solution $u(t,x) = J_0(t,x) + I(t,x)$ to* (2.1.1) *has the following sample path regularity:*

(1) If the initial data μ is an α-Hölder continuous function ($\alpha \in]0,1]$) over $\mathbb{R}$ satisfying (1.1.5)*, then*

$$J_0 \in C_{\alpha/2,\alpha}(\mathbb{R}_+ \times \mathbb{R}) \cap C^{+\infty}\left(\mathbb{R}_+^* \times \mathbb{R}\right).$$

(2) If the initial data μ is a continuous function satisfying (1.1.5)*, then*

$$J_0 \in C^{+\infty}\left(\mathbb{R}_+^* \times \mathbb{R}\right) \cap C(\mathbb{R}_+ \times \mathbb{R}).$$

(3) If the initial data μ is a signed Borel measure satisfying (1.1.5)*, then*

$$J_0 \in C^{+\infty}\left(\mathbb{R}_+^* \times \mathbb{R}\right),$$

and

$$I \in C_{\frac{1}{4}-,\frac{1}{2}-}\left(\mathbb{R}_+^* \times \mathbb{R}\right), \quad a.s.$$

Therefore,

$$u = J_0 + I \in C_{\frac{1}{4}-,\frac{1}{2}-}\left(\mathbb{R}_+^* \times \mathbb{R}\right), \quad a.s.$$

See Section 2.6.4 for the proof.

Remark 2.2.14. The common approach (e.g., that is used in [25, p.54 –55], [63], [65], etc.) to prove Hölder continuity does not work in our case. For example, let us consider the case where $\rho(u) = u$ and $\mu = \delta_0$. By the argument in [65, p. 432], for $p > 1$ and $q = p/(p-1)$, $\left|\left|I(t,x) - I(t',x')\right|\right|_{2p}^{2p}$ is bounded by

$$\leq C_{p,T}\left(\int_0^{t\vee t'}\int_{\mathbb{R}}\left(G_\nu(t-s,x-y)-G(t'-s,x'-y')\right)^2\mathrm{d}s\mathrm{d}y\right)^{p/q}$$
$$\times\int_0^{t\vee t'}\int_{\mathbb{R}}\left(G_\nu(t-s,x-y)-G(t'-s,x'-y')\right)^2\left(1+\left|\left|u(s,y)\right|\right|_{2p}^{2p}\right)\mathrm{d}s\mathrm{d}y\,.$$

By Hölder's inequality and (2.2.21), $||u(t,x)||_{2p}^2\geq ||u(t,x)||_2^2=\mathscr{K}(t,x)\geq G_{\nu/2}(t,x)\frac{1}{\sqrt{4\pi\nu t}}$. Hence, $||u(t,x)||_{2p}^{2p}\geq CG_{\nu/(2p)}(t,x)t^{1/2-p}$. The second term in the above bound is not integrable unless $p<3/2$.

Example 2.2.15 (Delta initial data)**.** Suppose $\rho(u)=\lambda u$ with $\lambda\neq 0$. If $\mu=\delta_0$, then neither $J_0(0,x)$ nor $\lim_{t\to 0_+}||I(t,x)||_2$ is continuous in x. For $J_0(0,x)=\delta_0(x)$, this is clear. As for $\lim_{t\to 0_+}||I(t,x)||_2$, by Corollary 2.2.6 (with $\varsigma=0$), we have

$$||I(t,x)||_2^2=\frac{1}{\lambda^2}\mathscr{K}(t,x)-G_\nu^2(t,x)=\frac{\lambda^2}{2\nu}e^{\frac{\lambda^4 t}{4\nu}}\Phi\left(\lambda^2\sqrt{\frac{t}{2\nu}}\right)G_{\nu/2}(t,x)\,.$$

Therefore,

$$\lim_{t\to 0_+}||I(t,x)||_2^2=\begin{cases}0 & \text{if } x\neq 0\,,\\ +\infty & \text{if } x=0\,.\end{cases}$$

Example 2.2.16 (Another unbounded initial data)**.** Suppose $\rho(u)=\lambda u$ with $\lambda\neq 0$. Let us consider the case where $\mu(\mathrm{d}x)=|x|^{-a}\mathrm{d}x$ with $0<a\leq 1/2$. Clearly, $J_0(0,x)=|x|^{-a}$ is not continuous. As for $I(t,x)$, unlike the case of the delta initial data, $\lim_{t\to 0_+}||I(t,x)||_p\equiv 0$ for $p\geq 2$ is a continuous function in x. But the function $t\mapsto I(t,0)$ from $\mathbb{R}_+$ to $L^p(\Omega)$ cannot be smoother than $\frac{1-2a}{4}$-Hölder continuous. Note that $\frac{1-2a}{4}\in[0,1/4[$. Some statements of this example are proved in Section 2.6.5.

2.3 Some Prerequisites

2.3.1 Space-time Convolutions of the Square of the Heat Kernel

Define the kernel function

$$\mathscr{L}_0(t,x;\nu,\lambda):=\lambda^2G_\nu^2(t,x)=\frac{\lambda^2}{\sqrt{4\pi\nu t}}G_{\nu/2}(t,x)$$

with $(t,x)\in\mathbb{R}_+^*\times\mathbb{R}$. For any $n\in\mathbb{N}^*$, define

$$\mathscr{L}_n(t,x;\nu,\lambda):=(\underbrace{\mathscr{L}_0\star\cdots\star\mathscr{L}_0}_{\substack{n+1\text{ times of}\\ \mathscr{L}_0(t,x;\nu,\lambda)}})(t,x)$$

with $(t,x)\in\mathbb{R}_+^*\times\mathbb{R}$. We use the same conventions on the kernel functions $\mathscr{L}_n(t,x;\nu,\lambda)$ as $\mathscr{K}(t,x;\nu,\lambda)$ regarding the parameters ν and λ.

Proposition 2.3.1 (Properties of the kernel functions)**.** *Let* $b=\frac{\lambda^2}{\sqrt{4\pi\nu}}$.

(i) $\mathscr{L}_n(t,x)$ *has the following explicit form*

$$\mathscr{L}_n(t,x) = G_{\nu/2}(t,x)\frac{\left(b\sqrt{\pi}\right)^{n+1}}{\Gamma\left(\frac{n+1}{2}\right)}\, t^{\frac{n-1}{2}} \tag{2.3.1}$$

for any $n \in \mathbb{N}$ *and* $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}$, *where* $\Gamma(\cdot)$ *is the Gamma function.*

(ii) The kernel functions $\mathscr{K}(t,x)$ *and* $\{\mathscr{L}_n(t,x) : n \in \mathbb{N}\}$ *satisfy the following relations: for any* $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}$,

$$\mathscr{K}(t,x) = \sum_{n=0}^{\infty} \mathscr{L}_n(t,x)\,, \tag{2.3.2}$$

and

$$(\mathscr{K} \star \mathscr{L}_0)(t,x) = \mathscr{K}(t,x) - \mathscr{L}_0(t,x)\,. \tag{2.3.3}$$

(iii) There are non-negative functions $B_n(t)$ *such that for all* $n \in \mathbb{N}$, *the function* $B_n(t)$ *is nondecreasing in t and*

$$\mathscr{L}_n(t,x) = \mathscr{L}_0(t,x)B_n(t), \quad \text{for all } (t,x) \in \mathbb{R}_+^* \times \mathbb{R}\,.$$

Moreover,

$$\sum_{n=0}^{\infty} \left(B_n(t)\right)^{1/m} < +\infty, \quad \text{for all } m \in \mathbb{N}^*\,.$$

In particular, we can choose

$$B_n(t) = \frac{\pi^{\frac{n+1}{2}} b^n t^{\frac{n}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}\,. \tag{2.3.4}$$

Remark 2.3.2. The above property (iii) will play a role similar to Gronwall's lemma. It is essentially an extension of the version used in [23, 24] in the sense that space-time convolution is involved instead of only the convolution in time variable: see Step 3 of the proof of the existence part of Theorem 2.2.2.

Proof. (i) We shall first prove (2.3.1). By induction, it clearly holds for $n = 0$ since $\Gamma(1/2) = \sqrt{\pi}$. Suppose that the equation holds for n. Now we shall evaluate $\mathscr{L}_{n+1}(t,x)$ from its definition. By the semigroup property of the heat kernel,

$$\begin{aligned}
\mathscr{L}_{n+1}(t,x) &= (\mathscr{L}_n \star \mathscr{L}_0)(t,x) \\
&= b\frac{\left(b\sqrt{\pi}\right)^{n+1}}{\Gamma\left(\frac{n+1}{2}\right)} \int_0^t \mathrm{d}s\, s^{-1/2}(t-s)^{\frac{n-1}{2}} \int_{\mathbb{R}} G_{\nu/2}\left(s,y\right) G_{\nu/2}\left(t-s,x-y\right) \mathrm{d}y \\
&= G_{\nu/2}(t,x)\, b\frac{\left(b\sqrt{\pi}\right)^{n+1}}{\Gamma\left(\frac{n+1}{2}\right)} \int_0^t s^{-1/2}(t-s)^{\frac{n-1}{2}} \mathrm{d}s\,.
\end{aligned}$$

Now using the Beta integral (see [51, (5.12.1), p. 142])

$$\int_0^t s^{-1/2}(t-s)^{\frac{n-1}{2}}\mathrm{d}s = t^{n/2}\int_0^1 u^{-1/2}(1-u)^{\frac{n-1}{2}}\mathrm{d}u = t^{n/2}\frac{\Gamma(1/2)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}, \quad t>0. \tag{2.3.5}$$

Therefore,

$$\mathscr{L}_{n+1}(t,x) = G_{\nu/2}(t,x)\frac{(b\sqrt{\pi})^{n+2}}{\Gamma\left(\frac{n+2}{2}\right)}t^{n/2},$$

which proves (2.3.1).

(ii) Using the explicit solutions of $\mathscr{L}_n(t,x)$, the equation (2.3.2) is equivalent to the following relation

$$\frac{b}{\sqrt{t}} + 2\pi b^2 e^{\pi b^2 t}\Phi\left(\sqrt{2\pi b^2 t}\right) = \sum_{n=0}^{\infty}\frac{(b\sqrt{\pi})^{n+1}}{\Gamma\left(\frac{n+1}{2}\right)}t^{\frac{n-1}{2}}.$$

The $n=0$ term in the sum on the right-hand side of the above equation is $b/\sqrt{t}$, so by removing this term, we reduce the above equation to the following relation

$$2e^{\pi b^2 t}\Phi\left(\sqrt{2\pi b^2 t}\right) = \sum_{n=1}^{\infty}\frac{(b\sqrt{\pi}\sqrt{t})^{n-1}}{\Gamma\left(\frac{n+1}{2}\right)}.$$

This equation holds by Lemma 2.3.4 below with $x=\sqrt{\pi b^2 t}$, which then proves (2.3.2). As a direct consequence, we have (2.3.3). Indeed, we only need to replace $\mathscr{K}$ in (2.3.3) by its series representation in (2.3.2) and then use the definition of $\mathscr{L}_n$.

(iii) Take $B_n(t)$ given in (2.3.4). Clearly, it is non-negative and nondecreasing in t, and $\mathscr{L}_n(t,x) = \mathscr{L}_0(t,x)B_n(t)$. Fix $m\in\mathbb{N}^*$. Apply the ratio test:

$$\frac{(B_n(t))^{1/m}}{(B_{n-1}(t))^{1/m}} = \left(\sqrt{\pi t}\,b\right)^{\frac{1}{m}}\left(\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)}\right)^{\frac{1}{m}} \approx \left(\sqrt{\pi t}\,b\right)^{\frac{1}{m}}\left(\frac{2}{n}\right)^{\frac{1}{2m}} \to 0, \quad \text{as } n\to\infty, \tag{2.3.6}$$

where we have used the fact ([51, 5.11.12, in p.141]) that

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} \sim z^{a-b}, \quad z\to+\infty, \quad |\arg z|<\pi, \quad a,b\in\mathbb{R}.$$

This completes the whole proof of the proposition. □

Lemma 2.3.3. *The following formula holds:*

$$\pi\int_0^t e^{\pi b^2 u}\Phi\left(\sqrt{2\pi b^2 u}\right)\mathrm{d}u = \frac{e^{\pi b^2 t}\Phi\left(\sqrt{2\pi b^2 t}\right)}{b^2} - \frac{1}{2b^2} - \frac{\sqrt{t}}{b}, \quad b\neq 0. \tag{2.3.7}$$

Proof. This can be obtained by integration by parts

$$\pi \int_0^t e^{\pi b^2 u} \Phi\left(\sqrt{2\pi b^2 u}\right) \mathrm{d}u = \left.\frac{e^{\pi b^2 u} \Phi\left(\sqrt{2\pi b^2 u}\right)}{b^2}\right|_{u=0}^{u=t} - \frac{1}{b^2} \int_0^t \frac{b}{2\sqrt{s}} \mathrm{d}s.$$

□

Lemma 2.3.4. *The following series expansion holds for all* $x \geq 0$

$$2e^{x^2} \Phi\left(\sqrt{2}x\right) = \sum_{n=1}^{\infty} \frac{x^{n-1}}{\Gamma\left(\frac{n+1}{2}\right)}.$$

Proof. Equivalently we need to prove that

$$e^{x^2}(1+\operatorname{erf}(x)) = \sum_{n=1}^{\infty} \frac{x^{n-1}}{\Gamma\left(\frac{n+1}{2}\right)}.$$

By [51, 7.6.2, in p.162], we know that

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} e^{-x^2} \sum_{n=0}^{\infty} \frac{2^n x^{2n+1}}{1 \cdot 3 \cdots (2n+1)}$$

which equals, since $\Gamma((2n+3)/2) = \frac{1 \cdot 3 \cdots (2n+1)}{2^{n+1}} \Gamma(1/2)$ and $\Gamma(1/2) = \sqrt{\pi}$,

$$e^{-x^2} \sum_{n=0}^{\infty} \frac{x^{2n+1}}{\Gamma\left(\frac{2n+3}{2}\right)} = e^{-x^2} \sum_{n=1}^{\infty} \frac{x^{2n-1}}{\Gamma\left(\frac{2n+1}{2}\right)}.$$

Then use the expansion

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{\Gamma(n+1)} = \sum_{n=1}^{\infty} \frac{x^{2(n-1)}}{\Gamma\left(\frac{2n}{2}\right)}.$$

Adding e^{x^2} and $e^{x^2}\operatorname{erf}(x)$ proves the lemma. □

2.3.2 Solutions to the Homogeneous Equation

Lemma 2.3.5. *The solution* $J_0(t,x)$ *to the homogeneous equation* (2.2.1) *with initial data* μ *satisfying Hypothesis* (1.1.5) *is smooth:* $J_0(t,x) \in C^{+\infty}\left(\mathbb{R}_+^* \times \mathbb{R}\right)$. *If, in addition,* μ *is an* α*-Hölder continuous function, then*

$$J_0(t,x) \in C^{+\infty}\left(\mathbb{R}_+^* \times \mathbb{R}\right) \cap C_{\alpha/2,\alpha}\left(\mathbb{R}_+ \times \mathbb{R}\right).$$

Note that the difficulties come from rapidly growing tails of μ. When the tails of μ are only of polynomial growth, which is the case for Schwartz distributions, it is well known that $J_0(t,x) \in C^{+\infty}\left(\mathbb{R}_+^* \times \mathbb{R}\right)$; see, for example, [67, Proposition 5.1, p. 217]. Borel measures satisfying (1.1.5) go beyond Schwartz distributions (for instance, $\mu(\mathrm{d}x) =$

$e^{|x|}\mathrm{d}x$). Nevertheless, $J_0(t,x)$ is still in $C^{+\infty}\left(\mathbb{R}_+^*\times\mathbb{R}\right)$. The proof needs more preparations and we will postpone it to Section 2.6.3; see Lemma 2.6.14.

The Hölder continuity result in this lemma will also be used in the proof of the Hölder continuity of the solution (see Theorem 2.2.13). For the proof of the existence and uniqueness (Theorem 2.2.2), we only need the function $(t,x)\mapsto J_0(t,x)$ to be continuous. For this purpose, one can also follow an argument similar to the proof of $L^p(\Omega)$-continuity in Proposition 2.4.2 (using Proposition 2.3.12).

2.3.3 A Lemma on Initial Data

When the initial data make $J_0^2(t,x)$ a constant, e.g. $\mu(\mathrm{d}x)=c\mathrm{d}x$, by the definition of $\mathscr{H}(t,x)$ in (2.2.5), we have

$$\left(v^2\star\mathscr{K}\right)(t,x)=v^2\mathscr{H}(t)\,.$$

Clearly,

$$\left(v^2\star\mathscr{L}_0\right)(t,x)=v^2\lambda^2\int_0^t\frac{1}{\sqrt{4\pi\nu s}}\mathrm{d}s\int_{\mathbb{R}}G_{\nu/2}\left(s,y\right)\mathrm{d}y=v^2\lambda^2\sqrt{t/\nu\pi}\,. \tag{2.3.8}$$

For general $J_0^2(t,x)$, we prove the following lemma.

Lemma 2.3.6. *For every signed measure μ such that* (1.1.5) *holds, let $\mu=\mu_+-\mu_-$ be its Jordan decomposition. Suppose $K(t,x)=G_{\nu/2}(t,x)h(t)$ for some non-negative function $h(t)$. Then*

$$\left(J_0^2\star K\right)(t,x)\le 2\sqrt{t}\,\left|J_0^*(2t,x)\right|^2\int_0^t\frac{h(t-s)}{\sqrt{s}}\mathrm{d}s\,,\quad \textit{for all } (t,x)\in\mathbb{R}_+^*\times\mathbb{R}\,, \tag{2.3.9}$$

where $J_0^*(t,x)=\left(G_\nu(t,\cdot)*|\mu|\right)(x)$ and $|\mu|=\mu_++\mu_-$. In particular, for all $(t,x)\in\mathbb{R}_+^*\times\mathbb{R}$,

$$\left(J_0^2\star\mathscr{K}\right)(t,x)\le\lambda^2\sqrt{\pi t/\nu}\,\left|J_0^*(2t,x)\right|^2\left(1+2\exp\left(\frac{\lambda^4 t}{4\nu}\right)\right)<+\infty, \tag{2.3.10}$$

and

$$\left(J_0^2\star\mathscr{L}_0\right)(t,x)\le\lambda^2\sqrt{\pi t/\nu}\,\left|J_0^*(2t,x)\right|^2<+\infty. \tag{2.3.11}$$

Proof. We first assume that μ is non-negative and denote $J_0(t,x)=\left(G_\nu(t,\cdot)*\mu\right)(x)$. Clearly,

$$\left(J_0^2\star K\right)(t,x)=\int_0^t\mathrm{d}s\int_{\mathbb{R}}J_0^2\left(s,y\right)G_{\nu/2}\left(t-s,x-y\right)h(t-s)\mathrm{d}y.$$

Since

$$J_0^2\left(s,y\right)=\iint_{\mathbb{R}^2}G_\nu(s,y-z_1)G_\nu(s,y-z_2)\mu(\mathrm{d}z_1)\mu(\mathrm{d}z_2)\,,$$

we have

$$\left(J_0^2 \star K\right)(t,x) = \int_0^t \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y \iint_{\mathbb{R}^2} G_\nu(s, y-z_1) G_\nu(s, y-z_2) G_{\nu/2}\left(t-s, x-y\right) \\ \times h(t-s)\, \mu(\mathrm{d}z_1)\mu(\mathrm{d}z_2)\,. \quad (2.3.12)$$

Notice that by Lemma 2.3.7,

$$G_\nu(s, y-z_1) G_\nu(s, y-z_2) = G_{\nu/2}\left(s, y - \frac{z_1+z_2}{2}\right) G_{2\nu}\left(s, z_2 - z_1\right)\,.$$

Now using the semigroup property of the heat kernel and Fubini's theorem, we integrate over y first in (2.3.12) to get

$$\left(J_0^2 \star K\right)(t,x) = \int_0^t \mathrm{d}s \iint_{\mathbb{R}^2} G_{2\nu}(s, z_2 - z_1) G_{\nu/2}\left(t, x - \frac{z_1+z_2}{2}\right) h(t-s)\, \mu(\mathrm{d}z_1)\mu(\mathrm{d}z_2)\,. \quad (2.3.13)$$

By Lemma 2.3.8 below, we have

$$G_{2\nu}(s, z_2 - z_1) G_{\nu/2}\left(t, x - \frac{z_1+z_2}{2}\right) \le 2\frac{\sqrt{t}}{\sqrt{s}} G_{2\nu}(t, x-z_1) G_{2\nu}(t, x-z_2)\,.$$

Finally, since $h(t)$ is nonnegative,

$$\left(J_0^2 \star K\right)(t,x) \le 2\sqrt{t}\left(G_{2\nu}(t,\cdot) * \mu\right)^2(x) \int_0^t \frac{h(t-s)}{\sqrt{s}} \mathrm{d}s = 2\sqrt{t}\, J_0^2(2t,x) \int_0^t \frac{h(t-s)}{\sqrt{s}} \mathrm{d}s\,,$$

which proves (2.3.9) for nonnegative measures. Now for a general signed measure μ, by Jordan decomposition, $\mu = \mu_+ - \mu_-$. Then

$$\begin{aligned} J_0^2(t,x) &= \left[(\mu_+ * G_\nu(t,\cdot))(x) - (\mu_- * G_\nu(t,\cdot))(x)\right]^2 \\ &\le \left[(\mu_+ * G_\nu(t,\cdot))(x) + (\mu_- * G_\nu(t,\cdot))(x)\right]^2 \\ &= \left[(|\mu| * G_\nu(t,\cdot))(x)\right]^2\,. \end{aligned}$$

Applying the above nonnegative case with $|\mu|$ proves (2.3.9).

The inequality (2.3.11) is proved by choosing $h(t) = \frac{\lambda^2}{\sqrt{4\pi\nu t}}$ and the Beta integral

$$\int_0^t \frac{\lambda^2}{\sqrt{4\pi\nu s(t-s)}} \mathrm{d} = \frac{1}{2}\lambda^2\sqrt{\pi/\nu}\,. \quad (2.3.14)$$

As for (2.3.10), notice that from the definition of $\mathscr{K}(t,x)$ in (2.2.4),

$$\mathscr{K}(t,x) \le G_{\nu/2}(t,x)\,\lambda^2\left(\frac{1}{\sqrt{4\pi\nu t}} + \frac{\lambda^2}{2\nu}\exp\left(\frac{\lambda^4 t}{4\nu}\right)\right)\,.$$

Then (2.3.10) follows from (2.3.9) by taking $h(t) = \frac{1}{\sqrt{4\pi\nu t}} + \frac{\lambda^2}{2\nu}\exp\left(\frac{\lambda^4 t}{4\nu}\right)$ and then using the Beta integral in (2.3.14) and the fact that

$$\int_0^t \frac{e^{a(t-s)}}{\sqrt{s}} \mathrm{d}s = 2\int_0^{\sqrt{t}} e^{a(t-u^2)} \mathrm{d}u = \sqrt{\pi/a}\, e^{at} \operatorname{erf}\left(\sqrt{at}\right) \le \sqrt{\pi/a}\, e^{at}, \quad a > 0. \tag{2.3.15}$$

This completes the whole proof. □

Comparing the proofs of (2.3.10) and (2.3.11), we can see that the following two conditions are equivalent:

$$\left(\mathcal{K} \star J_0^2\right)(t,x) < \infty \qquad \Longleftrightarrow \qquad \left(\mathcal{L}_0 \star J_0^2\right)(t,x) < \infty .$$

That is to say, the main issue is the integrability around $t = 0$ caused by the factor $\frac{1}{\sqrt{t}}$ in $\mathcal{L}_0$. We will see in the proof of Proposition 2.2.9 that in the case where $\mu = \delta_0'$, both (2.3.10) and (2.3.11) fail since a factor $t^{-3/2}$ ruins the integrability at zero.

Lemma 2.3.7. *For all $t, s > 0$ and $x, y \in \mathbb{R}$, we have*

$$G_\nu(t,x) G_\nu(s,y) = G_\nu\left(\frac{ts}{t+s}, \frac{sx+ty}{t+s}\right) G_\nu\left(t+s, x-y\right) .$$

In particular,

$$G_\nu^2(t,x) = \frac{1}{\sqrt{4\pi\nu t}} G_{\nu/2}(t,x) .$$

Proof. Clearly, we only need to verify that

$$\frac{x^2}{t} + \frac{y^2}{s} = \frac{\left(\frac{sx+ty}{t+s}\right)^2}{\frac{ts}{t+s}} + \frac{(x-y)^2}{t+s} ,$$

which is true by direct calculation. One can also prove this lemma using independent and conditional normal variables (see [26, Exercise 8.7, p. 119] for example). □

Lemma 2.3.8. *For all x, z_1 $z_2 \in \mathbb{R}$ and $t > 0$ and $s > 0$, we have*

$$G_1\left(t, x-\bar{z}\right) G_1\left(s, \Delta z\right) \le \frac{(4t)\vee s}{\sqrt{ts}} G_1\left((4t)\vee s, x-z_1\right) G_1\left((4t)\vee s, x-z_2\right) ,$$

where $\bar{z} = \frac{z_1+z_2}{2}$, $\Delta z = z_1 - z_2$ and $a \vee b := \max(a,b)$.

Proof. The proof is straightforward:

$$\begin{aligned} G_1\left(t, x-\bar{z}\right) G_1\left(s, \Delta z\right) =& \frac{1}{2\pi\sqrt{ts}} \exp\left(-\frac{[(x-z_1)+(x-z_2)]^2}{8t} - \frac{(z_1-z_2)^2}{2s}\right) \\ \le& \frac{1}{2\pi\sqrt{ts}} \exp\left(-\frac{[(x-z_1)+(x-z_2)]^2 + (z_1-z_2)^2}{2((4t)\vee s)}\right) . \end{aligned}$$

By the inequality $2(a^2+b^2) \geq (a+b)^2$, we have

$$(z_2-z_1)^2+[(x-z_1)+(x-z_2)]^2 \geq \frac{1}{2}(2x-2z_1)^2 = 2(x-z_1)^2 .$$

Similarly, we have

$$(z_2-z_1)^2+[(x-z_1)+(x-z_2)]^2 \geq 2(x-z_2)^2 .$$

Combining these two inequalities, we have

$$(z_2-z_1)^2+[(x-z_1)+(x-z_2)]^2 \geq (x-z_1)^2+(x-z_2)^2 .$$

Hence,

$$\begin{aligned} G_1(t,x-\bar{z})\,G_1(s,\Delta z) \leq & \frac{1}{2\pi\sqrt{ts}} \exp\left(-\frac{(x-z_1)^2+(x-z_2)^2}{2((4t)\vee s)}\right) \\ = & \frac{(4t)\vee s}{2\pi\sqrt{ts}} G_1((4t)\vee s, x-z_1)\,G_1((4t)\vee s, x-z_2) , \end{aligned}$$

which proves the lemma. □

2.3.4 Some Continuity Properties of the Heat Kernel

Proposition 2.3.9. *There are three universal constants*

$$C_1 = 1, \quad C_2 = \frac{\sqrt{2}-1}{\sqrt{\pi}}, \quad C_3 = \frac{1}{\sqrt{\pi}} ,$$

such that

(i) for all $t \geq 0$ and $x, y \in \mathbb{R}$,

$$\int_0^t \mathrm{d}r \int_{\mathbb{R}} \mathrm{d}z \left[G_\nu(t-r,x-z) - G_\nu(t-r,y-z)\right]^2 \leq \frac{C_1}{\nu}|x-y| ; \tag{2.3.16}$$

(ii) for all s, t with $0 \leq s \leq t$, and $x \in \mathbb{R}$,

$$\int_0^s \mathrm{d}r \int_{\mathbb{R}} \mathrm{d}z\,[G_\nu(t-r,x-z) - G_\nu(s-r,x-z)]^2 \leq \frac{C_2}{\sqrt{\nu}}\sqrt{t-s} \tag{2.3.17}$$

and

$$\int_s^t \mathrm{d}r \int_{\mathbb{R}} \mathrm{d}z\,[G_\nu(t-r,x-z)]^2 \leq \frac{C_3}{\sqrt{\nu}}\sqrt{t-s} . \tag{2.3.18}$$

The proof below uses the Fourier transform of the heat kernel:

$$\mathscr{F}(G_\nu(t,\cdot))(\xi) := \int_{\mathbb{R}} e^{-i\xi x} G_\nu(t,x)\mathrm{d}x = e^{-\frac{t\nu\xi^2}{2}} ,$$

and also uses Plancherel's theorem: For all $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$,

$$||g||_{L^2(\mathbb{R})}^2 = \frac{1}{2\pi} ||\mathscr{F} g||_{L^2(\mathbb{R})}^2 . \tag{2.3.19}$$

Similar estimates can be found in the proof of Theorem 6.7 in [42]. The above is a slight improvement because the constants are universal (independent of the finite time horizon T) and optimal. Note that in [42], the constant C_2 depends on T and $C_1 = 8/\pi$ is universal but not optimal [2].

Proof. (i) Assume first that $t > 0$. By Plancherel's theorem, the left-hand side of (2.3.16) is equal to

$$\begin{aligned}\frac{1}{2\pi}\int_0^t \mathrm{d}r \int_{\mathbb{R}} \mathrm{d}\xi \left| e^{-i\xi x - \frac{(t-r)\nu\xi^2}{2}} - e^{-i\xi y - \frac{(t-r)\nu\xi^2}{2}} \right|^2 &= \frac{1}{2\pi}\int_0^t \mathrm{d}r \int_{\mathbb{R}} \mathrm{d}\xi\, e^{-(t-r)\nu\xi^2} \left| e^{-i\xi x} - e^{-i\xi y}\right|^2 \\ &= \frac{1}{\pi}\int_0^t \mathrm{d}r \int_{\mathbb{R}} \mathrm{d}\xi\, e^{-(t-r)\nu\xi^2} \left(1 - \cos(\xi(x-y))\right) .\end{aligned}$$

Notice that for $a > 0$ and $b \in \mathbb{R}$, integration by parts gives

$$\int_{\mathbb{R}} e^{-a\xi^2} (1 - \cos(\xi b))\, \mathrm{d}\xi = \frac{\sqrt{\pi}\left(1 - e^{-\frac{b^2}{4a}}\right)}{\sqrt{a}} .$$

Applying this integral with $a = (t-r)\nu$ and $b = (x-y)$ to the above double integral shows that the left-hand side of (2.3.16) is equal to

$$\begin{aligned} &= \frac{1}{\sqrt{\nu\pi}} \int_0^t \frac{1 - e^{-\frac{(x-y)^2}{4\nu(t-r)}}}{\sqrt{t-r}} \mathrm{d}r \\ &= \frac{1}{\sqrt{\nu\pi}} \int_0^t \frac{1 - e^{-\frac{(x-y)^2}{4\nu s}}}{\sqrt{s}} \mathrm{d}s \\ &= \frac{2}{\sqrt{\nu\pi}} \left(\sqrt{s}\left(1 - e^{-\frac{(x-y)^2}{4\nu s}}\right)\Bigg|_{s=0}^{s=t} + \int_0^t \sqrt{s} e^{-\frac{(x-y)^2}{4\nu s}} \frac{(x-y)^2}{4\nu s^2} \mathrm{d}s \right) \\ &= \frac{2}{\sqrt{\nu\pi}} \left(\sqrt{t}\left(1 - e^{-\frac{(x-y)^2}{4\nu t}}\right) + \underbrace{\int_0^t e^{-\frac{(x-y)^2}{4\nu s}} \frac{(x-y)^2}{4\nu s^{3/2}} \mathrm{d}s}_{:=I} \right) . \end{aligned}$$

For the above integral I, we change the variable: $w = |x-y|/\sqrt{2\nu s}$, then $s = \frac{(x-y)^2}{2\nu w^2}$,

[2] See [42, (133) on p. 31] for the derivation for C_1. There should be a factor 8 on the right-hand side of (133) of [42]: The equality after (131) of [42] misses a factor 4; The inequality $1 - \cos(\theta) \le 1 \wedge \theta^2$ for $\theta \in \mathbb{R}$ should be $1 - \cos(\theta) \le 2\left(1 \wedge \theta^2\right)$. The diffusion parameter ν in this reference is equal to 2. Hence, the arguments there lead to a constant $C_1 = 8/\pi$.

$\mathrm{d}s = -\frac{(x-y)^2}{\nu w^3}\mathrm{d}w$ and so

$$I = \frac{\sqrt{\pi}|x-y|}{\sqrt{\nu}} \int_{\frac{|x-y|}{\sqrt{2\nu t}}}^{+\infty} \frac{e^{-\frac{w^2}{2}}}{\sqrt{2\pi}} \mathrm{d}w = \sqrt{\pi/\nu}\,|x-y|\left(1-\Phi\left(\frac{|x-y|}{\sqrt{2\nu t}}\right)\right).$$

Finally, we have

$$\begin{aligned}\int_0^t \mathrm{d}r \int_{\mathbb{R}} \mathrm{d}z \left[G_\nu(t-r,x-z) - G_\nu(t-r,y-z)\right]^2 \\ = \frac{2}{\sqrt{\nu\pi}}\left(\sqrt{t}\left(1-e^{-\frac{(x-y)^2}{4\nu t}}\right) + \sqrt{\pi/\nu}\,|x-y|\left(1-\Phi\left(\frac{|x-y|}{\sqrt{2\nu t}}\right)\right)\right).\end{aligned} \tag{2.3.20}$$

Now, denote $z = \frac{|x-y|}{\sqrt{2\nu t}}$. We need to prove that

$$\frac{1}{|x-y|}\int_0^t \mathrm{d}r \int_{\mathbb{R}} \mathrm{d}u \left[G_\nu(t-r,x-u) - G_\nu(t-r,y-u)\right]^2 = \frac{\sqrt{2}}{\nu\sqrt{\pi}}\frac{1-e^{-z^2/2}}{z} + \frac{2}{\nu}\left(1-\Phi(z)\right)$$

is bounded from above for $z \ge 0$. Denote the right-hand side by $f(z)$. Because

$$f'(z) = \frac{\sqrt{2}}{\nu\sqrt{\pi}\,z^2}\left(e^{-z^2/2}-1\right) \le 0,$$

we have that $f(z) \le \lim_{z\to 0_+} f(z) = 1/\nu$. Hence, the optimal constant is $C_1 = 1$. When t tends to zero, from (2.3.20), we know that the limit of the left-hand side of (2.3.16) is zero. This completes the proof of (i).

(ii) Assume $t > 0$. Apply Plancherel's theorem for the left-hand side of (2.3.17) and then apply Lemma 2.3.11 below:

$$\begin{aligned}\frac{1}{2\pi}\int_0^s \mathrm{d}r \int_{\mathbb{R}} \mathrm{d}\xi \left| e^{-i\xi x - \frac{(t-r)\nu\xi^2}{2}} - e^{-i\xi x - \frac{(s-r)\nu\xi^2}{2}}\right|^2 \\ = \frac{1}{2\pi}\int_0^s \mathrm{d}r \int_{\mathbb{R}} \mathrm{d}\xi \left(e^{-\frac{(t-r)\nu\xi^2}{2}} - e^{-\frac{(s-r)\nu\xi^2}{2}}\right)^2 \\ = \frac{1}{2\pi}\int_0^s \mathrm{d}r \int_{\mathbb{R}} \mathrm{d}\xi \left(e^{-(t-r)\nu\xi^2} + e^{-(s-r)\nu\xi^2} - 2e^{-\frac{(t+s-2r)\nu\xi^2}{2}}\right) \\ = \frac{1}{2\sqrt{\pi\nu}}\int_0^s \left(\frac{1}{\sqrt{t-r}} + \frac{1}{\sqrt{s-r}} - \frac{2}{\sqrt{(t+s)/2-r}}\right)\mathrm{d}r \\ \le \frac{\sqrt{2}-1}{\sqrt{\pi\nu}}\sqrt{t-s},\end{aligned} \tag{2.3.21}$$

which proves (2.3.17) with $C_2 = \frac{\sqrt{2}-1}{\sqrt{\pi}}$. As for (2.3.18), similarly, we have

$$\begin{aligned}\int_s^t \mathrm{d}r \int_{\mathbb{R}} \mathrm{d}z \left[G_\nu(t-r,x-z)\right]^2 = \frac{1}{2\pi}\int_s^t \mathrm{d}r \int_{\mathbb{R}} e^{-(t-r)\nu\xi^2}\mathrm{d}\xi \\ = \frac{1}{2\sqrt{\pi\nu}}\int_s^t \frac{1}{\sqrt{t-r}}\mathrm{d}r = \frac{1}{\sqrt{\pi\nu}}\sqrt{t-s},\end{aligned} \tag{2.3.22}$$

and we can take $C_3 = \frac{1}{\sqrt{\pi}}$ in this case. As for the case $t = 0$, by letting $s = 0$ in (2.3.21) and (2.3.22) and then sending t to zero, one can show that (2.3.17) and (2.3.18) continue to hold. This completes the whole proof. □

Corollary 2.3.10. *There exists a universal constant C (≈ 4.7201) such that for all (t,x) and $(s,y)\in\mathbb{R}_+\times\mathbb{R}$,*

$$\iint_{\mathbb{R}_+\times\mathbb{R}} \left(G_\nu(t-r,x-z)-G_\nu(s-r,y-z)\right)^2 \mathrm{d}r\mathrm{d}z \le C\left(\frac{|x-y|}{\nu}+\frac{\sqrt{|t-s|}}{\sqrt{\nu}}\right),$$

where we use the convention that $G_\nu(t,\cdot)\equiv 0$ if $t\le 0$.

Proof. It is clear that

$$\begin{aligned}
&\left(G_\nu(t-r,x-z)-G_\nu(s-r,y-z)\right)^2 \\
&\quad= \left([G_\nu(t-r,x-z)-G_\nu(s-r,x-z)]+\left[G_\nu(s-r,x-z)-G_\nu(s-r,y-z)\right]\right)^2 \\
&\quad\le 2\,[G_\nu(t-r,x-z)-G_\nu(s-r,x-z)]^2+2\left[G_\nu(s-r,x-z)-G_\nu(s-r,y-z)\right]^2 .
\end{aligned}$$

Then integrate both sides: apply (2.3.17) and (2.3.18) to the first integral, and (2.3.16) to the second one. Finally, since the three constants in Proposition 2.3.9 satisfy: $C_1 > C_3 > C_2$, this corollary is proved by choosing the largest constant $C = 2C_1$. □

Lemma 2.3.11. *For all $t\ge s\ge 0$, we have*

$$\int_0^s \left(\frac{1}{\sqrt{t-r}}+\frac{1}{\sqrt{s-r}}-\frac{2}{\sqrt{\frac{t+s}{2}-r}}\right)\mathrm{d}r \le 2\left(\sqrt{2}-1\right)\sqrt{t-s}\,.$$

Proof. Clearly,

$$\frac{1}{2}\int_0^s \left(\frac{1}{\sqrt{t-r}}+\frac{1}{\sqrt{s-r}}-\frac{2}{\sqrt{(t+s)/2-r}}\right)\mathrm{d}r = \sqrt{s}+\sqrt{t}-\sqrt{t-s}+\sqrt{2(t-s)}-\sqrt{2(t+s)}.$$

We need to prove that

$$\frac{\sqrt{s}+\sqrt{t}-\sqrt{t-s}+\sqrt{2(t-s)}-\sqrt{2(t+s)}}{\sqrt{t-s}}$$

is bounded from above for all $0\le s\le t$. Or equivalently, we need to show that

$$g(r) := \frac{\sqrt{r}+1-\sqrt{1-r}+\sqrt{2(1-r)}-\sqrt{2(1+r)}}{\sqrt{1-r}}$$

is bounded for all $r\in[0,1]$. Clearly, $g(0)=0$ and $\lim_{r\uparrow 1} g(r)=\sqrt{2}-1$. Hence $\sup_{r\in[0,1]} g(r) < \infty$. In fact,

$$g'(r) = \frac{\left(\sqrt{1+r}+\sqrt{1+1/r}\right)-2\sqrt{2}}{2(1-r)^{3/2}\sqrt{1+r}}$$

and notice that for all $r \in]0,1]$,

$$\sqrt{1+r} + \sqrt{1+1/r} \geq 2\left[(1+r)(1+1/r)\right]^{1/4} = 2\sqrt{\sqrt{r} + \frac{1}{\sqrt{r}}} \geq 2\sqrt{2}\,.$$

Hence $g'(r) \geq 0$ for $r \in [0,1[$ and $\sup_{r\in[0,1]} g(r) = g(1) = \sqrt{2}-1$. Therefore, the lemma is proved with $C = 2\left(\sqrt{2}-1\right)$. □

Proposition 2.3.12. *Fix $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}^d$. Set*

$$B_{t,x} := \left\{ \left(t',x'\right) \in \mathbb{R}_+^* \times \mathbb{R}^d : 0 < t' \leq t + \frac{1}{2},\ \left|x'-x\right| \leq 1 \right\}$$

Then there exists $a = a_{t,x} > 0$ such that for all $\left(t',x'\right) \in B_{t,x}$ and all $s \in [0,t']$ and $|y| \geq a$,

$$G_\nu(t'-s, x'-y) \leq G_\nu(t+1-s, x-y)\,.$$

In particular, this constant a can be chosen by

$$a = \sqrt{d}(4t+3)(|x|+1) + 2(t+1)\sqrt{d(1+\nu/e)}\,.$$

Proof. (i) We first consider the one dimensional case $d = 1$. Since $t+1-s$ is strictly larger than $t'-s$, the function $y \mapsto G_\nu(t+1-s, x-y)$ has heavier tails than the function $y \mapsto G_\nu(t'-s, x'-y)$. Solve the inequality

$$G_\nu(t+1-s, x-y) \geq G_\nu(t'-s, x'-y)$$

with t, t', x, x' and s fixed, which is a quadratic inequality for y as follows

$$-\frac{(x'-y)^2}{t'-s} + \frac{(x-y)^2}{t+1-s} \leq \nu \log\left(\frac{t'-s}{t+1-s}\right)\,.$$

Writing the above quadratic inequality explicitly in y, we have

$$\begin{aligned}(t'-t-1)y^2 &- 2\left[x(t'-s) - x'(t+1-s)\right]y \\ &+ x^2(t'-s) - x'^2(t+1-s) + (t+1-s)(t'-s)\nu\log\frac{t+1-s}{t'-s} \leq 0\,.\end{aligned}$$

Let $y_\pm(t,x,t',x',s)$ be the two solutions of the corresponding quadratic equation, which are

$$\frac{(t+1-s)x' - x(t'-s) \pm \sqrt{(t+1-s)(t'-s)\left((x-x')^2 + (t+1-t')\nu\log\left(\frac{t+1-s}{t'-s}\right)\right)}}{t+1-t'}\,.$$

Clearly, if $|y| \geq |y_+| \vee |y_-|$, then $G_\nu(t'-s, x'-y) \leq G_\nu(t+1-s, x-y)$. So we only need to show that

$$\sup_{(t',x')\in B_{t,x}} \ \sup_{s\in[0,t']} |y_+(t,x,t',x',s)| \vee |y_-(t,x,t',x',s)| < +\infty\,.$$

Note that

$$
\begin{aligned}
&|y_+(t,x,t',x',s)| \vee |y_-(t,x,t',x',s)| \\
&\quad \le \frac{(t+1-s)|x'|+|x|(t'-s)+\sqrt{(t+1-s)(t'-s)\left((x-x')^2+(t+1-t')\nu\log\left(\frac{t+1-s}{t'-s}\right)\right)}}{t+1-t'}.
\end{aligned}
$$

Now we first take supremum of the above upper bound over $s\in[0,t']$. By Lemma 2.3.13 below, we know that

$$
\begin{aligned}
&\sup_{s\in[0,t']}(t+1-s)(t'-s)\left((x-x')^2+(t+1-t')\nu\log\left(\frac{t+1-s}{t'-s}\right)\right)\\
&\qquad = t'(t+1)\left[(x-x')^2+(t+1-t')\nu\log\frac{t+1}{t'}\right]
\end{aligned}
$$

where the supremum, which is maximum, is taken at $s=0$. So after taking supremum over $s\in[0,t']$, we have

$$
\begin{aligned}
&|y_+(t,x,t',x',s)| \vee |y_-(t,x,t',x',s)| \\
&\qquad \le \frac{(t+1)|x'|+|x|t'+\sqrt{t'(t+1)\left((x-x')^2+(t+1-t')\nu\log\left(\frac{t+1}{t'}\right)\right)}}{t+1-t'}.
\end{aligned}
$$

Now, from the fact that $\left|x'-x\right|\le 1$, we have

$$
\begin{aligned}
&|y_+(t,x,t',x',s)| \vee |y_-(t,x,t',x',s)| \\
&\qquad \le \frac{(t+1)(|x|+1)+|x|t'+\sqrt{t'(t+1)\left(1+(t+1-t')\nu\log\left(\frac{t+1}{t'}\right)\right)}}{t+1-t'}.
\end{aligned}
$$

Finally, taking the supremum over t' with $0\le t'\le t+1/2$, we have

$$
\begin{aligned}
&|y_+(t,x,t',x',s)| \vee |y_-(t,x,t',x',s)| \\
&\quad \le 2(t+1)(|x|+1)+|x|(2t+1)+2\sqrt{(t+1)\left((t+1/2)+t'(t+1)\nu\log\left(\frac{t+1}{t'}\right)\right)}\\
&\quad < (4t+3)(|x|+1)+2(t+1)\sqrt{1+\nu/e},
\end{aligned}
$$

where we have used the fact that

$$
\sup_{s\ge 0} s\log\frac{t}{s} = s\log\frac{t}{s}\,\Big|_{s=t/e} = \frac{t}{e}, \quad \text{for all } t>0.
$$

Therefore, this case is proved by choosing a equal to the above bound.

(ii) As for the high dimensional case, by the same argument, we have the following inequality for y:

$$\sum_{i=1}^{d}\left(-\frac{(x_i'-y_i)^2}{t'-s}+\frac{(x_i-y_i)^2}{t+1-s}\right)\le \nu d\log\left(\frac{t'-s}{t+1-s}\right).$$

Hence, a sufficient condition for the above inequality is

$$-\frac{(x_i'-y_i)^2}{t'-s}+\frac{(x_i-y_i)^2}{t+1-s}\le \nu\log\left(\frac{t'-s}{t+1-s}\right),\quad \text{for all } i=1,\dots,d.$$

By (i), we can choose $|y_i|\ge a$ for the constant a obtained in (i). Let $\overline{B}_{t,x_i}$ be the set in the one-dimensional case. By definition, we have that

$$B_{t,x}\subset \overline{B}_{t,x_1}\times\overline{B}_{t,x_2}\times\cdots\times\overline{B}_{t,x_d}.$$

Finally, we can choose $|y|\ge\sqrt{d}\,a$, which completes the proof. □

Lemma 2.3.13. *For* $0<a<b$*, we have*

$$\frac{\log(b/a)}{b-a}\ge\frac{1}{b}. \tag{2.3.23}$$

The function $f(s)=(a-s)(b-s)\log\frac{b-s}{a-s}$ *is nonincreasing over* $s\in[0,a[$ *with*

$$\inf_{s\in[0,a[}f(s)=\lim_{s\to a}f(s)=(b-a)\log(b-a),$$
$$\sup_{s\in[0,a[}f(s)=f(0)=ab\log(b/a).$$

Proof. Note that (2.3.23) is equivalent to the following statements:

$$\frac{-\log s}{1-s}\ge 1,\quad s\in]0,1[\quad\Longleftrightarrow\quad s-\log s\ge 1,\quad s\in]0,1[.$$

Let $g(s)=s-\log s$ with $s\in]0,1[$. $g(s)$ is nonincreasing since $g'(s)=(s-1)/s<0$ for $s\in]0,1[$. So $g(s)\ge\lim_{s\to1}g(s)=1$. This proves (2.3.23). As for the function $f(s)$, we only need to show that

$$f'(s)=(b-a)-(a+b-2s)\log\frac{b-s}{a-s}\le 0,\quad \text{for all } s\in[0,a[.$$

Let $g(s)=\frac{b-a}{a+b-2s}-\log\frac{b-s}{a-s}$. Then the above statement is equivalent to the inequality $g(s)\le 0$ for all $s\in[0,a[$. By (2.3.23), we know that

$$g(0)=\frac{b-a}{a+b}-\log\frac{b}{a}\le(b-a)\left(\frac{1}{a+b}-\frac{1}{b}\right)\le 0.$$

So it suffices to show that

$$g'(s)=\frac{2(b-a)}{(a+b-2s)^2}+\frac{1}{b-s}-\frac{1}{a-s}\le 0,\quad \text{for all } s\in[0,a[.$$

After simplifications, this statement is equivalent to

$$s^2 - (a+b)s + \frac{a^2+b^2}{2} \geq 0 \quad \text{for all } s \in [0, a[\,,$$

which is clearly true since the discriminant is $-(a+b)^2 < 0$. This completes the proof. □

2.3.5 Some Criteria for Predictable Random Fields

A random field $\{Z(t,x)\}$ is called *elementary* if we can write $Z(t,x) = Y 1_{]a,b]}(t) 1_A(x)$, where $0 \leq a < b$, $A \subset \mathbb{R}$ is an interval, and Y is an $\mathscr{F}_a$–measurable random variable. A *simple* process is a finite sum of elementary random fields. The set of simple processes generates the *predictable* σ-field on $\mathbb{R}_+ \times \mathbb{R} \times \Omega$, denoted by $\mathscr{P}$. For $p \geq 2$ and $X \in L^2(\mathbb{R}_+ \times \mathbb{R}, L^p(\Omega))$, set

$$||X||_{M,p}^2 := \iint_{\mathbb{R}_+^* \times \mathbb{R}} ||X(s,y)||_p^2 \,\mathrm{d}s\mathrm{d}y < +\infty\,. \tag{2.3.24}$$

When $p = 2$, we write $||X||_M$ instead of $||X||_{M,2}$. In [68], $\iint X \mathrm{d}W$ is defined for predictable X such that $||X|| < +\infty$. However, the condition of predictability is not always so easy to check, and as in the case of ordinary Brownian motion [15, Chapter 3], it is convenient to be able to integrate elements X that are jointly measurable and adapted. For this, let $\mathscr{P}_p$ denote the closure in $L^2(\mathbb{R}_+ \times \mathbb{R}, L^p(\Omega))$ of simple processes. Clearly, $\mathscr{P}_2 \supseteq \mathscr{P}_p \supseteq \mathscr{P}_q$ for $2 \leq p \leq q < +\infty$, and according to Itô's isometry, $\iint X \mathrm{d}W$ is well defined for all elements of $\mathscr{P}_2$. The next two propositions give easily verifiable conditions for checking that $X \in \mathscr{P}_2$. In the following, we will use $\cdot$ and $\circ$ to denote the time and space dummy variables respectively.

Proposition 2.3.14. *Suppose for some $t > 0$ and $p \in [2, \infty[$, a random field*

$$X = \left\{X(s,y) : (s,y) \in \,]0,t[\times\mathbb{R}\right\}$$

has the following properties:

(i) X is adapted, i.e., for all $(s,y) \in \,]0,t[\times\mathbb{R}$, $X(s,y)$ is $\mathscr{F}_s$ measurable;

(ii) For all $(s,y) \in \,]0,t[\times\mathbb{R}$, $\left|\left|X(s,y)\right|\right|_p < +\infty$ and the function $(s,y) \mapsto X(s,y)$ from $]0,t[\times\mathbb{R}$ into $L^p(\Omega)$ is continuous;

(iii) $\left|\left|X(\cdot,\circ) 1_{]0,t[}(\cdot)\right|\right|_{M,p} < +\infty$.

Then $X(\cdot,\circ)\, 1_{]0,t[}(\cdot)$ belongs to $\mathscr{P}_p$.

Proof. Fix $\epsilon > 0$ with $\epsilon \leq t/3$. Since $\left|\left|X(\cdot,\circ) 1_{]0,t[}(\cdot)\right|\right|_{M,p} < +\infty$, choose $a = a(\epsilon) > \max(t, 2/t)$ large enough so that

$$\iint_{([1/a,t-1/a]\times[-a,a])^c} \left|\left|X(s,y)\right|\right|_p^2 \mathrm{d}s\mathrm{d}y < \epsilon\,.$$

Due to the $L^p(\Omega)$-continuity hypothesis in (ii), we can choose $n \in \mathbb{N}$ large enough so that, for all $(s_1, y_1), (s_2, y_2) \in [\epsilon, t-\epsilon] \times [-a, a]$,

$$\max\{|s_1 - s_2|, |y_1 - y_2|\} \le \frac{t-2/a}{n} \quad \Rightarrow \quad \left\|X(s_1, y_1) - X(s_2, y_2)\right\|_p < \frac{\epsilon}{a}.$$

Choose $m \in \mathbb{N}$ large enough so that $a/m \le (t-2/a)/n$. Set

$$t_j = \frac{j(t-2/a)}{n} + \frac{1}{a} \quad \text{with } j \in \{0, \dots, n\}$$

and

$$x_i = \frac{ia}{m} - a \quad \text{with } i \in \{0, \dots, 2m\}.$$

Then define

$$X_{n,m}(t,x) := \sum_{j=0}^{n-1} \sum_{i=0}^{2m-1} X(t_j, x_i) 1_{]t_j, t_{j+1}]}(t) 1_{]x_i, x_{i+1}]}(x) .$$

Since X is adapted, $X(t_j, x_i)$ is $\mathscr{F}_{t_j}$-measurable, and so $X_{n,m}$ is predictable, and clearly, $X_{n,m} \in \mathscr{P}_p$. Since $X_{n,m}(t,x)$ vanishes outside of the rectangle $[1/a, t-1/a] \times [-a, a]$, we have

$$\left\|X 1_{]0,t]} - X_{n,m}\right\|_{M,p}^2 = \iint_{([1/a, t-1/a] \times [-a,a])^c} \left\|X(s,y)\right\|_p^2 \mathrm{d}s\mathrm{d}y \\ + \sum_{j=0}^{n-1} \sum_{i=0}^{2m-1} \int_{t_j}^{t_{j+1}} \int_{x_i}^{x_{i+1}} \left\|X(t_j, x_i) - X(s,y)\right\|_p^2 \mathrm{d}s\mathrm{d}y$$

which is less than

$$\epsilon + \sum_{j=0}^{n-1} \sum_{i=0}^{2m-1} \int_{t_j}^{t_{j+1}} \int_{x_i}^{x_{i+1}} \frac{\epsilon^2}{a^2} \mathrm{d}s\mathrm{d}y = \epsilon + \frac{\epsilon^2}{a^2} \mathrm{Area}\left(\left[\frac{1}{a}, t - \frac{1}{a}\right] \times [-a, a]\right) = \epsilon + \epsilon^2 \frac{2at-4}{a^2} .$$

Since $a > t$, the above quantity is bounded by

$$\epsilon + \epsilon^2 \frac{2at-4}{a^2} \le \epsilon + \frac{2\epsilon^2 t}{a} \le \epsilon + 2\epsilon^2 .$$

We have therefore proved that $X(\cdot, \circ) 1_{]0,t[}(\cdot) \in \mathscr{P}_p$. □

Remark 2.3.15. The above proposition is an extension (but specialized to space-time white noise) of Dalang & Frangos's result in [27, Proposition 2] in the sense that the second moment of X can explode at $s = 0$ or $s = t$. The Condition (ii) requires $L^2(\Omega)$-continuity only on an open set $]0, t[\times \mathbb{R}$ instead of the whole space $[0, \infty) \times \mathbb{R}$.

Since the wave equation preserves the singularities, unlike the heat equation which has smoothing effect, we need a more general result as the following.

Proposition 2.3.16. *Suppose for some $t > 0$ and $p \ge 2$, a random field*

$$X = \left\{X(s,y) : (s,y) \in]0, t[\times \mathbb{R}\right\}$$

has the following properties:

(i) X *is adapted, i.e., for all* $(s,y) \in]0,t[\times\mathbb{R}$, $X(s,y)$ *is* $\mathscr{F}_s$ *measurable;*

(ii) X *is jointly measurable with respect to* $\mathscr{B}(\mathbb{R}^2)\times\mathscr{F}$*;*

(iii) $\left|\left|X(\cdot,\circ)1_{]0,t[}(\cdot)\right|\right|_{M,p} < +\infty$.

Then $X(\cdot,\circ)\,1_{]0,t[}(\cdot)$ *belongs to* $\mathscr{P}_2$.

Let $C_c^\infty(\mathbb{R}^n)$ be the test functions, i.e., functions in $C^\infty(\mathbb{R}^n)$ with compact support. The proof below is based on a proper smoothing of the random field X in such a way that the smoothed random field is still adapted with respect to the filtration $\{\mathscr{F}_t\}_{t\geq 0}$.

Proof. We first assume that X is bounded. Fix a non-negative test function $\psi \in C_c^\infty(\mathbb{R}^2)$, such that $\text{supp}(\psi) \subset]0,t[\times]-1,1[$ and $\iint_{\mathbb{R}^2}|\psi(s,y)|\mathrm{d}s\mathrm{d}y = 1$. Let $\psi_n(s,y) := n^2\psi(ns,ny)$ for each $n\in\mathbb{N}^*$, and $\widetilde{X}_n(s,y) := (\psi_n \star X)(s,y)$ for all $(s,y)\in]0,t[\times\mathbb{R}$. Note that when we do the convolution in time, $X(s,y)$ is understood to be zero for $s\not\in]0,t[$.

We shall first prove that

$$\widetilde{X}_n(\cdot,\circ)1_{]0,t[}(\cdot)\in\mathscr{P}_2,\quad \text{for all } n\in\mathbb{N}^*$$

and

$$\left|\left|\widetilde{X}_n(\cdot,\circ)\,1_{]0,t[}\right|\right|_{M,2} \leq \left|\left|X(\cdot,\circ)\,1_{]0,t[}\right|\right|_{M,2} < +\infty\,. \tag{2.3.25}$$

The inequality (2.3.25) is true since, by Hölder's inequality and the Fubini's theorem,

$$\begin{aligned}\left|\left|\widetilde{X}_n(\cdot,\circ)1_{]0,t[}(\cdot)\right|\right|_{M,2}^2 &= \iint_{[0,t]\times\mathbb{R}}\mathbb{E}\left(\left[\iint_{\mathbb{R}^2}\psi_n(s-u,y-z)X(u,z)\mathrm{d}u\mathrm{d}z\right]^2\right)\mathrm{d}s\mathrm{d}y\\ &\leq \iint_{[0,t]\times\mathbb{R}}\mathrm{d}s\mathrm{d}y\iint_{\mathbb{R}^2}\mathbb{E}\left(X^2(u,z)\right)\psi_n(s-u,y-z)\mathrm{d}u\mathrm{d}z\\ &= \left|\left|X(\cdot,\circ)1_{]0,t[}(\cdot)\right|\right|_{M,2}^2\,,\end{aligned}$$

which is finite by Property (iii).

The condition that $\text{supp}(\psi)\subset\mathbb{R}_+^*\times\mathbb{R}$, together with the joint measurability of X, ensures that $\widetilde{X}_n$ is still adapted. The sample path continuity of $\widetilde{X}_n$ in both space and time variables implies $L^2(\Omega)$-continuity, thanks to the boundedness of X. Hence, we can apply Proposition 2.3.14 to conclude that $\widetilde{X}_n(\cdot,\circ)1_{]0,t[}(\cdot)\in\mathscr{P}_2$, for all $n\in\mathbb{N}^*$.

Property (iii) implies that there is $\Omega'\subseteq\Omega$ such that $P(\Omega')=1$ and for all $\omega\in\Omega'$, $X(\cdot,\circ,\omega)\in L^2(]0,t[\times\mathbb{R})$. Now we restrict on the sample space Ω'. In particular, fix $\omega\in\Omega'$. Then, by a standard result in real analysis (see, e.g., [1, Theorem 2.29 (c)]), we have that

$$\begin{aligned}\lim_{n\to+\infty}\left|\left|\widetilde{X}_n(\cdot,\circ,\omega)-X(\cdot,\circ,\omega)\right|\right|_{L^2(]0,t[\times\mathbb{R})} = 0\,,\\ \left|\left|\widetilde{X}_n(\cdot,\circ,\omega)\right|\right|_{L^2(]0,t[\times\mathbb{R})} \leq ||X(\cdot,\circ,\omega)||_{L^2(]0,t[\times\mathbb{R})}\,.\end{aligned}$$

Thus, by Lebesgue's dominated convergence theorem,

$$\lim_{n\to\infty} \mathbb{E}\left[\left|\left| \widetilde{X}_n(\cdot,\circ) - X(\cdot,\circ) \right|\right|^2_{L^2(]0,t[\times\mathbb{R})} \right] = 0\,,$$

that is, $\widetilde{X}_n(\cdot,\circ)1_{]0,t[}(\cdot) \to X(\cdot,\circ)1_{]0,t[}(\cdot)$ in the norm $||\cdot||_2$. Hence $X(\cdot,\circ)1_{]0,t[}(\cdot)$ preserves the same measurability as $\widetilde{X}_n(\cdot,\circ)1_{]0,t[}(\cdot)$, which is predictability. Together with Property (iii), we conclude that $X(\cdot,\circ)1_{]0,t[}(\cdot) \in \mathscr{P}_2$.

Now we consider a general X. For $M > 0$, denote

$$X^M(s,y,\omega)1_{]0,t[}(s) = \begin{cases} X(s,y,\omega)1_{]0,t[}(s) & \text{if } \left|X(s,y,\omega)\right| \le M, \\ 0 & \text{otherwise.} \end{cases}$$

Since each $X^M(\cdot,\circ)1_{]0,t[}(\cdot)$ is predictable by the previous case, and

$$X^M(\cdot,\circ)1_{]0,t[}(\cdot) \to X(\cdot,\circ)1_{]0,t[}(\cdot), \quad \text{as } M \to +\infty, \text{ in } ||\cdot||_{M,2}$$

by Lebesgue's dominated convergence theorem, we have that $X(\cdot,\circ)1_{]0,t[}(\cdot)$ is also predictable. Therefore, together with Property (iii), we conclude that $X(\cdot,\circ)1_{]0,t[}(\cdot) \in \mathscr{P}_2$. This completes the whole proof. □

Remark 2.3.17. Proposition 2.3.14 is of Riemann's type, while Proposition 2.3.16 is of Lebesgue's type. The latter essentially generalizes the result of the Brownian motion case [15, Chapter 3].

2.3.6 A Lemma on Stochastic Convolutions

We first recall the following form of Burkholder's inequality, which is adapted from [19, Theorem 1.4].

Theorem 2.3.18 (The Burkholder-Davis-Gundy inequality)**.** *For every $k \in [1,+\infty[$, there is a constant z_k such that, for all continuous (local) martingale $\{M_t\}_{t\ge0}$ vanishing at zero,*

$$||M_t||_k \le z_k ||\langle M\rangle_t||^{1/2}_{k/2}\,,$$

where $\langle M\rangle$ denotes the quadratic variation of M. Moreover, the constant z_k can be chosen such that

$$z_2 = 1\,, \quad z_k \le 2\sqrt{k}, \quad \textit{for all } k \in [2,+\infty[.$$

Remark 2.3.19. The first part of the above theorem can be found in [60, Theorem 4.1, p. 160], which is proved easily by an application of Itô's lemma. The drawback of that proof is that we cannot get the best constants z_k. To get the best constants z_k, we refer to the Davis result [34, Theorem 1.1], which states that if X_t is a standard Brownian motion and T is a stopping time for X_t, then

$$\mathbb{E}\left[|X_T|^k\right] \le z_k^k \mathbb{E}\left[T^{k/2}\right], \quad \forall k \ge 2$$

where the best value z_k for $k \geq 2$ is the largest positive zero of the *parabolic cylinder function* $D_k(x)$ of parameter k (see [51, 12.2.4, p. 304] for a definition of this special function). Then the Burkholder-Davis-Gundy inequality of the above form can be readily obtained by applying a change of time for continuous local martingale (see, e.g., [15, Theorem 9.3, p. 188]) . As for the constants z_k, when $k \in \mathbb{N}$, zeros of $D_k(x)$ are identical to zeros of the modified Hermite polynomials $\mathrm{He}_n(x)$ due to [51, 12.7.1, p. 308]. Carlen and Krée [9, Appendix] proved that the largest positive zero z_k of $D_k(x)$ is bounded by $2\sqrt{k}$ for all $k \geq 2$.

We need a lemma, which is an extension of Lemma 2.4 of [19]. The arguments of this lemma also appear in [37, Lemma 3.4]. Suppose that for some $t > 0$, a process $Z = \left(Z\left(s, y\right) : \left(s, y\right) \in]0, t[\times\mathbb{R}\right)$ has the following properties:

(1) Z is adapted, i.e., for all $\left(s, y\right) \in]0, t[\times\mathbb{R}$, $Z\left(s, y\right)$ is $\mathscr{F}_s$ measurable;

(2) Z is jointly measurable with respect to $\mathscr{B}(\mathbb{R}^2) \times \mathscr{F}$;

(3) $\mathbb{E}\left[\iint_{[0,t]\times\mathbb{R}} G_\nu^2\left(t-s, x-y\right) |Z\left(s, y\right)|^2 \mathrm{d}s\mathrm{d}y\right] < \infty$, for all $x \in \mathbb{R}$.

Thanks to Proposition 2.3.16, for fixed $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$, the random field $\left(s, y\right) \in [0, t] \times \mathbb{R} \mapsto G_\nu\left(t-s, x-y\right) Z\left(s, y\right)$ belongs to $\mathscr{P}_2$. Hence the following stochastic convolution

$$\left(G_\nu \star Z\dot{W}\right)(t, x) := \iint_{[0,t]\times\mathbb{R}} G_\nu\left(t-s, x-y\right) Z\left(s, y\right) W\left(\mathrm{d}s, \mathrm{d}y\right), \tag{2.3.26}$$

is a well-defined Walsh integral.

Lemma 2.3.20. *Let Z be the random field that satisfies the above three properties. Then the stochastic convolution in* (2.3.26) *has the following moment estimates: For all even integers $p \geq 2$, and all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$, we have*

$$\left|\left|\left(G_\nu \star Z\dot{W}\right)(t, x)\right|\right|_p^2 \leq z_p^2 \iint_{[0,t]\times\mathbb{R}} G_\nu^2\left(t-s, x-y\right) \left|\left|Z\left(s, y\right)\right|\right|_p^2 \mathrm{d}s\mathrm{d}y$$

where z_p is the constant defined in Theorem 2.3.18.

See [19, Lemma 2.4] for the proof. We remark that in [19], Conus and Khoshnevisan proved this lemma under the assumption that Z is a predictable random field. We make only a small contribution here to allow all adapted, jointly measurable and integrable (Property (3) above) random fields.

2.4 Proof of the Existence Theorem (Theorem 2.2.2)

In this part, we prove the main Theorem 2.2.2 except the Hölder continuity part. Recall the definitions of $\mathscr{K}(t, x)$, $\overline{\mathscr{K}}(t, x)$, $\underline{\mathscr{K}}(t, x)$ and $\widehat{\mathscr{K}}_p(t, x)$ in (2.2.6) – (2.2.9). Note that $\widehat{\mathscr{K}}_p(t, x)$ depends on parameters p and $\overline{\varsigma}$ implicitly.

Similarly we apply the same conventions to the kernels $\mathscr{L}_n(t, x; \nu, \lambda)$, $n = 0, 1, \ldots$. For example,

$$
\begin{aligned}
\mathscr{L}_0(t,x) &:= \mathscr{L}_0(t,x;\nu,\lambda) = \lambda^2 G_\nu^2(t,x) = \frac{\lambda^2}{\sqrt{4\pi\nu t}} G_{\nu/2}(t,x)\,,\\
\underline{\mathscr{L}}_0(t,x) &:= \mathscr{L}_0\left(t,x;\nu,l_\rho\right)\,,\\
\overline{\mathscr{L}}_0(t,x) &:= \mathscr{L}_0\left(t,x;\nu,L_\rho\right)\,,\\
\widehat{\mathscr{L}}_0(t,x) &:= \mathscr{L}_0\left(t,x;\nu,a_{p,\overline{\varsigma}} z_p L_\rho\right)\,,\quad p\ge 2\,.
\end{aligned}
\tag{2.4.1}
$$

Note that, if $p=2$, then $\widehat{\mathscr{L}}_p(t,x) = \overline{\mathscr{L}}_p(t,x)$ and $\widehat{\mathscr{K}}_p(t,x) = \overline{\mathscr{K}}(t,x)$.

As a direct consequence of Proposition 2.3.1 and Lemma 2.3.6, we have that for all $n\in\mathbb{N}$, the condition (1.1.5) holds if and only if

$$
\left(J_0^2 \star \mathscr{L}_n\right)(t,x) \le \left(J_0^2 \star \mathscr{K}\right)(t,x) < +\infty, \quad \text{for all } (t,x)\in\mathbb{R}_+^*\times\mathbb{R}\,. \tag{2.4.2}
$$

Remark 2.4.1 (Existence v.s. moments)**.** According to the definition of random field solution (Definition 2.2.1), the existence of such a solution requires some estimates on its moments. On the other hand, if we assume existence, then one can readily obtain moment formulas. For example, for the Anderson model, if we denote by $f(t,x)$ the second moment, then $f(t,x)$ satisfies the integral equation: $f(t,x) = J_0^2(t,x) + \left(f\star\mathscr{L}_0\right)(t,x)$. Apply this relation recursively: $f(t,x) = J_0^2(t,x) + \sum_{i=0}^{n-1}\left(J_0^2\star\mathscr{L}_i\right)(t,x) + \left(f\star\mathscr{L}_n\right)(t,x)$. Then by a ratio test as in (2.3.6), one can show that $\left(f\star\mathscr{L}_n\right)(t,x)$ converges 0 as $n\to+\infty$. By (2.3.2), the sum converges to $\left(J_0^2\star\mathscr{K}\right)(t,x)$. Thus, we obtain the moment formula: $f(t,x) = J_0^2(t,x) + \left(J_0^2\star\mathscr{K}\right)(t,x)$. In fact, the existence and moment estimates are proved together in the Picard iteration scheme in Section 2.4.2.

In the following, the proof of the existence and moment estimates is in Section 2.4.2. The proof is based on the Picard iteration. Instead of taking a supremum over the space variable and then applying Gronwall's lemma, which is the standard method, we do an explicit calculation of the series. The arguments of the induction in the Picard iterations are summarized in Proposition 2.4.2 in Section 2.4.1. The estimates of two-point correlation functions and some special cases (the proofs of Corollaries 2.2.3 and 2.2.6) are listed in Sections 2.4.3 and 2.4.4. The Hölder continuity is proved later in a separate section – Section 2.6.

2.4.1 A Proposition for the Picard Iteration

When there are dummy variables in convolution, we use "$\cdot$" and "$\circ$" to denote the time and space variables respectively.

Proposition 2.4.2. *Suppose that for some even integer $p\ge 2$, a random field*

$$
Y = \left(Y(t,x):\ (t,x)\in\mathbb{R}_+^*\times\mathbb{R}\right)
$$

has the following three properties (i), (ii) and (iii):

(i) Y *is adapted, i.e., for all* $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}$, $Y(t,x)$ *is* $\mathscr{F}_t$*-measurable;*

(ii) Y *is jointly measurable with respect to* $\mathscr{B}\left(\mathbb{R}_+^* \times \mathbb{R}\right) \times \mathscr{F}$*;*

(iii) for all $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}$,

$$||Y(\cdot,\circ)G_\nu(t-\cdot,x-\circ)||_{M,p}^2 = \int_0^t \mathrm{d}s \int_{\mathbb{R}} \left|G_\nu\left(t-s,x-y\right)\right|^2 \left|\left|Y\left(s,y\right)\right|\right|_p^2 \mathrm{d}y < +\infty\,.$$

Then for all $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}$, $Y(\cdot,\circ)G_\nu(t-\cdot,x-\circ) \in \mathscr{P}_p$ *and*

$$w(t,x) = \iint_{]0,t[\times\mathbb{R}} Y\left(s,y\right)G_\nu\left(t-s,x-y\right)W(\mathrm{d}s,\mathrm{d}y), \quad \textit{for all } (t,x) \in \mathbb{R}_+^* \times \mathbb{R}$$

is well defined as a Walsh integral and the resulting random field w *is adapted to* $\{\mathscr{F}_s\}_{s\geq 0}$. *Moreover, the random field* $w = \{w(t,x) : (t,x) \in \mathbb{R}_+^* \times \mathbb{R}\}$ *has the following properties:*

(a) If Y *has locally bounded* p*-th moments, that is, for* $K \subseteq \mathbb{R}_+^* \times \mathbb{R}$ *compact,*

$$\sup_{(t,x)\in K} ||Y(t,x)||_p < +\infty\,, \tag{2.4.3}$$

which is the case in particular if Y *is* $L^p(\Omega)$*-continuous, then* w *is* $L^p(\Omega)$*-continuous on* $\mathbb{R}_+^* \times \mathbb{R}$*;*

(b) If (iii) holds for all even integers $p \geq 2$ *and* Y *is globally* $L^p(\Omega)$*-bounded in the sense that*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} ||Y(t,x)||_p < +\infty, \quad \textit{for all } T \geq 0\,,$$

then the above random field $w(t,x)$ *is also bounded in* $L^p(\Omega)$*, and*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} ||w(t,x)||_p \leq z_p \left(\frac{T}{\pi\nu}\right)^{1/4} \sup_{(t,x)\in[0,T]\times\mathbb{R}} ||Y(t,x)||_p < +\infty, \quad \textit{for all } T \geq 0$$

where z_p *is the universal constant in Burkholder's inequality (see Theorem 2.3.18). Moreover, it is a.s. Hölder continuous:* $w \in C_{1/4-,1/2-}(\mathbb{R}_+ \times \mathbb{R})$ *a.s..*

Proof of Proposition 2.4.2 (a). Fix $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}$. Since $G_\nu(t,x)$ is Borel measurable, deterministic and continuous, the random field

$$X = \left(X\left(s,y\right) : \left(s,y\right) \in \,]0,t[\times\mathbb{R}\right) \quad \text{with } X\left(s,y\right) := Y\left(s,y\right)G_\nu\left(t-s,x-y\right)$$

satisfies all conditions of Proposition 2.3.16. This implies that for all $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}$, $Y(\cdot,\circ)G_\nu(t-\cdot,x-\circ) \in \mathscr{P}_p$. Hence $w(t,x)$ is a well-defined Walsh integral and the resulting random field is adapted to the filtration $\{\mathscr{F}_s\}_{s\geq 0}$.

Now we shall prove the $L^p(\Omega)$-continuity. Fix $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}$. Let $B_{t,x}$ and a denote the set and the constant defined in Proposition 2.3.12, respectively. We assume that

$(t', x') \in B_{t,x}$. Denote

$$(t_*, x_*) = \begin{cases} (t', x') & \text{if } t' \le t, \\ (t, x) & \text{if } t' > t, \end{cases} \qquad \text{and} \qquad (\hat{t}, \hat{x}) = \begin{cases} (t, x) & \text{if } t' \le t. \\ (t', x') & \text{if } t' > t. \end{cases}$$

Set $K_a = [1/a, t+1] \times [-a, a]$. Let

$$A_a := \sup_{(s,y)\in K_a} ||Y(s,y)||_p^2 ,$$

which is finite by (2.4.3).

By Lemma 2.3.20, we have

$$\begin{aligned}
&||w(t,x) - w(t',x')||_p^p \\
&\le 2^{p-1}\mathbb{E}\left(\left|\int_0^{t_*}\int_{\mathbb{R}} Y(s,y)\left(G_\nu(t-s,x-y) - G_\nu(t'-s,x'-y)\right) W(\mathrm{d}s,\mathrm{d}y)\right|^p\right) \\
&\quad + 2^{p-1}\mathbb{E}\left(\left|\int_{t_*}^{\hat{t}}\int_{\mathbb{R}} Y(s,y)\, G_\nu(\hat{t}-s,\hat{x}-y)\, W(\mathrm{d}s,\mathrm{d}y)\right|^p\right) \\
&\le 2^{p-1} z_p^p \left(\int_0^{t_*}\int_{\mathbb{R}} ||Y(s,y)||_p^2 \left(G_\nu(t-s,x-y) - G_\nu(t'-s,x'-y)\right)^2 \mathrm{d}s\mathrm{d}y\right)^{p/2} \\
&\quad + 2^{p-1} z_p^p \left(\int_{t_*}^{\hat{t}}\int_{\mathbb{R}} ||Y(s,y)||_p^2 G_\nu^2(\hat{t}-s,\hat{x}-y)\,\mathrm{d}s\mathrm{d}y\right)^{p/2} \\
&\le 2^{p-1} z_p^p \left(L_1(t,t',x,x')\right)^{p/2} + 2^{p-1} z_p^p \left(L_2(t,t',x,x')\right)^{p/2} .
\end{aligned}$$

We first consider L_1. Decompose L_1 into two parts:

$$L_1(t,t',x,x') = \iint_{([0,t_*]\times\mathbb{R})\setminus K_a} \cdots \,\mathrm{d}s\mathrm{d}y + \iint_{([0,t_*]\times\mathbb{R})\cap K_a} \cdots \,\mathrm{d}s\mathrm{d}y = L_{1,1}(t,t',x,x') + L_{1,2}(t,t',x,x') .$$

One can apply Lebesgue's dominated convergence theorem to show that

$$\begin{aligned}
\lim_{(t',x')\to(t,x)} L_{1,1}(t,t',x,x') = \lim_{(t',x')\to(t,x)} \iint_{([0,t_*]\times\mathbb{R})\setminus K_a} &||Y(s,y)||_p^2 \\
&\times \left(G_\nu(t-s,x-y) - G_\nu(t'-s,x'-y)\right)^2 \mathrm{d}s\mathrm{d}y = 0 .
\end{aligned}$$

Indeed, Proposition 2.3.12 says that tails can be uniformly bounded:

$$\sup_{(t',x')\in B_{t,x}} \left(G_\nu(t-s,x-y) - G_\nu(t'-s,x'-y)\right)^2 \le 4G_\nu^2(t+1-s,x-y) , \tag{2.4.4}$$

for all $s \in [0, t']$ and $|y| \ge a$. Moreover,

$$\begin{aligned}
&\iint_{([0,t_*]\times\mathbb{R})\setminus K_a} ||Y(s,y)||_p^2 \; G_\nu^2(t+1-s,x-y)\mathrm{d}s\mathrm{d}y \\
&\qquad \le \iint_{[0,t+1]\times\mathbb{R}} ||Y(s,y)||_p^2 \; G_\nu^2(t+1-s,x-y)\mathrm{d}s\mathrm{d}y < +\infty .
\end{aligned}$$

As for $L_{1,2}$, we have that

$$
\begin{aligned}
L_{1,2}\left(t,t',x,x'\right) &\le A_a \iint_{([0,t_*]\times\mathbb{R})\cap K_a} \left(G_\nu\left(t-s,x-y\right)-G_\nu(t'-s,x'-y)\right)^2 \mathrm{d}s\mathrm{d}y \\
&\le A_a \iint_{[0,\hat{t}]\times\mathbb{R}} \left(G_\nu\left(t-s,x-y\right)-G_\nu(t'-s,x'-y)\right)^2 \mathrm{d}s\mathrm{d}y \\
&\le A_a C\left(\left|x-x'\right|+\sqrt{|t-t'|}\right) \to 0\,, \text{ as } \left(t',x'\right)\to(t,x)\,,
\end{aligned}
$$

where we have applied Corollary 2.3.10 with some constant $C>0$ depending only on ν. Hence, we have proved

$$
\lim_{(t',x')\to(t,x)} L_1\left(t',t,x,x'\right) = 0\,.
$$

Now let us consider L_2. Decompose it into two parts:

$$
L_2\left(t,t',x,x'\right) = \iint_{([t_*,\hat{t}]\times\mathbb{R})\setminus K_a} \cdots \,\mathrm{d}s\mathrm{d}y + \iint_{([t_*,\hat{t}]\times\mathbb{R})\cap K_a} \cdots \,\mathrm{d}s\mathrm{d}y = L_{2,1}\left(t,t',x,x'\right)+L_{2,2}\left(t,t',x,x'\right).
$$

The proof that $\lim_{(t',x')\to(t,x)} L_{2,1}\left(t,t',x,x'\right)=0$ is the same as for $L_{1,1}$, except that (2.4.4) must be replaced by

$$
\sup_{(t',x')\in B_{t,x}} G_\nu^2\left(\hat{t}-s,\hat{x}-y\right) \le G_\nu^2(t+1-s,x-y)\,.
$$

The proof for $L_{2,2}$ is similar to $L_{1,2}$:

$$
L_{2,2}\left(t,t',x,x'\right) \le A_a \int_{t_*}^{\hat{t}} \mathrm{d}s \int_{\mathbb{R}} G_\nu^2\left(\hat{t}-s,\hat{x}-y\right)\mathrm{d}y \le A_a C\sqrt{|t'-t|} \to 0\,, \text{ as } (t',x')\to(t,x)\,,
$$

where we have applied Corollary 2.3.10 with some constant C depending only on ν. Hence, we have proved

$$
\lim_{(t',x')\to(t,x)} L_2\left(t',t,x,x'\right) = 0\,.
$$

This completes the proof of (a). □

Proof of Proposition 2.4.2 (b). The $L^p(\Omega)$-boundedness is a direct consequence of Lemma 2.3.20: For $0\le t\le T$, we have that

$$
\begin{aligned}
||w(t,x)||_p^2 &\le z_p^2 \iint_{[0,t]\times\mathbb{R}} G_\nu^2\left(t-s,x-y\right)\left|\left|Y\left(s,y\right)\right|\right|_p^2 \mathrm{d}s\mathrm{d}y \\
&\le z_p^2 \sup_{(s,y)\in[0,T]\times\mathbb{R}} \left|\left|Y\left(s,y\right)\right|\right|_p^2 \iint_{[0,t]\times\mathbb{R}} G_\nu^2\left(t-s,x-y\right)\mathrm{d}s\mathrm{d}y \\
&\le \frac{z_p^2\sqrt{T}}{\sqrt{\pi\nu}} \sup_{(s,y)\in[0,T]\times\mathbb{R}} \left|\left|Y\left(s,y\right)\right|\right|_p^2\,,
\end{aligned}
$$

where the right-hand side does not depend on (t,x). Hence $w(t,x)$ is bounded in $L^p(\Omega)$.

Now we shall prove the Hölder continuity. The arguments are similar to the proof of (a). $L_1\left(t,t',x,x'\right)$ is bounded in the following way instead:

$$L_1\left(t,t',x,x'\right) \le A \iint_{[0,t_*]\times\mathbb{R}} \left(G_\nu\left(t-s,x-y\right)-G_\nu(t'-s,x'-y)\right)^2 \mathrm{d}s\mathrm{d}y$$

with $A := \sup_{(s,y)\in[0,\hat{t}]\times\mathbb{R}} \left|\left|Y\left(s,y\right)\right|\right|_p^2$. Then by Corollary 2.3.10, for some constant $C>0$ depending only on ν,

$$L_1\left(t,t',x,x'\right) \le AC\left(\left|x-x'\right|+\sqrt{|t'-t|}\right).$$

Similarly, we have that $L_2\left(t,t',x,x'\right) \le AC\sqrt{|t'-t|}$ with the same constants A and C. Therefore, by subadditivity of $x\mapsto|x|^{2/p}$ with $p\ge2$ and $x\ge0$, we have,

$$\begin{aligned}\left|\left|w(t,x)-w\left(t',x'\right)\right|\right|_p^2 &\le 2^{2(p-1)/p} z_p^2 AC\left[\left|x-x'\right|+2\sqrt{|t-t'|}\right]\\ &\le C_1\left|x-x'\right|+C_2|t-t'|^{1/2},\end{aligned}$$

for all $t,t'\ge0$ and $x,x'\in\mathbb{R}$, where

$$C_1 = 2^{2-2/p} z_p^2 AC, \quad \text{and} \quad C_2 = 2^{3-2/p} z_p^2 AC.$$

Finally, by Kolmogorov's continuity theorem (see, e.g., Proposition 2.6.4 below), we can conclude (b). □

We still need a lemma to transform the stochastic integral equation of the form (2.2.2) into deterministic integral inequalities for its moments. Define a constant

$$b_p = \begin{cases} 1 & \text{if } p=2, \\ 2 & \text{if } p>2. \end{cases} \tag{2.4.5}$$

Lemma 2.4.3. *Let $f(t,x)$ be some deterministic function. Suppose that ρ satisfies the growth condition* (1.4.1). *If the random fields w and v satisfy the following relations*

$$w(t,x) = f(t,x) + \iint_{[0,t]\times\mathbb{R}} G_\nu\left(t-s,x-y\right)\rho\left(v\left(s,y\right)\right)W(\mathrm{d}s,\mathrm{d}y), \quad \textit{for all } t>0 \textit{ and } x\in\mathbb{R},$$

where we assume that the Walsh integral is well defined, then for all even integers $p\ge2$, we have

$$\begin{aligned}\left|\left|\left(G_\nu \star \rho(v)\dot{W}\right)(t,x)\right|\right|_p^2 &\le z_p^2 \left|\left|G_\nu(t-\cdot,x-\circ)\rho(v(\cdot,\circ))\right|\right|_{M,p}^2\\ &\le \frac{1}{b_p}\left(\left(\overline{\varsigma}^2+||v||_p^2\right)\star\widehat{\mathscr{L}_0}\right)(t,x),\end{aligned}$$

where $\widehat{\mathscr{L}_0}(t,x)$ is defined in (2.4.1) *and the constant $a_{p,\overline{\varsigma}}$ is defined in* (1.4.4). *In particular, for all $(t,x)\in\mathbb{R}_+\times\mathbb{R}$,*

$$||w(t,x)||_p^2 \le b_p\, f^2(t,x) + \left(\left(\overline{\varsigma}^2+||v||_p^2\right)\star\widehat{\mathscr{L}_0}\right)(t,x),$$

and, assuming (1.4.2),

$$||w(t,x)||_2^2 \geq f^2(t,x) + \left(\left(\underline{\varsigma}^2 + ||v||_p^2\right) \star \underline{\mathscr{L}}_0\right)(t,x). \tag{2.4.6}$$

Proof. We first consider the case $p = 2$. By the Itô isometry and then the linear growth condition (1.4.1),

$$\begin{aligned}
||w(t,x)||_2^2 &= f^2(t,x) + \iint_{[0,t]\times\mathbb{R}} G_\nu^2(t-s,x-y)\,\mathbb{E}\left(\rho(v(s,y))^2\right) \mathrm{d}s\mathrm{d}y \\
&\leq f^2(t,x) + \iint_{[0,t]\times\mathbb{R}} L_\rho^2\, G_\nu^2(t-s,x-y)\left(\overline{\varsigma}^2 + \mathbb{E}\left[v^2(s,y)\right]\right) \mathrm{d}s\mathrm{d}y \\
&= b_2\, f^2(t,x) + \left(\left(\overline{\varsigma}^2 + ||v||_2^2\right) \star \widehat{\mathscr{L}}_0\right)(t,x)\,,
\end{aligned}$$

where we have used the facts that $a_{2,\overline{\varsigma}} = 1$ and $z_2 = 1$. The lower bound (2.4.6) is obtained similarly.

Now we consider the case $p > 2$. By the triangle inequality, we have

$$||w(t,x)||_p \leq |f(t,x)| + \left|\left|\left(G_\nu \star \rho(v)\dot{W}\right)(t,x)\right|\right|_p\,,$$

and hence

$$||w(t,x)||_p^2 \leq 2|f(t,x)|^2 + 2\left|\left|\left(G_\nu \star \rho(v)\dot{W}\right)(t,x)\right|\right|_p^2\,.$$

By Lemma 2.3.20,

$$\left|\left|\left(G_\nu \star \rho(v)\dot{W}\right)(t,x)\right|\right|_p^2 \leq z_p^2 \iint_{[0,t]\times\mathbb{R}} G_\nu^2(t-s,x-y)\left|\left|\rho(v(s,y))\right|\right|_p^2 \mathrm{d}s\mathrm{d}y\,.$$

If $\overline{\varsigma} = 0$, clearly $\left|\left|\rho(v(s,y))\right|\right|_p^2 \leq L_\rho^2 \left|\left|v(s,y)\right|\right|_p^2$. Otherwise, if $\overline{\varsigma} \neq 0$, by the linear growth condition (1.4.1), we know that

$$\mathbb{E}\left[|\rho(v(s,y))|^p\right] \leq L_\rho^p\, \mathbb{E}\left[\left(\overline{\varsigma}^2 + v(s,y)^2\right)^{p/2}\right] \leq L_\rho^p\, 2^{(p-2)/2}\left(\overline{\varsigma}^p + \mathbb{E}[|v(s,y)|^p]\right)\,.$$

By the sub-additivity of the function $|x|^{2/p}$ for $p \geq 2$ (that is, $(a+b)^{2/p} \leq a^{2/p} + b^{2/p}$ for all $a, b \geq 0$ and all $p \geq 2$), we have that

$$\left|\left|\rho(v(s,y))\right|\right|_p^2 \leq L_\rho^2\, 2^{(p-2)/p}\left(\overline{\varsigma}^2 + \left|\left|v(s,y)\right|\right|_p^2\right)\,, \quad \overline{\varsigma} \neq 0\,.$$

Combining these two cases, we have therefore proved that

$$\begin{aligned}
b_p \left|\left|\left(G_\nu \star \rho(v)\dot{W}\right)(t,x)\right|\right|_p^2 &\leq z_p^2\, L_\rho^2\, a_{p,\overline{\varsigma}}^2 \iint_{[0,t]\times\mathbb{R}} G_\nu^2(t-s,x-y)\left(\overline{\varsigma}^2 + \left|\left|v(s,y)\right|\right|_p^2\right) \mathrm{d}s\mathrm{d}y \\
&= \left(\left[\overline{\varsigma}^2 + ||v(\cdot,\circ)||_p^2\right] \star \widehat{\mathscr{L}}_0\right)(t,x)\,,
\end{aligned}$$

where we have used the facts that $a_{p,0}^2 = b_p$, and $a_{p,\overline{\varsigma}}^2 = 2^{\frac{p-2}{p}+1} = 2^{2(p-1)/p}$ for $\overline{\varsigma} \neq 0$ and $p > 2$. This completes the proof. □

Remark 2.4.4. If we work under the growth condition $|\rho(u)| \le L_\rho\left(\overline{\varsigma}+|u|\right)$ instead of (1.4.1), then from $\left\|\rho(v)\right\|_p^2 \le L_\rho^2\left(\overline{\varsigma}+\|v\|_p\right)^2 \le 2\,L_\rho^2\left(\overline{\varsigma}^2+\|v\|_p^2\right)$, we can get the same bound with the constant $a_{p,\overline{\varsigma}}$ replaced by $\sqrt{2}$.

2.4.2 Proof of Existence, Uniqueness and Moment Estimates

The proof is based on the standard Picard iteration. Throughout the proof, fix an arbitrary even integer $p \ge 2$.

Step 1. Define $u_0(t,x) = J_0(t,x)$. By Lemma 2.3.5, $u_0(t,x)$ is a well-defined and continuous function over $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}$. We shall now apply Proposition 2.4.2 with $Y = u_0$. We check the three properties that it requires. Properties (i) and (ii) are trivially satisfied since Y is deterministic and continuous over $\mathbb{R}_+^* \times \mathbb{R}$. Property (iii) is also true since, by Lemma 2.4.3,

$$b_p\, z_p^2 \left\|\rho\left(u_0(\cdot,\circ)\right) G_\nu(t-\cdot, x-\circ)\right\|_{M,p}^2 \le \left(\left[\overline{\varsigma}^2+J_0^2\right] \star \widehat{\mathscr{L}_0}\right)(t,x)\,, \qquad (2.4.7)$$

which is finite due to (2.3.8) and Lemma 2.3.6. Hence,

$$\rho\left(u_0(\cdot,\circ)\right) G_\nu(t-\cdot,x-\circ) \in \mathscr{P}_p\,, \quad \text{for all } (t,x) \in \mathbb{R}_+^* \times \mathbb{R}\,,$$

and for all $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}$,

$$I_1(t,x) = \iint_{[0,t]\times\mathbb{R}} \rho\left(u_0\left(s,y\right)\right) G_\nu\left(t-s,x-y\right) W\left(\mathrm{d}s,\mathrm{d}y\right)$$

is a well-defined Walsh integral. The random field I_1 is adapted. Clearly, the continuity of the deterministic function $\left(s,y\right) \mapsto \rho(u_0\left(s,y\right))$ implies its local $L^p(\Omega)$-boundedness (in the sense of Proposition 2.4.2 (a)). So $(t,x) \mapsto I_1(t,x)$ is also continuous in $L^p(\Omega)$. Define

$$u_1(t,x) := J_0(t,x) + I_1(t,x)\,.$$

The above $L^p(\Omega)$-continuity of $I_1(t,x)$ implies the $L^p(\Omega)$-continuity of $u_1(t,x)$ since $J_0(t,x)$ is continuous from $\mathbb{R}_+^* \times \mathbb{R}$ to $\mathbb{R}$.

Now we estimate its moments. We pay special attention to the second moment: The isometry property gives that

$$\|I_1(t,x)\|_2^2 = \left\|\rho\left(u_0(\cdot,\circ)\right) G_\nu(t-\cdot,x-\circ)\right\|_{M,2}^2$$

which equals $\left(\left[\varsigma^2+J_0^2\right] \star \mathscr{L}_0\right)(t,x)$ for the quasi-linear case (1.4.3), and is bounded from above (see (2.4.7) with $b_2 z_2^2 = 1$) and below (if ρ additionally satisfies (1.4.2)), in which case

$$\left(\left[\underline{\varsigma}^2+J_0^2\right] \star \underline{\mathscr{L}}_0\right)(t,x) \le \|I_1(t,x)\|_2^2 \le \left(\left[\overline{\varsigma}^2+J_0^2\right] \star \overline{\mathscr{L}}_0\right)(t,x)\,.$$

Since $J_0(t,x)$ is deterministic, $||u_1(t,x)||_2^2 = J_0^2(t,x) + ||I_1(t,x)||_2^2$, and by Lemma 2.4.3,

$$\begin{aligned}||u_1(t,x)||_p^2 &\le b_p J_0^2(t,x) + \left(\left(\overline{\varsigma}^2 + J_0^2\right) \star \widehat{\mathscr{L}_0}\right)(t,x) \\ &\le b_p J_0^2(t,x) + \left(\left(\overline{\varsigma}^2 + b_p\, J_0^2\right) \star \widehat{\mathscr{L}_0}\right)(t,x)\,.\end{aligned}$$

In summary, in this step we have proved that $\left\{u_1(t,x) : (t,x) \in \mathbb{R}_+^* \times \mathbb{R}\right\}$ is a well-defined random field such that

(1) it is adapted to the filtration $\{\mathscr{F}_t\}_{t>0}$;

(2) the function $(t,x) \mapsto u_1(t,x)$ from $\mathbb{R}_+^* \times \mathbb{R}$ into $L^p(\Omega)$ is continuous;

(3) $\mathbb{E}\left[u_1^2(t,x)\right] = J_0^2(t,x) + \left(\left[\varsigma^2 + J_0^2\right] \star \mathscr{L}_0\right)(t,x)$ for the quasi-linear case (1.4.3) and it is bounded from above and below (if ρ additionally satisfies (1.4.2)):

$$J_0^2(t,x) + \left(\left[\underline{\varsigma}^2 + J_0^2\left(s,y\right)\right] \star \underline{\mathscr{L}}_0\right)(t,x) \le \mathbb{E}\left[u_1^2(t,x)\right] \le J_0^2(t,x) + \left(\left[\overline{\varsigma}^2 + J_0^2\left(s,y\right)\right] \star \overline{\mathscr{L}}_0\right)(t,x)\,;$$

(4) $||u_1(t,x)||_p^2 \le b_p\, J_0^2(t,x) + \left(\left(\overline{\varsigma}^2 + b_p\, J_0^2\right) \star \widehat{\mathscr{L}_0}\right)(t,x)$.

Step 2. Assume by induction that for all $1 \le k \le n$ and all $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}$, the Walsh integral

$$I_k(t,x) = \iint_{[0,t]\times\mathbb{R}} \rho\left(u_{k-1}\left(s,y\right)\right) G_\nu\left(t-s, x-y\right) W\left(\mathrm{d}s, \mathrm{d}y\right)$$

is well defined such that

(1) u_k is adapted to the filtration $\{\mathscr{F}_t\}_{t>0}$, where

$$\left\{u_k(t,x) := J_0(t,x) + I_k(t,x) : (t,x) \in \mathbb{R}_+^* \times \mathbb{R}\right\}\,;$$

(2) the function $(t,x) \mapsto u_k(t,x)$ from $\mathbb{R}_+^* \times \mathbb{R}$ into $L^p(\Omega)$ is continuous;

(3) $\mathbb{E}\left[u_k^2(t,x)\right] = J_0^2(t,x) + \sum_{i=0}^{k-1}\left(\left[\varsigma^2 + J_0^2\right] \star \mathscr{L}_i\right)(t,x)$ for the quasi-linear case (1.4.3) and it is bounded from above and below (if ρ additionally satisfies (1.4.2)) by

$$J_0^2(t,x) + \sum_{i=0}^{k-1}\left(\left[\underline{\varsigma}^2 + J_0^2\right] \star \underline{\mathscr{L}}_i\right)(t,x) \le \mathbb{E}\left[u_k^2(t,x)\right] \le J_0^2(t,x) + \sum_{i=0}^{k-1}\left(\left[\overline{\varsigma}^2 + J_0^2\right] \star \overline{\mathscr{L}}_i\right)(t,x)\,.$$

(4) $||u_k(t,x)||_p^2 \le b_p\, J_0^2(t,x) + \sum_{i=0}^{k-1}\left(\left(\overline{\varsigma}^2 + b_p\, J_0^2\right) \star \widehat{\mathscr{L}_i}\right)(t,x)$.

Now let us consider the case $k = n+1$. We shall apply Proposition 2.4.2 again with $Y\left(s,y\right) = \rho\left(u_n\left(s,y\right)\right)$, by verifying the three properties that it requires. Properties (i) and (ii) are clearly satisfied by the induction assumptions (1) and (2). By Lemma 2.4.3 and the induction assumptions, we can show Property (iii) is also true:

$$b_p\, z_p^2 \left|\left|\rho\left(u_n(\cdot,\circ)\right) G_\nu(t-\cdot, x-\circ)\right|\right|_{M,p}^2 \le \left(\left[\overline{\varsigma}^2 + ||u_n||_p^2\right] \star \widehat{\mathscr{L}_0}\right)(t,x)$$

$$\leq \left(\left[\overline{\varsigma}^2 + b_p J_0^2\right] \star \widehat{\mathscr{L}_0}\right)(t,x) + \sum_{i=0}^{n-1} \left(\left[\overline{\varsigma}^2 + b_p J_0^2\right] \star \widehat{\mathscr{L}_i} \star \widehat{\mathscr{L}_0}\right)(t,x)$$

$$= \sum_{i=0}^{n} \left(\left[\overline{\varsigma}^2 + b_p J_0^2\right] \star \widehat{\mathscr{L}_i}\right)(t,x)\,, \tag{2.4.8}$$

where we have used the definition of $\widehat{\mathscr{L}_n}(t,x)$. Then by (2.3.2),

$$b_p\, z_p^2 \left|\left|\rho\left(u_n(\cdot,\circ)\right) G_\nu(t-\cdot, x-\circ)\right|\right|_{M,p}^2 \leq \left(\left[\overline{\varsigma}^2 + b_p J_0^2\right] \star \widehat{\mathscr{K}_p}\right)(t,x) < +\infty\,.$$

Hence,

$$\rho\left(u_n(\cdot,\circ)\right) G_\nu(t-\cdot, x-\circ) \in \mathscr{P}_p\,, \quad \text{for all } (t,x) \in \mathbb{R}_+^* \times \mathbb{R}\,,$$

and

$$I_{n+1}(t,x) = \iint_{[0,t]\times\mathbb{R}} \rho\left(u_n\left(s,y\right)\right) G_\nu\left(t-s, x-y\right) W\left(\mathrm{d}s, \mathrm{d}y\right)$$

is a well-defined Walsh integral. The random field I_{n+1} is adapted. Clearly, the $L^p(\Omega)$-continuity of the random field $\left(s,y\right) \mapsto \rho(u_n\left(s,y\right))$ (a direct consequence of the induction assumption (2)) implies its local $L^p(\Omega)$-boundedness (in the sense of Proposition 2.4.2 (a)). So $(t,x) \mapsto I_{n+1}(t,x)$ is also continuous in $L^p(\Omega)$. Define

$$u_{n+1}(t,x) := J_0(t,x) + I_{n+1}(t,x)\,.$$

Now we estimate the moments of $u_{n+1}(t,x)$. By Lemma 2.4.3 (see the bound in (2.4.8)), the p-th moment is bounded by

$$||u_{n+1}(t,x)||_p^2 \leq b_p J_0^2(t,x) + \sum_{i=0}^{n} \left(\left(\overline{\varsigma}^2 + b_p J_0^2\right) \star \widehat{\mathscr{L}_i}\right)(t,x)\,.$$

As for the second moment, the isometry property gives that

$$\mathbb{E}[I_{n+1}^2(t,x)] = \left|\left|\rho\left(u_n(\cdot,\circ)\right) G_\nu(t-\cdot, x-\circ)\right|\right|_{M,2}^2\,,$$

which equals $\sum_{i=0}^n \left(\left[\varsigma^2 + J_0^2\right] \star \mathscr{L}_n\right)(t,x)$ for the linear case, and is bounded from above (see (2.4.8) with $b_2 z_2^2 = 1$) and below (if ρ additionally satisfies (1.4.2)), in which case

$$\sum_{i=0}^{n} \left(\left[\underline{\varsigma}^2 + J_0^2\right] \star \underline{\mathscr{L}}_i\right)(t,x) \leq \mathbb{E}[I_{n+1}^2(t,x)] \leq \sum_{i=0}^{n} \left(\left[\overline{\varsigma}^2 + J_0^2\right] \star \overline{\mathscr{L}}_i\right)(t,x)\,.$$

The second moment of $u_{n+1}(t,x)$ is obtained since $J_0(t,x)$ is deterministic: $||u_{n+1}(t,x)||_2^2 = J_0^2(t,x) + ||I_{n+1}(t,x)||_2^2$.

Therefore, we have proved that the four properties (1) – (4) also hold for $k = n+1$. So we can conclude that for all $n \in \mathbb{N}$,

$$\left\{u_{n+1}(t,x) = J_0(t,x) + I_{n+1}(t,x) :\, (t,x) \in \mathbb{R}_+^* \times \mathbb{R}\right\}$$

is a well-defined random field such that

(1) it is adapted to the filtration $\{\mathscr{F}_t\}_{t>0}$;

(2) the function $(t,x)\mapsto u_{n+1}(t,x)$ from $\mathbb{R}_+^*\times\mathbb{R}$ into $L^p(\Omega)$ is continuous;

(3) $\mathbb{E}\left[u_{n+1}^2(t,x)\right]=J_0^2(t,x)+\sum_{i=0}^n\left(\left[\varsigma^2+J_0^2\right]\star\mathscr{L}_i\right)(t,x)$ for the quasi-linear case and it is bounded from above and below (if ρ satisfies (1.4.2) additionally):

$$J_0^2(t,x)+\sum_{i=0}^n\left(\left[\underline{\varsigma}^2+J_0^2\right]\star\underline{\mathscr{L}}_i\right)(t,x)\le\mathbb{E}\left[u_{n+1}^2(t,x)\right]\le J_0^2(t,x)+\sum_{i=0}^n\left(\left[\overline{\varsigma}^2+J_0^2\right]\star\overline{\mathscr{L}}_i\right)(t,x)\,.$$

(4) $||u_{n+1}(t,x)||_p^2\le b_p J_0^2(t,x)+\sum_{i=0}^n\left(\left(\overline{\varsigma}^2+b_p\,J_0^2\right)\star\widehat{\mathscr{L}}_i\right)(t,x)$ (according to Lemma 2.4.3).

Step 3. We claim that for all $(t,x)\in\mathbb{R}_+^*\times\mathbb{R}$, the series $\{u_n(t,x)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^p(\Omega)$ and we will use $u(t,x)$ to denote its limit. In order to prove this claim, define

$$F_n(t,x):=||u_{n+1}(t,x)-u_n(t,x)||_p^2\,.$$

For $n\ge1$, by Lemma 2.3.20,

$$F_n(t,x)\le z_p^2\iint_{[0,t]\times\mathbb{R}}G_\nu^2\left(t-s,x-y\right)\left|\left|\rho(u_n\left(s,y\right))-\rho(u_{n-1}\left(s,y\right))\right|\right|_p^2\mathrm{d}s\mathrm{d}y\,.$$

Then by the Lipschitz continuity of ρ, we have

$$\begin{aligned}F_n(t,x)&\le z_p^2\,\mathrm{Lip}_\rho^2\iint_{[0,t]\times\mathbb{R}}G_\nu^2\left(t-s,x-y\right)\left|\left|u_n\left(s,y\right)-u_{n-1}\left(s,y\right)\right|\right|_p^2\mathrm{d}s\mathrm{d}y\\&\le\left(F_{n-1}\star\widetilde{\mathscr{L}}_0\right)(t,x)\,,\end{aligned}$$

where

$$\widetilde{\mathscr{L}}_0(t,x):=\mathscr{L}_0\left(t,x;\nu,z_p\,\max\left(\mathrm{Lip}_\rho,a_{p,\overline{\varsigma}}\,L_\rho\right)\right)\,.$$

The functions $\widetilde{\mathscr{L}}_n(t,x)$ and $\widetilde{\mathscr{K}}(t,x)$ are defined by the same parameters as $\widetilde{\mathscr{L}}_0(t,x)$. For the case $n=0$, we need to use the linear growth condition (1.4.1) instead: Apply Lemma 2.4.3 (see also (2.2.11)),

$$F_0(t,x)=||u_1(t,x)-u_0(t,x)||_p^2\le\left(\left[\overline{\varsigma}^2+J_0^2\right]\star\widehat{\mathscr{L}}_0\right)(t,x)\le\left(\left[\overline{\varsigma}^2+J_0^2\right]\star\widetilde{\mathscr{L}}_0\right)(t,x)\,.$$

Then apply the above relation recursively:

$$\begin{aligned}F_n(t,x)&\le\left(F_{n-1}\star\widetilde{\mathscr{L}}_0\right)(t,x)\le\left(F_{n-2}\star\widetilde{\mathscr{L}}_1\right)(t,x)\\&\;\vdots\\&\le\left(F_0\star\widetilde{\mathscr{L}}_{n-1}\right)(t,x)\le\left(\left[\overline{\varsigma}^2+J_0^2\right]\star\widetilde{\mathscr{L}}_n\right)(t,x)\,.\end{aligned}$$

By Proposition 2.3.1 (iii), we have

$$\widetilde{\mathscr{L}}_n(t,x)=\widetilde{\mathscr{L}}_0(t,x)B_n(t)\,.$$

Since $B_n(t)$ is nondecreasing,

$$F_n(t,x) \le \left(\left[\overline{\varsigma}^2 + J_0^2\right] \star \widetilde{\mathscr{L}_0}\right)(t,x) B_n(t)\,.$$

Now by Proposition 2.3.1, for all $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}$ fixed and all $m \in \mathbb{N}^*$, we have

$$\begin{aligned}\sum_{i=0}^{\infty} |F_i(t,x)|^{1/m} &\le \sum_{i=0}^{\infty} \left|\left(\left[\overline{\varsigma}^2 + J_0^2\right] \star \widetilde{\mathscr{L}_0}\right)(t,x) B_i(t)\right|^{1/m} \\ &= \left|\left(\left[\overline{\varsigma}^2 + J_0^2\right] \star \widetilde{\mathscr{L}_0}\right)(t,x)\right|^{1/m} \sum_{i=0}^{\infty} |B_i(t)|^{1/m} < +\infty\,.\end{aligned}$$

In particular, by taking $m = 2$, we have $\sum_{n=0}^{\infty} |F_n(t,x)|^{1/2} < +\infty$, which proves that $\{u_n(t,x)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^p(\Omega)$.

The moments estimates (2.2.11), (2.2.13) and (2.2.15) can be obtained simply by sending n to $+\infty$ in the conclusions (3) and (4) of the previous step and using Proposition 2.3.1. For example,

$$\begin{aligned}||u(t,x)||_p^2 &\le \lim_{n\to+\infty} \left(b_p\, J_0^2(t,x) + \sum_{i=0}^{n} \left(\left(\overline{\varsigma}^2 + b_p\, J_0^2\right) \star \widehat{\mathscr{L}_i}\right)(t,x)\right) \\ &= b_p\, J_0^2(t,x) + \sum_{i=0}^{\infty} \left(\left(\overline{\varsigma}^2 + b_p\, J_0^2\right) \star \widehat{\mathscr{L}_i}\right)(t,x) \\ &= b_p\, J_0^2(t,x) + \left(\left(\overline{\varsigma}^2 + b_p\, J_0^2\right) \star \widehat{\mathscr{K}_p}\right)(t,x)\,.\end{aligned}$$

Now let us prove the $L^p(\Omega)$-continuity of $(t,x) \mapsto u(t,x)$ over $\mathbb{R}_+^* \times \mathbb{R}$. For all $a > 0$, denote the compact set $K_a := [1/a, a] \times [-a, a]$. The above $L^p(\Omega)$ limit is uniform over K_a since

$$\sum_{i=0}^{\infty} \sup_{(t,x)\in K_a} |F_i(t,x)|^{1/m} \le \left(\sum_{i=0}^{\infty} |B_i(a)|^{1/m}\right) \sup_{(t,x)\in K_a} \left|\left(\left[\overline{\varsigma}^2 + J_0^2\right] \star \widetilde{\mathscr{L}_0}\right)(t,x)\right|^{1/m}$$

from the fact that $B_n(t)$ is nondecreasing. By Lemma 2.3.6 (in particular (2.3.11)), for some constant $C > 0$, depending only on ν, L_ρ and $\overline{\varsigma}$, we have

$$\left|\left(\left[\overline{\varsigma}^2 + J_0^2\right] \star \widetilde{\mathscr{L}_0}\right)(t,x)\right|^{1/m} \le C\, t^{1/(2m)} \left|J_0^*(2t,x)\right|^{2/m}, \quad \text{for all } (t,x) \in \mathbb{R}_+^* \times \mathbb{R},$$

where $J_0^*(2t,x) = \left(|\mu| * G_\nu(2t,\cdot)\right)(x)$. Since the function $(t,x) \mapsto J_0^*(2t,x)$ is continuous over $\mathbb{R}_+^* \times \mathbb{R}$ by Lemma 2.3.5,

$$\sup_{(t,x)\in K_a} \left|\left(\left[\overline{\varsigma}^2 + J_0^2\right] \star \widetilde{\mathscr{L}_0}\right)(t,x)\right|^{1/m} \le C\, a^{1/(2m)} \sup_{(t,x)\in K_a} \left|J_0^*(2t,x)\right|^{2/m} < \infty\,.$$

Hence $\sum_{i=0}^{\infty} \sup_{(t,x)\in K_a} |F_i(t,x)|^{1/m} < +\infty$, which implies that the function $(t,x) \mapsto u(t,x)$ from $\mathbb{R}_+^* \times \mathbb{R}$ into $L^p(\Omega)$ is continuous over K_a since each $u_n(t,x)$ is so. As a can be arbitrarily large, we have then proved the $L^p(\Omega)$-continuity of $(t,x) \mapsto u(t,x)$ over $\mathbb{R}_+^* \times \mathbb{R}$.

Finally, we conclude that $\{u_n(t,x)\}_{n\in\mathbb{N}}$ converges in $L^p(\Omega)$ to $u(t,x)$ such that

(1) $u(t,x)$ is adapted to the filtration $\{\mathscr{F}_t\}_{t>0}$;

(2) the function $(t,x)\mapsto u(t,x)$ from $\mathbb{R}_+^*\times\mathbb{R}$ into $L^p(\Omega)$ is continuous;

(3) For the quasi-linear case (1.4.3), the second moment equals

$$\mathbb{E}[u^2(t,x)] = J_0^2(t,x) + \sum_{i=0}^{\infty}\left(\left[\varsigma^2 + J_0^2\right]\star\mathscr{L}_i\right)(t,x) = J_0^2(t,x) + \left(\left[\varsigma^2 + J_0^2\right]\star\mathscr{K}\right)(t,x),$$

which proves (2.2.15), and it is bounded from above and below (if ρ additionally satisfies (1.4.2)) by

$$\begin{aligned} J_0^2(t,x) + \left(\left[\underline{\varsigma}^2 + J_0^2\right]\star\underline{\mathscr{K}}\right)(t,x) &= J_0^2(t,x) + \sum_{i=0}^{\infty}\left(\left[\underline{\varsigma}^2 + J_0^2\right]\star\underline{\mathscr{L}}_i\right)(t,x) \le \mathbb{E}[u^2(t,x)] \\ &\le J_0^2(t,x) + \sum_{i=0}^{\infty}\left(\left[\overline{\varsigma}^2 + J_0^2\right]\star\overline{\mathscr{L}}_i\right)(t,x) = J_0^2(t,x) + \left(\left[\overline{\varsigma}^2 + J_0^2\right]\star\overline{\mathscr{K}}\right)(t,x), \end{aligned}$$

which proves (2.2.11) (for $p=2$) and (2.2.13).

(4) $\|u(t,x)\|_p^2 \le b_p J_0^2(t,x) + \left(\left(\overline{\varsigma}^2 + b_p J_0^2\right)\star\widehat{\mathscr{K}}_p\right)(t,x)$, which proves (2.2.11) (for $p>2$).

As a direct consequence of the above upper bound and (2.3.3), we have

$$\begin{aligned} \left(\left[\overline{\varsigma}^2 + \|u\|_p^2\right]\star\widehat{\mathscr{L}}_0\right)(t,x) &\le \left(\left[\overline{\varsigma}^2 + b_p J_0^2\right]\star\widehat{\mathscr{L}}_0\right)(t,x) + \left(\left[\overline{\varsigma}^2 + b_p J_0^2\right]\star\widehat{\mathscr{K}}_p\star\widehat{\mathscr{L}}_0\right)(t,x) \\ &= \left(\left[\overline{\varsigma}^2 + b_p J_0^2\right]\star\widehat{\mathscr{K}}_p\right)(t,x). \end{aligned} \tag{2.4.9}$$

This inequality will be used in Step 4.

Step 4(Verifications). Now we shall verify that $\{u(t,x): (t,x)\in\mathbb{R}_+^*\times\mathbb{R}\}$ defined in the previous step is indeed a solution to the stochastic integral equation (2.2.2) in the sense of Definition 2.2.1. Clearly, u is adapted and jointly-measurable and hence it satisfies (1) and (2) of Definition 2.2.1. The function $(t,x)\mapsto u(t,x)$ from $\mathbb{R}_+^*\times\mathbb{R}$ into $L^2(\mathbb{R})$ proved in Step 3 implies (3) of Definition 2.2.1. So we only need to verify that u satisfies (4) of Definition 2.2.1, that is, $u(t,x)$ satisfies (2.1.1) (or (2.2.2)) a.s., for all $(t,x)\in\mathbb{R}_+^*\times\mathbb{R}$.

We shall apply Proposition 2.4.2 with $Y(s,y)=\rho(u(s,y))$ by verifying the three properties that it requires. Properties (i) and (ii) are satisfied by (1) and (2) in the conclusion part of Step 3. Property (iii) is also true since, by Lemma 2.4.3 and also (2.4.9),

$$b_p z_p^2 \left\|\rho(u(\cdot,\circ))\,G_\nu(t-\cdot,x-\circ)\right\|_{M,p}^2 \le \left(\left(\overline{\varsigma}^2 + \|u\|_p^2\right)\star\widehat{\mathscr{L}}_0\right)(t,x) \le \left(\left[\overline{\varsigma}^2 + b_p J_0^2\right]\star\widehat{\mathscr{K}}_p\right)(t,x),$$

which is finite due to Lemma 2.3.6. Hence,

$$\rho(u(\cdot,\circ))\,G_\nu(t-\cdot,x-\circ)\in\mathscr{P}_p, \quad \text{for all } (t,x)\in\mathbb{R}_+^*\times\mathbb{R},$$

and the following Walsh integral is well defined

$$I(t,x) := \iint_{[0,t]\times\mathbb{R}} \rho\left(u\left(s,y\right)\right) G_\nu\left(t-s,x-y\right) W(\mathrm{d}s,\mathrm{d}y)\,.$$

The random field $I(t,x)$ is adapted to $\{\mathscr{F}_t\}_{t>0}$. Furthermore, $(t,x)\mapsto I(t,x)$ is $L^p(\Omega)$-continuous, since by Conclusion (2) of Step 3, $(t,x)\mapsto u(t,x)$ is $L^p(\Omega)$-continuous.

By Step 3, we know that

$$u_n(t,x) = J_0(t,x) + I_n(t,x) = J_0(t,x) + \iint_{[0,t]\times\mathbb{R}} \rho\left(u_{n-1}\left(s,y\right)\right) G_\nu\left(t-s,x-y\right) W\left(\mathrm{d}s,\mathrm{d}y\right)$$

with $u_n(t,x)$ converging to $u(t,x)$ in $L^p(\Omega)$. We only need to show that the right-hand side converges in $L^p(\Omega)$ to $J_0(t,x)+I(t,x)$. In fact, by Lemma 2.3.20 and the Lipschitz continuity of ρ,

$$\begin{aligned}
&\left|\left|\iint_{[0,t]\times\mathbb{R}} \left[\rho\left(u\left(s,y\right)\right)-\rho\left(u_n\left(s,y\right)\right)\right] G_\nu\left(t-s,x-y\right) W\left(\mathrm{d}s,\mathrm{d}y\right)\right|\right|_p^2 \\
&\qquad\qquad \le z_p^2\, \mathrm{Lip}_\rho^2 \iint_{[0,t]\times\mathbb{R}} G_\nu^2\left(t-s,x-y\right)\left|\left|u\left(s,y\right)-u_n\left(s,y\right)\right|\right|_p^2 \mathrm{d}s\mathrm{d}y\,.
\end{aligned}$$

Now apply Lebesgue's dominated convergence theorem to conclude that the above integral tends to zero as $n\to\infty$ since:

(i) for all $\left(s,y\right)\in\,]0,t]\times\mathbb{R}$, $\left|\left|u_n\left(s,y\right)-u\left(s,y\right)\right|\right|_p^2 \to 0$ as $n\to+\infty$;

(ii) the integrand can be bounded in the following way:

$$\begin{aligned}
\left|\left|u_n\left(s,y\right)-u\left(s,y\right)\right|\right|_p^2 &\le 2\left|\left|u_n\left(s,y\right)\right|\right|_p^2 + 2\left|\left|u\left(s,y\right)\right|\right|_p^2 \\
&\le 4 b_p\, J_0^2(s,y) + 4\left(\left[\overline{\varsigma}^2 + b_p\, J_0^2\right]\star\widehat{\mathscr{K}_p}\right)(s,y)\,,
\end{aligned}$$

where the last inequality is true because by Step 2,

$$\begin{aligned}
\left|\left|u_n(s,y)\right|\right|_p^2 &\le b_p\, J_0^2(s,y) + \sum_{i=0}^{n}\left(\left[\overline{\varsigma}^2 + b_p\, J_0^2\right]\star\widehat{\mathscr{L}_i}\right)(s,y) \\
&\le b_p\, J_0^2(s,y) + \left(\left[\overline{\varsigma}^2 + b_p\, J_0^2\right]\star\widehat{\mathscr{K}_p}\right)(s,y)\,,
\end{aligned}$$

and also by Step 3, $\left|\left|u(s,y\right|\right|_p^2 \le b_p\, J_0^2(s,y) + \left(\left[\overline{\varsigma}^2 + b_p\, J_0^2\right]\star\widehat{\mathscr{K}_p}\right)\left(s,y\right)$. Hence by (2.3.3),

$$\begin{aligned}
&4\, a_{p,\overline{\varsigma}}^2 z_p^2\, L_\rho^2 \iint_{[0,t]\times\mathbb{R}} \left(b_p\, J_0^2\left(s,y\right) + \left(\left[\overline{\varsigma}^2 + b_p\, J_0^2\right]\star\widehat{\mathscr{K}_p}\right)(s,y)\right) G_\nu^2(t-s,x-y)\mathrm{d}s\mathrm{d}y \\
&\qquad = 4\left(b_p\, J_0^2 \star\widehat{\mathscr{L}_0}\right)(t,x) + \left(\left[\overline{\varsigma}^2 + b_p\, J_0^2\right]\star\widehat{\mathscr{L}_0}\star\widehat{\mathscr{K}_p}\right)(t,x) \\
&\qquad \le 4\left(\left[\overline{\varsigma}^2 + b_p\, J_0^2\right]\star\widehat{\mathscr{K}_p}\right)(t,x)\,,
\end{aligned}$$

which is finite due to Lemma 2.3.6.

Hence we have proved that

$$J_0(t,x)+I_n(t,x)\overset{L^p(\Omega)}{\longrightarrow} J_0(t,x)+I(t,x),\quad \text{as } n\to\infty.$$

These two $L^p(\Omega)$-limits of $J_0(t,x)+I_n(t,x)$ must be equal a.s., i.e., for all $(t,x)\in\mathbb{R}_+^*\times\mathbb{R}$,

$$u(t,x)=J_0(t,x)+\iint_{[0,t]\times\mathbb{R}}\rho\left(u\left(s,y\right)\right)G_\nu\left(t-s,x-y\right)W\left(\mathrm{d}s,\mathrm{d}y\right),\quad \text{a.s.}$$

We have therefore proved that $u(t,x)$ satisfies the required integral equation for all $(t,x)\in\mathbb{R}_+^*\times\mathbb{R}$. This completes the proof of the existence part of Theorem 2.2.2 with the moment estimates.

Step 5 (Uniqueness). Let u_1 and u_2 be two solutions to (2.2.2) (in the sense of Definition 2.2.1) with the same initial data, and denote $v(t,x):=u_1(t,x)-u_2(t,x)$. The $L^2(\Omega)$-continuity– Property (3) of Definition 2.2.1 – guarantees that both $(t,x)\mapsto u_1(t,x)$ and $(t,x)\mapsto u_2(t,x)$ are $L^2(\Omega)$-continuous since $(t,x)\mapsto J_0(t,x)$ is continuous by Lemma 2.3.5. Then $v(t,x)$ is well defined and the function $(t,x)\mapsto v(t,x)$ is $L^2(\Omega)$-continuous. Writing $v(t,x)$ explicitly

$$v(t,x)=\iint_{[0,t]\times\mathbb{R}}\left[\rho\left(u_1\left(s,y\right)\right)-\rho\left(u_2\left(s,y\right)\right)\right]G_\nu\left(t-s,x-y\right)W(\mathrm{d}s,\mathrm{d}y)$$

and then taking the second moment, by the isometry property and Lipschitz condition of ρ, we have

$$\mathbb{E}[v(t,x)^2]\le\left(\mathbb{E}[v^2]\star\widetilde{\mathscr{L}_0}\right)(t,x)\,,$$

where

$$\widetilde{\mathscr{L}_0}(t,x):=\mathscr{L}_0\left(t,x;\nu,\mathrm{Lip}_\rho\right).$$

Now we convolve both sides with respect to $\widetilde{\mathscr{K}}$ and use the fact in (2.3.3) to get

$$(\mathbb{E}[v^2]\star\widetilde{\mathscr{K}})(t,x)\le(\mathbb{E}[v^2]\star\widetilde{\mathscr{L}_0}\star\widetilde{\mathscr{K}})(t,x)=(\mathbb{E}[v^2]\star\widetilde{\mathscr{K}})(t,x)-(\mathbb{E}[v^2]\star\widetilde{\mathscr{L}_0})(t,x)\,.$$

So we have

$$(\mathbb{E}[v^2]\star\widetilde{\mathscr{L}_0})(t,x)\equiv 0\,,$$

which implies that $\mathbb{E}[v(t,x)^2]=0$ for all $(t,x)\in\mathbb{R}_+^*\times\mathbb{R}$ since (i) the kernel $\widetilde{\mathscr{L}_0}$ is non-negative and has support on $[0,\infty)\times\mathbb{R}$; (ii) the function $(t,x)\mapsto\mathbb{E}[v(t,x)^2]$ is non-negative and continuous on the domain $\mathbb{R}_+^*\times\mathbb{R}$ as a consequence of the $L^2(\Omega)$-continuity of $v(t,x)$. Therefore, we conclude that for all $(t,x)\in\mathbb{R}_+^*\times\mathbb{R}$, $u_1(t,x)=u_2(t,x)$ a.s., i.e., u_1 and u_2 are versions of each other. This proves the uniqueness. □

2.4.3 Estimates of Two-point Correlation Functions

In this part, we prove the estimates ((2.2.12), (2.2.14) and (2.2.16)) of the two-point correlation functions. We only need to prove the formula (2.2.16) for the quasi-linear case. The other two cases follow the same arguments.

Proof of (2.2.16). Assume that $|\rho(u)|^2 = \lambda^2 \left(\varsigma^2 + u^2\right)$. Let $u(t,x)$ be the solution to (2.1.1). Fix $t \in \mathbb{R}_+^*$ and $x, y \in \mathbb{R}$. Consider the $L^2(\Omega)$-martingale $\{U(\tau; t, x) : \tau \in [0, t]\}$ defined by

$$U(\tau; t, x) := J_0(t,x) + \int_0^\tau \int_{\mathbb{R}} \rho(u(s,z)) G_\nu(t-s, x-z) W(\mathrm{d}s, \mathrm{d}z) \;.$$

Then $\mathbb{E}\left[U(\tau; t, x)\right] = J_0(t,x)$. Similarly, we can define the martingale $\{U(\tau; t, y) : \tau \in [0,t]\}$. The mutual variation process of these two martingales is

$$\left[U(\cdot; t, x), U(\cdot; t, y)\right]_\tau = \lambda^2 \int_0^\tau \mathrm{d}s \int_{\mathbb{R}} \left(\varsigma^2 + |u(s,z)|^2\right) G_\nu(t-s, x-z) G_\nu\left(t-s, y-z\right) \mathrm{d}z\,,$$

for all $\tau \in [0,t]$. Hence, by Itô's lemma, for every $\tau \in [0,t]$,

$$\begin{aligned}\mathbb{E}\left[U(\tau; t, x) U(\tau; t, y)\right] &= J_0(t,x) J_0\left(t, y\right) \\ &\quad + \lambda^2 \int_0^\tau \mathrm{d}s \int_{\mathbb{R}} \left(\varsigma^2 + \mathbb{E}[|u(s,z)|^2]\right) G_\nu(t-s, x-z) G_\nu\left(t-s, y-z\right) \mathrm{d}z\,.\end{aligned}$$

Finally, we choose $\tau = t$ to get

$$\begin{aligned}\mathbb{E}\left[u(t,x) u\left(t,y\right)\right] &= J_0(t,x) J_0\left(t,y\right) + \lambda^2 \varsigma^2 \int_0^t \mathrm{d}s \int_{\mathbb{R}} G_\nu(t-s, x-z) G_\nu\left(t-s, y-z\right) \mathrm{d}z \\ &\quad + \lambda^2 \int_0^t \mathrm{d}s \int_{\mathbb{R}} ||u(s,z)||_2^2 \, G_\nu(t-s, x-z) G_\nu\left(t-s, y-z\right) \mathrm{d}z\,. \quad (2.4.10)\end{aligned}$$

Notice that

$$\int_0^t \mathrm{d}s \int_{\mathbb{R}} G_\nu(t-s, x-z) G_\nu\left(t-s, y-z\right) \mathrm{d}z$$

can be calculated explicitly by (2.4.12). Putting back the above quantity, we have then proved (2.2.16). □

Lemma 2.4.5. *For $\nu > 0$ and $t > 0$, we have*

$$\int_0^t G_\nu(s,x) \mathrm{d}s = \frac{2|x|}{\nu} \left(\Phi\left(\frac{|x|}{\sqrt{\nu t}}\right) - 1\right) + 2t G_\nu(t,x)\,, \qquad (2.4.11)$$

and

$$\begin{aligned}\int_0^t \mathrm{d}s \int_{\mathbb{R}} & G_\nu(t-s, x-z) G_\nu\left(t-s, y-z\right) \mathrm{d}z \\ &= \frac{|x-y|}{\nu} \left(\Phi\left(\frac{|x-y|}{\sqrt{2\nu t}}\right) - 1\right) + 2t\, G_{2\nu}\left(t, x-y\right)\,. \quad (2.4.12)\end{aligned}$$

Proof. (i) We first prove (2.4.11). If $x = 0$, then

$$\int_0^t G_\nu(s,0) \mathrm{d}s = \int_0^t \frac{1}{\sqrt{2\pi\nu s}} \mathrm{d}s = \sqrt{\frac{2t}{\pi\nu}}\,,$$

which equals the right-hand side of (2.4.11) with $x = 0$. From now on, we assume that

$x \neq 0$. We first change variable $u = \frac{|x|}{\sqrt{\nu s}}$ and so

$$\int_0^t G_\nu(s,x)\mathrm{d}s = \frac{2|x|}{\nu}\int_{|x|/\sqrt{\nu t}}^{+\infty} \frac{1}{\sqrt{2\pi}u^2} e^{-u^2/2}\mathrm{d}u\,.$$

After integration by parts,

$$\begin{aligned}\int_0^t G_\nu(s,x)\mathrm{d}s &= \frac{2|x|}{\nu}\left(\frac{e^{-u^2/2}}{\sqrt{2\pi}u}\Bigg|_{+\infty}^{|x|/\sqrt{\nu t}} - \int_{|x|/\sqrt{\nu t}}^{+\infty}\frac{e^{-u^2/2}}{\sqrt{2\pi}}\mathrm{d}u\right)\\ &= \frac{2\sqrt{t}}{\sqrt{2\nu\pi}}e^{-\frac{x^2}{2\nu t}} + \frac{2|x|}{\nu}\left(\Phi\left(\frac{|x|}{\sqrt{\nu t}}\right)-1\right)\end{aligned}$$

which is equal to the right-hand side of (2.4.11).

(ii) As for (2.4.12), notice that by Lemma 2.3.7,

$$G_\nu(t-s,x-z)G_\nu\left(t-s,y-z\right) = G_{2\nu}\left(t-s,x-y\right)G_{\nu/2}\left(t-s,z-\frac{x+y}{2}\right)$$

So after integrating first over z, we have

$$\int_0^t \mathrm{d}s\int_{\mathbb{R}} G_\nu(t-s,x-z)G_\nu\left(t-s,y-z\right)\mathrm{d}z = \int_0^t G_{2\nu}\left(t-s,x-y\right)\mathrm{d}s.$$

Then (2.4.12) is proved by (2.4.11) with 2ν. This finishes the whole proof. □

Remark 2.4.6 (Consistency of two-point correlation functions with second moments)**.** We finally remark that the two-point correlation function (2.2.16) is consistent with the second moment (2.2.15), in the sense that (2.2.16) with $x=y$ gives (2.2.15). Indeed, by letting $x=y$, the right-hand side of (2.2.16) gives

$$\begin{aligned}h(t,x) := J_0^2(t,x) &+ \frac{\lambda^2\varsigma^2\sqrt{t}}{\sqrt{\pi\nu}}\\ &+ \lambda^2\int_0^t\mathrm{d}s\int_{\mathbb{R}}\left[J_0^2\left(s,y\right)+\left((\varsigma^2+J_0^2)\star\mathscr{K}\right)\left(s,y\right)\right]G_{2\nu}(t-s,0)G_{\nu/2}(t-s,x-z)\mathrm{d}z\,.\end{aligned}$$

Notice that

$$\lambda^2 G_{2\nu}(t-s,0)G_{\nu/2}(t-s,x-z) = \mathscr{L}_0\left(t-s,x-y\right)\,.$$

So

$$\begin{aligned}h(t,x) =& J_0^2(t,x) + \frac{\lambda^2\varsigma^2\sqrt{t}}{\sqrt{\pi\nu}} + \left(J_0^2\star\mathscr{L}_0\right)(t,x) + \left((J_0^2+\varsigma^2)\star\mathscr{K}\star\mathscr{L}_0\right)(t,x)\\ =& J_0^2(t,x) + \frac{\lambda^2\varsigma^2\sqrt{t}}{\sqrt{\pi\nu}} + \left((J_0^2+\varsigma^2)\star\mathscr{K}\right)(t,x) - \left(\varsigma^2\star\mathscr{L}_0\right)(t,x)\\ =& J_0^2(t,x) + \left((J_0^2+\varsigma^2)\star\mathscr{K}\right)(t,x)\,,\end{aligned}$$

where we have used the facts (2.3.3) and (2.3.8). The last line of the above equalities is exactly the formula of the second moment (2.2.15).

2.4.4 Special Cases: the Dirac delta and the Lebesgue Initial Data

In this part, we prove two Corollaries 2.2.3 and 2.2.6. We need two lemmas.

Lemma 2.4.7. *For all* $t \geq 0$,

$$\int_0^t (\mathscr{H}(s)+1)\, G_{2\nu}(t-s,x)\mathrm{d}s = \frac{1}{\lambda^2}\left(\exp\left(\frac{\lambda^4 t - 2\lambda^2|x|}{4\nu}\right)\mathrm{erfc}\left(\frac{|x|-\lambda^2 t}{2\sqrt{\nu t}}\right) - \mathrm{erfc}\left(\frac{|x|}{2\sqrt{\nu t}}\right)\right).$$

Proof. Denote the convolution by $I(t)$. By [35, (27), Chapter 4.5, p. 146], we have the the Laplace transform

$$\mathscr{L}\left[G_{2\nu}(\cdot,x)\right](z) = \frac{\exp\left(-\sqrt{z/\nu}\,|x|\right)}{2\sqrt{z\nu}}.$$

Notice that

$$\mathscr{H}(t)+1 = e^{\frac{\lambda^4 t}{4\nu}}\left(\mathrm{erf}\left(\frac{1}{2}\lambda^2\sqrt{\frac{t}{\nu}}\right)+1\right).$$

Clearly,

$$\mathscr{L}\left[e^{\frac{\lambda^4 t}{4\nu}}\right](z) = \frac{1}{z-\lambda^4/(4\nu)}.$$

By [35, (5), Chapter 4.12, p. 176],

$$\mathscr{L}\left[e^{\frac{\lambda^4 t}{4\nu}}\mathrm{erf}\left(\frac{1}{2}\lambda^2\sqrt{\frac{t}{\nu}}\right)\right](z) = \frac{\lambda^2}{2\sqrt{\nu z}\left(z-\lambda^4/(4\nu)\right)}.$$

Hence, we have

$$\mathscr{L}[I](z) = \mathscr{L}\left[G_{2\nu}(\cdot,x)\right](z)\cdot\mathscr{L}\left[\mathscr{H}(\cdot)+1\right](z) = \frac{\exp\left(-\frac{|x|}{\sqrt{\nu}}\sqrt{z}\right)}{\sqrt{4\nu}z\left(\sqrt{z}-\frac{\lambda^2}{\sqrt{4\nu}}\right)}.$$

As for the inverse Laplace transform, we apply [35, (14) in Chapter 5.6, p. 246], namely,

$$\mathscr{L}^{-1}\left[\beta z^{-1}(\sqrt{z}+\beta)^{-1}e^{-a\sqrt{z}}\right](t) = \mathrm{erfc}\left(\frac{a}{2\sqrt{t}}\right) - e^{a\beta+\beta^2 t}\mathrm{erfc}\left(\frac{a}{2\sqrt{t}}+\beta\sqrt{t}\right),\quad \Re(a^2)\geq 0,$$

with $a = |x|/\sqrt{\nu}$ and $\beta = -\lambda^2/\sqrt{4\nu}$, which finishes the proof. □

Proof of Corollary 2.2.3. In this case, $J_0(t,x)\equiv 1$. The second moment (2.2.18) is clear by (2.2.15):

$$\mathbb{E}\left[|u(t,x)|^2\right] = 1 + (1\star\mathscr{K})(t,x) + \varsigma^2\mathscr{H}(t) = 1 + \left(1+\varsigma^2\right)\mathscr{H}(t)$$

where we have used the definition of $\mathscr{H}(t)$. Then, by the two-point correlation function

(2.2.16) (see also (2.4.10)) and semigroup property of the heat kernel, we have

$$\begin{aligned}\mathbb{E}\left[u(t,x)u(t,y)\right] &= 1+\lambda^2\int_0^t \mathrm{d}s\int_{\mathbb{R}}\left[\varsigma^2+1+(1+\varsigma^2)\mathcal{H}(s)\right]G_\nu(t-s,x-z)G_\nu(t-s,y-z)\,\mathrm{d}z\\ &= 1+\lambda^2(1+\varsigma^2)\int_0^t(\mathcal{H}(s)+1)\,G_{2\nu}(t-s,x-y)\,\mathrm{d}s,\end{aligned}$$

where the integral can be evaluated by Lemma 2.4.7. This completes the proof. □

The next lemma was used in Remark 2.2.4.

Lemma 2.4.8. *For all $t\ge 0$ and $x\ne 0$, we have*

$$\int_0^t \mathrm{d}s\frac{|x|}{\sqrt{\pi\nu s^3}}\exp\left\{-\frac{x^2}{4\nu s}+\frac{t-s}{4\nu}\right\}\Phi\left(\sqrt{\frac{t-s}{2\nu}}\right)=\exp\left(\frac{t-2|x|}{4\nu}\right)\operatorname{erfc}\left(\frac{|x|-t}{\sqrt{4\nu t}}\right),$$

and in particular,

$$\lim_{x\to 0}\int_0^t \mathrm{d}s\frac{|x|}{\sqrt{\pi\nu s^3}}\exp\left\{-\frac{x^2}{4\nu s}+\frac{t-s}{4\nu}\right\}\Phi\left(\sqrt{\frac{t-s}{2\nu}}\right)=\exp\left(\frac{t}{4\nu}\right)\operatorname{erfc}\left(\frac{-t}{\sqrt{4\nu t}}\right).$$

Proof. Suppose that $x\ne 0$. Denote the integral by $I(t)$ and introduce two functions:

$$f(t):=\frac{|x|}{\sqrt{\pi\nu t^3}}\exp\left(-\frac{x^2}{4\nu t}\right),\qquad g(t):=\exp\left(\frac{t}{4\nu}\right)\Phi\left(\sqrt{\frac{t}{2\nu}}\right).$$

Clearly, $I(t)$ is the convolution of f and g. By [35, (28), Chapter 4.5, p. 146],

$$\mathcal{L}[f](z)=2\exp\left(-|x|\sqrt{z/\nu}\right).$$

Notice $g(t)=(H(t)+1)/2$ with $H(t)=\mathcal{H}(t;\nu/2,1/\sqrt{4\pi\nu})$. By the calculation in Lemma 2.4.7,

$$\mathcal{L}[g](z)=\frac{1}{2(z-1/(4\nu))}+\frac{1}{4\sqrt{\nu z}\,(z-1/(4\nu))}.$$

Hence,

$$\mathcal{L}[I](z)=\mathcal{L}[f](z)\mathcal{L}[g](z)=\frac{e^{-|x|\sqrt{z/\nu}}}{\sqrt{z}\left(\sqrt{z}-\frac{1}{2\sqrt{\nu}}\right)}.$$

As for the inverse Laplace transform, we apply [35, (16) in Chapter 5.6, p. 247], namely,

$$\mathcal{L}^{-1}\left[z^{-1/2}\left(z^{1/2}+\beta\right)^{-1}e^{-az^{1/2}}\right](t)=\exp\left(a\beta+\beta^2 t\right)\operatorname{erfc}\left(\frac{a}{2\sqrt{t}}+\beta\sqrt{t}\right), \tag{2.4.13}$$

for $\Re(a^2)>0$, with $a=|x|/\sqrt{\nu}$ and $\beta=-1/\sqrt{4\nu}$, which completes the proof. □

Proof of Corollary 2.2.6. In this case, $J_0(t,x)=G_\nu(t,x)$. Notice that $\lambda^2 J_0^2(t,x)=\mathcal{L}_0(t,x)$.

So, by (2.2.15) and also (2.3.3), we have

$$\begin{aligned}\mathbb{E}\left[|u(t,x)|^2\right] &= J_0^2(t,x) + \frac{1}{\lambda^2}\left(\mathscr{L}_0 \star \mathscr{K}\right)(t,x) + \varsigma^2\,\mathscr{H}(t)\\ &= \frac{1}{\lambda^2}\mathscr{K}(t,x) + \varsigma^2\,\mathscr{H}(t)\,.\end{aligned}$$

Then, by the two-point correlation function (2.2.16) (see also (2.4.10)), we have

$$\begin{aligned}\mathbb{E}\left[u(t,x)u\left(t,y\right)\right] &= J_0(t,x)J_0\left(t,y\right)\\ &+\lambda^2\int_0^t \mathrm{d}s\, G_{2\nu}\left(t-s,x-y\right)\int_{\mathbb{R}}\left(\varsigma^2+\frac{1}{\lambda^2}\mathscr{K}(s,z)+\varsigma^2\,\mathscr{H}(s)\right)G_{\frac{\nu}{2}}\left(t-s,z-\frac{x+y}{2}\right)\mathrm{d}z\,.\end{aligned}$$

By the semigroup property of the heat kernel (note that z appears in a heat kernel in $\mathscr{K}(s,z)$: see (2.2.4)), integration over z gives

$$\varsigma^2\left(\mathscr{H}(s)+1\right)+G_{\frac{\nu}{2}}\left(t,\frac{x+y}{2}\right)\left(\frac{1}{\sqrt{4\pi\nu s}}+\frac{\lambda^2}{2\nu}e^{\frac{\lambda^4 s}{4\nu}}\Phi\left(\lambda^2\sqrt{\frac{s}{2\nu}}\right)\right)$$

which equals, by the definition of $\mathscr{H}(t)$,

$$\begin{aligned}&=\varsigma^2\left(\mathscr{H}(s)+1\right)+G_{\frac{\nu}{2}}\left(t,\frac{x+y}{2}\right)\left(\frac{1}{\sqrt{4\pi\nu s}}+\frac{\lambda^2}{4\nu}\left(\mathscr{H}(s)+1\right)\right)\\ &=\left(\varsigma^2+\frac{\lambda^2}{4\nu}G_{\frac{\nu}{2}}\left(t,\frac{x+y}{2}\right)\right)\left(\mathscr{H}(s)+1\right)+G_{\frac{\nu}{2}}\left(t,\frac{x+y}{2}\right)\frac{1}{\sqrt{4\pi\nu s}}\,.\end{aligned}$$

Then multiply the above quantity by $\lambda^2 G_{2\nu}\left(t-s,x-y\right)$ and integrate over s:

$$\begin{aligned}\mathbb{E}\left[u(t,x)u\left(t,y\right)\right] =&G_\nu(t,x)G_\nu\left(t,y\right)\\ &+\lambda^2\left(\varsigma^2+\frac{\lambda^2}{4\nu}G_{\frac{\nu}{2}}\left(t,\frac{x+y}{2}\right)\right)\int_0^t G_{2\nu}\left(t-s,x-y\right)\left(\mathscr{H}(s)+1\right)\mathrm{d}s\\ &+\frac{\lambda^2}{\sqrt{4\pi\nu}}G_{\frac{\nu}{2}}\left(t,\frac{x+y}{2}\right)\int_0^t\frac{G_{2\nu}\left(t-s,x-y\right)}{\sqrt{s}}\mathrm{d}s\,.\end{aligned}$$

The first integral can be calculated by Lemma 2.4.7. The last integral can be evaluated by Lemma 2.6.5:

$$\frac{\lambda^2}{\sqrt{4\pi\nu}}G_{\frac{\nu}{2}}\left(t,\frac{x+y}{2}\right)\int_0^t\frac{G_{2\nu}\left(t-s,x-y\right)}{\sqrt{s}}\mathrm{d}s=\frac{\lambda^2}{4\nu}G_{\frac{\nu}{2}}\left(t,\frac{x+y}{2}\right)\operatorname{erfc}\left(\frac{|x-y|}{2\sqrt{\nu t}}\right).$$

This completes the whole proof after some simplifications. □

The next lemma was used in Remark 2.2.7.

Lemma 2.4.9. *The integral in* (2.2.22) *equals*

$$G_\nu(t,x)G_\nu\left(t,y\right)+\frac{1}{4\nu}G_{\frac{\nu}{2}}\left(t,\frac{x+y}{2}\right)\exp\left(\frac{t-2|x-y|}{4\nu}\right)\operatorname{erfc}\left(\frac{|x-y|-t}{\sqrt{4\nu t}}\right).$$

Proof. Denote the integral in (2.2.22) by $F(t,x,y)$. After simplifications, we have

$$F(t,x,y) = \frac{1}{4\pi\nu t} G_{\frac{\nu}{2}}\left(t, \frac{x+y}{2}\right)$$
$$\times \int_0^1 \frac{|x-y|}{\sqrt{s^3}} \exp\left(-\frac{(x-y)^2}{4\nu t s}\right)\left(\frac{1}{\sqrt{1-s}} + \sqrt{\pi t/\nu}\exp\left(\frac{t(1-s)}{4\nu}\right)\Phi\left(\sqrt{\frac{t(1-s)}{2\nu}}\right)\right)\mathrm{d}s\,.$$

Denote the above integral by $I_1(1)+I_2(1)$. Suppose that $x \neq y$ and let

$$f(s) := \frac{|x-y|}{s^{3/2}} \exp\left(-\frac{(x-y)^2}{4\nu t s}\right), \qquad g(s) := \frac{1}{\sqrt{s}}\,,$$

and

$$h(s) := \sqrt{\pi t/\nu}\exp\left(\frac{ts}{4\nu}\right)\Phi\left(\sqrt{\frac{ts}{2\nu}}\right).$$

By [35, (28), Chapter 4.5, p. 146], we have

$$\mathscr{L}[f](z) = 2\sqrt{\pi\nu t}\exp\left(-\frac{|x-y|\sqrt{z}}{\sqrt{\nu t}}\right).$$

Notice that

$$h(s) = \frac{\sqrt{\pi t}}{2\sqrt{\nu}} e^{\frac{ts}{4\nu}}\left(1+\operatorname{erf}\left(\frac{\sqrt{t}}{2}\sqrt{s/\nu}\right)\right) = \frac{\sqrt{\pi t}}{2\sqrt{\nu}}\left(\widetilde{\mathscr{H}}(s)+1\right)$$

where $\widetilde{\mathscr{H}}(s) = \mathscr{H}\left(s;\nu/2, \frac{\sqrt{t}}{\sqrt{4\pi\nu}}\right)$. So by the calculation in Lemma 2.4.7, we have

$$\mathscr{L}[h](z) = \frac{\sqrt{\pi t}}{2\sqrt{\nu}}\left(\frac{1}{z-t/(4\nu)} + \frac{\sqrt{t}}{2\sqrt{\nu z}\,(z-t/(4\nu))}\right).$$

Clearly, $\mathscr{L}[g](z) = \sqrt{\pi}/\sqrt{z}$. Hence,

$$\mathscr{L}[I_1](z) = \mathscr{L}[f](z)\mathscr{L}[g](z) = 2\pi\sqrt{\nu t}\,\frac{\exp\left(-\frac{|x-y|\sqrt{z}}{\sqrt{\nu t}}\right)}{\sqrt{z}}\,.$$

Then by the inverse Laplace transform [35, (6), Chapter 5.6, p. 246], we have

$$I_1(T) = \frac{2\sqrt{\pi\nu t}}{\sqrt{T}}\exp\left(-\frac{(x-y)^2}{4\nu T t}\right), \qquad T>0\,.$$

As for $I_2(T)$, we have

$$\mathscr{L}[I_2](z) = \mathscr{L}[f](z)\mathscr{L}[h](z) = \pi t \exp\left(-\frac{|x-y|\sqrt{z}}{\sqrt{\nu t}}\right)\frac{1}{\sqrt{z}\left(\sqrt{z}-\sqrt{t/(4\nu)}\right)}\,.$$

Then apply (2.4.13) with $a = |x-y|/\sqrt{\nu t}$ and $\beta = -\sqrt{t/(4\nu)}$ to get

$$I_2(T) = \pi t \exp\left(\frac{tT - 2|x-y|}{4\nu}\right) \operatorname{erfc}\left(\frac{|x-y| - tT}{\sqrt{4\nu t T}}\right), \quad T > 0\,.$$

Therefore, by letting $T = 1$, we have

$$\begin{aligned} &F(t,x,y) \\ =&G_{\frac{\nu}{2}}\left(t, \frac{x+y}{2}\right)\left(\frac{1}{\sqrt{4\pi\nu t}}\exp\left(-\frac{(x-y)^2}{4\nu t}\right) + \frac{1}{4\nu}\exp\left(\frac{t-2|x-y|}{4\nu}\right)\operatorname{erfc}\left(\frac{|x-y|-t}{\sqrt{4\nu t}}\right)\right) \\ =&G_\nu(t,x)G_\nu\left(t,y\right) + \frac{1}{4\nu}G_{\frac{\nu}{2}}\left(t,\frac{x+y}{2}\right)\exp\left(\frac{t-2|x-y|}{4\nu}\right)\operatorname{erfc}\left(\frac{|x-y|-t}{\sqrt{4\nu t}}\right). \end{aligned}$$

The case $x \to y$ can be simply obtained from the above formula. □

2.4.5 Initial Value δ' (Proof of Proposition 2.2.9)

Proof of Proposition 2.2.9. Clearly, $J_0(t,x) = \frac{\partial}{\partial x}G_\nu(t,x) = -\frac{x}{\nu t}G_\nu(t,x)$. Suppose that (2.2.2) has a random field solution $u(t,x)$. Then $u(t,x)$ satisfies the stochastic integral equation (2.2.2) with $\rho(u) = \lambda u$. Hence, by the isometry of the Walsh integral,

$$||u(t,x)||_2^2 = J_0^2(t,x) + \left(G_\nu^2 \star \left|\left|\rho(u)\right|\right|_2^2\right)(t,x) \geq J_0^2(t,x), \quad \text{for all } (t,x) \in \mathbb{R}_+^* \times \mathbb{R}\,.$$

Thus,

$$\left(G_\nu^2 \star \left|\left|\rho(u)\right|\right|_2^2\right)(t,x) = \lambda^2\left(G_\nu^2 \star ||u||_2^2\right)(t,x) \geq \lambda^2\left(G_\nu^2 \star J_0^2\right)(t,x)\,.$$

To calculate

$$f(t,x) = \left(J_0^2 \star G_\nu^2\right)(t,x) = \iint_{[0,t]\times\mathbb{R}} \frac{y^2}{\nu^2 s^2} G_\nu^2\left(s,y\right) G_\nu^2\left(t-s, x-y\right) \mathrm{d}s\mathrm{d}y\,,$$

we use Lemma 2.3.7 to write G_ν^2 in the form of $G_{\nu/2}$ and then combine the two $G_{\nu/2}$'s:

$$\begin{aligned} G_\nu^2\left(s,y\right) G_\nu^2\left(t-s,x-y\right) &= \frac{1}{4\pi\nu\sqrt{s(t-s)}} G_{\nu/2}\left(s,y\right) G_{\nu/2}(t-s,y-x) \\ &= \frac{G_{\nu/2}(t,x)}{4\pi\nu\sqrt{s(t-s)}} G_{\nu/2}\left(s\left(1-\frac{s}{t}\right), y - \frac{s}{t}x\right). \end{aligned}$$

To integrate over y, after the change variable $z = y - sx/t$, one can see that

$$f(t,x) = \frac{G_{\nu/2}(t,x)}{4\pi\nu^3}\int_0^t \frac{1}{s^2\sqrt{s(t-s)}}\mathbb{E}\left[Z^2 + \frac{s^2x^2}{t^2}\right]\mathrm{d}s$$

where $Z \sim N\left(0, \frac{\nu s(t-s)}{2t}\right)$ is a normal random variable. Hence,

$$
\begin{aligned}
f(t,x) &= \frac{G_{\nu/2}(t,x)}{8\pi t^2 \nu^3} \int_0^t \frac{\nu t(t-s) + 2sx^2}{\sqrt{s^3(t-s)}} \mathrm{d}s \\
&= \frac{G_{\nu/2}(t,x)}{8\pi t^2 \nu^3} \left(\int_0^t \frac{\nu t^2}{\sqrt{s^3(t-s)}} \mathrm{d}s + \pi \left(2x^2 - \nu t\right) \right) = \infty,
\end{aligned}
$$

for all $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}$, where we have used the Beta integral. This violates Property (3) of Definition 2.2.1. Therefore, there is no random field solution $u(t,x)$ in the sense of Definition 2.2.1. □

2.5 Proof of Exponential Growth Indices (Theorem 2.2.10)

We prove Theorem 2.2.10 in this part. We first give a property of these growth indices.

Lemma 2.5.1 ([19]). *For $a, b \in [2, \infty[$, $a \le b$, we have $\overline{\lambda}(a) \le \overline{\lambda}(b)$ and $\underline{\lambda}(a) \le \underline{\lambda}(b)$.*

We first note that the quasi-linear case ($|\rho(u)|^2 = \lambda^2 \left(\varsigma^2 + u^2\right)$) corresponds to the case where $L_\rho = l_\rho = |\lambda|$ and $\overline{\varsigma} = \underline{\varsigma} = \varsigma$. (3) is a direct consequence of (1) and (2). Hence, in the following, we only need to prove (1) and (2).

2.5.1 Proof of the Lower Bound

By the moment formula (2.2.13), we can bound the second moment from below by finding a proper lower bound of $J_0(t,x)$. This is done by the following lemma.

Lemma 2.5.2. *Assume that $\mu \in \mathcal{M}_+(\mathbb{R})$ and $\mu \neq 0$. For any $\epsilon > 0$ and $\xi \in]0, \nu[$, there exists a constant $a_{\epsilon,\xi,\nu} > 0$ such that*

$$
J_0(t,x) \ge a_{\epsilon,\xi,\nu} 1_{[\epsilon,+\infty[}(t)\, G_\xi(t,x), \quad \text{for all } t \ge \epsilon \text{ and } x \in \mathbb{R}.
$$

Proof. Equivalently, we need to prove that the function

$$
g(t,x) := \frac{J_0(t,x)}{G_\xi(t,x)} = \sqrt{\xi/\nu} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{2\nu t} + \frac{x^2}{2\xi t}} \mu(\mathrm{d}y)
$$

is strictly bounded away from zero for $t \in [\epsilon, +\infty[$ and $x \in \mathbb{R}$. Notice that for $\xi \neq \nu$,

$$
\begin{aligned}
-\frac{(x-y)^2}{2\nu t} + \frac{x^2}{2\xi t} &= -\frac{\xi(x-y)^2 - \nu x^2}{2\nu\xi t} \\
&= -\frac{(\xi-\nu)\left[x - \frac{\xi y}{\xi-\nu}\right]^2 - \frac{\nu\xi y^2}{\xi-\nu}}{2\nu\xi t} \\
&= -\frac{(\xi-\nu)\left[x - \frac{\xi y}{\xi-\nu}\right]^2}{2\nu\xi t} + \frac{y^2}{2(\xi-\nu)t}.
\end{aligned}
$$

So if we choose $\xi < \nu$, then

$$-\frac{(x-y)^2}{2\nu t} + \frac{x^2}{2\xi t} \geq -\frac{y^2}{2(\nu-\xi)t}$$

and thus for $t \in [\epsilon, +\infty[$,

$$\begin{aligned} g(t,x) &\geq \sqrt{\xi/\nu}\int_{\mathbb{R}} e^{-\frac{y^2}{2(\nu-\xi)t}}\mu(\mathrm{d}y) \geq \sqrt{\xi/\nu}\int_{\mathbb{R}} e^{-\frac{y^2}{2(\nu-\xi)\epsilon}}\mu(\mathrm{d}y) \\ &= \sqrt{2\pi(\nu-\xi)\xi\epsilon/\nu}\left(G_{\nu-\xi}(\epsilon,\cdot)*\mu\right)(0) := a_{\epsilon,\xi,\nu}\,, \end{aligned}$$

which proves the lemma. We finally remark that $\left(G_{\nu-\xi}(\epsilon,\cdot)*\mu\right)(0)$ is strictly positive and finite because $\mu \in \mathcal{M}_+(\mathbb{R})$, $\mu \neq 0$, and $G_{\nu-\xi}(\epsilon, y) > 0$. □

Proof of Theorem 2.2.10 (2). Due to Lemma 2.5.1, we only need to calculate the lower growth index of order 2. Denote the second moment by $f(t,x)$. Let us first assume that $\underline{\varsigma} = 0$. Fix $\epsilon > 0$. Choose $\xi \in]0,\nu[$ and $a = a_{\epsilon,\xi,\nu} > 0$ according to Lemma 2.5.2 such that

$$J_0(t,x) \geq I_{0,l}(t,x) := a\, 1_{[\epsilon,+\infty[}(t)\, G_\xi(t,x)\,.$$

Notice from (2.2.4) and (2.2.5) that the kernel $\underline{\mathcal{K}}(t,x)$ is bounded from below by

$$\underline{\mathcal{K}}(t,x) \geq \frac{l_\rho^4}{4\nu}K(t,x), \qquad \text{with } K(t,x) := G_{\frac{\nu}{2}}(t,x)\, e^{\frac{l_\rho^4\, t}{4\nu}}.$$

Then using the lower bound (2.2.13) of the second moments and the above two inequalities, we have

$$f(t,x) \geq J_0^2(t,x) + \left(J_0^2 \star \underline{\mathcal{K}}\right)(t,x) \geq \frac{l_\rho^4}{4\nu}\left(I_{0,l}^2 \star K\right)(t,x)\,.$$

Now we need to bound $\left(I_{0,l}^2 \star K\right)(t,x)$. Notice that $I_{0,l}^2(t,x) = \frac{a^2}{2\sqrt{\pi\xi t}}\, 1_{[\epsilon,+\infty[}(t)\, G_{\frac{\xi}{2}}(t,x)$. So by the semigroup property of the heat kernel,

$$\begin{aligned} \left(I_{0,l}^2 \star K\right)(t,x) &= \frac{a^2}{2}\int_\epsilon^t \mathrm{d}s\, \frac{e^{\frac{l_\rho^4(t-s)}{4\nu}}}{\sqrt{\pi\xi s}}\int_{\mathbb{R}} G_{\frac{\nu}{2}}\left(t-s, x-y\right) G_{\frac{\xi}{2}}\left(s,y\right)\mathrm{d}y \\ &= \frac{a^2}{2\sqrt{\pi\xi}} e^{\frac{l_\rho^4\, t}{4\nu}}\int_\epsilon^t G_{\frac{\nu}{2}}\left(t - \frac{(\nu-\xi)s}{\nu}, x\right)\frac{e^{-\frac{l_\rho^4\, s}{4\nu}}}{\sqrt{s}}\mathrm{d}s\,. \end{aligned}$$

Notice that for $s \in [\epsilon, t]$,

$$G_{\frac{\nu}{2}}\left(t - \frac{(\nu-\xi)s}{\nu}, x\right) = \frac{\exp\left\{-\frac{x^2}{\nu\left(t-\frac{(\nu-\xi)}{\nu}s\right)}\right\}}{\sqrt{\pi\nu\left(t-\frac{(\nu-\xi)s}{\nu}\right)}} \geq \frac{\exp\left\{-\frac{x^2}{\nu\left(t-\frac{(\nu-\xi)}{\nu}t\right)}\right\}}{\sqrt{\pi\nu\left(t-\frac{(\nu-\xi)\epsilon}{\nu}\right)}}$$

$$= G_{\frac{\xi}{2}}(t,x)\sqrt{\frac{\xi t}{\nu t-(\nu-\xi)\epsilon}}$$

and also

$$\int_\epsilon^t \frac{e^{-\frac{l_\rho^4 s}{4\nu}}}{\sqrt{s}}\mathrm{d}s \geq \frac{1}{\sqrt{t}}\int_\epsilon^t e^{-\frac{l_\rho^4 s}{4\nu}}\mathrm{d}s = \frac{4\nu}{l_\rho^4\sqrt{t}}\left(e^{-\frac{l_\rho^4 \epsilon}{4\nu}} - e^{-\frac{l_\rho^4 t}{4\nu}}\right).$$

So for large t,

$$\left(I_{0,l}^2 \star K\right)(t,x) \geq \frac{2a^2\nu}{l_\rho^4\sqrt{\pi\xi t}} G_{\frac{\xi}{2}}(t,x)\sqrt{\frac{\xi t}{\nu t-(\nu-\xi)\epsilon}}\left(e^{\frac{l_\rho^4(t-\epsilon)}{4\nu}}-1\right).$$

Thus

$$\begin{aligned}\lim_{t\to+\infty}\frac{1}{t}\sup_{|x|>\alpha t}\log f(t,x) &\geq \lim_{t\to+\infty}\frac{1}{t}\sup_{|x|>\alpha t}\log\left(e^{\frac{l_\rho^4(t-\epsilon)}{4\nu}}G_{\frac{\xi}{2}}(t,x)\right)\\ &= \lim_{t\to+\infty}\frac{1}{t}\log\left(e^{\frac{l_\rho^4(t-\epsilon)}{4\nu}}G_{\frac{\xi}{2}}(t,\alpha t)\right)\\ &= \lim_{t\to+\infty}\frac{1}{t}\log\left(e^{\frac{l_\rho^4(t-\epsilon)}{4\nu}-\frac{\alpha^2 t^2}{\xi t}}\right)\\ &= \frac{l_\rho^4}{4\nu}-\frac{\alpha^2}{\xi}.\end{aligned}$$

Therefore,

$$\begin{aligned}\underline{\lambda}(2) &= \sup\left\{\alpha>0: \lim_{t\to+\infty}\frac{1}{t}\sup_{|x|>\alpha t}\log f(t,x)>0\right\}\\ &\geq \sup\left\{\alpha>0: \frac{l_\rho^4}{4\nu}-\frac{\alpha^2}{\xi}>0\right\} = \sqrt{\xi/\nu}\,\frac{l_\rho^2}{2},\end{aligned}$$

for all $\xi \in]0,\nu[$, and so $\underline{\lambda}(2) \geq l_\rho^2/2$.

As for the case $\underline{\varsigma} \neq 0$, for all $\mu \in \mathscr{M}_+(\mathbb{R})$, the second moment is bounded from below by

$$f(t,x) \geq \underline{\varsigma}^2\mathscr{H}(t) = \underline{\varsigma}^2\exp\left\{\frac{l_\rho^4 t}{4\nu}\right\}\Phi\left(l_\rho^2\sqrt{\frac{t}{2\nu}}\right)-\underline{\varsigma}^2,$$

and hence

$$\lim_{t\to\infty}\frac{1}{t}\sup_{|x|\geq\alpha t}\log f(t,x) \geq \lim_{t\to\infty}\frac{1}{t}\log\left(\underline{\varsigma}^2\mathscr{H}(t)\right) = \frac{l_\rho^4}{4\nu}>0, \quad \text{for all } \alpha>0.$$

Therefore, $\underline{\lambda}(2)=\infty$, which implies $\overline{\lambda}(2)=\infty$. This completes the proof of (2). □

2.5.2 Proof of the Upper Bound

For $a > 0$ and $\beta \in \mathbb{R}$, define

$$E_{a,\beta}(x) := e^{-\beta x}\Phi\left(\frac{a\beta - x}{\sqrt{a}}\right) + e^{\beta x}\Phi\left(\frac{a\beta + x}{\sqrt{a}}\right).$$

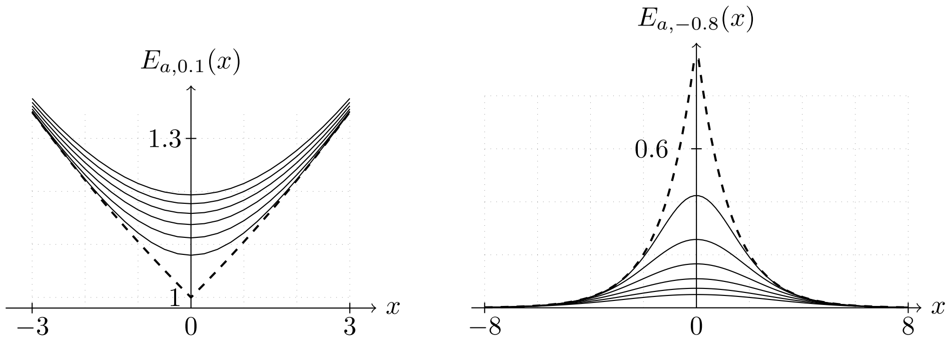
This is a smooth version of the continuous function $e^{\beta|x|}$: see Figure 2.3 below. Simple calculations show that

$$\begin{aligned}\left(e^{\beta|\cdot|} * G_\nu(t,\cdot)\right)(x) &= \int_{-\infty}^{0} e^{-\beta y} G_\nu(t, x-y)\,\mathrm{d}y + \int_{0}^{\infty} e^{\beta y} G_\nu(t, x-y)\,\mathrm{d}y \\ &= e^{\frac{\beta^2 \nu t}{2}}\left(e^{-\beta x}\Phi\left(\frac{\beta\nu t - x}{\sqrt{\nu t}}\right) + e^{\beta x}\Phi\left(\frac{\beta\nu t + x}{\sqrt{\nu t}}\right)\right) \\ &= e^{\frac{\beta^2 \nu t}{2}} E_{\nu t,\beta}(x), \end{aligned} \tag{2.5.1}$$

and so this function can be equivalently defined to be

$$E_{a,\beta}(x) = e^{-\beta^2 a/2}\left(e^{\beta|\cdot|} * G_a(1,\cdot)\right)(x), \tag{2.5.2}$$

where $G_a(t,x)$ is the one-dimensional heat kernel function (1.1.1). Note that the function $\left(e^{\beta|\cdot|} * G_\nu(t,\cdot)\right)(x)$ is the solution to the homogeneous heat equation (2.2.1) with initial condition $\mu(\mathrm{d}x) = e^{\beta|x|}\mathrm{d}x$.



(a) The case $\beta > 0$ ($\beta = 0.1$). (b) The case $\beta < 0$ ($\beta = -0.8$)

Figure 2.3 – Graphs of the function $E_{a,\beta}(x)$ for various parameters: The dashed lines in both figures denote the graph of $e^{\beta|x|}$. The solid lines from top to bottom are $E_{a,\beta}(x)$ with the parameter a ranging from 6 to 1. The parameter β controls the asymptotic behavior near infinity while both a and β determine how the function $e^{\beta|x|}$ is smoothed at zero. The larger a is, the closer $E_{a,\beta}(0)$ is to 1.

We need some properties of this function $E_{a,\beta}(x)$ which are summarized in the following proposition.

Proposition 2.5.3 (Properties of $E_{a,\beta}(x)$)**.** *For $a > 0$ and $\beta \in \mathbb{R}$,*

(i) $E_{a,0}(x) = 1$*;*

(ii) *for* $\nu > 0$, $\left(e^{\beta|\cdot|} * G_\nu(t,\cdot)\right)(x) = e^{\frac{\beta^2 \nu t}{2}} E_{\nu t,\beta}(x)$;

(iii) *we have the derivatives of* $E_{a,\beta}(x)$:

$$E'_{a,\beta}(x) = -\beta e^{-\beta x}\Phi\left(\frac{a\beta - x}{\sqrt{a}}\right) + \beta e^{\beta x}\Phi\left(\frac{a\beta + x}{\sqrt{a}}\right)$$
$$E''_{a,\beta}(x) = \beta\sqrt{\frac{2}{\pi a}}\, e^{-\frac{a^2\beta^2 + x^2}{2a}} + \beta^2 E_{a,\beta}(x)\,;$$

(iv) *for all* $\beta > 0$,

$$e^{\beta|x|} \le E_{a,\beta}(x) < e^{\beta x} + e^{-\beta x}\,;$$

and for $\beta < 0$,

$$\Phi\left(\sqrt{a}\,\beta\right) E^{1/2}_{a,2\beta}(x) \le E_{a,\beta}(x) \le e^{-|\beta x|}\,;$$

(v) *for* $\beta > 0$, *the function* $E_{a,\beta}(x)$ *is convex and has only one global minimum at zero:*

$$\inf_{x\in\mathbb{R}} E_{a,\beta}(x) = E_{a,\beta}(0) = 2\Phi(\beta\sqrt{a}) > 1$$

with $E''_{a,\beta}(0) = \beta\sqrt{\frac{2}{\pi a}} e^{-\frac{\beta^2 a}{2}} + 2\beta^2\Phi(\beta\sqrt{a}) > 0$; *for* $\beta < 0$, *the function* $E_{a,\beta}(x)$ *is decreasing for* $x \ge 0$ *and increasing for* $x < 0$, *and it therefore achieves its global maximum at zero*

$$\sup_{x\in\mathbb{R}} E_{a,\beta}(x) = E_{a,\beta}(0) = 2\Phi(\beta\sqrt{a}) < 1$$

with $E''_{a,\beta}(0) = \beta\sqrt{\frac{2}{\pi a}} e^{-\frac{\beta^2 a}{2}} + 2\beta^2\Phi(\beta\sqrt{a}) \le 0$;

(vi) *If* $E_{a,\beta}(x)$ *is viewed as a function mapping* $(a,\beta,x) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}$ *to* $\mathbb{R}$, *then*

$$\frac{\partial E_{a,\beta}(x)}{\partial a} = \beta\,\frac{\exp\left\{-\frac{a^2\beta^2 + x^2}{2a}\right\}}{\sqrt{2\pi a}}\,. \tag{2.5.3}$$

Hence, for all $x \in \mathbb{R}$, *then the function* $a \mapsto E_{a,\beta}(x)$ *is nondecreasing for* $\beta > 0$ *and nonincreasing for* $\beta < 0$.

Proof. (i) is trivial. (ii) is clear from (2.5.2). (iii) is routine. Now we prove (iv). Suppose that $\beta < 0$. We first prove the upper bound. Since $x \mapsto E_{a,\beta}(x)$ is an even function, we shall only consider $x \ge 0$. We need to show that for $x \ge 0$

$$e^{-\beta x}\Phi\left(\frac{a\beta - x}{\sqrt{a}}\right) + e^{\beta x}\Phi\left(\frac{a\beta + x}{\sqrt{a}}\right) \le e^{\beta x}$$

or equivalently from the fact that $1 - \Phi\left(\frac{a\beta + x}{\sqrt{a}}\right) = \Phi\left(\frac{-a\beta - x}{\sqrt{a}}\right)$,

$$F(x) := e^{\beta x}\Phi\left(\frac{-a\beta - x}{\sqrt{a}}\right) - e^{-\beta x}\Phi\left(\frac{a\beta - x}{\sqrt{a}}\right) \ge 0\,.$$

This is true since

$$F'(x) = \beta e^{\beta x}\Phi\left(\frac{-a\beta - x}{\sqrt{a}}\right) + \beta e^{-\beta x}\Phi\left(\frac{a\beta - x}{\sqrt{a}}\right) \le 0$$

and $\lim_{x\to+\infty} F(x) = 0$ by applying l'Hôpital's rule. Note that $F(0) = \Phi(-\sqrt{a}\beta) - \Phi(\sqrt{a}\beta) > 0$ since $\beta < 0$. As for the lower bound, when $\beta < 0$, we have that

$$E_{a,\beta}^2(x) = \left(e^{-\beta x}\Phi\left(\frac{a\beta - x}{\sqrt{a}}\right) + e^{\beta x}\Phi\left(\frac{a\beta + x}{\sqrt{a}}\right)\right)^2 \ge e^{-2|\beta x|}\Phi^2\left(\frac{a\beta + |x|}{\sqrt{a}}\right) \ge e^{-2|\beta x|}\Phi^2\left(\sqrt{a}\beta\right).$$

Then the lower bound follows from the fact that $e^{-2|\beta x|} \ge E_{a,2\beta}(x)$. As for the first part of (iv) where $\beta > 0$, the upper bound follows from $\Phi(\cdot) \le 1$. The derivation for the lower bound is exactly the same as the upper bound with $\beta < 0$ except changing some signs.

Now we shall prove (v). We first consider the case $\beta > 0$. By (iii), $E''_{a,\beta}(x) \ge 0$ for all $x \in \mathbb{R}$, hence $E_{a,\beta}(x)$ is globally convex. By (2.5.2), we have

$$\frac{\mathrm{d}}{\mathrm{d}x} E_{a,\beta}(x) = \beta e^{-a\beta^2/2} \int_0^\infty e^{\beta|y|}\left(G_a(1, x-y) - G_a(1, x+y)\right)\mathrm{d}y\,.$$

Clearly, if $x \ge (\le)0$, for all $y \ge 0$, $G_a(1, x-y) - G_a(1, x+y) \ge (\le)0$. Hence, $\frac{\mathrm{d}}{\mathrm{d}x} E_{a,\beta}(x) \ge (\le)0$ if $x \ge (\le)0$ and the global minimum is taken at $x = 0$. Similarly, for $\beta < 0$, we have $\frac{\mathrm{d}}{\mathrm{d}x} E_{a,\beta}(x) \le (\ge)0$ if $x \ge (\le)0$ and the global maximum is taken at $x = 0$, which then implies that $E''_{a,\beta}(0) \le 0$ (note that by (iii), $E''_{a,\beta}(x)$ exists).

As for (vi),

$$\begin{aligned}
\frac{\partial}{\partial a} e^{-\beta x}\Phi\left(\frac{a\beta - x}{\sqrt{a}}\right) &= e^{-\beta x}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(a\beta - x)^2}{2a}\right)\frac{\partial}{\partial a}\frac{a\beta - x}{\sqrt{a}} \\
&= \frac{a\beta + x}{2a^{3/2}\sqrt{2\pi}}\exp\left(-\frac{a^2\beta^2 + x^2}{2a}\right),
\end{aligned}$$

and similarly,

$$\frac{\partial}{\partial a} e^{\beta x}\Phi\left(\frac{a\beta + x}{\sqrt{a}}\right) = \frac{a\beta - x}{2a^{3/2}\sqrt{2\pi}}\exp\left(-\frac{a^2\beta^2 + x^2}{2a}\right).$$

Adding these two terms proves (2.5.3). The rest is clear. This completes the proof. □

Lemma 2.5.4. *For all $t > 0$, $s > 0$, $\beta > 0$ and $x \in \mathbb{R}$, denote*

$$H(x;\beta,t,s) := \sup_{(z_1,z_2)\in\mathbb{R}^2} G_{2\nu}(s, z_2 - z_1)\, G_{\frac{\nu}{2}}\left(t, x - \frac{z_1 + z_2}{2}\right)\exp\left(-\beta|z_1| - \beta|z_2|\right).$$

Then

$$H(x;\beta,t,s) \le \begin{cases} \dfrac{1}{2\pi\nu\sqrt{ts}}\exp\left(-\dfrac{x^2}{\nu t}\right) & \text{if } |x| \le \nu\beta t\,, \\ \\ \dfrac{1}{2\pi\nu\sqrt{ts}}\exp\left(-2\beta|x| + \nu\beta^2 t\right) & \text{if } |x| \ge \nu\beta t\,. \end{cases}$$

In particular,

$$H(x;\beta,t,s) \le \frac{1}{2\pi\nu\sqrt{ts}}\exp\left(-2\beta|x| + \nu\beta^2 t\right)\,, \tag{2.5.4}$$

for all $x \in \mathbb{R}$, $\beta > 0$, $t > 0$ and $s > 0$.

Proof. We only need to maximize the exponent

$$-\frac{(z_1 - z_2)^2}{4\nu s} - \frac{\left(x - \frac{z_1+z_2}{2}\right)^2}{\nu t} - \beta|z_1| - \beta|z_2|\,,$$

over $(z_1, z_2) \in \mathbb{R}^2$. By the change of variables $u = \frac{z_1 - z_2}{2}$, $w = \frac{z_1+z_2}{2}$, we need to minimize the expression

$$\frac{u^2}{\nu s} + \frac{(x-w)^2}{\nu t} + \beta\left(|u+w| + |u-w|\right)\,, \tag{2.5.5}$$

over $(u, w) \in \mathbb{R}^2$. Notice that $2|w| = |(u+w) - (u-w)| \le |u+w| + |u-w|$. So (2.5.5) is bounded from below by

$$\frac{u^2}{\nu s} + \frac{(x-w)^2}{\nu t} + 2\beta|w| \ge \frac{(x-w)^2}{\nu t} + 2\beta|w| := f(w)\,.$$

To minimize $f(w)$, we consider two cases:

$$f(w) = \begin{cases} \frac{1}{\nu t}\left(w - (x - \nu\beta t)\right)^2 + 2\beta x - \nu t\beta^2 & \text{if } w \ge 0\,, \\ \frac{1}{\nu t}\left(w - (x + \nu\beta t)\right)^2 - 2\beta x - \nu t\beta^2 & \text{if } w \le 0\,. \end{cases}$$

Hence,

$$\min_{w\in\mathbb{R}} f(w) = \begin{cases} \dfrac{x^2}{\nu t} & \text{if } |x| \le \nu\beta t\,, \\ 2\beta|x| - \nu t\beta^2 & \text{if } |x| \ge \nu\beta t\,. \end{cases}$$

This also implies (2.5.4) since $\frac{x^2}{\nu t} \ge 2\beta|x| - \nu t\beta^2$ for all $x \in \mathbb{R}$. □

Lemma 2.5.5. *Suppose $\mu \in \mathcal{M}_G^\beta(\mathbb{R})$ (recall (2.2.10)) with $\beta > 0$. Set $C = \int_{\mathbb{R}} e^{\beta|x|}|\mu|(\mathrm{d}x)$. Let $K(t,x) = G_{\nu/2}(t,x)h(t)$ for some non-negative function $h(t)$. Then we have*

$$J_0^2(t,x) \le \frac{C^2}{2\pi\nu t}\exp\left(-2\beta|x| + \nu\beta^2 t\right)\,, \tag{2.5.6}$$

$$\left(J_0^2 \star K\right)(t,x) \le \frac{C^2}{2\pi\nu\sqrt{t}}\exp\left(-2\beta|x| + \nu\beta^2 t\right)\int_0^t \frac{h(t-s)}{\sqrt{s}}\mathrm{d}s\,. \tag{2.5.7}$$

Proof. We first prove (2.5.6):

$$|J_0(t,x)| \le \int_{\mathbb{R}} G_\nu\left(t,x-y\right)|\mu|(\mathrm{d}y) \le \left(\sup_{y\in\mathbb{R}} G_\nu\left(t,x-y\right)e^{-\beta|y|}\right)\int_{\mathbb{R}} e^{\beta|x|}|\mu|(\mathrm{d}y)\,.$$

To find the supremum in the above inequality, it is equivalent to minimize

$$f(y) := \frac{(x-y)^2}{2\nu t} + \beta|y|\,,$$

over $y \in \mathbb{R}$. This has been done in the proof of Lemma 2.5.4. The proof of (2.5.7) is similar to Lemma 2.3.6. From (2.3.13) and Lemma 2.5.4, we have that

$$\begin{aligned}\left(J_0^2 \star K\right)(t,x) &\le \int_0^t \mathrm{d}s\, H(x;\beta,t,s)h(t-s) \iint_{\mathbb{R}^2} \exp\left(\beta|z_1|+\beta|z_2|\right)|\mu|(\mathrm{d}z_1)|\mu|(\mathrm{d}z_2)\\ &= \left(\int_{\mathbb{R}} e^{\beta|x|}|\mu|(\mathrm{d}x)\right)^2 \int_0^t H(x;\beta,t,s)h(t-s)\mathrm{d}s\,.\end{aligned}$$

Then apply (2.5.4). This completes the proof. □

Before the main proof, we remark that one can apply the bound in (2.3.10), which does not assume $\mu \in \mathcal{M}_G^\beta(\mathbb{R})$, to the upper bound (2.2.11) of the second moments, together with the above lemma, to get an estimate: $\bar{\lambda}(2) \le L_\rho^2/\sqrt{2}$. But we need a better estimate with $\sqrt{2}$ replaced by 2. This gap is due to the factor 2 in $J_0^*(2t,x)$ of (2.3.10) coming from an application of Lemma 2.3.8.

Proof of Theorem 2.2.10 (1). Assume that $\overline{\varsigma} = 0$.

Second order. We first consider the growth index of order 2. Set $f(t,x) = \mathbb{E}(u(t,x)^2)$. Without loss of generality, we can assume that μ is non-negative; otherwise, we can just replace all μ below by $|\mu|$.

Since

$$\overline{\mathcal{K}}(t,x) \le h(t)G_{\frac{\nu}{2}}(t,x)\,, \quad \text{with } h(t) := \frac{L_\rho^2}{\sqrt{4\pi\nu t}} + \frac{L_\rho^4}{2\nu}\exp\left(\frac{L_\rho^4\, t}{4\nu}\right)\,,$$

from (2.2.11), we have that

$$f(t,x) \le J_0^2(t,x) + \left(J_0^2(\cdot,\circ) \star G_{\frac{\nu}{2}}(\cdot,\circ)h(\cdot)\right)(t,x)\,.$$

Notice that

$$\int_0^t \frac{h(t-s)}{\sqrt{s}}\mathrm{d}s \le \frac{L_\rho^2\sqrt{\pi/\nu}}{2} + L_\rho^2\sqrt{\pi/\nu}\,\exp\left(\frac{L_\rho^4\, t}{4\nu}\right),$$

where we have used the Beta integral and the inequality (2.3.15). Apply Lemma 2.5.5 for $\mu \in \mathcal{M}_G^\beta(\mathbb{R})$ with $\beta > 0$:

$$f(t,x) \le \frac{C^2}{2\pi\nu t}\exp\left(\beta^2\nu t - 2\beta|x|\right)$$

$$+\frac{C^2 L_\rho^2}{2\pi^{1/2}\nu^{3/2}\sqrt{t}}\left(\frac{1}{2}+\exp\left(\frac{L_\rho^4\, t}{4\nu}\right)\right)\exp\left(-2\beta|x|+\nu\beta^2 t\right),$$

where $C=\int_{\mathbb{R}} e^{\beta|x|}|\mu|(\mathrm{d}x)$. Then, for $\alpha>0$,

$$\begin{aligned}\sup_{|x|>\alpha t} f(t,x) \le & \frac{C^2}{2\pi\nu t}\exp\left(\beta^2\nu t-2\beta\alpha t\right) \\ & +\frac{C^2 L_\rho^2}{2\pi^{1/2}\nu^{3/2}\sqrt{t}}\left(\frac{1}{2}+\exp\left(\frac{L_\rho^4\, t}{4\nu}\right)\right)\exp\left(-2\beta\alpha t+\nu\beta^2 t\right).\end{aligned}$$

Now, it is clear that the two exponents have the properties that

$$\begin{aligned}\beta^2\nu t-2\beta\alpha t<0 \quad &\Longleftrightarrow \quad \alpha>\frac{\beta\nu}{2} \quad \text{and,}\\ \frac{L_\rho^4\, t}{4\nu}-2\beta\alpha t+\nu\beta^2 t<0 \quad &\Longleftrightarrow \quad \alpha>\frac{\beta\nu}{2}+\frac{L_\rho^4}{8\nu\beta}.\end{aligned}$$

Hence,

$$\alpha>\frac{\beta\nu}{2}+\frac{L_\rho^4}{8\nu\beta} \quad \Longrightarrow \quad \lim_{t\to\infty}\frac{1}{t}\sup_{|x|>\alpha t}\log f(t,x)<0.$$

Therefore,

$$\bar{\lambda}(2)=\inf\left\{\alpha>0:\ \lim_{t\to\infty}\frac{1}{t}\sup_{|x|>\alpha t}\log f(t,x)<0\right\}\le\frac{\beta\nu}{2}+\frac{L_\rho^4}{8\nu\beta}.$$

Since the function $\beta\mapsto\frac{\beta\nu}{2}+\frac{L_\rho^4}{8\nu\beta}$ is decreasing for $\beta\le\frac{L_\rho^2}{2\nu}$ and increasing for $\beta\ge\frac{L_\rho^2}{2\nu}$, with minimum value $\frac{L_\rho^2}{2\nu}$, and $\mathcal{M}_G^{\beta}(\mathbb{R})\subseteq\mathcal{M}_G^{L_\rho^2/(2\nu)}(\mathbb{R})$ for $\beta\ge\frac{L_\rho^2}{2\nu}$, we have that

$$\overline{\lambda}(2)\le\begin{cases}\dfrac{\beta\nu}{2}+\dfrac{L_\rho^4}{8\nu\beta}, & \text{if}\quad 0\le\beta<\dfrac{L_\rho^2}{2\nu},\\ \dfrac{1}{2}L_\rho^2, & \text{if}\quad \beta\ge\dfrac{L_\rho^2}{2\nu}.\end{cases}$$

This completes the proof of the upper bound of $\overline{\lambda}(2)$.

Higher order. Due to Lemma 2.5.1, for all $p\ge 2$, we can bound $\overline{\lambda}(p)$ from above by $\overline{\lambda}(\lceil p\rceil_2)$ where $\lceil p\rceil_2:=2\lceil p/2\rceil$ is the smallest even integer not less than p. So in the following, we shall assume that p is an even integer greater than 2.

Notice that

$$\begin{aligned}\bar{\lambda}(p)&=\inf\left\{\alpha>0:\ \lim_{t\to\infty}\frac{1}{t}\sup_{|x|>\alpha t}\log||u(t,x)||_p^p<0\right\}\\ &=\inf\left\{\alpha>0:\ \lim_{t\to\infty}\frac{1}{t}\sup_{|x|>\alpha t}\log||u(t,x)||_p^2<0\right\}.\end{aligned}$$

The remainder of the proof is similar to the previous case. We only need to make the following changes:

1. replace the second moment $f(t,x)$ by $||u(t,x)||_p^2$;
2. replace $J_0^2(t,x)$ by $2J_0^2(t,x)$;
3. replace the kernel function $\overline{\mathcal{K}}(t,x)$ by $\widehat{\mathcal{K}}_p(t,x)$. This is equivalent to replacing L_ρ everywhere by $\sqrt{2}\, z_p\, L_\rho$, where we have used the fact that $a_{p,0} = \sqrt{2}$.

This completes the whole proof of (1). $\square$

2.5.3 Proof of Proposition 2.2.12

Lemma 2.5.6. *We have the following approximations*

$$\Phi(x) \to \begin{cases} 1 - \dfrac{e^{-x^2/2}}{\sqrt{2\pi}\, x} & x \to +\infty\,, \\[2ex] \dfrac{e^{-x^2/2}}{\sqrt{2\pi}\, |x|} & x \to -\infty\,. \end{cases}$$

Proof. Notice that $\Phi(x) = \frac{1}{2}\left(1+\mathrm{erf}\left(x/\sqrt{2}\right)\right)$. Then use the asymptotic expansions of $\mathrm{erfc}(x) = 1-\mathrm{erf}(x)$ function: see [51, 7.12.1, on p. 164], or [50, 40:9:1, in p. 409]. $\square$

Proof of Proposition 2.2.12. For the initial data $\mu(\mathrm{d}x) = e^{-\beta|x|}\mathrm{d}x$ with $\beta > 0$, by (2.5.2), we have

$$J_0(t,x) = \left(\mu * G_\nu(t,\cdot)\right)(x) = e^{\beta^2 \nu t/2} E_{\nu t,-\beta}(x)\,.$$

Then by Proposition 2.5.3 (iv)

$$e^{\beta^2 \nu t}\Phi^2\left(-\beta\sqrt{\nu t}\right) E_{\nu t,-2\beta}(x) \le J_0^2(t,x) \le e^{\beta^2 \nu t - 2|\beta x|}\,. \tag{2.5.8}$$

In the following, we use $f(t,x)$ to denote the second moment.

Upper bound. The proof of the upper bound is straightforward. By the moment formula (2.2.15) and the upper bound in (2.5.8),

$$f(t,x) \le e^{\beta^2 \nu t - 2\beta|x|} + \int_0^t e^{\beta^2 \nu(t-s)}\left(\frac{\lambda^2}{\sqrt{4\pi\nu s}} + \frac{\lambda^4}{2\nu} e^{\frac{\lambda^4 s}{4\nu}}\Phi\left(\lambda^2\sqrt{\frac{s}{2\nu}}\right)\right)\left(e^{-2\beta|\cdot|} * G_{\nu/2}(s,\cdot)\right)(x)\mathrm{d}s.$$

Since by Proposition 2.5.3 (iv) the convolution part can be bounded by

$$\left(e^{-2\beta|\cdot|} * G_{\nu/2}(s,\cdot)\right)(x) = e^{\beta^2 \nu s} E_{\frac{\nu s}{2},-2\beta}(x) \le e^{\beta^2 \nu s - 2\beta|x|}\,,$$

there is some constant C such that

$$f(t,x) \le e^{\beta^2 \nu t - 2\beta|x|} + e^{\beta^2 \nu t - 2\beta|x|}\int_0^t \left(\frac{\lambda^2}{\sqrt{4\pi\nu s}} + \frac{\lambda^4}{2\nu} e^{\frac{\lambda^4 s}{4\nu}}\Phi\left(\lambda^2\sqrt{\frac{s}{2\nu}}\right)\right)\mathrm{d}s$$

$$\leq C\, e^{\beta^2 \nu t - 2\beta|x| + \frac{\lambda^4 t}{4\nu}}\,.$$

Therefore, for $\alpha \geq 0$,

$$\lim_{t\to\infty} \frac{1}{t} \sup_{|x|>\alpha t} \log f(t,x) \leq -2\beta\alpha + \beta^2\nu + \frac{\lambda^4}{4\nu}\,,$$

and

$$\overline{\lambda}(2) \leq \inf\left\{\alpha > 0, -2\beta\alpha + \beta^2\nu + \frac{\lambda^4}{4\nu} < 0\right\} = \frac{\nu\beta}{2} + \frac{\lambda^4}{8\nu\beta}\,.$$

This yields all the upper bounds in Proposition 2.2.12.

Lower bound. Now we consider the lower bound. By (2.5.2),

$$E_{\nu t,-2\beta}(x) = e^{-2\beta^2\nu t}\left(e^{-2\beta|\cdot|} * G_\nu(t,\cdot)\right)(x)\,,$$

and hence by the lower bound on $J_0^2(t,x)$ in (2.5.8),

$$J_0^2(t,x) \geq e^{-\beta^2\nu t}\Phi^2\left(-\beta\sqrt{\nu t}\right)\left(e^{-2\beta|\cdot|} * G_\nu(t,\cdot)\right)(x)\,.$$

So, by the moment formula (2.2.15) and the fact that $\mathcal{K}(t,x) \geq \frac{\lambda^4}{2\nu} G_{\nu/2}(t,x)\exp\left(\frac{\lambda^4 t}{4\nu}\right)$, we have

$$\begin{aligned} f(t,x) &\geq J_0^2(t,x) + \int_0^t e^{-\beta^2\nu(t-s)}\Phi^2\left(-\beta\sqrt{\nu(t-s)}\right)\frac{\lambda^4}{4\nu}e^{\frac{\lambda^4 s}{4\nu}}\left(e^{-2\beta|\cdot|} * G_\nu(t-s,\cdot) * G_{\nu/2}(s,\cdot)\right)(x)\mathrm{d}s \\ &\geq \int_0^t e^{-\beta^2\nu(t-s)}\Phi^2\left(-\beta\sqrt{\nu(t-s)}\right)\frac{\lambda^4}{4\nu}e^{\frac{\lambda^4 s}{4\nu}}\left(e^{-2\beta|\cdot|} * G_\nu(t-s/2,\cdot)\right)(x)\mathrm{d}s\,, \end{aligned}$$

where we have applied the semigroup property of the heat kernel in the last step. Noticing that by Proposition 2.5.3 (ii) and (vi),

$$\begin{aligned} \left(e^{-2\beta|\cdot|} * G_\nu(t-s/2,\cdot)\right)(x) &= e^{2\beta^2\nu(t-s/2)}E_{\nu(t-s/2),-2\beta}(x) \\ &\geq e^{2\beta^2\nu(t-s/2)}E_{\nu t/2,-2\beta}(x)\,, \end{aligned}$$

we have

$$f(t,x) \geq E_{\nu t/2,-2\beta}(x)\, e^{\beta^2\nu t}\,\frac{\lambda^4}{4\nu}\int_0^t \Phi^2\left(-\beta\sqrt{\nu(t-s)}\right)e^{\frac{\lambda^4 s}{4\nu}}\mathrm{d}s.$$

Choose an arbitrary constant $c \in [0,1[$. The above integral is bounded by

$$\begin{aligned} \frac{\lambda^4}{4\nu}\int_0^t \Phi^2\left(-\beta\sqrt{\nu(t-s)}\right)e^{\frac{\lambda^4 s}{4\nu}}\mathrm{d}s &\geq \Phi^2\left(-\beta\sqrt{\nu(1-c)t}\right)\int_{ct}^t \frac{\lambda^4}{4\nu}e^{\frac{\lambda^4 s}{4\nu}}\mathrm{d}s \\ &= \Phi^2\left(-\beta\sqrt{\nu(1-c)t}\right)\left(e^{\frac{\lambda^4 t}{4\nu}} - e^{\frac{c\lambda^4 t}{4\nu}}\right). \end{aligned}$$

Hence,

$$f(t,x) \ge E_{\nu t/2,-2\beta}(x)\, e^{\beta^2 \nu t}\Phi^2\left(-\beta\sqrt{\nu(1-c)t}\right)\left(e^{\frac{\lambda^4 t}{4\nu}} - e^{\frac{c\lambda^4 t}{4\nu}}\right).$$

By Proposition 2.5.3 (v), for $\alpha > 0$,

$$\sup_{|x|>\alpha t} E_{\nu t/2,-2\beta}(x) = E_{\nu t/2,-2\beta}(\alpha t)\,.$$

Notice that

$$E_{\nu t/2,-2\beta}(\alpha t) = e^{2\beta\alpha t}\Phi\left(-\left[2\beta\sqrt{\nu/2} + \frac{\alpha}{\sqrt{\nu/2}}\right]\sqrt{t}\right) + e^{-2\beta\alpha t}\Phi\left(\left[\frac{\alpha}{\sqrt{\nu/2}} - 2\beta\sqrt{\nu/2}\right]\sqrt{t}\right).$$

If $\frac{\alpha}{\sqrt{\nu/2}} - 2\beta\sqrt{\nu/2} \ge 0$, i.e., $\alpha \ge \beta\nu$, then by Lemma 2.5.6 the second term dominates and

$$E_{\nu t/2,-2\beta}(\alpha t) \ge e^{-2\beta\alpha t}\Phi\left(\left[\frac{\alpha}{\sqrt{\nu/2}} - 2\beta\sqrt{\nu/2}\right]\sqrt{t}\right) \ge \frac{1}{2}e^{-2\beta\alpha t}\,.$$

Otherwise, if $\alpha < \beta\nu$, then by Lemma 2.5.6, for large t,

$$e^{2\beta\alpha t}\Phi\left(-\left[\frac{\alpha}{\sqrt{\nu/2}} + 2\beta\sqrt{\nu/2}\right]\sqrt{t}\right) \approx \frac{\sqrt{\nu}\,\exp\left\{-\left(\beta^2\nu + \frac{\alpha^2}{\nu}\right)t\right\}}{2\sqrt{\pi}\,|\alpha+\beta\nu|\sqrt{t}},$$

and

$$e^{-2\beta\alpha t}\Phi\left(\left[\frac{\alpha}{\sqrt{\nu/2}} - 2\beta\sqrt{\nu/2}\right]\sqrt{t}\right) \approx \frac{\sqrt{\nu}\,\exp\left\{-\left(\beta^2\nu + \frac{\alpha^2}{\nu}\right)t\right\}}{2\sqrt{\pi}\,|\alpha-\beta\nu|\sqrt{t}}.$$

So $E_{\nu t/2,-2\beta}(\alpha t)$ has lower bounds with the following exponents

$$\begin{cases} -2\beta\alpha t & \text{if } \alpha \ge \beta\nu, \\ -\left(\beta^2\nu + \frac{\alpha^2}{\nu}\right)t & \text{if } \alpha < \beta\nu. \end{cases}$$

For large t, by Lemma 2.5.6, the function $t \mapsto \Phi^2\left(-\beta\sqrt{\nu(1-c)t}\right)$ contributes to an exponent $\beta^2\nu(c-1)t$. Therefore,

$$\lim_{t\to\infty}\frac{1}{t}\sup_{|x|>\alpha t}\log f(t,x) \ge \begin{cases} c\beta^2\nu + \dfrac{\lambda^4}{4\nu} - 2\beta\alpha, & \text{if } \alpha \ge \beta\nu\,, \\[2ex] (c-1)\beta^2\nu + \dfrac{\lambda^4}{4\nu} - \dfrac{\alpha^2}{\nu}, & \text{if } \alpha < \beta\nu\,. \end{cases}$$

If $\alpha \ge \beta\nu$, then

$$\underline{\lambda}(2) \ge \sup\left\{\alpha > 0 : c\beta^2\nu + \frac{\lambda^4}{4\nu} - 2\beta\alpha > 0\right\} = \frac{c\nu\beta}{2} + \frac{\lambda^4}{8\nu\beta},$$

which is valid if

$$\frac{c\nu\beta}{2} + \frac{\lambda^4}{8\nu\beta} \ge \beta\nu \quad\Longleftrightarrow\quad \beta \le \frac{\lambda^2}{2\nu\sqrt{2-c}}\,.$$

If $\alpha \le \beta\nu$, then

$$\underline{\lambda}(2) \ge \sup\left\{\alpha > 0 : (c-1)\beta^2\nu + \frac{\lambda^4}{4\nu} - \frac{\alpha^2}{\nu} > 0\right\} = \sqrt{\frac{\lambda^4}{4} + (c-1)\beta^2\nu^2}\,,$$

which is valid if

$$\sqrt{\frac{\lambda^4}{4} + (c-1)\beta^2\nu^2} \le \beta\nu \quad\Longleftrightarrow\quad \beta \ge \frac{\lambda^2}{2\nu\sqrt{2-c}}\,.$$

Finally, since the constant c can be arbitrarily close to 1, this completes the proof. □

2.6 Hölder Continuity

If the initial data is bounded, then the solution u is bounded in $L^p(\Omega)$ for all $p \ge 2$ by the moment estimates (2.2.11) in the sense that

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} ||u(t,x)||_p < +\infty\,, \quad \text{for all } T > 0\,.$$

Then Proposition 2.4.2 (b) implies u is jointly a.s. Hölder continuous:

$$u \in C_{1/4-,1/2-}\left(\mathbb{R}_+^* \times \mathbb{R}\right)\,, \quad \text{a.s.}$$

We will extend this classical result to the case where the initial data can be unbounded either at one point, like δ_0, or at $\pm\infty$, like $\mu(\mathrm{d}x) = e^{|x|}\mathrm{d}x$. The only requirement on the initial data is the hypothesis (1.1.5).

2.6.1 Kolmogorov's Continuity Theorem

This part is a completion of the corresponding part of the mini-course [42, Section 4.2]. Let τ be a metric on $\mathbb{R}^N$. Recall that $\tau : \mathbb{R}^N \times \mathbb{R}^N \mapsto \mathbb{R}_+$ is called a *metric* if

1. $\tau(x,y) \ge 0$,
2. $\tau(x,y) = 0$ if and only if $x = y$,
3. $\tau(x,y) = \tau(y,x)$,
4. $\tau(x,z) \le \tau(x,y) + \tau(x,z)$.

Clearly, any l^p-norm $p \in [1,+\infty]$ on $x \in \mathbb{R}^N$ induces a metric:

$$\tau(x,y) := \begin{cases} \left(|x_1 - y_1|^p + \cdots |x_N - y_N|^p\right)^{1/p} & \text{if } p \in [0,+\infty[\,, \\ \max_{i=1,\dots,N} |x_i - y_i| & \text{if } p = +\infty\,. \end{cases}$$

The following metric

$$\tau_{\alpha_1,\ldots,\alpha_N}(x,y) := \sum_{i=1}^{N} \left|x_i - y_i\right|^{\alpha_i}, \quad \text{with } \alpha_1,\ldots,\alpha_N \in \,]0,1], \tag{2.6.1}$$

is not induced from a norm except the case where all $\alpha_i = 1$.

Theorem 2.6.1. *([42, Theorem 4.3]) Suppose $\{X(\mathbf{t})\}_{\mathbf{t}\in T}$ is a stochastic process indexed by a compact cube $T := [a_1,b_1]\times\cdots\times[a_N,b_N]\subset\mathbb{R}^N$. Suppose also that there exist constants $C>0$, $p>0$, and $\gamma>N$ such that uniformly for all $\mathbf{s},\mathbf{t}\in T$,*

$$\mathbb{E}\left(|X(\mathbf{t})-X(\mathbf{s})|^p\right) \le C\,\tau^{\gamma}(\mathbf{t},\mathbf{s})\ .$$

Then X has a continuous modification $\overline{X}$. Moreover, if $0\le\theta<(\gamma-N)/p$, then

$$\left\|\sup_{\mathbf{s}\neq\mathbf{t}} \frac{\left|\overline{X}(\mathbf{s})-\overline{X}(\mathbf{t})\right|}{\tau^{\theta}(\mathbf{s},\mathbf{t})}\right\|_p < +\infty. \tag{2.6.2}$$

For the proof of this theorem, we refer the interested readers to [42, Theorem 4.3] or [60, Theorem 2.1, in p. 62] for the isotropic cases (τ is induced by an l^p-norm or is of form (2.6.1) with $\alpha_1=\cdots=\alpha_N$)[3].

For the anisotropic case (τ is of the form 2.6.1 where α_i are not identical), we refer to [43, Theorem 1.4.1, p. 31] and [28, Corollary A.3, p. 34]. Since we are interested in the case where the random field is indexed by the open domain $\mathbb{R}_+^*\times\mathbb{R}^d$ and it has all p-th moments, we formulate a convenient version – Proposition 2.6.4 – for our applications.

Definition 2.6.2. (Hölder continuity) A function $f: D\mapsto\mathbb{R}$ with $D\subseteq\mathbb{R}^N$ is said to be *(globally and uniformly) Hölder continuous* with indices $(\alpha_1,\ldots,\alpha_N)$, $\alpha_i>0$, $i=1,\ldots,N$, if there exists a constant A such that

$$\left|f(x)-f(y)\right| \le A\sum_{i=1}^{N}|x_i-y_i|^{\alpha_i}, \quad \text{for all } x,y\in D.$$

It is said to be *locally (and uniformly) Hölder continuous* with indices $(\alpha_1,\ldots,\alpha_N)$ if for

[3] Here we point out two typos in the proof of [42, Theorem 4.3]. In particular, (39) should be

$$\left\|\sup_{\substack{u,v\in\mathscr{D}_\infty\\ \tau(u,v)\le 2^{-k}}} |X(u)-X(v)|\right\|_{L^p(P)} \le \frac{\widetilde{C}}{2^{k(\gamma-1)/p}}$$

and (42) should be

$$\left\|\sup_{\substack{0\le s\neq t\le 1:\\ 2^{-k}<\tau(s,t)\le 2^{-k+1}}} \frac{\left|\overline{X}(s)-\overline{X}(t)\right|}{\tau^{\theta}(s,t)}\right\|_{L^p(P)} \le \frac{\widetilde{C}}{2^{k[(\gamma-1)/p-\theta]}}\ .$$

all compact sets $K \subseteq D$ there exists a constant A_K such that

$$\left|f(x)-f(y)\right| \le A_K \sum_{i=1}^{N} |x_i - y_i|^{\alpha_i}, \quad \text{for all } x, y \in K.$$

The following elementary result relates the moment statement in (2.6.2) with the definition of Hölder continuity. It comes from [42, Exercise 4.7, on p. 12].

Proposition 2.6.3. *Under the conditions of Theorem 2.6.1 with the metric $\tau_{\alpha_1,\dots,\alpha_N}$ defined in* (2.6.1) *where $\alpha_1,\dots,\alpha_N \in]0,1]$, X has a modification which is pathwise locally Hölder continuous with index $(\beta\alpha_1,\dots,\beta\alpha_N)$ for all $\beta \in \left]0,(\gamma-N)/p\right[$.*

Proof. Fix an arbitrary $\beta \in \left]0,(\gamma-N)/p\right[$. We only need to prove that the continuous version $\overline{X}$ in Theorem 2.6.1 has a modification of $(\beta\alpha_1,\dots,\beta\alpha_N)$-Hölder continuity. (2.6.2) implies that

$$\sup_{\mathbf{s}\neq\mathbf{t}} \frac{\left|\overline{X}(\mathbf{s})-\overline{X}(\mathbf{t})\right|}{\tau^{\beta}_{\alpha_1,\dots,\alpha_N}(\mathbf{s},\mathbf{t})} < +\infty, \quad a.s..$$

So for some sample space Ω_0 with $P(\Omega_0)=1$, the above inequality is true for each $\omega \in \Omega_0$. Hence, we can define

$$\widetilde{X}(\omega) = \begin{cases} \overline{X}(\omega) & \text{if } \omega \in \Omega_0, \\ 0 & \text{otherwise}, \end{cases} \qquad \text{and} \qquad A(\omega) := 1_{\Omega_0}(\omega) \times \sup_{\mathbf{s}\neq\mathbf{t}} \frac{\left|\overline{X}(\mathbf{s},\omega)-\overline{X}(\mathbf{t},\omega)\right|}{\tau^{\beta}_{\alpha_1,\dots,\alpha_N}(\mathbf{s},\mathbf{t})}.$$

If $\beta \in]0,1]$, by the subadditivity of the function $x \mapsto |x|^\beta$, we have, for each $\omega \in \Omega_0$,

$$\left|\overline{X}(\mathbf{s},\omega)-\overline{X}(\mathbf{t},\omega)\right| \le A(\omega) \sum_{i=1}^{N} |s_i - t_i|^{\beta\alpha_i}, \quad \text{for all } \mathbf{s},\mathbf{t} \in K;$$

otherwise, if $\beta > 1$, by the convexity of the function $x \mapsto |x|^\beta$, we have

$$\left|\overline{X}(\mathbf{s},\omega)-\overline{X}(\mathbf{t},\omega)\right| \le N^{\beta-1} A(\omega) \sum_{i=1}^{N} |s_i - t_i|^{\beta\alpha_i}, \quad \text{for all } \mathbf{s},\mathbf{t} \in K.$$

By the definition 2.6.2, $\widetilde{X}$ is pathwise $(\beta\alpha_1,\dots,\beta\alpha_N)$-Hölder continuous. Clearly, $\widetilde{X}$ is a modification of X. This completes the proof. □

Proposition 2.6.4. *Let $\left\{X(t,x) : (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d\right\}$ be a random field indexed by $\mathbb{R}_+ \times \mathbb{R}^d$. Suppose that there exist $d+1$ constants $\alpha_i \in]0,1]$ with $i = 0,1,\dots,d$ such that for all $p > 2(d+1)$ and all $n > 1$, there is a constant $C_{p,n}$ such that*

$$\left|\left|X(t,x)-X(s,y)\right|\right|_p^2 \le C_{p,n}\tau_{\alpha_0,\dots,\alpha_d}\left((t,x),(s,y)\right) \tag{2.6.3}$$

for all $(t,x),(s,y) \in K_n := [1/n,n]\times[-n,n]^d$, where the metric $\tau_{\alpha_0,\dots,\alpha_d}$ is defined in (2.6.1)*. Then X has a modification which is locally Hölder continuous with indices $(\beta\alpha_0,\dots,\beta\alpha_d)$ for all $\beta \in \left]0,1/2\right[$ over the domain $\mathbb{R}_+^* \times \mathbb{R}^d$. Moreover, if the compact sets K_n can be chosen as $[0,n]\times[-n,n]^d$, then the same Hölder continuity of X can be extended to the domain*

$\mathbb{R}_+ \times \mathbb{R}^d$.

Proof. For all compact sets $K \in \mathbb{R}_+^* \times \mathbb{R}^d$, there exists a K_n with $n > 1$ such that $K \subset K_n$. The condition (2.6.3) is equivalent to

$$\mathbb{E}\left[\left|X(t,x) - X\left(s,y\right)\right|^p\right] \le C_{p,n}^{p/2} \tau_{\alpha_0,\dots,\alpha_d}^{p/2}\left((t,x),\left(s,y\right)\right) .$$

By Proposition 2.6.3, X restricted on K_n has a modification $\overline{X}_n$ which is pathwise $\left(\beta\alpha_0,\dots,\beta\alpha_d\right)$-Hölder continuous for all $\beta \in \left]0,\left(p/2-d-1\right)/p\right[$. Since p can be arbitrarily large, β can be chosen to be any values in $]0,1/2[$. Clearly, two modifications $\overline{X}_n^{(1)}$ and $\overline{X}_n^{(2)}$ on K_n are indistinguishable since they are pathwise continuous. Denote the sample space by Ω_n (clearly, $P(\Omega_n) = 1$) on which the modification $\overline{X}_n$ is defined. Let $\Omega_0 := \bigcap_{n\in N, n>1} \Omega_n$. Clearly, $P(\Omega_0) = 1$. Hence, the following random field is well-defined:

$$\overline{X}(t,x,\omega) := \begin{cases} \overline{X}_n(t,x,\omega) & \text{if } \omega \in \Omega_0 \text{ and } (t,x) \in K_n \setminus K_{n-1}\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

$\overline{X}$ is a modification of X because for each $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}^d$, there is a $K_n \ni (t,x)$ such that

$$P\left(X(t,x) = \overline{X}_n(t,x)\right) = 1, \quad P\left(\overline{X}(t,x) = \overline{X}_n(t,x)\right) = 1 \quad \Rightarrow \quad P\left(X(t,x) = \overline{X}(t,x)\right) = 1\,.$$

Now fix $\beta \in \ ]0,1/2[$. We need to show that $\overline{X}$ is pathwise locally $\left(\beta\alpha_0,\dots,\beta\alpha_d\right)$-Hölder continuous. Clearly, if $\omega \in \Omega \setminus \Omega_0$, then $\overline{X}(t,x,\omega) \equiv 0$ is trivially continuous. Otherwise, fix $\omega \in \Omega_0$. For all compact set $K \in \mathbb{R}_+^* \times \mathbb{R}^d$, choose $K_n \supseteq K$. By definition, $\overline{X}(t,x,\omega) = \overline{X}_n(t,x,\omega)$ for all $(t,x) \in K_n$. Then $\left(\beta\alpha_0,\dots,\beta\alpha_d\right)$-Hölder continuity of $\overline{X}_n$ implies that for some constant $A_K(\omega) < +\infty$,

$$\left|\overline{X}(t,x,\omega) - \overline{X}(s,y,\omega)\right| = \left|\overline{X}_n(t,x,\omega) - \overline{X}_n(s,y,\omega)\right| \le A_K(\omega)\left(|t-s|^{\beta\alpha_0} + \sum_{i=1}^d \left|x_i - y_i\right|^{\beta\alpha_i}\right),$$

for all $(t,x), \left(s,y\right) \in K \subseteq K_n$. This completes the proof. □

2.6.2 Some Technical Lemmas

Lemma 2.6.5. *For* $0 \le s \le t$ *and* $x, y \in \mathbb{R}$, *we have*

$$\int_0^t G_\nu(s,x) G_\sigma(t-s,y) \mathrm{d}s = \frac{1}{2\sqrt{\nu\sigma}} \operatorname{erfc}\left(\frac{1}{\sqrt{2t}}\left(\frac{|x|}{\sqrt{\nu}} + \frac{|y|}{\sqrt{\sigma}}\right)\right),$$

where ν *and* σ *are strictly positive. In particular, by letting* $x = 0$, *we have*

$$\int_0^t \frac{G_\sigma(t-s,y)}{\sqrt{2\pi\nu s}} \mathrm{d}s = \frac{1}{2\sqrt{\nu\sigma}} \operatorname{erfc}\left(\frac{|y|}{\sqrt{2\sigma t}}\right) \le \frac{\sqrt{\pi t}}{\sqrt{2\nu}} G_\sigma\left(t,y\right).$$

Proof. Denote the convolution by $I(t)$. By the Laplace transform (see [35, (27), Chapter

4.5, p. 146]) $\mathscr{L}\left[G_\nu(\cdot,x)\right](z) = \frac{\exp\left(-\sqrt{2z/\nu}\,|x|\right)}{\sqrt{2z\nu}}$, we have

$$\mathscr{L}[I](z) = \mathscr{L}\left[G_\nu(\cdot,x)\right](z)\cdot\mathscr{L}\left[G_\sigma(\cdot,y)\right](z) = \frac{\exp\left(-\sqrt{2z}\left(\frac{|x|}{\sqrt{\nu}}+\frac{|y|}{\sqrt{\sigma}}\right)\right)}{2\sqrt{\nu\sigma z^2}}\,.$$

Then the lemma is proved by applying the inverse Laplace transform (see [35, (3), Chapter 5.6, p. 245]). As for the special case $x=0$, we only need to prove the inequality. By [51, (7.7.1), p. 162], we have

$$\operatorname{erfc}(x) = \frac{2}{\pi}e^{-x^2}\int_0^\infty \frac{e^{-x^2t^2}}{1+t^2}\mathrm{d}t \le \frac{2}{\pi}e^{-x^2}\int_0^\infty\frac{1}{1+t^2}\mathrm{d}t = e^{-x^2}\,,$$

and so

$$\operatorname{erfc}\left(\frac{|x|}{\sqrt{2\sigma t}}\right) \le \exp\left(-\frac{x^2}{2\sigma t}\right) = \sqrt{2\pi\sigma t}\,G_\sigma(t,x)\,, \tag{2.6.4}$$

which finishes the proof. □

Lemma 2.6.6 (Bellman-Gronwall inequality, Lemma 10.2.2 of [44])**.** *If $\psi\in L^1[a,b]$ and*

$$\psi(t)\le f(t)+\beta\int_a^t\psi(s)\mathrm{d}s\,,\quad \textit{for all } t\in[a,b]\,,$$

where f is measurable, then

$$\psi(t)\le f(t)+\beta\int_a^t f(s)\,e^{\beta(t-s)}\mathrm{d}s\,.$$

In particular, when $f(t)$ is a constant C, we have

$$\psi(t)\le C\,e^{\beta(t-a)},\quad \textit{for all } t\in[a,b]\,.$$

Lemma 2.6.7. $\sup_{z\in\mathbb{R}}|1-e^{-z^2/2}|/|z|\approx 0.451256.$

Proof. Let $f(z)=\frac{1-e^{-z^2/2}}{z}$ for $z\ne 0$ and $f(0):=\lim_{z\to0}\frac{1-e^{-z^2/2}}{z}=0$. It is clear that $z\mapsto f(z)$ is continuous over the extended real line $\mathbb{R}\cup\{\pm\infty\}$ with $\lim_{z\to0}f(z)=\lim_{z\to\pm\infty}f(z)=0$. Hence, $\sup_{z\in\mathbb{R}}\left|f(z)\right|<+\infty$. We cannot calculate this supremum explicitly. Some numerics show that $\sup_{z\in\mathbb{R}}f(z)\approx f(\pm1.5852)\approx 0.451256$. This completes the proof. □

Proposition 2.6.8. *For all $L>0$, $\beta\in\,]0,1[$ and $t>0$, there are two constants $C'_{L,\beta,\nu t}>0$ and $C''_{L,\beta,\nu t}>0$ such that for all $x\in\mathbb{R}$, $\nu>0$ and all h with $|h|\le\beta L$, we have*

$$\begin{aligned}|G_\nu(t,x+h)-G_\nu(t,x)|&\\ \le\Big[C'_{L,\beta,\nu t}G_\nu(t,x)&+C''_{L,\beta,\nu t}\left(G_\nu(t,x-2L)+G_\nu(t,x+2L)\right)\Big]\,|h|\,,\end{aligned} \tag{2.6.5}$$

and

$$|G_\nu(t,x+h)+G_\nu(t,x-h)-2G_\nu(t,x)| \le \left[C'_{L,\beta,\nu t}G_\nu(t,x)+C''_{L,\beta,\nu t}\left(G_\nu\left(t,x-2L\right)+G_\nu\left(t,x+2L\right)\right)\right]|h|\,. \quad (2.6.6)$$

In particular, these constants can be taken as

$$C'_{L,\beta,\nu t}=\frac{C}{\sqrt{2\nu t}}+\frac{1}{(1-\beta)L}\,,\qquad C''_{L,\beta,\nu t}=C'_{L,\beta,\nu t}\exp\left(\frac{2L^2}{\nu t}\right),$$

and $C:=\sup_{x\in\mathbb{R}}\frac{1}{|x|}\left|e^{-x^2/2}-1\right|\approx 0.451256.$

Proof. Fix $L>0$ and $\beta\in\,]0,1[$. Assume that $|h|\le\beta L$. Define

$$f(t,x,h)=G_\nu(t,x+h)+G_\nu(t,x-h)-2G_\nu(t,x)\,,$$

and

$$I(t,x,h)=\begin{cases} h^{-1}\,G_\nu^{-1}(t,x-L)\left[G_\nu(t,x+h)-G_\nu(t,x)\right] & \text{if } x\ge 0,\\ h^{-1}\,G_\nu^{-1}(t,x+L)\left[G_\nu(t,x+h)-G_\nu(t,x)\right] & \text{if } x\le 0.\end{cases}$$

Clearly,

$$\left|\frac{f(t,x,h)}{h\,\left(G_\nu(t,x+L)+G_\nu(t,x-L)\right)}\right|\le|I(t,x,h)|+|I(t,x,-h)|\,. \quad (2.6.7)$$

So we only need to bound $|I(t,x,h)|$ for $-\beta L\le h\le\beta L$. If $x\ge 0$, we have

$$I(t,x,h)=\frac{1}{h}\left(\exp\left(-\frac{(x+h)^2}{2\nu t}+\frac{(x-L)^2}{2\nu t}\right)-\exp\left(-\frac{x^2}{2\nu t}+\frac{(x-L)^2}{2\nu t}\right)\right)$$

and so

$$\frac{\partial}{\partial x}I(t,x,h)=-\frac{1}{\nu t}\exp\left(-\frac{(x+h)^2}{2\nu t}+\frac{(x-L)^2}{2\nu t}\right)-\frac{L}{\nu t}I(t,x,h)\,.$$

Hence, after writing the above differential equation in the integral form and taking absolute value on both sides, we have

$$|I(t,x,h)|\le\int_0^x(\nu t)^{-1}\exp\left(-\frac{(y+h)^2}{2\nu t}+\frac{(y-L)^2}{2\nu t}\right)\mathrm{d}y+\frac{L}{\nu t}\int_0^x\left|I(t,y,h)\right|\mathrm{d}y+|I(t,0,h)|\,.$$

By Lemma 2.6.7,

$$|I(t,0,h)|=e^{\frac{L^2}{2\nu t}}\left|\frac{e^{-\frac{h^2}{2\nu t}}-1}{h}\right|\le\frac{e^{\frac{L^2}{2\nu t}}}{\sqrt{2\nu t}}\sup_{x\in\mathbb{R}}\left|\frac{e^{-x^2/2}-1}{x}\right|\le\frac{C}{\sqrt{2\nu t}}e^{\frac{L^2}{2\nu t}},\quad\text{for all } h\in\mathbb{R},$$

where $C\approx 0.451256$. Since $|h|\le\beta L$, we have

$$\begin{aligned}\int_0^x\frac{1}{\nu t}\exp\left(-\frac{(y+h)^2}{2\nu t}+\frac{(y-L)^2}{2\nu t}\right)\mathrm{d}y&\le\int_0^\infty\frac{1}{\nu t}\exp\left(-\frac{(y+h)^2}{2\nu t}+\frac{(y-L)^2}{2\nu t}\right)\mathrm{d}y\\&=\frac{\exp\left(\frac{L^2-h^2}{2\nu t}\right)}{L+h}\le\frac{\exp\left(\frac{L^2}{2\nu t}\right)}{(1-\beta)L}\,,\end{aligned}$$

and hence we can apply Bellman-Gronwall's lemma 2.6.6 to $|I(t,x,h)|$ over the interval $[0,x]$, since $|I(t,x,h)|$ satisfies

$$|I(t,x,h)| \leq C_{t,L,\beta} + \frac{L}{\nu t}\int_0^x \left|I(t,y,h)\right| \mathrm{d}y\,, \quad C_{t,L,\beta} := \left(\frac{C}{\sqrt{2\nu t}} + \frac{1}{(1-\beta)L}\right)\exp\left(\frac{L^2}{2\nu t}\right),$$

to get

$$|I(t,x,h)| \leq C_{t,L,\beta}\exp\left(\frac{L(x-0)}{\nu t}\right) \leq C_{t,L,\beta}\exp\left(\frac{L|x|}{\nu t}\right).$$

By symmetry, for $x \leq 0$, we get the same bound for $|I(t,x,h)|$. Hence, from (2.6.7), we get the same bound for f:

$$\left|f(t,x,h)\right| \leq C_{t,L,\beta}|h|\left(G_\nu(t,x+L) + G_\nu(t,x-L)\right)\exp\left(\frac{L|x|}{\nu t}\right).$$

Finally, some calculations show that

$$\begin{aligned}&\Big(G_\nu(t,x+L)+G_\nu(t,x-L)\Big)\exp\left(\frac{L|x|}{\nu t}\right)\\ &\quad= G_\nu(t,x)e^{-\frac{L^2}{2\nu t}} + G_\nu(t,x-2L)e^{\frac{3L^2}{2\nu t}}1_{\{x\geq 0\}} + G_\nu(t,x+2L)e^{\frac{3L^2}{2\nu t}}1_{\{x\leq 0\}}\\ &\quad\leq G_\nu(t,x)e^{-\frac{L^2}{2\nu t}} + \Big(G_\nu(t,x-2L)+G_\nu(t,x+2L)\Big)e^{\frac{3L^2}{2\nu t}}.\end{aligned}$$

Therefore, the common upper bound for $I(t,x,h)$ and $f(t,x,h)$ is bounded by

$$|h|\left(C_{t,L,\beta}\exp\left(\frac{-L^2}{2\nu t}\right)G_\nu(t,x) + C_{t,L,\beta}\exp\left(\frac{3L^2}{2\nu t}\right)\Big(G_\nu(t,x-2L)+G_\nu(t,x+2L)\Big)\right),$$

which completes the proof. □

Lemma 2.6.9. *For $\nu > 0$, $t > 0$, $n > 1$ and $x \in \mathbb{R}$, we have*

$$\left|\frac{G_{\nu/2}(t+r,x)}{G_{\nu/2}(t,x)} - 1\right| \leq \frac{3r}{t+r}\exp\left(\frac{n^2x^2}{\nu t\left(1+n^2\right)}\right) \tag{2.6.8}$$

$$\leq \frac{3}{2}\frac{\sqrt{r}}{\sqrt{t}}\exp\left(\frac{n^2x^2}{\nu t\left(1+n^2\right)}\right), \tag{2.6.9}$$

for all $r \in \left[0, n^2 t\right]$.

Proof. Fix $t > 0$, $x \in \mathbb{R}$, $\nu > 0$ and $n > 1$. Define

$$g_{t,x}(r) := \frac{G_{\nu/2}(t+r,x)}{G_{\nu/2}(t,x)} - 1 = \frac{\sqrt{t}}{\sqrt{t+r}}\exp\left(\frac{x^2}{\nu t}\frac{r}{t+r}\right) - 1, \quad r \in \left[0, n^2 t\right].$$

Clearly $g_{t,x}(0) = 0$. Notice that

$$\left|g_{t,x}(r)\right| \leq \left|\exp\left(\frac{x^2}{\nu t}\frac{r}{t+r}\right) - 1\right| + \exp\left(\frac{x^2}{\nu t}\frac{r}{t+r}\right)\left|\frac{\sqrt{t}}{\sqrt{t+r}} - 1\right|.$$

The second part can be simply bounded as follows:

$$\begin{aligned}\exp\left(\frac{x^2}{\nu t}\frac{r}{t+r}\right)\left|\frac{\sqrt{t}}{\sqrt{t+r}}-1\right| &= \exp\left(\frac{x^2}{\nu t}\frac{r}{t+r}\right)\frac{r}{\sqrt{t+r}\left(\sqrt{t}+\sqrt{t+r}\right)}\\ &\le \exp\left(\frac{n^2x^2}{\nu(1+n^2)t}\right)\frac{r}{t+r}, \quad \text{for all } r\in\left[0,n^2t\right],\end{aligned}$$

where we have used the fact that

$$r\in\left[0,n^2t\right] \quad\Rightarrow\quad \frac{r}{r+t}\in\left[0,\frac{n^2}{1+n^2}\right].$$

To bound the first part, we use the following fact: For fixed $a>0$ and $b>0$,

$$0\le e^{ah}-1\le e^{ab}\frac{h}{b}, \quad \text{for all } h\in[0,b];$$

see Figure 2.4 for an explanation. Apply this fact to $\exp\left(\frac{x^2}{\nu t}\frac{r}{t+r}\right)-1$ with $a=\frac{x^2}{\nu t}$, $h=\frac{r}{r+t}$ and $b=\frac{n^2}{1+n^2}$ to obtain

$$\begin{aligned}\left|\exp\left(\frac{x^2}{\nu t}\frac{r}{t+r}\right)-1\right| &\le \exp\left(\frac{n^2x^2}{\nu t\left(1+n^2\right)}\right)\frac{r}{r+t}\frac{1+n^2}{n^2}\\ &\le 2\exp\left(\frac{n^2x^2}{\nu t\left(1+n^2\right)}\right)\frac{r}{r+t}, \quad \text{for all } r\in\left[0,n^2t\right].\end{aligned}$$

Then adding these two bounds proves (2.6.8). Finally, (2.6.9) is proved by applying $t+r\ge 2\sqrt{tr}$. □

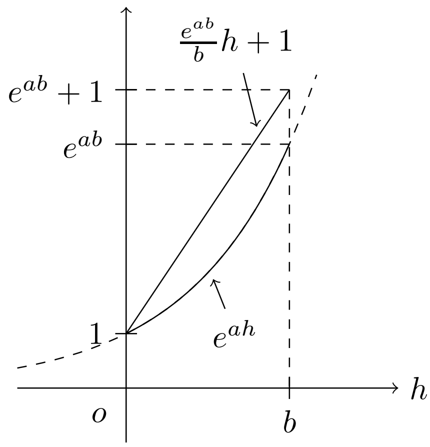


Figure 2.4 – For fixed $a>0$ and $b>0$, $e^{ah}-1\le e^{ab}h/b$ for all $h\in[0,b]$.

2.6.3 Solution to the Homogeneous Equation

In this part, we will prove a result – Lemma 2.6.14 – which is more general than what we need in this section. This general result will be used later. We first define some spaces of distributions and functions. Let $\mathscr{D}'(\mathbb{R})$ and $\mathscr{S}'(\mathbb{R})$ be the set of distributions over $\mathbb{R}$ and the set of Schwartz (or tempered) distributions (see [64] or [61]), respectively. Recall that $C_c^{+\infty}(\mathbb{R})$ is the set of smooth functions on $\mathbb{R}$ with compact support and $\mathscr{M}_H(\mathbb{R})$ is the set of signed Borel measures over $\mathbb{R}$ satisfying (1.1.5).

Definition 2.6.10. For $k \in \mathbb{N}$, define

$$\mathscr{D}'_k(\mathbb{R}) := \left\{ \mu \in \mathscr{D}'(\mathbb{R}) \,:\, \exists \mu_0 \in \mathscr{M}_H(\mathbb{R}),\ \text{s.t.},\ \mu = \mu_0^{(k)} \right\}$$

where $\mu_0^{(k)}$ denotes the k-th distributional derivative. Define $\mathscr{D}'_{+\infty}(\mathbb{R}) := \bigcup_{k\in\mathbb{N}} \mathscr{D}'_k(\mathbb{R})$.

Clearly, if $0 \le r \le s$, then $\mathscr{D}'_r(\mathbb{R}) \subseteq \mathscr{D}'_s(\mathbb{R})$. We have the following relations:

$$\begin{aligned} \frac{1}{|x|^a} &\in \mathscr{D}'_0(\mathbb{R}), \quad \text{for all } a \in [0,1[\,; \\ \delta_0^{(n)} &\in \mathscr{D}'_n(\mathbb{R}), \quad \text{for all } n = 0,1,\dots; \\ \mathscr{S}'(\mathbb{R}) &\subseteq \mathscr{D}'_{+\infty}(\mathbb{R}), \end{aligned}$$

where $\delta_0^{(n)}$ is the n-th distributional derivative of the Dirac delta function.

Let $\mathrm{He}_n(x;t)$ be the *Hermite polynomials*:

$$\mathrm{He}_n(x;\,t) := \sum_{k=0}^{\lfloor n/2\rfloor} \binom{n}{2k} (2k-1)!!\,(-t)^k\, x^{n-2k}, \quad \text{for all } t>0 \text{ and } x \in \mathbb{R},$$

where $\lfloor n/2\rfloor$ is the largest integer not bigger than $n/2$ and $n!!$ is the double factorial

$$n!! := \begin{cases} n\cdot(n-2)\dots 5\cdot 3\cdot 1, & \text{if } n>0 \text{ odd}, \\ n\cdot(n-2)\dots 6\cdot 4\cdot 2, & \text{if } n>0 \text{ even}, \\ 1, & \text{if } n=-1,0. \end{cases}$$

Note that $\mathrm{He}_n(x;t)$ is a polynomial in x of degree n with leading coefficient 1. In particular, $\mathrm{He}_0(x) = 1$ and $\mathrm{He}_1(x) = x$. Clearly, $\mathrm{He}_n(x;t)$ has the following scaling property:

$$\mathrm{He}_n(x;t) = t^{n/2}\,\mathrm{He}_n\left(\frac{x}{\sqrt{t}};1\right) = (t/2)^{n/2} H_n\left(\frac{x}{\sqrt{2t}}\right),$$

where the $H_n(x)$ are the *standard Hermite polynomials*: $H_n(x) := (-1)^n e^{x^2}\frac{\mathrm{d}^n}{\mathrm{d}x^n} e^{-x^2}$. On the other hand,

$$H_n(x) = 2^{n/2}\mathrm{He}_n\left(\sqrt{2}\,x;\,1\right).$$

See [51, 18.7.11 and 18.7.12, on p. 444] for the relations between these two Hermite polynomials. Denote

$$|\mathrm{He}|_n(x;\,t) := \sum_{k=0}^{\lfloor n/2\rfloor} \binom{n}{2k} (2k-1)!!\, t^k\, |x|^{n-2k}, \quad \text{for all } t>0 \text{ and } x\in\mathbb{R}.$$

In the following, we use ∂_y^n and ∂_t^n to denote the n-th partial derivatives with respect to y and t, respectively. In particular, we use the following convention:

$$\partial_y^k \left[G_\nu\left(t,x-y\right)\right] = (-1)^k \frac{\partial^k}{\partial z^k} G_\nu(t,z)\Big|_{z=x-y} = (-1)^k \partial_x^k G_\nu\left(t,x-y\right) .$$

Lemma 2.6.11 (Theorem 9.3.3 of [44])**.** *For each $n\in\mathbb{N}$,*

$$\partial_y^n \left[G_\nu\left(t,x-y\right)\right] = G_\nu\left(t,x-y\right)(\nu t)^{-n}\mathrm{He}_n\left(x-y;\nu t\right) .$$

Lemma 2.6.12. *If $\mu\in\mathcal{M}_H(\mathbb{R})$, then for all functions $P_n(x)=|x|^n$ with $n\in\mathbb{R}_+$,*

$$\left(|\mu| * [G_\nu(t,\cdot)P_n(\cdot)]\right)(x) \le \sqrt{2}\left(\frac{2\nu t n}{e}\right)^{n/2}\left(|\mu| * G_{2\nu}(t,\cdot)\right)(x) < +\infty , \tag{2.6.10}$$

$$\left(\left[|\mu| P_n(\cdot)\right] * G_\nu(t,\cdot)\right)(x) \le 2^{n-1/2}\left(\left(\frac{2\nu t n}{e}\right)^{n/2}\left(|\mu| * G_{2\nu}(t,\cdot)\right)(x) + |x|^n\left(|\mu| * G_\nu(t,\cdot)\right)(x)\right) < +\infty , \tag{2.6.11}$$

for all $(t,x)\in\mathbb{R}_+^*\times\mathbb{R}$. Moreover, for all exponential functions $E_a(x) := \exp(a|x|)$ with $a>0$,

$$\left(|\mu| * [G_\nu(t,\cdot)E_a(\cdot)]\right)(x) \le \sqrt{2}\, e^{\nu t a^2}\left(|\mu| * G_{2\nu}(t,\cdot)\right)(x) < +\infty , \tag{2.6.12}$$

for all $(t,x)\in\mathbb{R}_+^*\times\mathbb{R}$.

Proof. Fix an arbitrary $n\ge 0$ and a real number $a\in\mathbb{R}$. Denote the left-hand sides of (2.6.10), (2.6.11) and (2.6.12) by $I_1(t,x)$, $I_2(t,x)$, and $I_3(t,x)$, respectively.

(1) We first prove (2.6.10). Clearly

$$I_1(t,x) = \int_{\mathbb{R}} G_\nu(2t,x-y)\frac{G_\nu\left(t,x-y\right)|x-y|^n}{G_\nu(2t,x-y)}|\mu|(\mathrm{d}y) .$$

Notice that

$$\begin{aligned}\sup_{y\in\mathbb{R}} \frac{G_\nu\left(t,x-y\right)|x-y|^n}{G_\nu(2t,x-y)} &= \sup_{y\in\mathbb{R}} \sqrt{2}\exp\left(-\frac{|x-y|^2}{4\nu t}\right)|x-y|^n \\ &= 2^{n+1/2}\nu^{n/2}t^{n/2}\sup_{y\ge 0}\exp\left(-|y|\right)|y|^{n/2} .\end{aligned}$$

Clearly, the function $f(y)=e^{-|y|}|y|^a$ with $a>0$ has two symmetric bumps and it achieves its global maximum at $y=\pm a$ since $f'(\pm a)=0$. Hence,

$$\sup_{y\in\mathbb{R}} \frac{G_\nu\left(t,x-y\right)|x-y|^n}{G_\nu(2t,x-y)} \le 2^{n+1/2}\nu^{n/2}t^{n/2}e^{-n/2}\left(\frac{n}{2}\right)^{n/2} = \sqrt{2}\left(\frac{2\nu t n}{e}\right)^{n/2} ,$$

and so we obtain the upper bound in (2.6.10) which is finite by (1.1.5).

(2) As for $I_2(t,x)$, we have that, by (1),

$$\begin{aligned} I_2(t,x) &= \int_{\mathbb{R}} G_\nu\left(t,x-y\right)|y|^n|\mu|(\mathrm{d}y) \\ &\le 2^{n-1}\int_{\mathbb{R}} G_\nu\left(t,x-y\right)\left(|x-y|^n+|x|^n\right)|\mu|\left(\mathrm{d}y\right) \\ &= 2^{n-1}|x|^n\left(\left|\mu\right| * G_\nu(t,\cdot)\right)(x) + 2^{n-1}\left(\left|\mu\right| * [G_\nu(t,\cdot)P_n(\cdot)]\right)(x) < +\infty\,. \end{aligned}$$

Then use the bound in (2.6.10). This proves (2.6.11).

(3) Now let us prove (2.6.12). Clearly

$$I_3(t,x) = \int_{\mathbb{R}} G_\nu(2t,x-y)\frac{G_\nu\left(t,x-y\right)\exp\left(a|x-y|\right)}{G_\nu(2t,x-y)}|\mu|(\mathrm{d}y)\,.$$

Notice that

$$\begin{aligned} \sup_{y\in\mathbb{R}}\frac{G_\nu\left(t,x-y\right)\exp\left(a|x-y|\right)}{G_\nu(2t,x-y)} &= \sup_{y\in\mathbb{R}}\sqrt{2}\exp\left(-\frac{|x-y|^2}{4\nu t}+a|x-y|\right) \\ &= \sup_{y\in\mathbb{R}}\sqrt{2}\exp\left(-\frac{\left(|x-y|-2\nu t a\right)^2}{4\nu t}+\nu t a^2\right) \\ &= \sqrt{2}\, e^{\nu t a^2}\,. \end{aligned}$$

Therefore,

$$I_3(t,x) \le \sqrt{2}\, e^{\nu t a^2}\left(|\mu| * G_{2\nu}(t,\cdot)\right)(x) < +\infty\,.$$

This completes the proof. ☐

Lemma 2.6.13. *Suppose $\mu\in\mathcal{M}_H(\mathbb{R})$. For all $n,m,a,b\in\mathbb{N}$, we have that*

$$\partial_t^a\partial_x^b\int_{\mathbb{R}}\partial_t^n\partial_x^m G_\nu\left(t,x-y\right)\mu(\mathrm{d}y) = \int_{\mathbb{R}}\partial_t^{n+a}\partial_x^{m+b}G_\nu\left(t,x-y\right)\mu(\mathrm{d}y)\,,$$

for all $t>0$ and $x\in\mathbb{R}$.

Proof. We only need to consider two cases: $a=1$, $b=0$ and $a=0$, $b=1$. Let us first consider the case where $a=0$ and $b=1$. Fix $t>0$. Because $G_\nu(t,x)$ solves the heat equation (2.2.1), we have that

$$\partial_t^n G_\nu(t,x-y) = \left(\frac{\nu}{2}\right)^n\partial_x^{2n}G_\nu(t,x-y)\,.$$

Then, by Lemma 2.6.11,

$$\begin{aligned} \partial_t^n\partial_x^{m+1}G_\nu\left(t,x-y\right) &= \left(\frac{\nu}{2}\right)^n\partial_x^{2n+m+1}G_\nu\left(t,x-y\right) \\ &= \left(\frac{\nu}{2}\right)^n(-\nu t)^{-(2n+m+1)}G_\nu\left(t,x-y\right)\mathrm{He}_{2n+m+1}\left(x-y;\nu t\right)\,. \end{aligned}$$

Hence, for a neighborhood $[x_0-h,x_0+h]$ of x_0 with $h>0$, there are two constants $C>0$

and $a > 0$, depending only on t, x_0 and h, such that

$$\left|\partial_t^n \partial_x^{m+1} G_\nu\left(t, x-y\right)\right| \le C\, G_{2\nu}(t, x_0 - y) \times |\mathrm{He}|_{2n+m+1}\left(|x_0| + h + |y|; \nu t\right) \exp\left(a|y|\right), \tag{2.6.13}$$

for all $x \in [x_0 - h, x_0 + h]$ and $y \in \mathbb{R}$. In fact,

$$\begin{aligned}\frac{G_\nu\left(t, x-y\right)}{G_{2\nu}(t, x_0 - y)} &= \sqrt{2}\,\exp\left(\frac{-y^2 + 2(2x - x_0)y - 2x^2 + x_0^2}{4\nu t}\right)\\ &\le \sqrt{2}\,\exp\left(\frac{2|2x - x_0|\,|y| + x_0^2}{4\nu t}\right)\\ &\le \sqrt{2}\,\exp\left(\frac{2\left(|x_0| + 2h\right)|y| + x_0^2}{4\nu t}\right),\end{aligned}$$

where we have used the fact that $|2x - x_0| \le |x - x_0| + |x| \le h + |x_0| + h$. Notice that

$$\begin{aligned}\left|\mathrm{He}_{2n+m+1}(x - y; \nu t)\right| &\le |\mathrm{He}|_{2n+m+1}\left(x - y; \nu t\right) \le |\mathrm{He}|_{2n+m+1}\left(|x| + |y|; \nu t\right)\\ &\le |\mathrm{He}|_{2n+m+1}\left(|x_0| + h + |y|; \nu t\right).\end{aligned}$$

Therefore, we have proved (2.6.13) with

$$C = \sqrt{2}\left(\frac{\nu}{2}\right)^n (\nu t)^{-(2n+m+1)}\, e^{\frac{x_0^2}{4\nu t}}, \quad \text{and} \quad a = \frac{|x_0| + 2h}{2\nu t}.$$

Clearly, the function $y \in \mathbb{R} \mapsto \partial_t^n \partial_x^{m+1} G_\nu\left(t, x-y\right)$ is continuous for $x \in [x_0 - h, x_0 + h]$. The function $C G_\nu(t, x_0 - y)\, |\mathrm{He}|_{2n+m+1}\left(|x_0| + h + |y|; \nu t\right) \exp\left(a|y|\right)$ is integrable with respect to $|\mu|(\mathrm{d}y)$ by Lemma 2.6.12. Therefore, we can switch the differential and the integral signs (see [4, Theorem 16.8, on p. 212]).

Now let us consider the case where $a = 1$ and $b = 0$. Fix $x \in \mathbb{R}$. By the same arguments, we have

$$\begin{aligned}\partial_t^{n+1} \partial_x^m G_\nu\left(t, x-y\right) &= \left(\frac{\nu}{2}\right)^{n+1} \partial_x^{2(n+1)+m} G_\nu\left(t, x-y\right)\\ &= \left(\frac{\nu}{2}\right)^{n+1} (-\nu t)^{-(2(n+1)+m)} G_\nu\left(t, x-y\right) \mathrm{He}_{2(n+1)+m}\left(x - y; \nu t\right).\end{aligned}$$

Fix $t_0 > 0$. For $t \in [t_0/2, 2t_0]$, we have

$$G_\nu\left(t, x\right) = \frac{1}{\sqrt{2\pi\nu t}} \exp\left(-\frac{x^2}{2\nu t}\right) \le \frac{1}{\sqrt{\pi\nu t_0}} \exp\left(-\frac{x^2}{4\nu t_0}\right) = 2G_{2\nu}(t_0, x).$$

Hence, we have that

$$\left|\partial_t^{n+1} \partial_x^m G_\nu\left(t, x-y\right)\right| \le \left(\frac{\nu}{2}\right)^{n+1} \left(\frac{2}{t_0}\right)^{2(n+1)+m} 2G_{2\nu}(t_0, x - y)\, |\mathrm{He}|_{2(n+1)+m}\left(x - y; 2\nu t_0\right),$$

for all $t \in [t_0/2, 2t_0]$. Clearly, the function $y \in \mathbb{R} \mapsto \partial_t^{n+1} \partial_x^m G_\nu\left(t, x-y\right)$ for $t \in [t_0/2, 2t_0]$

is continuous. The function $G_{2\nu}(t_0, x-y)\,|\mathrm{He}|_{2(n+1)+m}\left(x-y;2\nu t_0\right)$ is integrable with respect to $|\mu|(\mathrm{d}y)$ by Lemma 2.6.12. Therefore, we can switch the differential and the integral signs. This completes the whole proof. □

Now define

$$J_0(t,x) := (-1)^k \left(\mu_0 * \partial_y^k \left[G_1(\nu t,\cdot)\right]\right)(x)\,, \quad \text{for all } (t,x)\in\mathbb{R}_+^*\times\mathbb{R}\,, \tag{2.6.14}$$

which can be equivalently written as

$$J_0(t,x) = (-\nu t)^{-k}\left(\mu_0 * \left[\mathrm{He}_k(\cdot;\nu t)G_\nu(t,\cdot)\right]\right)(x)\,, \tag{2.6.15}$$

by Lemma 2.6.11.

Lemma 2.6.14. *For all $\mu\in\mathscr{D}_k'(\mathbb{R})$, the function $(t,x)\in\mathbb{R}_+^*\times\mathbb{R}\mapsto J_0(t,x)$ in (2.6.14) is smooth, i.e., $J_0\in C^{+\infty}\left(\mathbb{R}_+^*\times\mathbb{R}\right)$. If, in addition, μ is an α-Hölder continuous function ($\alpha\in\,]0,1]$), then*

$$J_0(t,x)\in C^{+\infty}\left(\mathbb{R}_+^*\times\mathbb{R}\right)\cap C_{\alpha/2,\alpha}\left(\mathbb{R}_+\times\mathbb{R}\right)\,. \tag{2.6.16}$$

Proof. Let μ_0 be the signed Borel measure associated to μ. Notice that

$$J_0(t,x) = (-1)^k\int_{\mathbb{R}}\partial_y^k\left[G_\nu(t,x-y)\right]\mu_0(\mathrm{d}y) = \int_{\mathbb{R}}\partial_x^k G_\nu(t,x-y)\mu_0(\mathrm{d}y)\,,\quad t>0.$$

Hence, by Lemma 2.6.13, for all $n,m\in\mathbb{N}$,

$$\partial_t^n\partial_x^m J_0(t,x) = \int_{\mathbb{R}}\partial_t^n\partial_x^{k+m}G_\nu(t,x-y)\mu_0(\mathrm{d}y)\,,\quad \text{for } t>0,$$

which proves that $J_0(t,x)\in C^{+\infty}\left(\mathbb{R}_+^*\times\mathbb{R}\right)$.

Now assume that μ is an α-Hölder continuous function. Let us show that $J_0(t,x)\in C_{\alpha/2,\alpha}\left(\mathbb{R}_+\times\mathbb{R}\right)$. Denote $\mu(\mathrm{d}x) = f(x)\mathrm{d}x$ where $f(x)$ is α-Hölder continuous. Then for some constant $C>0$,

$$\left|f(x)-f(y)\right|\le C|x-y|^\alpha\,,\quad \text{for all } x,y\in\mathbb{R}\,.$$

Fix (t,x) and $(t',x')\in\mathbb{R}_+\times\mathbb{R}$ with $t'>t$. Decompose the difference into two parts:

$$\begin{aligned}\left|J_0(t,x)-J_0(t',x')\right| &\le \left|J_0(t,x)-J_0(t',x)\right| + \left|J_0(t',x)-J_0(t',x')\right| \\ &:= I_1\left(t,t';x\right) + I_2\left(t';x,x'\right)\,.\end{aligned}$$

We first consider $I_1\left(t,t';x\right)$, which equals

$$\begin{aligned}I_1\left(t,t';x\right) &= \left|\int_{\mathbb{R}}\left(G_\nu\left(t,x-y\right)-G_\nu(t',x-y)\right)\,f(y)\mathrm{d}y\right| \\ &= \left|\int_{\mathbb{R}}G_\nu\left(1,z\right)\left(f\left(x-\sqrt{t}\,z\right)-f\left(x-\sqrt{t'}\,z\right)\right)\mathrm{d}z\right|\,,\end{aligned}$$

Then by the Hölder continuity of f, we have that

$$I_1\left(t,t';x\right) \le C\left|\sqrt{t}-\sqrt{t'}\right|^{\alpha}\int_{\mathbb{R}} G_\nu(1,z)|z|^{\alpha}\mathrm{d}z \le C'\left|t'-t\right|^{\alpha/2},$$

with $C' = C\int_{\mathbb{R}}|z|^{\alpha}G_\nu(1,z)\mathrm{d}z$, where we have used the inequality $\left|\sqrt{t'}-\sqrt{t}\right| \le \left|t'-t\right|^{1/2}$.

The arguments for $I_2\left(t';x,x'\right)$ are similar. By the Hölder continuity of f, we have

$$\begin{aligned} I_2\left(t';x,x'\right) &= \left|\int_{\mathbb{R}} G_\nu\left(t',y\right) f(x-y)\mathrm{d}y - \int_{\mathbb{R}} G_\nu\left(t',y\right) f(x'-y)\mathrm{d}y\right| \\ &\le \int_{\mathbb{R}} G_\nu\left(t',y\right)\left|f(x-y)-f(x'-y)\right|\mathrm{d}y \\ &\le C\left|x-x'\right|^{\alpha}\int_{\mathbb{R}} G_\nu\left(t',y\right)\mathrm{d}y = C\left|x-x'\right|^{\alpha}. \end{aligned}$$

Combining the above two cases, we have therefore proved that

$$\left|J(t,x)-J\left(t',x'\right)\right| \le \left(C'\vee C\right)\left(\left|t'-t\right|^{\alpha/2}+\left|x'-x\right|^{\alpha}\right),$$

for all (t,x) and $\left(t',x'\right)\in\mathbb{R}_+\times\mathbb{R}$, which completes the proof. □

Lemma 2.6.15. *Suppose that* $\mu\in\mathscr{D}_k'(\mathbb{R})$, $k\in\mathbb{N}$. *Let* $\mu_0\in\mathscr{M}_H(\mathbb{R})$ *be the signed Borel measure associated to* μ *such that* $\mu=\mu_0^{(k)}$. *Then the function* $J_0(t,x)$ *defined in* (2.6.14) *solves the heat equation* (2.2.1) *for* $t>0$ *and*

$$\lim_{t\to 0_+}\left\langle\psi,J_0(t,\cdot)\right\rangle = \left\langle\psi,\mu\right\rangle, \quad \textit{for all } \psi\in C_c^{+\infty}(\mathbb{R}). \tag{2.6.17}$$

Proof. By Lemma 2.6.13, we can differentiate under the integral signs:

$$\begin{aligned} \left(\frac{\partial}{\partial t}-\frac{\nu}{2}\frac{\partial^2}{\partial x^2}\right)J_0(t,x) &= (-1)^k\int_{\mathbb{R}}\left(\frac{\partial}{\partial t}-\frac{\nu}{2}\frac{\partial^2}{\partial x^2}\right)\partial_y^k\left[G_\nu\left(t,x-y\right)\right]\mu_0(\mathrm{d}y) \\ &= (-1)^k\int_{\mathbb{R}}\partial_y^k\underbrace{\left[\left(\frac{\partial}{\partial t}-\frac{\nu}{2}\frac{\partial^2}{\partial x^2}\right)G_\nu\left(t,x-y\right)\right]}_{=0}\mu_0(\mathrm{d}y) \\ &= 0. \end{aligned}$$

Now let us prove (2.6.17). Let $\psi\in C_c^{+\infty}(\mathbb{R})$ and suppose that $\operatorname{supp}\left(\psi\right)\in[-n,n]$ for some $n>0$. By Lemma 2.6.12 and (2.6.15), we know that for some constant C depending on t, k and ν,

$$(\nu t)^{-k}\int_{\mathbb{R}}\left|\mathrm{He}_k\left(x-y;\nu t\right)\right| G_\nu(t,x-y)|\mu_0|(\mathrm{d}y) \le C\left(|\mu_0| * G_{2\nu}(t,\cdot)\right)(x),$$

which implies that

$$(\nu t)^{-k}\int_{\mathbb{R}}\mathrm{d}x\,|\psi(x)|\int_{\mathbb{R}}\left|\mathrm{He}_k\left(x-y;\nu t\right)\right| G_\nu(t,x-y)|\mu_0|(\mathrm{d}y)$$

$$\leq \sup_{|y|\leq n} C\left(|\mu_0| * G_{2\nu}(t,\cdot)\right)(y) \int_{\mathbb{R}} |\psi(x)| \mathrm{d}x < +\infty\,,$$

where we have used the fact that the function $y \mapsto \left(|\mu_0| * G_{2\nu}(t,\cdot)\right)(y)$ is continuous (see Lemma 2.6.14). So we can apply Fubini's theorem to get

$$\begin{aligned}
\left\langle \psi, J_0(t,\cdot)\right\rangle(x) &= (-\nu t)^k \int_{\mathbb{R}} \mathrm{d}x\, 1\psi(x) \int_{\mathbb{R}} G_\nu\left(t, x-y\right) \mathrm{He}_k\left(x-y;\nu t\right) \mu_0(\mathrm{d}y)\\
&= (-\nu t)^k \int_{\mathbb{R}} \mu_0(\mathrm{d}y) \int_{\mathbb{R}} G_\nu\left(t,x-y\right) \mathrm{He}_k\left(x-y;\nu t\right)\psi(x)\mathrm{d}x\\
&= (-1)^k \int_{\mathbb{R}} \mu_0(\mathrm{d}y) \int_{\mathbb{R}} \psi(x)(-1)^k \partial_x^k\left[G_\nu\left(t,x-y\right)\right]\mathrm{d}x\\
&= (-1)^k \int_{\mathbb{R}} \mu_0(\mathrm{d}y) \int_{\mathbb{R}} \psi^{(k)}(x)\, G_\nu\left(t,x-y\right)\mathrm{d}x\,,
\end{aligned}$$

where in the last step we have applied the integration by parts formula. Denote $F_t(y) = \int_{\mathbb{R}} \psi^{(k)}(x)\, G_\nu\left(t,x-y\right)\mathrm{d}x$. Clearly,

$$\lim_{t\to 0_+} F_t(y) = \psi^{(k)}(y)\,, \quad \text{for all } y\in\mathbb{R}.$$

Since $\psi^{(k)} \in C_c^{+\infty}(\mathbb{R})$, there is some constant $C>0$ such that

$$\left|\psi^{(k)}(x)\right| \leq C\, G_\nu(1,x)\,, \quad \text{for all } x\in\mathbb{R}.$$

Hence, for all $t\in[0,1]$,

$$\begin{aligned}
\left|F_t(y)\right| &\leq C\int_{\mathbb{R}} G_\nu(1,x)G_\nu\left(t,x-y\right)\mathrm{d}x = CG_\nu(1+t,y)\\
&= \frac{C}{\sqrt{2\pi\nu(1+t)}}\exp\left\{-\frac{y^2}{2\nu(1+t)}\right\}\\
&\leq \frac{C}{\sqrt{2\pi\nu}}\exp\left\{-\frac{y^2}{4\nu}\right\} = \sqrt{2}\, C\, G_{2\nu}(1,y)\,.
\end{aligned}$$

Because $\mu_0 \in \mathcal{M}_H(\mathbb{R})$, the function $\sqrt{2}\, C\, G_{2\nu}(1,y)$ is integrable with respect to $|\mu_0|(\mathrm{d}y)$. Therefore, by the Lebesgue dominated convergence theorem,

$$\lim_{t\to 0_+} \left\langle \psi, J_0(t,\cdot)\right\rangle(x) = (-1)^k \left\langle \psi^{(k)}, \mu_0\right\rangle\,.$$

Finally, (2.6.17) is proved by passing the derivatives from ψ to μ_0. This completes the whole proof. □

2.6.4 Proof of Hölder Continuity

Proposition 2.6.16. *Given $\overline{\varsigma} \in \mathbb{R}$ and any initial data μ satisfying* (1.1.5)*, let $J_0^*(t,x) = \left(|\mu| * G_\nu(t,\cdot)\right)(x)$. Then for all $n>1$, there exist constants $C_{n,i}$, $i=1,3,5$, such that for all*

$t, t' \in [1/n, n]$ with $t < t'$ and $x, x' \in [-n, n]$,

$$\iint_{[0,t]\times\mathbb{R}} \left(\overline{\varsigma}^2 + 2\left|J_0^*(s,y)\right|^2\right)\left(G_\nu(t-s, x-y) - G_\nu(t'-s, x-y)\right)^2 \mathrm{d}s\mathrm{d}y \le C_{n,1}\sqrt{t'-t}\,, \tag{2.6.18}$$

$$\iint_{[0,t]\times\mathbb{R}} \left(\overline{\varsigma}^2 + 2\left|J_0^*(s,y)\right|^2\right)\left(G_\nu(t-s, x-y) - G_\nu(t-s, x'-y)\right)^2 \mathrm{d}s\mathrm{d}y \le C_{n,3}\left|x-x'\right|\,, \tag{2.6.19}$$

and

$$\iint_{[t,t']\times\mathbb{R}} \left(\overline{\varsigma}^2 + 2\left|J_0^*(s,y)\right|^2\right) G_\nu^2(t'-s, x'-y)\mathrm{d}s\mathrm{d}y \le C_{n,5}\sqrt{t'-t}\,. \tag{2.6.20}$$

Note that $J_0^*(t,x)$ may grow exponentially as $|x| \to \infty$, so Fourier transform cannot be used.

Proof of (2.6.18) *and* (2.6.19). We consider the contribution by $\left|J_0^*(t,x)\right|^2$. Denote

$$I(t,x;t',x') := \iint_{[0,t]\times\mathbb{R}} \left|J_0^*(s,y)\right|^2 \left(G_\nu(t-s, x-y) - G_\nu(t'-s, x'-y)\right)^2 \mathrm{d}s\mathrm{d}y\,.$$

Replace the $\left|J_0^*(s,y)\right|^2$ by the following double integral

$$\left|J_0^*(s,y)\right|^2 = \iint_{\mathbb{R}^2} G_\nu(s, y-z_1)G_\nu(s, y-z_2)|\mu|(\mathrm{d}z_1)|\mu|(\mathrm{d}z_2)\,,$$

and use Lemma 2.3.7:

$$G_\nu(s, y-z_1)G_\nu(s, y-z_2) = G_{\nu/2}\left(s, y - \frac{z_1+z_2}{2}\right)G_{2\nu}(s, z_1-z_2)\,.$$

Thus

$$\begin{aligned} I(t,x;t',x') = &\int_0^t \mathrm{d}s \iint_{\mathbb{R}^2} |\mu|(\mathrm{d}z_1)|\mu|(\mathrm{d}z_2)\; G_{2\nu}(s, z_1-z_2) \\ &\times \int_{\mathbb{R}} G_{\nu/2}\left(s, y - \frac{z_1+z_2}{2}\right)\left(G_\nu(t-s, x-y) - G_\nu(t'-s, x'-y)\right)^2 \mathrm{d}y\,. \end{aligned} \tag{2.6.21}$$

In the following, we use $\int G(G-G)^2\mathrm{d}y$ to denote the integral over y in (2.6.21) and set $\bar{z} := (z_1+z_2)/2$. Expand $(G-G)^2 = G^2 - 2GG + G^2$ and apply Lemma 2.3.7 to each term:

$$\begin{aligned} &\left(G_\nu(t-s, x-y) - G_\nu(t'-s, x'-y)\right)^2 = \\ &\qquad \frac{1}{\sqrt{4\pi\nu(t-s)}} G_{\nu/2}(t-s, x-y) + \frac{1}{\sqrt{4\pi\nu(t'-s)}} G_{\nu/2}(t'-s, x'-y) \\ &\qquad -2G_{2\nu}\left(\frac{t+t'}{2} - s, x-x'\right) G_{\nu/2}\left(\frac{2(t-s)(t'-s)}{t+t'-2s}, y - \frac{(t-s)x' + (t'-s)x}{t+t'-2s}\right)\,. \end{aligned}$$

Then integrate over y using the semigroup property of the heat kernel:

$$\begin{aligned}\int G(G-G)^2 \mathrm{d}y = & \frac{1}{\sqrt{4\pi\nu(t-s)}} G_{\nu/2}(t, x-\bar{z}) + \frac{1}{\sqrt{4\pi\nu(t'-s)}} G_{\nu/2}\left(t', x'-\bar{z}\right) \\ & -2G_{2\nu}\left(\frac{t+t'}{2}-s, x-x'\right) G_{\nu/2}\left(\frac{2(t-s)(t'-s)}{t+t'-2s}+s, \frac{(t-s)x'+(t'-s)x}{t+t'-2s}-\bar{z}\right). \end{aligned} \tag{2.6.22}$$

Property (2.6.18)**.** We first prove (2.6.18). Set $x = x'$ in (2.6.21). Denote $h = t'-t$. Clearly, $h \in [0, n^2 t]$. Then

$$\frac{2(t-s)(t'-s)}{t+t'-2s} + s = t + \frac{(t-s)h}{2(t-s)+h}$$

and (2.6.22) becomes

$$\begin{aligned}\int G(G-G)^2 \mathrm{d}y = & \left(\frac{1}{\sqrt{4\pi\nu(t-s)}} + \frac{1}{\sqrt{4\pi\nu(t'-s)}}\right) G_{\nu/2}(t, x-\bar{z}) \\ & + \frac{1}{\sqrt{4\pi\nu(t'-s)}} \left(G_{\nu/2}(t', x-\bar{z}) - G_{\nu/2}(t, x-\bar{z})\right) \\ & - \frac{1}{\sqrt{\pi\nu\left(\frac{t+t'}{2}-s\right)}} G_{\nu/2}\left(t + \frac{(t-s)h}{2(t-s)+h}, x-\bar{z}\right) \\ = & \left(\frac{1}{\sqrt{4\pi\nu(t-s)}} + \frac{1}{\sqrt{4\pi\nu(t'-s)}} - \frac{1}{\sqrt{\pi\nu\left(\frac{t+t'}{2}-s\right)}}\right) G_{\nu/2}(t, x-\bar{z}) \\ & + \frac{1}{\sqrt{4\pi\nu(t'-s)}} \left(\frac{G_{\nu/2}\left(t', x-\bar{z}\right)}{G_{\nu/2}(t, x-\bar{z})} - 1\right) G_{\nu/2}(t, x-\bar{z}) \\ & - \frac{1}{\sqrt{\pi\nu\left(\frac{t+t'}{2}-s\right)}} \left(\frac{G_{\nu/2}\left(t + \frac{(t-s)h}{2(t-s)+h}, x-\bar{z}\right)}{G_{\nu/2}(t, x-\bar{z})} - 1\right) G_{\nu/2}(t, x-\bar{z}) \\ := & I_1 + I_2 - I_3 . \end{aligned}$$

Let us first consider I_2. By Lemma 2.6.9,

$$\begin{aligned} |I_2| \le & \frac{3}{4\sqrt{\pi\nu t(t'-s)}} G_{\nu/2}(t, x-\bar{z}) \exp\left(\frac{n^2 (x-\bar{z})^2}{\nu t\left(1+n^2\right)}\right) \sqrt{h} \\ = & \frac{3}{4\pi\nu t\sqrt{t'-s}} \exp\left(-\frac{(x-\bar{z})^2}{\nu t\left(1+n^2\right)}\right) \sqrt{h} \\ = & \frac{3\sqrt{1+n^2}}{4\sqrt{\pi\nu t(t'-s)}} G_{\nu(1+n^2)/2}(t, x-\bar{z}) \sqrt{h}. \end{aligned}$$

Hence

$$\int_0^t G_{2\nu}(s,z_1-z_2)|I_2|\mathrm{d}s \le \sqrt{h}\int_0^t \frac{3\sqrt{1+n^2}}{4\sqrt{\pi\nu t(t'-s)}} G_{\nu(1+n^2)/2}(t,x-\bar{z})\, G_{2\nu}(s,z_1-z_2)\mathrm{d}s.$$

By Lemma 2.3.8, we have

$$G_{\nu(1+n^2)/2}(t,x-\bar{z})\, G_{2\nu}(s,z_1-z_2) \le \frac{2\sqrt{\left(1+n^2\right)t}}{\sqrt{s}} G_{2\nu(1+n^2)}(t,x-z_1) G_{2\nu(1+n^2)}(t,x-z_2)\,,$$

and so,

$$\iint_{\mathbb{R}^2} |\mu|(\mathrm{d}z_1)|\mu|(\mathrm{d}z_2)\int_0^t G_{2\nu}(s,z_1-z_2)|I_2|\mathrm{d}s \\ \le \frac{3\left(1+n^2\right)\sqrt{h}}{2\sqrt{\pi\nu}}\left(|\mu| * G_{2\nu(1+n^2)}(t,\cdot)\right)^2(x)\int_0^t \frac{1}{\sqrt{s(t'-s)}}\mathrm{d}s\,.$$

Clearly,

$$\int_0^t \frac{1}{\sqrt{s(t'-s)}}\mathrm{d}s \le \int_0^{t'} \frac{1}{\sqrt{s(t'-s)}}\mathrm{d}s = \pi$$

Therefore,

$$\iint_{\mathbb{R}^2} |\mu|(\mathrm{d}z_1)|\mu|(\mathrm{d}z_2)\int_0^t G_{2\nu}(s,z_1-z_2)|I_2|\mathrm{d}s \\ \le \frac{3\left(1+n^2\right)\sqrt{\pi}}{2\sqrt{\nu}}\left(|\mu| * G_{2\nu(1+n^2)}(t,\cdot)\right)^2(x)\sqrt{h}\,. \quad (2.6.23)$$

As for I_3, notice that since $s \in [0,t]$,

$$\frac{(t-s)h}{2(t-s)+h} = \frac{h}{2+\dfrac{h}{t-s}} \le \frac{h}{2+\dfrac{h}{t}} = \frac{t}{\dfrac{2t}{h}+1} \le t \le n^2 t\,, \quad \text{for all } h \ge 0\,.$$

Apply Lemma 2.6.9 with $r = \frac{(t-s)h}{2(t-s)+h}$ to obtain

$$\left|\frac{G_{\nu/2}\left(t+\frac{(t-s)h}{2(t-s)+h},x-\bar{z}\right)}{G_{\nu/2}(t,x-\bar{z})}-1\right| \le \frac{3}{2}\exp\left(\frac{n^2(x-\bar{z})^2}{\nu t\left(1+n^2\right)}\right)\sqrt{\frac{(t-s)h}{2(t-s)+h}}\frac{1}{\sqrt{t}} \\ \le \frac{3}{2\sqrt{2}}\exp\left(\frac{n^2(x-\bar{z})^2}{\nu t\left(1+n^2\right)}\right)\frac{\sqrt{h}}{\sqrt{t}}, \quad \text{for all } h \ge 0\,,$$

where the second inequality is due to the fact that

$$\frac{(t-s)h}{2(t-s)+h} \le \frac{(t-s)h}{2(t-s)} = \frac{h}{2}\,.$$

Hence,

$$
\begin{aligned}
|I_3| &\leq \frac{3}{2\pi\nu t\sqrt{2\left(\frac{t+t'}{2}-s\right)}} \exp\left(-\frac{(x-\bar{z})^2}{\nu t\left(1+n^2\right)}\right)\sqrt{h} \\
&\leq \frac{3}{2\pi\nu t\sqrt{2\,(t-s)}} \exp\left(-\frac{(x-\bar{z})^2}{\nu t\left(1+n^2\right)}\right)\sqrt{h} \\
&= \frac{3\sqrt{1+n^2}}{2\sqrt{2\pi\nu t(t-s)}} G_{\nu(1+n^2)/2}\,(t,x-\bar{z})\,\sqrt{h}\,.
\end{aligned}
$$

Then by the same arguments as I_2, we have that

$$
\begin{aligned}
&\iint_{\mathbb{R}^2} |\mu|(\mathrm{d}z_1)|\mu|(\mathrm{d}z_2)\int_0^t G_{2\nu}(s,z_1-z_2)|I_3|\mathrm{d}s \\
&\qquad = \frac{3\left(1+n^2\right)\sqrt{\pi}}{\sqrt{2\nu}}\left(|\mu| * G_{2\nu(1+n^2)}(t,\cdot)\right)^2(x)\,\sqrt{h}\,. \qquad (2.6.24)
\end{aligned}
$$

Now let us consider I_1. Apply Lemma 2.3.8 over $G_{2\nu}\,(s,z_1-z_2)\,G_{\nu/2}\,(t,x-\bar{z})$ to obtain

$$
\begin{aligned}
\int_0^t G_{2\nu}(s,z_1-z_2)|I_1|\mathrm{d}s \leq{}& \frac{\sqrt{t}}{\sqrt{\pi\nu}} G_{2\nu}(t,x-z_1)G_{2\nu}(t,x-z_2) \\
&\times \int_0^t \left|\frac{1}{\sqrt{s(t-s)}}+\frac{1}{\sqrt{s(t'-s)}}-\frac{2}{\sqrt{s((t+t')/2-s)}}\right|\mathrm{d}s\,.
\end{aligned}
$$

Notice that

$$
\begin{aligned}
&\left|\frac{1}{\sqrt{s(t-s)}}+\frac{1}{\sqrt{s(t'-s)}}-\frac{2}{\sqrt{s((t+t')/2-s)}}\right| \\
&\quad\leq \left|\frac{1}{\sqrt{s(t-s)}}-\frac{1}{\sqrt{s((t+t')/2-s)}}\right| + \left|\frac{1}{\sqrt{s(t'-s)}}-\frac{1}{\sqrt{s((t+t')/2-s)}}\right| \\
&\quad= \frac{1}{\sqrt{s(t-s)}}-\frac{1}{\sqrt{s((t+t')/2-s)}}+\frac{1}{\sqrt{s((t+t')/2-s)}}-\frac{1}{\sqrt{s(t'-s)}} \\
&\quad= \frac{1}{\sqrt{s(t-s)}}-\frac{1}{\sqrt{s(t'-s)}}\,.
\end{aligned}
$$

Integrate the right-hand side of the above inequality using the integral

$$
\int_0^t \frac{1}{\sqrt{s(t'-s)}}\mathrm{d}s = 2\arctan\left(\frac{\sqrt{t}}{\sqrt{t'-t}}\right), \quad \text{for all } t' > t \geq 0,
$$

which can be verified easily by differentiating. Note that it reduces to the Beta integral when $t' \to t$. So

$$
\int_0^t \left|\frac{1}{\sqrt{s(t-s)}}+\frac{1}{\sqrt{s(t'-s)}}-\frac{2}{\sqrt{s((t+t')/2-s)}}\right|\mathrm{d}s \leq \pi - 2\arctan\left(\sqrt{t/h}\right).
$$

We claim that the function

$$f_a(x) := x\left(\pi - 2\arctan(ax)\right), \quad \text{for all } x \ge 0 \text{ and } a > 0$$

is non-negative and bounded from above. Indeed, it is easy to see that $\lim_{x\to+\infty} f_a(x) = 2$ and we only need to show that

$$f_a'(x) = -\frac{2ax}{a^2x^2+1} - 2\arctan(ax) + \pi \ge 0\,.$$

This is true since $\lim_{x\to+\infty} f_a'(x) = 0$ and $f_a''(x) = -\frac{4a}{(a^2x^2+1)^2} \le 0$. Therefore, we have proved that $f_a(x) \le \lim_{x\to+\infty} f_a(x) = 2$. Hence,

$$\pi - 2\arctan\left(\sqrt{t/h}\right) \le 2\sqrt{h/t}\,.$$

Therefore,

$$\iint_{\mathbb{R}^2} |\mu|(\mathrm{d}z_1)|\mu|(\mathrm{d}z_2)\int_0^t G_{2\nu}(s,z_1-z_2)|I_1|\mathrm{d}s \le \frac{2\sqrt{h}}{\sqrt{\pi\nu}}\left(|\mu| * G_{2\nu}(t,\cdot)\right)^2(x)\,. \tag{2.6.25}$$

We conclude from (2.6.23), (2.6.24) and (2.6.25) that for all $(t,x),(t',x) \in [1/n,n]\times[-n,n]$ with $t' > t$,

$$I(t,x;t',x) \le \left(C_\nu^\star\left(|\mu| * G_{2\nu}(t,\cdot)\right)^2(x) + C_{n,\nu}^*\left(|\mu| * G_{2\nu(1+n^2)}(t,\cdot)\right)^2(x)\right)\sqrt{h}\,,$$

where

$$C_\nu^\star := \frac{2}{\sqrt{\pi\nu}}\,, \qquad C_{n,\nu}^* := \frac{3\left(1+\sqrt{2}\right)\left(1+n^2\right)}{2}\sqrt{\pi/\nu}\,.$$

As for the contribution of the constant $\overline{\varsigma}$, it corresponds to the initial data $\mu(\mathrm{d}x) \equiv \overline{\varsigma}\,\mathrm{d}x$ and we apply Proposition 2.3.9, in particular (2.3.17). Finally, by the smoothing effect of the heat kernel (Lemma 2.3.5), we can choose the following constant for (2.6.18)

$$\begin{aligned} C_{n,1} &= \overline{\varsigma}^2\,\frac{\sqrt{2}-1}{\sqrt{\pi\nu}} \\ &\quad + \sup_{\substack{t\in[1/n,n]\\ x\in[-n,n]}} 2\left(C_\nu^\star\left(|\mu| * G_{2\nu}(t,\cdot)\right)^2(x) + C_{n,\nu}^*\left(|\mu| * G_{2\nu(1+n^2)}(t,\cdot)\right)^2(x)\right) < +\infty \end{aligned} \tag{2.6.26}$$

Now let us consider the case where $\mu(\mathrm{d}x) = f(x)\mathrm{d}x$ and $t, t' \in [0,n]$. By multiplying and diving $G_{2\nu(2+n^2)}(n,\cdot)$, (2.6.21) is bounded by

$$\begin{aligned} I(t,x;t',x') \le C_{f,n}\int_0^t \mathrm{d}s \iint_{\mathbb{R}^2} \mathrm{d}z_1\mathrm{d}z_2\, G_{2\nu(2+n^2)}^{-1}(n,z_1) G_{2\nu(2+n^2)}^{-1}(n,z_1) G_{2\nu}(s,z_1-z_2) \\ \times \int_{\mathbb{R}} \mathrm{d}y\, G_{\nu/2}\left(s, y - \frac{z_1+z_2}{2}\right)\left(G_\nu\left(t-s,x-y\right) - G_\nu(t'-s,x'-y)\right)^2, \end{aligned} \tag{2.6.27}$$

where

$$C_{f,n} = \left(\sup_{x\in\mathbb{R}} |f(x)| G_{2\nu(2+n^2)}(n,x)\right)^2$$

which is finite since $\mu \in \mathcal{M}_H(\mathbb{R})$. Now follow the same argument as before, we simply replace This completes the proof of (2.6.18).

Property (2.6.19)**.** Now we prove (2.6.19). Set $t = t'$ in (2.6.21). Let us consider the integral over $\mathrm{d}s\mathrm{d}y$ in (2.6.21):

$$\int_0^t \mathrm{d}s\, G_{2\nu}(s, z_1 - z_2) \int G(G-G)^2 \mathrm{d}y\,,$$

which is denoted by $\int G\mathrm{d}s \int G(G-G)^2\mathrm{d}y$ for convenience. Using the semigroup property to integrate over $\mathrm{d}y$ gives, as in (2.6.22),

$$\int G(G-G)^2\mathrm{d}y = \frac{1}{\sqrt{4\pi\nu(t-s)}}\left(G_{\nu/2}(t, x-\bar{z}) + G_{\nu/2}\left(t, x'-\bar{z}\right)\right) \\ - 2G_{2\nu}\left(t-s, x-x'\right) G_{\nu/2}\left(t, \frac{x+x'}{2} - \bar{z}\right).$$

Then apply Lemma 2.6.5 to integrate over s,

$$\int G\mathrm{d}s \int G(G-G)^2\mathrm{d}y = \frac{1}{4\nu}\left(G_{\nu/2}(t, x-\bar{z}) + G_{\nu/2}\left(t, x'-\bar{z}\right)\right)\operatorname{erfc}\left(\frac{|z_1-z_2|}{\sqrt{4\nu t}}\right) \\ - \frac{1}{2\nu} G_{\nu/2}\left(t, \frac{x+x'}{2} - \bar{z}\right)\operatorname{erfc}\left(\frac{1}{\sqrt{2t}}\left(\frac{|z_1-z_2|}{\sqrt{2\nu}} + \frac{\left|x-x'\right|}{\sqrt{2\nu}}\right)\right).$$

Since for all $x \ge 0$,

$$\frac{\mathrm{d}}{\mathrm{d}x}\operatorname{erfc}(x) = -\frac{2e^{-x^2}}{\sqrt{\pi}} < 0, \quad \text{and} \quad \frac{\mathrm{d}^2}{\mathrm{d}x^2}\operatorname{erfc}(x) = \frac{4xe^{-x^2}}{\sqrt{\pi}} > 0\,,$$

we know that for $h \ge 0$

$$\operatorname{erfc}(|x| + h) \ge \operatorname{erfc}(|x|) - \frac{2e^{-x^2}}{\sqrt{\pi}} h\,.$$

Applying the above inequality to $\operatorname{erfc}\left(\frac{1}{\sqrt{2t}}\left(\frac{|z_1-z_2|}{\sqrt{2\nu}} + \frac{|x-x'|}{\sqrt{2\nu}}\right)\right)$, we have

$$\int G\mathrm{d}s \int G(G-G)^2\mathrm{d}y \le \frac{1}{\nu} G_{\nu/2}\left(t, \frac{x+x'}{2} - \bar{z}\right) \frac{\left|x-x'\right|}{\sqrt{\pi\nu t}} \exp\left(-\frac{(z_1-z_2)^2}{4\nu t}\right) \\ + \frac{1}{4\nu}\left(G_{\nu/2}(t, x-\bar{z}) + G_{\nu/2}\left(t, x'-\bar{z}\right) - 2G_{\nu/2}\left(t, \frac{x+x'}{2} - \bar{z}\right)\right)\operatorname{erfc}\left(\frac{|z_1-z_2|}{\sqrt{4\nu t}}\right).$$

Now apply Proposition 2.6.8 with $h = \frac{x'-x}{2}$, $L = 2n$ and $\beta = 1/2$: there are two constants

$$C_n' = \sup_{t\in[1/n,n]} C_{2n,1/2,\nu t} = \frac{C\sqrt{n}}{\sqrt{2\nu}} + \frac{1}{n}, \quad C \approx 0.451256\,,$$

and

$$C_n'' = \sup_{t\in[1/n,n]} C_{2n,1/2,\nu t}'' = C_n' \exp\left(\frac{8n^3}{\nu}\right)\,,$$

where $C_{L,\beta,\nu t}'$ and $C_{L,\beta,\nu t}''$ are defined in Proposition 2.6.8, such that

$$\begin{aligned}\left|G_{\nu/2}(t, x-\bar{z}) + G_{\nu/2}(t, x'-\bar{z}) - 2G_{\nu/2}\left(t, \frac{x+x'}{2} - \bar{z}\right)\right| \le \\ \Bigg(C_n'' \left[G_{\nu/2}\left(t, \frac{x+x'}{2} - \bar{z} - 2L\right) + G_{\nu/2}\left(t, \frac{x+x'}{2} - \bar{z} + 2L\right)\right] \\ + C_n' G_{\nu/2}\left(t, \frac{x+x'}{2} - \bar{z}\right)\Bigg) |x-x'|, \quad \text{for all } \left|\frac{x-x'}{2}\right| \le \beta L = n\,.\end{aligned}$$

Note that $t \ge 1/n$ is essential for the two constants C_n' and C_n'' to be finite. By (2.6.4), we have

$$\operatorname{erfc}\left(\frac{|z_1 - z_2|}{\sqrt{4\nu t}}\right) \le \sqrt{4\pi\nu t}\, G_{2\nu}(t, z_1 - z_2)\,,$$

and so

$$\begin{aligned}\left|\int G\mathrm{d}s \int G(G-G)^2 \mathrm{d}y\right| \le& \left(\frac{2}{\nu} + \frac{\sqrt{\pi t}}{\sqrt{4\nu}} C_n'\right) |x-x'|\, G_{\nu/2}\left(t, \frac{x+x'}{2} - \bar{z}\right) G_{2\nu}(t, z_1 - z_2) \\ &+ \frac{\sqrt{\pi t}\, C_n''}{\sqrt{4\nu}} |x-x'|\, G_{\nu/2}\left(t, \frac{x+x'}{2} - \bar{z} - 2L\right) G_{2\nu}(t, z_1 - z_2) \\ &+ \frac{\sqrt{\pi t}\, C_n''}{\sqrt{4\nu}} |x-x'|\, G_{\nu/2}\left(t, \frac{x+x'}{2} - \bar{z} + 2L\right) G_{2\nu}(t, z_1 - z_2)\,.\end{aligned}$$

Now apply Lemma 2.3.8:

$$\begin{aligned}\left|\int G\mathrm{d}s \int G(G-G)^2 \mathrm{d}y\right| \le& \left(\frac{2}{\nu} + \frac{\sqrt{\pi n}}{\sqrt{4\nu}} C_n'\right) |x-x'|\, G_{2\nu}(t, \widetilde{x}_1 - z_1)\, G_{2\nu}(t, \widetilde{x}_1 - z_2) \\ &+ \frac{\sqrt{\pi n}\, C_n''}{\sqrt{4\nu}} |x-x'|\, G_{2\nu}(t, \widetilde{x}_2 - z_1)\, G_{2\nu}(t, \widetilde{x}_2 - z_2) \\ &+ \frac{\sqrt{\pi n}\, C_n''}{\sqrt{4\nu}} |x-x'|\, G_{2\nu}(t, \widetilde{x}_3 - z_1)\, G_{2\nu}(t, \widetilde{x}_3 - z_2)\end{aligned}$$

where

$$\widetilde{x}_1 = \frac{x+x'}{2}, \qquad \widetilde{x}_2 = \frac{x+x'}{2} - 2L, \qquad \widetilde{x}_3 = \frac{x+x'}{2} + 2L\,,$$

and we have use the fact that $t \le n$. Clearly, $\widetilde{x}_i \in [-9n, 9n]$ for all $i = 1, 2, 3$. Finally, after integrating over $|\mu|(\mathrm{d}z_1)$ and $|\mu|(\mathrm{d}z_2)$, we get

$$I(t,x;t,x') \le C'_{n,3}\left|x-x'\right|, \quad \text{for all } (t,x),(t,x') \in [1/n,n]\times[-n,n]\,,$$

where

$$C'_{n,3} = \sup_{\substack{t\in[1/n,n]\\ x\in[-9n,9n]}} \left(\left(\frac{2}{\nu}+\frac{\sqrt{\pi n}}{\sqrt{4\nu}}\left(C'_n+2C''_n\right)\right)\left(|\mu|*G_{2\nu}(t,\cdot)\right)^2(x)\right).$$

As for the contribution of the constant $\overline{\varsigma}$, it corresponds to the initial data $|\mu|(\mathrm{d}x)\equiv\overline{\varsigma}\,\mathrm{d}x$ and we apply Proposition 2.3.9, in particular (2.3.16). Finally, we can choose

$$C_{n,3} = C_1\overline{\varsigma}^2 + \sup_{\substack{t\in[1/n,n]\\ x\in[-9n,9n]}} \left(\frac{4}{\nu}+\frac{\sqrt{\pi n}}{\sqrt{\nu}}\left(C'_n+2C''_n\right)\right)\left(|\mu|*G_{2\nu}(t,\cdot)\right)^2(x), \quad C_1\approx 1.36005\,,$$

for (2.6.19). This constant $C_{n,3}$ is finite by the same reason as before. This finishes the proof of (2.6.19). □

The following proof needs the following integral

$$\int_t^{t'} \frac{1}{\sqrt{s(t'-s)}}\mathrm{d}s = 2\arcsin\left(\sqrt{\frac{t'-t}{t'}}\right), \quad \text{for all } t'>t\ge 0\,. \tag{2.6.28}$$

It is true for $t=0$ since the left-hand side reduces to the Beta integral (2.3.5). For the case where $t\in\,]0,t']$, this equality can be seen by differentiating with respect to t on both sides.

Proof of (2.6.20). We first consider the contribution of $J_0^*(t,x)$. Let

$$I\left(t,x;t',x'\right) = \iint_{[t,t']\times\mathbb{R}} \left|J_0^*\left(s,y\right)\right|^2 G_\nu^2(t'-s,x'-y)\mathrm{d}s\mathrm{d}y\,.$$

Similar to the arguments leading to (2.6.21), we have

$$\begin{aligned}I\left(t,x;t',x'\right) = \int_t^{t'}\mathrm{d}s\iint_{\mathbb{R}^2}|\mu|(\mathrm{d}z_1)|\mu|(\mathrm{d}z_2)\;G_{2\nu}(s,z_1-z_2)&\\ \times\int_{\mathbb{R}} G_{\nu/2}\left(s,y-\frac{z_1+z_2}{2}\right)G_\nu^2\left(t'-s,x'-y\right)\mathrm{d}y\,.&\end{aligned} \tag{2.6.29}$$

Applying Lemma 2.3.7 on $G_\nu^2\left(t'-s,x'-y\right)$ and then integrating over y using the semi-group property of the heat kernel, we have

$$I\left(t,x;t',x'\right) = \int_t^{t'}\mathrm{d}s\iint_{\mathbb{R}^2}\frac{1}{\sqrt{4\pi\nu(t'-s)}}G_{2\nu}(s,z_1-z_2)G_{\nu/2}\left(t',x'-\frac{z_1+z_2}{2}\right)|\mu|(\mathrm{d}z_1)|\mu|(\mathrm{d}z_2)\,.$$

Now apply Lemma 2.3.8,

$$G_{2\nu}(s,z_1-z_2)G_{\nu/2}\left(t',x'-\frac{z_1+z_2}{2}\right)\le\frac{2\nu t'}{\sqrt{\nu^2 st'}}G_{2\nu}(t',x'-z_1)G_{2\nu}(t',x'-z_2)\,.$$

Hence

$$\begin{aligned}I\left(t,x;t',x'\right)&\le\left|J_0^*\left(2t',x'\right)\right|^2\int_t^{t'}\frac{\sqrt{t'}}{\sqrt{\pi\nu s(t'-s)}}\mathrm{d}s\\&=\left|J_0^*\left(2t',x'\right)\right|^2\frac{2\sqrt{t'}}{\sqrt{\pi\nu}}\arcsin\left(\sqrt{\frac{t'-t}{t'}}\right)\\&\le\left|J_0^*\left(2t',x'\right)\right|^2\frac{\sqrt{\pi}}{\sqrt{\nu}}\sqrt{t'-t}\,,\end{aligned}$$

where we have used the integral (2.6.28) and the fact that $\arcsin(x)\le\pi x/2$ for $x\in[0,1]$. Therefore,

$$I\left(t,x;t',x'\right)\le C_{n,5}'\sqrt{t'-t}$$

with the constant

$$C_{n,5}'=\sqrt{\pi/\nu}\sup_{\substack{t\in[1/n,n]\\x\in[-n,n]}}\left|J_0^*(2t,x)\right|^2<+\infty\,.$$

As for the contribution of $\overline{\varsigma}$, it corresponds to the initial data $|\mu|(\mathrm{d}x)\equiv\overline{\varsigma}\,\mathrm{d}x$ and we apply Proposition 2.3.9, in particular (2.3.18). Finally, we can choose

$$C_{n,5}=\frac{\overline{\varsigma}^2}{\sqrt{\pi\nu}}+2\sqrt{\pi/\nu}\sup_{\substack{t\in[1/n,n]\\x\in[-n,n]}}\left|J_0^*(2t,x)\right|^2 \tag{2.6.30}$$

for (2.6.20). This completes the proof of (2.6.20). □

Proposition 2.6.17. *Given $\overline{\varsigma}\in\mathbb{R}$ and any initial data μ satisfying* (1.1.5)*, let $J_0^*(t,x)=\left(|\mu|*G_\nu(t,\cdot)\right)(x)$. Then for all $n>1$, there exist constants $C_{n,i}$, $i=2,4,6$, such that for all $t,t'\in[1/n,n]$ with $t<t'$ and $x,x'\in[-n,n]$,*

$$\left|\left(\left(\overline{\varsigma}^2+2\left|J_0^*\right|^2\right)\star G_\nu^2\star\left(G_\nu(\cdot,\circ)-G_\nu(\cdot+t'-t,\circ)\right)^2\right)(t,x)\right|\le C_{n,2}\sqrt{t'-t}\,, \tag{2.6.31}$$

$$\left|\left(\left(\overline{\varsigma}^2+2\left|J_0^*\right|^2\right)\star G_\nu^2\star\left(G_\nu(\cdot,\circ)-G_\nu(\cdot,\circ+x'-x)\right)^2\right)(t,x)\right|\le C_{n,4}|x'-x|\,, \tag{2.6.32}$$

and

$$\iint_{[t,t']\times\mathbb{R}}\left(\left(\overline{\varsigma}^2+2\left|J_0^*\right|^2\right)\star G_\nu^2\right)(s,y)\,G_\nu^2(t'-s,x'-y)\mathrm{d}s\mathrm{d}y\le C_{n,6}\sqrt{t'-t}\,. \tag{2.6.33}$$

Remark 2.6.18. If (2.6.18) – (2.6.20) holds for $0<t<t'\le n$ instead of $1/n\le t<t'\le n$,

then (2.6.31) – (2.6.33) can be easily proved using (2.6.18) – (2.6.20). For example,

$$\begin{aligned}\left|\left(\left(\overline{\varsigma}^2+2\left|J_0^*\right|^2\right)\star G_\nu^2\star\left(G_\nu(\cdot,\circ)-G_\nu(\cdot+t'-t,\circ)\right)^2\right)(t,x)\right| &\le C_{n,1}\sqrt{t'-t}\left(1\star G_\nu^2\right)(t,x)\\ &= C_{n,1}\frac{\sqrt{t}}{\sqrt{\pi\nu}}\sqrt{t'-t}\\ &\le C_{n,1}\frac{\sqrt{n}}{\sqrt{\pi\nu}}\sqrt{t'-t}\,.\end{aligned}$$

Proof of Proposition 2.6.17. We first prove (2.6.31) and (2.6.32). Denote

$$\begin{aligned}I(t,x;t',x') &:= \iint_{[0,t]\times\mathbb{R}}\left(\left|J_0^*\right|^2\star G_\nu^2\right)(s,y)\left(G_\nu\left(t-s,x-y\right)-G_\nu(t'-s,x'-y)\right)^2\mathrm{d}s\mathrm{d}y\\ &= \iint_{[0,t]\times\mathbb{R}}\mathrm{d}s\mathrm{d}y\left(G_\nu\left(t-s,x-y\right)-G_\nu\left(t'-s,x'-y\right)\right)^2\\ &\qquad\times\iint_{[0,s]\times\mathbb{R}}\left|J_0^*(u,z)\right|^2 G_\nu^2(s-u,y-z)\mathrm{d}u\mathrm{d}z\,.\end{aligned}$$

Denote $\bar{z}=(z_1+z_2)/2$. As in (2.6.21), replace $\left|J_0^*(u,z)\right|^2$ by the double integral

$$\left|J_0^*(u,z)\right|^2=\iint_{\mathbb{R}^2}G_{2\nu}(u,z_1-z_2)G_{\nu/2}\left(u,z-\bar{z}\right)|\mu|(\mathrm{d}z_1)|\mu|(\mathrm{d}z_2)\,.$$

Then the convolutions become (after permuting the integrals and using Lemma 2.3.7)

$$\begin{aligned}I\left(t,x;t',x'\right) &= \iint_{\mathbb{R}^2}|\mu|(\mathrm{d}z_1)|\mu|(\mathrm{d}z_2)\int_0^t\mathrm{d}s\int_0^s\mathrm{d}u\,\frac{1}{\sqrt{4\nu\pi(s-u)}}G_{2\nu}(u,z_1-z_2)\\ &\times\iint_{\mathbb{R}^2}G_{\nu/2}\left(u,z-\bar{z}\right)G_{\nu/2}(s-u,y-z)\left(G_\nu\left(t-s,x-y\right)-G_\nu\left(t'-s,x'-y\right)\right)^2\mathrm{d}y\mathrm{d}z\,.\end{aligned}$$

We first integrate over $\mathrm{d}z$ using the semigroup property:

$$\begin{aligned}I\left(t,x;t',x'\right) &= \iint_{\mathbb{R}^2}|\mu|(\mathrm{d}z_1)|\mu|(\mathrm{d}z_2)\int_0^t\mathrm{d}s\int_0^s\mathrm{d}u\,\frac{1}{\sqrt{4\nu\pi(s-u)}}G_{2\nu}(u,z_1-z_2)\\ &\qquad\times\int_{\mathbb{R}}G_{\nu/2}\left(s,y-\bar{z}\right)\left(G_\nu\left(t-s,x-y\right)-G_\nu\left(t'-s,x'-y\right)\right)^2\mathrm{d}y\,.\end{aligned}$$

Then integrate over $\mathrm{d}u$ using Lemma 2.6.5 and the fact that $s\le t\le n$ to obtain

$$\begin{aligned}I\left(t,x;t',x'\right) &\le \frac{\sqrt{\pi n}}{\sqrt{4\nu}}\int_0^t\mathrm{d}s\iint_{\mathbb{R}^2}|\mu|(\mathrm{d}z_1)|\mu|(\mathrm{d}z_2)\,G_{2\nu}(s,z_1-z_2)\\ &\qquad\times\int_{\mathbb{R}}G_{\nu/2}\left(s,y-\bar{z}\right)\left(G_\nu\left(t-s,x-y\right)-G_\nu\left(t'-s,x'-y\right)\right)^2\mathrm{d}y\,.\end{aligned}\tag{2.6.34}$$

Comparing this upper bound with (2.6.21), we can apply Proposition 2.6.16 to conclude that (2.6.31) and (2.6.32) are true with the corresponding constants

$$C_{n,2}=\frac{\sqrt{\pi n}}{\sqrt{4\nu}}C_{n,1}\,,\quad\text{and}\quad C_{n,4}=\frac{\sqrt{\pi n}}{\sqrt{4\nu}}C_{n,3}\,.\tag{2.6.35}$$

As for (2.6.33), let

$$I\left(t,x;t',x'\right)=\iint_{[t,t']\times\mathbb{R}}\left(\left|J_0^*\right|^2\star G_\nu^2\right)(s,y)\,G_\nu^2(t'-s,x'-y)\mathrm{d}s\mathrm{d}y\,.$$

By similar arguments leading to (2.6.34), we have

$$\begin{aligned}I\left(t,x;t',x'\right)\le\frac{\sqrt{\pi n}}{\sqrt{4\nu}}\int_t^{t'}\mathrm{d}s\iint_{\mathbb{R}^2}|\mu|(\mathrm{d}z_1)|\mu|(\mathrm{d}z_2)\,G_{2\nu}(s,z_1-z_2)&\\ \times\int_{\mathbb{R}}G_{\nu/2}\left(s,y-\frac{z_1+z_2}{2}\right)G_\nu^2(t'-s,x'-y)\mathrm{d}y\,.&\end{aligned}$$

Comparing this upper bound with (2.6.29), we can apply Proposition 2.6.16 to conclude that (2.6.33) is true with the corresponding constant

$$C_{n,6}=\frac{\sqrt{\pi n}}{\sqrt{4\nu}}C_{n,5}\,. \tag{2.6.36}$$

This completes the whole proof. □

Proof of Theorem 2.2.13. Hölder continuity of $J_0(t,x)$ in the three cases is covered by Lemma 2.3.5. So we only need to prove the Hölder continuity of the stochastic integral part $I(t,x)$. Without loss of generality, we assume that $\mu\ge 0$. Otherwise, we simply replace the μ's in the following arguments by $|\mu|$. Fix $n>1$. By Propositions 2.6.16 and 2.6.17, there exist $C_{n,i}>0$ for $i=1,\dots,6$ such that for all (t,x) and $\left(t',x'\right)\in[1/n,n]\times[-n,n]$ with $t'>t$, the six inequalities in Propositions 2.6.16 and 2.6.17 hold. By Lemma 2.3.20 and the linear growth(1.4.1) of ρ, for all even integers $p>2$,

$$\begin{aligned}&\left|\left|I(t,x)-I\left(t',x'\right)\right|\right|_p^p\\ &\le 2^{p-1}\mathbb{E}\left(\left|\int_0^t\int_{\mathbb{R}}\rho(u(s,y))\left(G_\nu\left(t-s,x-y\right)-G_\nu(t'-s,x'-y)\right)W(\mathrm{d}s,\mathrm{d}y)\right|^p\right)\\ &\quad+2^{p-1}\mathbb{E}\left(\left|\int_t^{t'}\int_{\mathbb{R}}\rho(u(s,y))G_\nu\left(t'-s,x'-y\right)W(\mathrm{d}s,\mathrm{d}y)\right|^p\right)\\ &=2^{p-1}z_p^pL_\rho^p\left(L_1\left(t,t',x,x'\right)\right)^{p/2}+2^{p-1}z_p^pL_\rho^p\left(L_2\left(t,t';x'\right)\right)^{p/2}\,,\end{aligned}$$

where

$$L_1\left(t,t',x,x'\right)=\iint_{[0,t]\times\mathbb{R}}\left(G_\nu\left(t-s,x-y\right)-G_\nu(t'-s,x'-y)\right)^2\left(\overline{\varsigma}^2+\left|\left|u\left(s,y\right)\right|\right|_p^2\right)\mathrm{d}s\mathrm{d}y$$

and

$$L_2\left(t,t';x'\right)=\iint_{[t,t']\times\mathbb{R}}G_\nu^2\left(t'-s,x'-y\right)\left(\overline{\varsigma}^2+\left|\left|u\left(s,y\right)\right|\right|_p^2\right)\mathrm{d}s\mathrm{d}y\,.$$

Then by the subadditivity of the function $x\mapsto|x|^{2/p}$, we have

$$\left|\left|I(t,x)-I\left(t',x'\right)\right|\right|_p^2\le 4z_p^2L_\rho^2\left(L_1\left(t,t',x,x'\right)+L_2\left(t,t';x'\right)\right),$$

where we have used the fact $2^{2(p-1)/p} \le 4$.

Notice that the kernel function $\mathscr{K}$ defined in (2.2.4) can be written as

$$\mathscr{K}(t,x;\nu,\lambda) = \Upsilon(t;\nu,\lambda)\; G_\nu^2(t,x)\,,$$

with

$$\Upsilon(t;\nu,\lambda) := \lambda^2 \left(1+\lambda^2\sqrt{\pi t/\nu}\,\exp\left(\frac{\lambda^4 t}{4\nu}\right)\Phi\left(\lambda^2\sqrt{\frac{t}{2\nu}}\right)\right).$$

For simplicity, denote

$$\Upsilon_*(t) := \Upsilon\left(t\,;\,\nu, a_{p,\overline{\varsigma}} z_p\, L_\rho\right) < +\infty, \quad \text{for all } t\in\mathbb{R}_+\,.$$

Clearly, $\Upsilon_*(t) \le \Upsilon_*(n)$ for $t\le n$. Hence, the upper bound on the p-th moments in (2.2.11) can be bounded further by

$$\begin{aligned}\left|\left|u(s,y)\right|\right|_p^2 &\le 2\,J_0^2(s,y) + \left(\left(\overline{\varsigma}^2 + 2\,J_0^2\right)\star\widehat{\mathscr{K}_p}\right)(s,y)\\ &\le 2\,J_0^2(s,y) + \Upsilon_*(n)\left(\left(\overline{\varsigma}^2+2\,J_0^2\right)\star G_\nu^2\right)(s,y), \quad \text{for } s\le t\le n.\end{aligned}$$

Then we shall use this bound on $\left|\left|u(s,y)\right|\right|_p^2$ to estimate L_1 and L_2.

Case I. We first consider the case where $x = x'$. Denote $s = t'-t$. By Propositions 2.6.16 and 2.6.17,

$$\begin{aligned}L_1(t,t',x,x) \le &\left(\left(\overline{\varsigma}^2+2J_0^2\right)\star\left(G_\nu(\cdot,\circ)-G_\nu(\cdot+s,\circ)\right)^2\right)(t,x)\\ &+\Upsilon_*(n)\left(\left(\overline{\varsigma}^2+2J_0^2\right)\star G_\nu^2\star\left(G_\nu(\cdot,\circ)-G_\nu(\cdot+s,\circ)\right)^2\right)(t,x)\\ \le &\left(C_{n,1}+\Upsilon_*(n)C_{n,2}\right)|s|^{1/2},\end{aligned}$$

and

$$\begin{aligned}L_2\left(t,t'\,;\,x'\right) \le &\iint_{[t,t']\times\mathbb{R}} G_\nu^2\left(t'-s,x'-y\right)\\ &\qquad\times\left(\overline{\varsigma}^2+2J_0^2(s,y)+\left(\left(\overline{\varsigma}^2+2\,J_0^2\right)\star\widehat{\mathscr{K}_p}\right)(s,y)\right)\mathrm{d}s\mathrm{d}y\\ \le &\left(C_{n,5}+\Upsilon_*(n)C_{n,6}\right)|s|^{1/2}.\end{aligned}$$

Hence, for all $x\in[-n,n]$ and $1/n\le t<t'\le n$,

$$\left|\left|I(t,x)-I(t',x)\right|\right|_p^2 \le 4z_p^2\,L_\rho^2\left(C_{n,1}+C_{n,5}+\Upsilon_*(n)\left(C_{n,2}+C_{n,6}\right)\right)\left|t'-t\right|^{1/2}. \tag{2.6.37}$$

Case II. Similarly, in the case where $t = t' > 0$, denote $h = x'-x$. We only have the term L_1. By Propositions 2.6.16 and 2.6.17:

$$\begin{aligned}\left|\left|I(t,x)-I\left(t,x'\right)\right|\right|_p^2 &\le 4z_p^2\,L_\rho^2\;L_1(t,t,x,x')\\ &\le \quad 4z_p^2\,L_\rho^2\left(\left(\overline{\varsigma}^2+2J_0^2\right)\star\left(G_\nu(\cdot,\circ)-G_\nu(\cdot,\circ+h)\right)^2\right)(t,x)\end{aligned}$$

$$+4z_p^2 L_\rho^2 \Upsilon_*(n) \left(\left(\overline{\varsigma}^2 + 2J_0^2\right) \star G_\nu^2 \star (G_\nu(\cdot,\circ) - G_\nu(\cdot,\circ+h))^2\right)(t,x)$$
$$\leq 4z_p^2 L_\rho^2 \left[C_{n,3} + \Upsilon_*(n) C_{n,4}\right] |h| .$$

Finally, combining these two cases gives

$$\begin{aligned}\left|\left|I(t,x) - I\left(t',x'\right)\right|\right|_p^2 &\leq 2\left|\left|I(t,x) - I\left(t',x\right)\right|\right|_p^2 + 2\left|\left|I\left(t',x\right) - I\left(t',x'\right)\right|\right|_p^2 \\ &\leq \widetilde{C}_{p,n}\left(\left|t'-t\right|^{1/2} + \left|x'-x\right|\right),\end{aligned}$$

for all $1/n \leq t < t' \leq n$, $x, x' \in [-n,n]$, where

$$\widetilde{C}_{p,n} = 8z_p^2 L_\rho^2 \left(C_{n,1} + C_{n,3} + C_{n,5} + \Upsilon_*(n)\left(C_{n,2} + C_{n,4} + C_{n,6}\right)\right).$$

Then the Hölder continuity is proved by an application of Kolmogorov's continuity theorem (see Proposition 2.6.4). This completes the whole proof. □

2.6.5 Proof of the Example 2.2.16 where $\mu = |x|^{-a}$

We need a lemma. Recall that a Schwartz distribution $\mu \in \mathscr{S}'(\mathbb{R})$ is called *non-negative definite*, if $\langle \mu, \phi * \phi^* \rangle \geq 0$ for every $\phi \in \mathscr{S}(\mathbb{R})$, where $\left(\phi * \phi^*\right)(x)$ denotes the convolution of the functions $\phi(x)$ and $\phi^*(x) := \phi(-x)$,

$$\left(\phi * \phi^*\right)(x) = \int_{\mathbb{R}} \phi(y)\phi(x-y)\mathrm{d}y .$$

Lemma 2.6.19. *If $\mu \in \mathscr{S}'(\mathbb{R})$ is non-negative definite, then*

$$\left|\left(\mu * G_\nu(t,\cdot)\right)(x)\right| \leq \left(\mu * G_\nu(t,\cdot)\right)(0), \quad \textit{for all } x \in \mathbb{R} \textit{ and } t > 0 .$$

Proof. Let $\hat{\mu}$ be the Fourier transform of μ. By a version of Bochner's theorem (see [38, Theorem 1, on p.152]), $\hat{\mu}$ is a positive tempered measure and hence

$$\begin{aligned}\left|\left(\mu * G_\nu(t,\cdot)\right)(x)\right| &= \left|\int_{\mathbb{R}} \exp\left(-i\xi x - \frac{\nu t}{2}\xi^2\right)\hat{\mu}(\mathrm{d}\xi)\right| \\ &\leq \int_{\mathbb{R}} \exp\left(-\frac{\nu t}{2}\xi^2\right)\hat{\mu}(\mathrm{d}\xi) = \left(\mu * G_\nu(t,\cdot)\right)(0) .\end{aligned}$$

This completes the proof. □

Proof of Example 2.2.16. By our moment formula, we only need to show the case where $p = 2$. This proof consists of the following two parts.

Part I. We first show that $\lim_{t\to 0_+} ||I(t,x)||_p \equiv 0$. For some constant $C_a > 0$, the Fourier transform of μ is $C_a|x|^{-1+a}$ (see [66, Lemma 2 (a), on p. 117]), which is a non-negative measure. Hence Bochner's theorem (see, e.g., [38, Theorem 1, on p.152]) implies that μ

is non-negative definite. Then apply Lemma 2.6.19,

$$0 < J_0(t,x) \le J_0(t,0) = \int_{\mathbb{R}} \frac{1}{|y|^a} G_\nu(t,y)\mathrm{d}y = 2\int_0^\infty \frac{e^{-y^2/(2\nu t)}}{y^a\sqrt{2\pi\nu t}}\mathrm{d}y.$$

Then by the change of variable $u = \frac{y^2}{2\nu t}$ and Euler's integral (or the definition of the Gamma functions, see, e.g, [51, 5.2.1, p. 136]), we have

$$J_0(t,0) = 2\int_0^{+\infty} \frac{e^{-u}}{(2\nu t u)^{a/2}\sqrt{2\pi\nu t}} \frac{\sqrt{2\nu t}}{2\sqrt{u}}\mathrm{d}u = \frac{\Gamma\left(\frac{1-a}{2}\right)}{\sqrt{\pi}(2\nu t)^{a/2}}. \tag{2.6.38}$$

By the moment formula (2.2.15) and the above bound,

$$\begin{aligned}
||I(t,x)||_2^2 &= \left(J_0^2 \star \mathcal{K}\right)(t,x) \\
&\le \int_0^t \mathrm{d}s \left(\frac{\lambda^2}{\sqrt{4\pi\nu(t-s)}} + \frac{\lambda^4}{2\nu} e^{\frac{\lambda^4(t-s)}{4\nu}} \Phi\left(\lambda^2\sqrt{\frac{t-s}{2\nu}}\right)\right) \frac{C}{s^a} \int_{\mathbb{R}} G_{\nu/2}\left(t-s, x-y\right)\mathrm{d}y \\
&\le \int_0^t \left(\frac{\lambda^2}{\sqrt{4\pi\nu(t-s)}} + \frac{\lambda^4}{2\nu} e^{\frac{\lambda^4(t-s)}{4\nu}}\right) \frac{C}{s^a}\mathrm{d}s
\end{aligned} \tag{2.6.39}$$

where $C = \frac{\Gamma^2\left(\frac{1-a}{2}\right)}{\pi(2\nu)^a}$. The integrand of (2.6.39) is integrable if and only if $a < 1$. Hence, as t tends to zero, the integral goes to zero too. Finally, $\lim_{t\to 0_+} ||I(t,x)||_2^2 = 0$.

Part II. Now let us consider the function $t \mapsto I(t,0)$ from $\mathbb{R}_+$ to $L^p(\Omega)$. Since $(x-y)^2 \le 2(x^2+y^2)$, we have

$$\begin{aligned}
J_0(t,x) = \int_{\mathbb{R}} \frac{1}{|y|^a} G_\nu\left(t, x-y\right)\mathrm{d}y &\ge \frac{1}{\sqrt{2}} \exp\left(-\frac{x^2}{\nu t}\right) \int_{\mathbb{R}} \frac{1}{|y|^a} G_{\nu/2}(t,y)\mathrm{d}y \\
&= \frac{1}{\sqrt{2}} \exp\left(-\frac{x^2}{\nu t}\right) \frac{\Gamma\left(\frac{1-a}{2}\right)}{\sqrt{\pi}} \frac{1}{(\nu t)^{a/2}},
\end{aligned}$$

where in the last step we have used the integral (2.6.38). Hence,

$$J_0^2(t,x) \ge C G_{\nu/2}(t/2, x) t^{1/2-a}, \quad C = \frac{1}{2^{3/2}\sqrt{\pi}} \Gamma^2\left(\frac{1-a}{2}\right) \nu^{1/2-a}.$$

Then, since $\mathcal{K}(t,x) \ge G_{\nu/2}(t,x)\frac{\lambda^2}{\sqrt{4\pi\nu t}}$, we have

$$\begin{aligned}
||I(t,x)||_2^2 &\ge \int_0^t \mathrm{d}s \frac{C\lambda^2 s^{1/2-a}}{\sqrt{4\pi\nu(t-s)}} \int_{\mathbb{R}} G_{\nu/2}\left(t-s, x-y\right) G_{\nu/2}(s/2, y)\mathrm{d}y \\
&= \int_0^t \frac{C\lambda^2 s^{1/2-a}}{\sqrt{4\pi\nu(t-s)}} G_{\nu/2}\left(t - \frac{s}{2}, x\right)\mathrm{d}s \\
&\ge \frac{C\lambda^2 \exp\left(-\frac{2x^2}{\nu t}\right)}{2\pi\nu t} \int_0^t s^{1/2-a}\mathrm{d}s = \frac{C\,\lambda^2 \exp\left(-\frac{2x^2}{\nu t}\right)}{2\pi\,\nu\,(3/2-a)} t^{\frac{1-2a}{2}}.
\end{aligned}$$

Now if $x = 0$, then for all integers $p \geq 2$, since $I(0,x) \equiv 0$, we have that

$$||I(t,0) - I(0,0)||_p^2 = ||I(t,0)||_p^2 \geq ||I(t,0)||_2^2 \geq C' t^{\frac{1-2a}{2}}, \qquad C' = \frac{C\lambda^2}{2\pi\nu(3/2-a)}.$$

Therefore, the function $t \mapsto I(t,0)$ from $\mathbb{R}_+$ to $L^p(\Omega)$ cannot be smoother than η-Hölder continuous at $t = 0$ with $\eta = \frac{1-2a}{4}$. Finally, $a \in]0,1/2]$ implies that $\eta \in [0,1/4[$. This completes the whole proof. □

2.7 Finding the Second Moment via Integral Transforms

Assume that $\rho(u) = \lambda u$. Then the second moment $\mathbb{E}[u(t,x)^2]$ denoted by $f(t,x)$ satisfies the following integral equation

$$f(t,x) = J_0^2(t,x) + \lambda^2 \left(G_\nu^2 \star f\right)(t,x), \tag{2.7.1}$$

where $J_0(t,x) = \left(\mu * G_\nu(t,\cdot)\right)(x)$ is the solution to the homogeneous equation and μ is the initial condition.

Assumption 2.7.1. Assume that the double transform – the Fourier transform in x and the Laplace transform in t – exists for $J_0^2(t,x)$.

Note that Assumption 2.7.1 is rather strong. If the initial data has exponential growth, for example, $\mu(\mathrm{d}x) = e^{\beta|x|}\mathrm{d}x$ with $\beta > 0$, then $J_0(t,x)$ has two exponential growing tails (see (2.5.1)), and hence the Fourier transform of $J_0^2(t,x)$ in x does not exists.

Now let us assume that Assumption 2.7.1 holds. Apply the Fourier transform over x on both sides of (2.7.1),

$$\mathscr{F}\left[f(t,\cdot)\right](\xi) = \mathscr{F}\left[J_0^2(t,\cdot)\right](\xi) + \lambda^2 \int_0^t \mathscr{F}\left[G_\nu^2(t-s,\cdot)\right](\xi)\mathscr{F}\left[f(s,\cdot)\right](\xi)\mathrm{d}s.$$

Then apply the Laplace transform on t, we have

$$\mathscr{L}\mathscr{F}\left[f\right](z,\xi) = \mathscr{L}\mathscr{F}\left[J_0^2\right](z,\xi) + \lambda^2 \mathscr{L}\mathscr{F}\left[G_\nu^2\right](z,\xi)\mathscr{L}\mathscr{F}\left[f\right](z,\xi).$$

Hence

$$\mathscr{L}\mathscr{F}\left[f\right](z,\xi) = \mathscr{L}\mathscr{F}\left[J_0^2\right](z,\xi) + \frac{\lambda^2 \mathscr{L}\mathscr{F}\left[G_\nu^2\right](z,\xi)}{1-\lambda^2 \mathscr{L}\mathscr{F}\left[G_\nu^2\right](z,\xi)} \mathscr{L}\mathscr{F}\left[J_0^2\right](z,\xi).$$

Now let us calculate $\mathscr{L}\mathscr{F}[G_\nu^2](z,\xi)$. Clearly,

$$G_\nu^2(t,x) = \frac{1}{\sqrt{4\pi\nu t}} G_{\nu/2}(t,x).$$

Hence,

$$\mathscr{F}\left[G_\nu^2(t,\cdot)\right](\xi) = \frac{\exp\left(-\nu t|\xi|^2/4\right)}{\sqrt{4\pi\nu t}}\,.$$

Then use the following Laplace transform (see [35, (15) and (16) of Section 4.2, p. 135])

$$\mathscr{L}\left[\frac{1}{\sqrt{t}}\right](z) = \frac{\sqrt{\pi}}{\sqrt{z}}, \quad \Re[z] > 0$$

to conclude

$$\mathscr{L}\mathscr{F}[G_\nu^2](z,\xi) = \frac{1}{\sqrt{4\nu z + |\xi|^2\nu^2}}, \qquad \Re[z] > 0\,.$$

Hence

$$\frac{\lambda^2\mathscr{L}\mathscr{F}\left[G_\nu^2\right](z,\xi)}{1-\lambda^2\mathscr{L}\mathscr{F}\left[G_\nu^2\right](z,\xi)} = \frac{\lambda^2}{\sqrt{4\nu z + |\xi|^2\nu^2}-\lambda^2}\,.$$

Now we need to calculate the inverse Laplace and inverse Fourier transforms of the above formulas. First, we use the inverse Laplace transform (see [35, (4) of Section 5.3, p. 233])

$$\mathscr{L}^{-1}\left[\frac{1}{\sqrt{z}+a}\right] = \frac{1}{\sqrt{\pi t}} - a e^{a^2 t}\operatorname{erfc}(a\sqrt{t})\,.$$

For $a > 0$, this inverse transform can be written as

$$\begin{aligned}\mathscr{L}^{-1}\left[\frac{1}{\sqrt{z}-a}\right] &= \frac{1}{\sqrt{\pi t}} + a e^{a^2 t}\left(1+\operatorname{erf}(a\sqrt{t})\right)\\ &= \frac{1}{\sqrt{\pi t}} + 2a e^{a^2 t}\Phi(a\sqrt{2t})\,.\end{aligned}$$

Thus apply this transform with $a = \frac{\lambda^2}{\sqrt{4\nu}}$ to get

$$\mathscr{L}^{-1}\left[\frac{\lambda^2}{\sqrt{4\nu z + |\xi|^2\nu^2}-\lambda^2}\right](t) = \exp\left(-\frac{\nu|\xi|^2}{4}\right)\left(\frac{\lambda^2}{\sqrt{4\nu\pi t}} + \frac{\lambda^4}{2\nu}\exp\left(\frac{\lambda^4 t}{4\nu}\right)\Phi\left(\lambda^2\sqrt{\frac{t}{2\nu}}\right)\right)\,.$$

Finally, by the inverse Fourier transform over ξ, we get the $\mathscr{K}(t,x)$ function

$$\begin{aligned}\mathscr{K}\left(t,x;\nu/2,\frac{\lambda^2}{\sqrt{4\pi\nu}}\right) &= \mathscr{F}^{-1}\mathscr{L}^{-1}\left[\frac{\lambda^2}{\sqrt{4\nu z + |\xi|^2\nu^2}-\lambda^2}\right](t,x)\\ &= G_{\nu/2}(t,x)\left(\frac{\lambda^2}{\sqrt{4\nu\pi t}} + \frac{\lambda^4}{2\nu}\exp\left(\frac{\lambda^4 t}{4\nu}\right)\Phi\left(\lambda^2\sqrt{\frac{t}{2\nu}}\right)\right)\\ &= \mathscr{K}(t,x)\,.\end{aligned}$$

Therefore, the second moment equals

$$f(t,x) = J_0^2(t,x) + \left(J_0^2 \star \mathscr{K}\right)(t,x)\,.$$

3 Stochastic Integral Equations of Space-time Convolution Type

3.1 Introduction

In the previous chapter, we have studied the stochastic heat equation. In order to study later the stochastic wave equation, we first investigate a stochastic integral equation of space-time convolution type, and then apply it to the stochastic wave and heat equations by verifying the required assumptions. Other SPDE's could be included in this framework.

More precisely, we will consider the following stochastic integral equation in $\mathbb{R}_+^* \times \mathbb{R}^d$ with $d \geq 1$,

$$u(t,x) = J_0(t,x) + I(t,x)\,, \tag{3.1.1}$$

where

$$I(t,x) := \iint_{\mathbb{R}_+\times\mathbb{R}^d} G\left(t-s, x-y\right)\theta\left(s,y\right)\rho\left(u\left(s,y\right)\right)W\left(\mathrm{d}s,\mathrm{d}y\right)\,.$$

Let $\{\Omega, \mathscr{F}, \{\mathscr{F}_t : t \geq 0\}, P\}$ be a filtered probability space, which is the same as the one used in Chapter 2 except that the spatial domain here is $\mathbb{R}^d$. Here are the specifications of this equation:

(1) $\dot{W}$ is the space-time white noise on $\mathbb{R}_+ \times \mathbb{R}^d$.

(2) The kernel function $G(t,x)$ is a Borel measurable function from $\mathbb{R}_+ \times \mathbb{R}^d$ to $\mathbb{R}$ with some tail and continuity properties (see Assumptions 3.2.9, 3.2.10, 3.2.11 below) Note that $G(t,x)$ is usually, but not necessarily, the fundamental solution of a partial differential operator. We use the convention that $G(t,\cdot) \equiv 0$ if $t < 0$. Therefore, the stochastic integral over $\mathbb{R}_+ \times \mathbb{R}^d$ is actually over $[0,t] \times \mathbb{R}^d$.

(3) The function $J_0(t,x)$ is a real-valued Borel measurable function with certain integrability properties (see Assumption 3.2.12 below).

(4) $\theta(t,x)$ is a real-valued deterministic function.

The main results are stated in Section 3.2: We first define the random field solution in Section 3.2.1, and then we list all the required assumptions and some notation in

Section 3.2.2. The main theorem – Theorem 3.2.16 – on the existence, uniqueness, moment estimates and sample path Hölder continuity is stated in Section 3.2.3. A direct application to the stochastic heat equation with distribution-valued initial data (Theorem 3.2.17) is presented in Section 3.2.4, where certain properties of the function $\theta(t,x)$ will play a key role. Theorem 3.2.16 is proved in Section 3.3. Theorem 3.2.17 is proved in Section 3.4. Another application to the stochastic wave equation in $\mathbb{R}_+ \times \mathbb{R}$ driven by nonlinear multiplicative space-time white noise is studied in Chapter 4.

3.2 Main Results

3.2.1 Notion of Random Field Solution

Note that (3.1.1) can be equivalently written as

$$I(t,x) = \iint_{\mathbb{R}_+\times\mathbb{R}^d} G\left(t-s, x-y\right)\theta\left(s,y\right)\rho\left(I\left(s,y\right)+J_0\left(s,y\right)\right) W\left(\mathrm{d}s,\mathrm{d}y\right). \tag{3.2.1}$$

We define the random field solution to (3.1.1) as follows:

Definition 3.2.1. A solution $u(t,x) = J_0(t,x)+I(t,x)$ is called a *random field solution* to (3.1.1) (or (3.2.1)) if

(1) $u(t,x)$ is adapted, i.e., for all $(t,x)\in\mathbb{R}_+^*\times\mathbb{R}^d$, $u(t,x)$ is $\mathscr{F}_t$-measurable;

(2) $u(t,x)$ is jointly measurable with respect to $\mathscr{B}\left(\mathbb{R}_+^*\times\mathbb{R}^d\right)\times\mathscr{F}$;

(3) For all $(t,x)\in\mathbb{R}_+^*\times\mathbb{R}^d$,

$$\left(G^2(\cdot,\circ)\star\left[\left|\left|\rho(u(\cdot,\circ))\right|\right|_2^2\theta^2(\cdot,\circ)\right]\right)(t,x) < +\infty,$$

and the function $(t,x)\mapsto I(t,x)$ from $\mathbb{R}_+\times\mathbb{R}^d$ into $L^2(\Omega)$ is continuous;

(4) $I(t,x)$ satisfies (3.2.1), a.s., for all $(t,x)\in\mathbb{R}_+\times\mathbb{R}^d$.

We call $I(t,x)$ the *stochastic integral part* of the random field solution.

Remark 3.2.2. To see why we reformulate the problem (3.1.1) in the form (3.2.1) in the above definition, let us consider the stochastic wave equation in the spatial domain $\mathbb{R}$. The solution to the homogeneous equation

$$\begin{cases} \left(\dfrac{\partial^2}{\partial t^2}-\kappa^2\dfrac{\partial^2}{\partial x^2}\right)u(t,x)=0\,, & x\in\mathbb{R},\ t\in\mathbb{R}_+^*\,,\\ u(0,\cdot)=g(\cdot)\in L^2_{loc}(\mathbb{R})\,,\quad \dfrac{\partial u}{\partial t}(0,\cdot)=0\,, \end{cases}$$

is $J_0(t,x) = 1/2\left(g(\kappa t+x)+g(\kappa t-x)\right)$. Since the initial position g is only a locally square integrable function, for each fixed $t>0$, the function $x\mapsto J_0(t,x)$ is also defined in $L^2_{loc}(\mathbb{R})$. Therefore, for $(t,x)\in\mathbb{R}_+\times\mathbb{R}$ fixed, $u(t,x)$ is not well-defined. Nevertheless, as we will show later, $I(t,x)$ is always well defined for each $(t,x)\in\mathbb{R}_+\times\mathbb{R}$, and in most cases (when Assumption 3.2.14 below holds), it has a continuous version. Finally, we remark that for the stochastic heat equation with deterministic initial conditions studied in the previous chapter, there is no need to transform (3.1.1) into (3.2.1) because $(t,x) \mapsto J_0(t,x)$ is a continuous function over $\mathbb{R}_+^* \times \mathbb{R}$ thanks to the smoothing effect of the heat kernel (see Proposition 2.3.5).

3.2.2 Assumptions, Conventions and Notation

According to Dalang's theory [23], a very first assumption to check is whether the linear case – the case where $\rho(u) \equiv 1$ – admits a random field solution. Define, for $t \in \mathbb{R}_+$, and $x, y \in \mathbb{R}^d$,

$$\Theta(t,x,y) := \iint_{[0,t]\times\mathbb{R}^d} G(t-s,x-z)G\big(t-s,y-z\big)\theta^2(s,z)\mathrm{d}s\mathrm{d}z\,. \qquad (3.2.2)$$

Clearly, $2\Theta(t,x,y) \le \Theta(t,x,x) + \Theta\big(t,y,y\big)$. This function will also be used for the two-point correlation functions.

Assumption 3.2.3 (Dalang's condition)**.** Assume that $G(t,x)$ is a deterministic and Borel measurable function such that for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, $\Theta(t,x,x) < +\infty$.

If $\theta(t,x) \equiv 1$, $d=1$ and the underlying differential operator is the generator of a real-valued Lévy process with the Lévy exponent $\Psi(\xi)$, then this condition is equivalent to

$$\frac{1}{2\pi}\int_{\mathbb{R}} \frac{\mathrm{d}\xi}{\beta + 2\Re\Psi(\xi)} < +\infty\,, \quad \text{for all } \beta > 0\,,$$

where $\Re\Psi(\xi)$ is the real part of $\Psi(\xi)$; see [23, 37]. For the one-dimensional stochastic heat equation studied in Chapter 2, this condition is clearly satisfied since

$$\frac{1}{2\pi}\int_{\mathbb{R}} \frac{\mathrm{d}\xi}{\beta + \xi^2} < +\infty\,, \quad \text{for all } \beta > 0\,,$$

which is equivalent to (1.1.2). For the one-dimensional stochastic wave equation, this is also true; see (1.3.2) and (4.2.5).

The next assumption plays the role of Bellman-Gronwall's lemma. We need some notation. For two functions $f, g : \mathbb{R}_+ \times \mathbb{R}^d \mapsto \mathbb{R}_+$, define the *$\theta$-weighted convolution* as follows:

$$\big(f \rhd g\big)(t,x) = \big(\big(\theta^2 f\big) \star g\big)(t,x)\,, \quad \text{for all } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d\,.$$

In the following, $f(t,x)$ will play the role of $J_0^2(t,x)$, and $g(t,x)$ of $G^2(t,x)$. In the Picard iteration scheme, we need to calculate

$$\big(\big(\cdots\big(\big(f \rhd g_1\big) \rhd g_2\big) \rhd \cdots\big) \rhd g_n\big)(t,x)\,,$$

where $g_i = g$. We would like to write this as $\int_{\mathbb{R}} f(x)h(x)\mathrm{d}x$, for some function $h(x)$.

Remark 3.2.4 (Non-associativity of $\rhd$)**.** It would be nice to have

$$\left(\left(\cdots\left(\left(f \rhd g_1\right) \rhd g_2\right) \rhd \cdots\right) \rhd g_n\right)(t,x) \stackrel{?}{=} \left(f \rhd \left(g_1 \rhd \left(\cdots \rhd \left(g_{n-1} \rhd g_n\right)\cdots\right)\right)\right)(t,x)\,. \quad (3.2.3)$$

This is not true since $\rhd$ is *not* associative. In fact,

$$\begin{aligned}\left(\left[f \rhd g_1\right] \rhd g_2\right)(t,x) &= \int_0^t \int_{\mathbb{R}^d} g_2\left(t-s_2, x-y_2\right)\theta^2\left(s_2,y_2\right)\left[f \rhd g_1\right]\left(s_2,y_2\right)\mathrm{d}s_2\mathrm{d}y_2 \\ &= \int_0^t \int_{\mathbb{R}^d} \mathrm{d}s_2\mathrm{d}y_2\, g_2\left(t-s_2,x-y_2\right)\theta^2\left(s_2,y_2\right) \\ &\quad \times \int_0^{s_2}\int_{\mathbb{R}^d} g_1\left(s_2-s_1,y_2-y_1\right)\theta^2\left(s_1,y_1\right) f\left(s_1,y_1\right)\mathrm{d}s_1\mathrm{d}y_1\,.\end{aligned}$$

On the other hand,

$$\begin{aligned}\left(f \rhd \left[g_1 \rhd g_2\right]\right)(t,x) &= \int_0^t \int_{\mathbb{R}^d} f\left(t-\tau_2, x-z_2\right)\theta^2\left(t-\tau_2, x-z_2\right)\left[g_1 \rhd g_2\right]\left(\tau_2,z_2\right)\mathrm{d}\tau_2\mathrm{d}z_2 \\ &= \int_0^t \int_{\mathbb{R}^d} \mathrm{d}\tau_2\mathrm{d}z_2\, f\left(t-\tau_2,x-z_2\right)\theta^2\left(t-\tau_2,x-z_2\right) \\ &\quad \times \int_0^{\tau_2}\int_{\mathbb{R}^d} g_1\left(\tau_2-\tau_1,z_2-z_1\right)\theta^2\left(\tau_2-\tau_1,z_2-z_1\right) g_2\left(\tau_1,z_1\right)\mathrm{d}\tau_1\mathrm{d}z_1\,.\end{aligned}$$

Then by the change of variables

$$\tau_1 = t-s_2\,,\quad \tau_2 = t-s_1\,,\quad z_1 = x-y_2\,,\quad z_2 = x-y_1\,, \quad (3.2.4)$$

and Fubini's theorem, we have

$$\begin{aligned}\left(f \rhd \left[g_1 \rhd g_2\right]\right)(t,x) &= \int_0^t \int_{\mathbb{R}^d} \mathrm{d}s_2\mathrm{d}y_2\, g_2\left(t-s_2,x-y_2\right) \\ &\quad \times \int_0^{s_2}\int_{\mathbb{R}^d} \theta^2\left(s_2-s_1,y_2-y_1\right) g_1\left(s_2-s_1,y_2-y_1\right)\theta^2\left(s_1,y_1\right) f\left(s_1,y_1\right)\mathrm{d}s_1\mathrm{d}y_1 \\ &\neq \left(\left[f \rhd g_1\right] \rhd g_2\right)(t,x)\,.\end{aligned}$$

Clearly, when $\theta \equiv 1$, $\rhd$ reduces to the space-time convolution $\star$, which is associative. Writing the left-hand side of (3.2.3) carefully and changing the variables leads to the following definition.

Definition 3.2.5. Let $g_k : \mathbb{R}_+ \times \mathbb{R}^d \mapsto \mathbb{R}_+$, $k = 1,\ldots,n$, be n nonnegative functions with $n \geq 2$. Define the *θ-weighted multiple space-time convolution* by

$$\begin{aligned}&\rhd_n\left(g_1,g_2,\ldots,g_n\right)\left(t,x;s,y\right) \\ &\quad = \int_0^s \int_{\mathbb{R}^d} \mathrm{d}s_{n-1}\mathrm{d}y_{n-1}\, g_n\left(s-s_{n-1},y-y_{n-1}\right)\theta^2\left(t-s+s_{n-1},x-y+y_{n-1}\right) \\ &\quad\quad \times \int_0^{s_{n-1}}\int_{\mathbb{R}^d} \mathrm{d}s_{n-2}\mathrm{d}y_{n-2} g_{n-1}\left(s_{n-1}-s_{n-2},y_{n-1}-y_{n-2}\right)\theta^2\left(t-s+s_{n-2},x-y+y_{n-2}\right) \\ &\quad\quad \times \cdots\cdots \end{aligned} \quad (3.2.5)$$

$$\times \int_0^{s_3}\int_{\mathbb{R}^d} \mathrm{d}s_2\mathrm{d}y_2\; g_3\left(s_3-s_2, y_3-y_2\right)\theta^2\left(t-s+s_2, x-y+y_2\right)$$
$$\times \int_0^{s_2}\int_{\mathbb{R}^d} g_2\left(s_2-s_1, y_2-y_1\right)\theta^2\left(t-s+s_1, x-y+y_1\right) g_1\left(s_1, y_1\right)\mathrm{d}s_1\mathrm{d}y_1.$$

for (t,x) and $\left(s,y\right)\in\mathbb{R}_+\times\mathbb{R}^d$ with $0\le s\le t$.

Notice that

$$\rhd_n\left(g_1,\ldots,g_n\right)(t,x;t,x) = \left(\left(\cdots\left(\left(g_1\rhd g_2\right)\rhd g_3\right)\rhd\cdots\right)\rhd g_n\right)(t,x),$$

where the right-hand side has $n-1$ convolutions.

The multiple convolution $\rhd_n$ has an equivalent definition: By the change of variables

$$\tau_1 = s-s_{n-1},\quad \tau_2 = s-s_{n-2},\quad \cdots,\quad \tau_{n-1}=s-s_1,\quad \text{and}$$
$$z_1 = y-y_{n-1},\quad z_2 = y-y_{n-2},\quad \cdots,\quad z_{n-1}=y-y_1,$$

and Fubini's theorem,

$$\begin{aligned}
&\rhd_n\left(g_1,g_2,\ldots,g_n\right)\left(t,x;s,y\right)\\
&\quad=\int_0^s\int_{\mathbb{R}^d}\mathrm{d}\tau_{n-1}\mathrm{d}z_{n-1}\,\theta^2\left(t-\tau_{n-1},x-z_{n-1}\right)g_1\left(s-\tau_{n-1},y-z_{n-1}\right)\\
&\quad\times\int_0^{\tau_{n-1}}\int_{\mathbb{R}^d}\mathrm{d}\tau_{n-2}\mathrm{d}z_{n-2}\,\theta^2\left(t-\tau_{n-2},x-z_{n-2}\right)g_2\left(\tau_{n-1}-\tau_{n-2},z_{n-1}-z_{n-2}\right)\\
&\quad\times\cdots\cdots\\
&\quad\times\int_0^{\tau_3}\int_{\mathbb{R}^d}\mathrm{d}\tau_2\mathrm{d}z_2\,\theta^2\left(t-\tau_2,x-z_2\right)g_{n-2}\left(\tau_3-\tau_2,z_3-z_2\right)\\
&\quad\times\int_0^{\tau_2}\int_{\mathbb{R}^d}\theta^2\left(t-\tau_1,x-z_1\right)g_{n-1}\left(\tau_2-\tau_1,z_2-z_1\right)g_n\left(\tau_1,z_1\right)\mathrm{d}\tau_1\mathrm{d}z_1\,.
\end{aligned}\tag{3.2.6}$$

Lemma 3.2.6. *Let $f,g_k:\mathbb{R}_+\times\mathbb{R}^d\mapsto\mathbb{R}_+$, $k=1,\ldots,n+1$, and $n\ge 2$. Then for all $(t,x)\in\mathbb{R}_+\times\mathbb{R}^d$, we have*

$$\left(\left(\cdots\left(\left(f\rhd g_1\right)\rhd g_2\right)\rhd\cdots\right)\rhd g_n\right)(t,x) = \left(f\rhd\rhd_n\left(g_1,\ldots,g_n\right)(t,x;\cdot,\circ)\right)(t,x), \tag{3.2.7}$$
$$= \left(\left(f\rhd g_1\right)\rhd\rhd_{n-1}\left(g_2,\ldots,g_n\right)(t,x;\cdot,\circ)\right)(t,x), \tag{3.2.8}$$

and

$$\int_0^t\int_{\mathbb{R}^d}\left(f\rhd\rhd_n\left(g_1,\ldots,g_n\right)(s,y;\cdot,\circ)\right)(s,y)\,\theta^2(s,y)\,g_{n+1}\left(t-s,x-y\right)\mathrm{d}s\mathrm{d}y$$
$$=\left(f\rhd\rhd_{n+1}\left(g_1,\ldots,g_{n+1}\right)(t,x;\cdot,\circ)\right)(t,x)\,. \tag{3.2.9}$$

,

Note that the relation in (3.2.8) can be generalized to the more general relation

$$
\begin{aligned}
&\Big(\Big(\cdots\Big(\big(f \rhd g_1\big) \rhd g_2\Big) \rhd \cdots\Big) \rhd g_n\Big)(t,x) \\
&\quad = \Big(\Big(\cdots\big(f \rhd g_1\big) \rhd \cdots \rhd g_m\Big) \rhd \rhd_{n-m}\big(g_{m+1},\dots,g_n\big)(t,x;\cdot,\circ)\Big)(t,x)\,, \quad 1 \le m \le n-2\,,
\end{aligned}
$$

and (3.2.8) is enough for our use later.

Proof. We first prove (3.2.7). The left-hand side of (3.2.7) equals

$$
\begin{aligned}
&\int_0^t \int_{\mathbb{R}^d} \mathrm{d}s_{n-1}\mathrm{d}y_{n-1}\, g_n\big(t-s_{n-1}, x-y_{n-1}\big)\theta^2\big(s_{n-1},y_{n-1}\big) \\
&\times \int_0^{s_{n-1}}\int_{\mathbb{R}^d} \mathrm{d}s_{n-2}\mathrm{d}y_{n-2}\, g_{n-1}\big(s_{n-1}-s_{n-2}, y_{n-1}-y_{n-2}\big)\theta^2\big(s_{n-2},y_{n-2}\big) \\
&\times \cdots\cdots \\
&\times \int_0^{s_2}\int_{\mathbb{R}^d} \mathrm{d}s_1\mathrm{d}y_1\, 1 g_2\big(s_2-s_1, y_2-y_1\big)\theta^2\big(s_1,y_1\big) \\
&\times \int_0^{s_1}\int_{\mathbb{R}^d} g_1\big(s_1-s_0, y_1-y_0\big)\theta^2\big(s_0,y_0\big) f\big(s_0,y_0\big)\,\mathrm{d}s_0\mathrm{d}y_0 \\
&= \rhd_{n+1}\big(f, g_1,\dots,g_n\big)(t,x;t,x)\,.
\end{aligned}
$$

As in (3.2.6), by the change of variables

$$
\begin{aligned}
&\tau_0 = t - s_{n-1}, \quad \tau_1 = t - s_{n-2}, \quad \cdots, \quad \tau_{n-1} = t - s_0\,, \quad \text{and} \\
&z_0 = x - y_{n-1}, \quad z_1 = x - y_{n-2}, \quad \cdots, \quad z_{n-1} = x - y_0\,,
\end{aligned}
$$

the above equation equals

$$
\begin{aligned}
&\int_0^t \int_{\mathbb{R}^d} \mathrm{d}\tau_{n-1}\mathrm{d}z_{n-1}\,\theta^2(t-\tau_{n-1}, x-z_{n-1})\, f\big(s-\tau_{n-1}, y-z_{n-1}\big) \\
&\times \Bigg(\int_0^{\tau_{n-1}}\int_{\mathbb{R}^d} \mathrm{d}\tau_{n-2}\mathrm{d}z_{n-2}\,\theta^2(t-\tau_{n-2}, x-z_{n-2})\, g_1(\tau_{n-1}-\tau_{n-2}, z_{n-1}-z_{n-2}) \\
&\times \cdots\cdots \\
&\times \int_0^{\tau_2}\int_{\mathbb{R}^d} \mathrm{d}\tau_1\mathrm{d}z_1\,\theta^2(t-\tau_1, x-z_1)\, g_{n-2}(\tau_2-\tau_1, z_2-z_1) \\
&\times \int_0^{\tau_1}\int_{\mathbb{R}^d} \theta^2(t-\tau_0, x-z_0)\, g_{n-1}(\tau_1-\tau_0, z_1-z_0)\, g_n(\tau_0,z_0)\,\mathrm{d}\tau_0\mathrm{d}z_0\Bigg)\,.
\end{aligned}
$$

The part in the parentheses is indeed $\rhd_n\big(g_1,\dots,g_n\big)(t,x;\tau_{n-1},z_{n-1})$; see (3.2.6). This proves (3.2.7).

As for (3.2.8), apply (3.2.7) with n replaced by $n-1$ and $f(t,x)$ by $\big(f \rhd g_1\big)(t,x)$:

$$
\Big(\big(f \rhd g_1\big) \rhd \rhd_{n-1}\big(g_2,\dots,g_n\big)(t,x;\cdot,\circ)\Big)(t,x) = \Big(\Big(\cdots\Big(\big(f \rhd g_1\big) \rhd g_2\Big) \rhd \cdots\Big) \rhd g_n\Big)(t,x)\,.
$$

Now let us prove (3.2.9). By (3.2.7), the left-hand side of (3.2.9) equals

$$
\Big(\Big(\Big(\cdots\Big(\big(f \rhd g_1\big) \rhd g_2\Big) \rhd \cdots\Big) \rhd g_n\Big) \rhd g_{n+1}\Big)(t,x)
$$

which is equal to the right-hand side of (3.2.9) by (3.2.7). This completes the proof. □

When $n=2$, for $f,g:\mathbb{R}_+\times\mathbb{R}^d\mapsto\mathbb{R}_+$, we have

$$\rhd_2(f,g)(t,x;t,x)=\left(f\rhd g\right)(t,x),$$

and

$$\rhd_2\left(f,g\right)(t,x;s,y)=\int_0^s\int_{\mathbb{R}^d}g\left(s-s_0,y-y_0\right)\theta^2\left(t-s+s_0,x-y+y_0\right)f\left(s_0,y_0\right)\mathrm{d}s_0\mathrm{d}y_0\,. \tag{3.2.10}$$

By the change of variables $\tau_0=s-s_0$ and $z_0=y-y_0$, and Fubini's theorem, we have

$$\rhd_2\left(f,g\right)(t,x;s,y)=\int_0^s\int_{\mathbb{R}^d}\theta^2\left(t-\tau_0,x-z_0\right)f\left(s-\tau_0,y-z_0\right)g\left(\tau_0,z_0\right)\mathrm{d}\tau_0\mathrm{d}z_0\,. \tag{3.2.11}$$

In particular, if $\theta(t,x)\equiv 1$, then the θ-weighted convolution $\rhd_2$ reduces to the standard space-time convolution $\star$ (as is the case for $\rhd$), in which case the first two variables (t,x) do not play a role. We call (3.2.10) and (3.2.5) the *forward* formulas, and (3.2.11) and (3.2.6) the *backward* formulas.

Define the kernel function

$$\mathscr{L}_0\left(t,x;\lambda\right):=\lambda^2G^2(t,x),\quad \text{for all } (t,x)\in\mathbb{R}_+^*\times\mathbb{R}^d,$$

with a parameter $\lambda\in\mathbb{R}$. For all $n\in\mathbb{N}^*$, define

$$\mathscr{L}_n\left(t,x;s,y;\lambda\right):=\rhd_{n+1}\left(\mathscr{L}_0(\cdot,\circ;\lambda),\dots,\mathscr{L}_0(\cdot,\circ;\lambda)\right)\left(t,x;s,y\right),$$

for all $(t,x),\left(s,y\right)\in\mathbb{R}_+^*\times\mathbb{R}^d$ with $s\le t$. We will use the convention that

$$\mathscr{L}_0\left(t,x;s,y;\lambda\right)=\lambda^2G^2\left(s,y\right).$$

Define, for all $n\in\mathbb{N}$,

$$\begin{aligned}\mathscr{H}_n\left(t,x;\lambda\right)&:=\left(1\rhd\mathscr{L}_n(t,x;\cdot,\circ;\lambda)\right)(t,x)\\&=\int_0^t\int_{\mathbb{R}^d}\theta^2\left(t-\tau_0,x-z_0\right)\mathscr{L}_n(t,x;\tau_0,z_0;\lambda)\mathrm{d}\tau_0\mathrm{d}z_0\,.\end{aligned}$$

Clearly, we have the following scaling property:

$$\mathscr{L}_n(t,x;s,y;\lambda)=\lambda^{2n+2}\mathscr{L}_n(t,x;s,y;1),\quad\text{and}\quad\mathscr{H}_n(t,x;\lambda)=\lambda^{2n+2}\mathscr{H}_n(t,x;1)\,.$$

By definition, these kernel functions $\mathscr{L}_n$ and $\mathscr{H}_n$ are non-negative.

We use the following conventions:

$$\mathscr{L}_n\left(t,x;s,y\right):=\mathscr{L}_n\left(t,x;s,y;\lambda\right),$$

$$\overline{\mathscr{L}}_n(t,x;s,y) := \mathscr{K}(t,x;s,y;\,L_\rho)\,,$$
$$\underline{\mathscr{L}}_n(t,x;s,y) := \mathscr{L}_n(t,x;s,y;\,l_\rho)\,,$$
$$\widehat{\mathscr{L}}_n(t,x;s,y) := \mathscr{L}_n(t,x;s,y;\,a_{p,\overline{\varsigma}}\,z_p\,L_\rho)\,,\quad \text{for all } p\geq 2\,,$$

where z_p is the optimal universal constant in the Burkholder-Davis-Gundy inequality (see Theorem 2.3.18) and $a_{p,\overline{\varsigma}}$ is defined in (1.4.4). Note that the kernel function $\widehat{\mathscr{L}}_n(t,x;s,y)$ depends on the parameters p and $\overline{\varsigma}$, which is usually clear from the context. Similarly, define $\overline{\mathscr{H}}_n(t,x)$, $\underline{\mathscr{H}}_n(t,x)$ and $\widehat{\mathscr{H}}_n(t,x)$. The same conventions will apply to the kernel functions $\mathscr{K}(t,x;s,y)$, $\overline{\mathscr{K}}(t,x;s,y)$, $\underline{\mathscr{K}}(t,x;s,y)$ and $\widehat{\mathscr{K}}(t,x;s,y)$ below.

Assumption 3.2.7. Assume that all the kernel functions $\mathscr{L}_n(t,x;s,y;\lambda)$ and functions $\mathscr{H}_n(t,x;s;\lambda)$, with $n\in\mathbb{N}$ and $\lambda\in\mathbb{R}$, are well defined and the sum of the kernel functions $\mathscr{L}_n(t,x;s,y;\lambda)$ converges for all (t,x) and $(s,y)\in\mathbb{R}_+^*\times\mathbb{R}^d$ with $s\leq t$. Denote this sum by

$$\mathscr{K}(t,x;s,y;\lambda) := \sum_{n=0}^{\infty}\mathscr{L}_n(t,x;s,y;\lambda)\,.$$

The next assumption is a convenient assumption which will guarantee the continuity of the function $(t,x)\mapsto I(t,x)$ from $\mathbb{R}_+\times\mathbb{R}^d$ into $L^p(\Omega)$ for $p\geq 2$. Compare Assumptions 3.2.7 and 3.2.8 with Proposition 2.3.1 for the heat equation and Proposition 4.3.5 for the wave equation.

Assumption 3.2.8. Assume that there are non-negative functions $B_n(t) := B_n(t;\lambda)$ such that

(i) $B_n(t)$ is nondecreasing in t;

(ii) $\mathscr{L}_n(t,x;s,y)\leq\mathscr{L}_0(s,y)\,B_n(t)$, for all (t,x) and $(s,y)\in\mathbb{R}_+^*\times\mathbb{R}^d$ with $s\leq t$ and all $n\in\mathbb{N}$ (set $B_0(t)\equiv 1$);

(iii) $\sum_{n=0}^{\infty}\sqrt{B_n(t)}<+\infty$, for all $t>0$.

The above assumption guarantees that the following function

$$\Upsilon(t;\lambda) := \sum_{n=0}^{\infty}B_n(t;\lambda)\,,\quad t\geq 0\,, \tag{3.2.12}$$

is well defined. We use the same conventions on the parameter λ for the function $\Upsilon(t;\lambda)$. Clearly,

$$\mathscr{K}(t,x;s,y)\leq\Upsilon(t)\mathscr{L}_0(s,y)\,,\quad \text{for all } (t,x) \text{ and } (s,y)\in\mathbb{R}_+\times\mathbb{R}^d \text{ with } s\leq t\,. \tag{3.2.13}$$

Another implication of this assumption is that

$$\sum_{n=0}^{\infty}\mathscr{H}_n(t,x)\leq\mathscr{H}_0(t,x)\Upsilon(t)<+\infty\,,\quad \text{for all } (t,x)\in\mathbb{R}_+\times\mathbb{R}^d \text{ and } 0\leq s\leq t\,,$$

and so the function

$$\mathcal{H}(t,x) := (1 \rhd \mathcal{K}(t,x;\cdot,\circ))(t,x) \\ = \int_0^t \int_{\mathbb{R}^d} \theta^2(t-\tau_0, x-z_0)\, \mathcal{K}(t,x;\tau_0,z_0)\mathrm{d}\tau_0 \mathrm{d}z_0$$

is well defined and equals $\sum_{n=0}^{\infty} \mathcal{H}_n(t,x)$ by the monotone convergence theorem.

The next three assumptions are used to prove the $L^p(\Omega)$-continuity in each Picard iteration. In order to apply Lebesgue's dominated convergence theorem, we need to treat the heat equation and the wave equation separately. In particular, Assumption 3.2.9 is for the kernel functions similar to the wave kernel function (see also Proposition 4.3.6) and Assumptions 3.2.10 and 3.2.11 are for those similar to the heat kernel function (see also Proposition 2.3.12 and Corollary 2.3.10). We need some notation: For $\beta \in]0,1[$, $\tau > 0$, $\alpha > 0$ and $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}^d$, denote the set

$$B_{t,x,\beta,\tau,\alpha} := \left\{ (t',x') \in \mathbb{R}_+^* \times \mathbb{R}^d : \beta t \le t' \le t+\tau,\ |x-x'| \le \alpha \right\}. \tag{3.2.14}$$

Assumption 3.2.9 (Uniformly bounded kernel functions)**.** Assume that $G(t,x)$ has the following two properties:

(i) there exist three constants $\beta \in]0,1[$, $\tau > 0$ and $\alpha > 0$ such that for all $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}^d$, for some constant $C > 0$, we have for all $(t',x') \in B_{t,x,\beta,\tau,\alpha}$ and all $(s,y) \in [0,t'[\times \mathbb{R}^d$,

$$G(t'-s, x'-y) \le C\, G(t+1-s, x-y).$$

(ii) for almost all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, $\lim_{(t',x')\to(t,x)} G(t',x') = G(t,x)$.

Assumption 3.2.10 (Tail control of kernel functions)**.** Assume that there exists $\beta \in]0,1[$ such that for all $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}^d$, for some constant $a > 0$, we have for all $(t',x') \in B_{t,x,\beta,1/2,1}$ and all $s \in [0,t'[$ and $y \in \mathbb{R}^d$ with $|y| \ge a$,

$$G(t'-s, x'-y) \le G(t+1-s, x-y).$$

Assumption 3.2.11. Assume that for all $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}^d$,

$$\lim_{(t',x')\to(t,x)} \iint_{\mathbb{R}_+ \times \mathbb{R}^d} \left(G(t'-s,x'-y) - G(t-s,x-y)\right)^2 \mathrm{d}s\mathrm{d}y = 0,$$

and for almost all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, $\lim_{(t',x')\to(t,x)} G(t',x') = G(t,x)$.

Note that this assumption can be more explicitly expressed in the following way:

$$\lim_{(t',x')\to(t,x)} \Bigg(\iint_{]0,t_*]\times\mathbb{R}^d} \left(G(t'-s,x'-y) - G(t-s,x-y)\right)^2 \mathrm{d}s\mathrm{d}y \\ + \iint_{]t_*,\hat{t}]\times\mathbb{R}^d} G^2(\hat{t}-s, \hat{x}-y)\mathrm{d}s\mathrm{d}y \Bigg) = 0, \tag{3.2.15}$$

where

$$(t_*, x_*) = \begin{cases} (t', x') & \text{if } t' \le t, \\ (t, x) & \text{if } t' > t, \end{cases} \qquad \text{and} \qquad (\hat{t}, \hat{x}) = \begin{cases} (t, x) & \text{if } t' \le t. \\ (t', x') & \text{if } t' > t. \end{cases} \tag{3.2.16}$$

The next assumption is a basic assumption on the the function $J_0(t,x)$, related to (1.3.8) and (2.3.11).

Assumption 3.2.12. Assume that the function $J_0 : \mathbb{R}_+ \times \mathbb{R}^d \mapsto \mathbb{R}$ is a Borel measurable function such that for all compact sets $K \subseteq \mathbb{R}_+^* \times \mathbb{R}^d$, $v \in \mathbb{R}$ and all integers $p \ge 2$,

$$\sup_{(t,x)\in K} \left(\left[v^2 + J_0^2 \right] \rhd G^2 \right)(t,x) < +\infty .$$

The following chain of inequalities is a direct consequence of this assumption and (3.2.13): for all $n \in \mathbb{N}$, and all (t,x) and $(s,y) \in \mathbb{R}_+^* \times \mathbb{R}^d$ with $s \le t$,

$$\begin{aligned} \left(J_0^2 \rhd \mathscr{L}_n(t,x;\cdot,\circ) \right)(t,x) &\le \left(J_0^2 \rhd \mathscr{K}(t,x;\cdot,\circ) \right)(t,x) \\ &\le \Upsilon(t) \left(J_0^2 \rhd \mathscr{L}_0 \right)(t,x) < +\infty . \end{aligned} \tag{3.2.17}$$

When the kernel function $G(t,x)$ has smoothing effects, as is the case for the heat kernel (see Lemma 2.6.14), the following assumption comes for free.

Assumption 3.2.13. Assume that for all compact sets $K \subseteq \mathbb{R}_+^* \times \mathbb{R}^d$, we have that

$$\sup_{(t,x)\in K} |J_0(t,x)| < +\infty .$$

Finally, the last assumption is a set of sufficient conditions for Hölder continuity. This assumption has been verified for the heat equation in Propositions 2.6.16 and 2.6.17 under the settings $d = 1$ and $\theta(t,x) \equiv 1$.

Assumption 3.2.14. (Sufficient conditions for Hölder continuity) Given $J_0(t,x)$ and $v \in \mathbb{R}$, assume that there are $d+1$ constants $\gamma_i \in]0,1]$, $i = 0,\dots,d$ such that for all $n > 1$, one can find a finite constant $C_n < +\infty$, such that for all integers $p \ge 2$, all (t,x) and $(t',x') \in K_n := [1/n, n] \times [-n,n]^d$ with $t < t'$, we have that

$$\begin{aligned} \iint_{\mathbb{R}_+\times\mathbb{R}^d} \left(v^2 + 2J_0^2(s,y) \right) \left(G(t-s, x-y) - G(t'-s, x'-y) \right)^2 \theta^2(s,y) \, \mathrm{d}s\mathrm{d}y \\ \le C_n \, \tau_{\gamma_0,\dots,\gamma_d}\left((t,x), (t',x') \right) , \end{aligned} \tag{3.2.18}$$

and

$$\begin{aligned} \iint_{\mathbb{R}_+\times\mathbb{R}^d} \left((v^2 + 2J_0^2) \rhd G^2 \right)(s,y) \left(G(t-s, x-y) - G(t'-s, x'-y) \right)^2 \theta^2(s,y) \, \mathrm{d}s\mathrm{d}y \\ \le C_n \, \tau_{\gamma_0,\dots,\gamma_d}\left((t,x), (t',x') \right) , \end{aligned} \tag{3.2.19}$$

where $\tau_{\gamma_0,\dots,\gamma_d}\left((t,x),(t',x')\right) := \left|t-t'\right|^{\gamma_0} + \sum_{i=1}^d \left|x_i - x_i'\right|^{\gamma_i}$.

The following lemma is useful for verifying Assumption 3.2.14.

Lemma 3.2.15. *Assumption 3.2.14 is equivalent to the following statement: Given J_0 and $v \in \mathbb{R}$, assume that there are $d+1$ constants $\gamma_i \in]0,1]$, $i = 0,\dots,d$ such that for all $n > 1$, one can find six finite constants $C_{n,i} < +\infty$, $i = 1,\dots,6$, such that for all integers $p \geq 2$, all (t,x) and $(t+s, x+h) \in K_n := [1/n, n] \times [-n,n]^d$ with $s > 0$, we have,*

$$\left(\left(v^2 + 2J_0^2\right) \triangleright \left(G(\cdot,\circ) - G(\cdot+s,\circ)\right)^2\right)(t,x) \leq C_{n,1}\,|s|^{\gamma_0}\,, \tag{3.2.20}$$

$$\left(\left(v^2 + 2J_0^2\right) \triangleright \left(G(\cdot,\circ) - G(\cdot,\circ+h)\right)^2\right)(t,x) \leq C_{n,3}\sum_{i=1}^d |h_i|^{\gamma_i}\,, \tag{3.2.21}$$

$$\iint_{[t,t+s]\times\mathbb{R}^d} \left(v^2 + 2J_0^2(u,y)\right) G^2(t+s-u, x+h-y)\theta^2(u,y)\mathrm{d}u\mathrm{d}y \leq C_{n,5}\,|s|^{\gamma_0}\,, \tag{3.2.22}$$

$$\left(\left[\left(v^2 + 2J_0^2\right) \triangleright G^2\right] \triangleright \left(G(\cdot,\circ) - G(\cdot+s,\circ)\right)^2\right)(t,x) \leq C_{n,2}\,|s|^{\gamma_0}\,,$$

$$\left(\left[\left(v^2 + 2J_0^2\right) \triangleright G^2\right] \triangleright \left(G(\cdot,\circ) - G(\cdot,\circ+h)\right)^2\right)(t,x) \leq C_{n,4}\sum_{i=1}^d |h_i|^{\gamma_i}\,,$$

$$\iint_{[t,t+s]\times\mathbb{R}^d} \left(\left(v^2 + 2J_0^2\right) \triangleright G^2\right)(u,y)\; G^2(t+s-u, x+h-y)\theta^2(u,y)\mathrm{d}u\mathrm{d}y \leq C_{n,6}\,|s|^{\gamma_0}.$$

The proof of this lemma is straightforward and we leave it to the interested readers.

3.2.3 Main Theorem

To state the main theorem in a clear way, we group various conditions as follows:

Cond(G) (General conditions)

- (a) $G(t,x)$ satisfies Assumptions 3.2.3, 3.2.7, and 3.2.8;
- (b) $J_0(t,x)$, satisfy Assumption 3.2.12;
- (c) the function $\rho(u)$ is Lipschitz continuous with Lipschitz constant $\mathrm{Lip}_\rho > 0$ such that it satisfies the growth condition (1.4.1).

Cond(W) (Wave equation case) $G(t,x)$ satisfies Assumptions 3.2.9.

Cond(H) (Heat equation case)

- (a) $G(t,x)$ satisfies Assumptions 3.2.10 and 3.2.11;
- (b) $\sup_{(t,x)\in K} |\theta(t,x)| < +\infty$, for all compact sets $K \subseteq \mathbb{R}_+ \times \mathbb{R}^d$;
- (c) $J_0(t,x)$, satisfy Assumption 3.2.13.

Theorem 3.2.16. *If Cond(G), and at least one of Cond(W) and Cond(H) hold, then the stochastic integral equation* (3.1.1) *has a random field solution*

$$\left\{u(t,x) = J_0(t,x) + I(t,x) : t > 0, x \in \mathbb{R}^d\right\}$$

in the sense of Definition 3.2.1. This solution has the following properties

(1) $I(t,x)$ *is unique (in the sense of versions);*

(2) *The function* $(t,x)\mapsto I(t,x)$ *from* $\mathbb{R}_+\times\mathbb{R}^d$ *into* $L^p(\Omega)$ *is continuous for all integers* $p\geq 2$*;*

(3) *For all even integer* $p\geq 2$*, the* p*-th moment of the solution* $u(t,x)$ *satisfies the upper bounds*

$$||u(t,x)||_p^2 \leq \begin{cases} J_0^2(t,x)+\left(J_0^2\rhd\overline{\mathcal{K}}(t,x;\cdot,\circ)\right)(t,x)+\overline{\varsigma}^2\,\overline{\mathcal{H}}(t,x) & \text{if } p=2,\\[2ex] 2J_0^2(t,x)+\left(2J_0^2\rhd\widehat{\mathcal{K}}(t,x;\cdot,\circ)\right)(t,x)+\overline{\varsigma}^2\,\widehat{\mathcal{H}}(t,x) & \text{if } p>2,\end{cases} \tag{3.2.23}$$

for all $t>0$, $x\in\mathbb{R}^d$*. And the two-point correlation satisfies the upper bound*

$$\begin{aligned}\mathbb{E}\left[u(t,x)u(t,y)\right] &\leq J_0(t,x)J_0(t,y)+L_\rho^2\,\overline{\varsigma}^2\,\Theta(t,x,y)\\ &+L_\rho^2\int_0^t \mathrm{d}s\int_{\mathbb{R}^d}\overline{f}(s,z)\,\theta^2(s,z)\,G(t-s,x-z)G\big(t-s,y-z\big)\,\mathrm{d}z\,,\end{aligned} \tag{3.2.24}$$

for all $t>0$, $x,y\in\mathbb{R}^d$*, where* $\overline{f}(s,z)$ *denotes the right hand side of* (3.2.23) *for* $p=2$*;*

(4) *If* ρ *satisfies* (1.4.2)*, then the second moment satisfies the lower bound*

$$||u(t,x)||_2^2 \geq J_0^2(t,x)+\left(J_0^2\rhd\underline{\mathcal{K}}(t,x;\cdot,\circ)\right)(t,x)+\underline{\varsigma}^2\,\underline{\mathcal{H}}(t,x) \tag{3.2.25}$$

for all $t>0$, $x\in\mathbb{R}^d$*. And the two-point correlation satisfies the lower bound*

$$\begin{aligned}\mathbb{E}\left[u(t,x)u(t,y)\right] &\geq J_0(t,x)J_0(t,y)+l_\rho^2\,\underline{\varsigma}^2\,\Theta(t,x,y)\\ &+l_\rho^2\int_0^t \mathrm{d}s\int_{\mathbb{R}^d}\underline{f}(s,z)\,\theta^2(s,z)\,G(t-s,x-z)G\big(t-s,y-z\big)\,\mathrm{d}z\,,\end{aligned} \tag{3.2.26}$$

for all $t>0$, $x,y\in\mathbb{R}^d$*, where* $\underline{f}(s,z)$ *denotes the right hand side of* (3.2.25)*;*

(5) *In particular, for the quasi-linear case* $|\rho(u)|^2=\lambda^2\left(\varsigma^2+u^2\right)$*, the second moment has an explicit expression:*

$$||u(t,x)||_2^2 = J_0^2(t,x)+\left(J_0^2\rhd\mathcal{K}(t,x;\cdot,\circ)\right)(t,x)+\varsigma^2\,\mathcal{H}(t,x) \tag{3.2.27}$$

for all $t>0$, $x\in\mathbb{R}^d$*. And the two-point correlation is given by*

$$\begin{aligned}\mathbb{E}\left[u(t,x)u(t,y)\right] &= J_0(t,x)J_0(t,y)+\lambda^2\,\varsigma^2\,\Theta(t,x,y)\\ &+\lambda^2\int_0^t \mathrm{d}s\int_{\mathbb{R}^d} f(s,z)\,\theta^2(s,z)\,G(t-s,x-z)G\big(t-s,y-z\big)\,\mathrm{d}z\,,\end{aligned} \tag{3.2.28}$$

for all $t>0$, $x,y\in\mathbb{R}^d$*, where* $f(s,z)=||u(s,z)||_2^2$ *is defined in* (3.2.27)*;*

(6) If, in addition, Assumption 3.2.14 holds, then $I(t,x)$ is a.s. Hölder continuous:

$$I(t,x) \in C_{\frac{\gamma_0}{2}-,\frac{\gamma_1}{2}-,\ldots,\frac{\gamma_d}{2}-}\left(\mathbb{R}_+^* \times \mathbb{R}^d\right), \text{ a.s.}$$

Moreover, if the compact sets K_n in Assumption 3.2.14 can be chosen as $[0,n] \times [-n,n]^d$, then

$$I(t,x) \in C_{\frac{\gamma_0}{2}-,\frac{\gamma_1}{2}-,\ldots,\frac{\gamma_d}{2}-}\left(\mathbb{R}_+ \times \mathbb{R}^d\right), \text{ a.s.}$$

3.2.4 Application: Stochastic Heat Equation with Distribution-Valued Initial Data

We apply Theorem 3.2.16 to study the following stochastic heat equation

$$\begin{cases} \left(\dfrac{\partial}{\partial t} - \dfrac{\nu}{2}\dfrac{\partial^2}{\partial x^2}\right) u(t,x) = \theta(t,x)\, \rho(u(t,x))\, \dot{W}(t,x), & x \in \mathbb{R},\ t \in \mathbb{R}_+^*, \\ u(0,\cdot) = \mu(\cdot), \end{cases} \tag{3.2.29}$$

where $\dot{W}$ is the space-time noise, $\rho(u)$ is Lipschitz continuous, μ is some deterministic initial data, and $\theta(t,x)$ is some deterministic function. We will focus on this equation with general initial data, and we will study how certain properties of $\theta(t,x)$ function affect the *admissible initial data* – the initial data starting from which the stochastic heat equation (3.2.29) admits a random field solution. Recall that Proposition 2.2.9 shows that if $\theta(t,x) \equiv 1$, then the initial data cannot go beyond measures.

As for the properties of $\theta(t,x)$, we will not pursue the full generality here. Instead, we only make some easy assumptions on $\theta(t,x)$ to show the balance between certain properties of $\theta(t,x)$ and the set of the admissible initial data. For $r \ge 0$, define

$$\Xi_r := \left\{ \theta : \mathbb{R}_+ \times \mathbb{R} \mapsto \mathbb{R} : \sup_{(t,x)\in\mathbb{R}_+\times\mathbb{R}} \frac{|\theta(t,x)|}{t^r \wedge 1} < +\infty \right\}, \quad \text{and} \quad \Xi_\infty := \bigcap_{n\in\mathbb{N}} \Xi_n .$$

Clearly, if $0 \le m \le n$, then $\Xi_m \supseteq \Xi_n$. Here are some simple examples

$$\begin{aligned} t^k \wedge 1 &\in \Xi_k, \quad \text{for all } k \ge 0; \\ \exp(-1/t) &\in \Xi_{+\infty}. \end{aligned}$$

Recall the definition of the space $\mathscr{D}_k'(\mathbb{R})$ in Definition 2.6.10.

Theorem 3.2.17. *Suppose that the function ρ is Lipschitz continuous. If the function $\theta(t,x) \in \Xi_r$ for some $0 \le r \le +\infty$, then the stochastic heat equation* (3.2.29) *has a random field solution*

$$\{u(t,x) :\ t > 0, x \in \mathbb{R}\},$$

for any initial data $\mu \in \mathscr{D}_k'(\mathbb{R})$ with $k \in \mathbb{N}$ and $0 \le k < r + 1/4$. Furthermore, this random field solution has the following properties:

(1) $u(t,x)$ is unique (versions of each other);

(2) The function $(t,x)\mapsto u(t,x)$ from $\mathbb{R}_+^*\times\mathbb{R}$ into $L^p(\Omega)$ is continuous for all $p\ge 2$.

Example 3.2.18. If $\theta(t,x)\equiv 1$, then the largest $r\ge 0$ such that $\theta\in\Xi_r$ is 0. The largest integer between $[0,r+1/4[$ with $r=0$ is 0. Hence, by Theorem 3.2.17, the admissible initial data are $\mathscr{D}_0'(\mathbb{R})$, which reduces to the condition (1.1.5) in the previous chapter.

Example 3.2.19 (Derivatives of the Dirac delta functions)**.** If $\theta(t,x)=t^r\wedge 1$, then initial data can be $\delta_0^{(k)}$ with $0\le k<r+1/4$. In particular, if $\theta(t,x)\equiv 1$ only δ_0 itself can be the initial data. This is consistent with Proposition 2.2.9. If we choose $\theta(t,x)=\exp(-1/t)$, then all derivatives of δ_0 can be the initial data.

Example 3.2.20 (Schwartz distribution-valued initial data and beyond)**.** If we choose $\theta(t,x)\in\Xi_{+\infty}$, for example $\theta(t,x)=\exp(-1/t)$, then the initial data can be any Schwartz distributions (see the structure theorem of $\mathscr{S}'(\mathbb{R})$ in [64, Theorem VI, p. 239]). Actually, the admissible initial data $\mathscr{D}_{+\infty}'(\mathbb{R})$ can go beyond Schwartz distributions. Here are some simple examples: $\mu(\mathrm{d}x)=\mu_0^{(k)}(\mathrm{d}x)$ for all $k\in\mathbb{N}$ where $\mu_0(\mathrm{d}x)=e^{|x|}\mathrm{d}x$.

3.3 Proof of the Existence Result (Theorem 3.2.16)

3.3.1 Some Criteria for Predictable Random Fields

A random field $\{Z(t,x)\}$ is called *elementary* if we can write $Z(t,x)=Y1_{]a,b]}(t)1_A(x)$, where $0\le a<b$, $A\subset\mathbb{R}^d$ is a rectangle, and Y is an $\mathscr{F}_a$–measurable random variable. A *simple* process is a finite sum of elementary random fields. The set of simple processes generates the *predictable* σ-field on $\mathbb{R}_+\times\mathbb{R}^d\times\Omega$, denoted by $\mathscr{P}$. For $p\ge 2$ and $X\in L^2\left(\mathbb{R}_+\times\mathbb{R}^d,L^p(\Omega)\right)$, set

$$||X||_{M,p}^2:=\iint_{\mathbb{R}_+^*\times\mathbb{R}^d}\left|\left|X\left(s,y\right)\right|\right|_p^2\mathrm{d}s\mathrm{d}y<+\infty\,. \tag{3.3.1}$$

When $p=2$, we write $||X||_M$ instead of $||X||_{M,2}$. In [68], $\iint X\mathrm{d}W$ is defined for predictable X such that $||X||_M<+\infty$. However, the condition of predictability is not always so easy to check, and as in the case of ordinary Brownian motion [15, Chapter 3], it is convenient to be able to integrate elements X that are jointly measurable and adapted. For this, let $\mathscr{P}_p$ denote the closure in $L^2\left(\mathbb{R}_+\times\mathbb{R}^d,L^p(\Omega)\right)$ of simple processes. Clearly, $\mathscr{P}_2\supseteq\mathscr{P}_p\supseteq\mathscr{P}_q$ for $2\le p\le q<+\infty$, and according to Itô's isometry, $\iint X\mathrm{d}W$ is well-defined for all elements of $\mathscr{P}_2$. The next two propositions give easily verifiable conditions for checking that $X\in\mathscr{P}_2$. In the following, we will use $\cdot$ and $\circ$ to denote the time and space dummy variables respectively.

Proposition 3.3.1. *Suppose that for some $t>0$ and $p\in[2,\infty[$, a random field*

$$X=\left\{X\left(s,y\right):\left(s,y\right)\in\,]0,t[\times\mathbb{R}^d\right\}$$

has the following properties:

(i) X *is adapted, i.e., for all* $(s, y) \in]0, t[\times\mathbb{R}^d$, $X(s, y)$ *is* $\mathscr{F}_s$*-measurable;*

(ii) *For all* $(s, y) \in]0, t[\times\mathbb{R}^d$, $\left\|X(s, y)\right\|_p < +\infty$ *and the function* $(s, y) \mapsto X(s, y)$ *from* $]0, t[\times\mathbb{R}^d$ *into* $L^p(\Omega)$ *is continuous;*

(iii) $||X||_{M,p} < +\infty$.

Then $X(\cdot, \circ)\, 1_{]0,t[}(\cdot)$ *belongs to* $\mathscr{P}_p$.

The following proposition is a direct extension of Proposition 2.3.16. We leave the proof to the interested readers.

Proposition 3.3.2. *Suppose that for some* $t > 0$ *and* $p \geq 2$, *a random field*

$$X = \left\{ X(s, y) : (s, y) \in]0, t[\times\mathbb{R}^d \right\}$$

has the following properties:

(i) X *is adapted, i.e., for all* $(s, y) \in]0, t[\times\mathbb{R}^d$, $X(s, y)$ *is* $\mathscr{F}_s$*-measurable;*

(ii) X *is jointly measurable with respect to* $\mathscr{B}\left(\mathbb{R}^{1+d}\right) \times \mathscr{F}$*;*

(iii) $||X||_{M,p} < +\infty$.

Then $X(\cdot, \circ)\, 1_{]0,t[}(\cdot)$ *belongs to* $\mathscr{P}_2$.

3.3.2 A Lemma on Stochastic Convolutions

We need a lemma, which is an extension Lemma 2.3.20. Let $G(t, x)$ be a deterministic measurable function on $]0,\infty[\times\mathbb{R}^d$. Suppose that for some $t > 0$, a process $Z = \left(Z(s, y) : (s, y) \in]0, t[\times\mathbb{R}^d\right)$ has the following properties:

(1) Z is adapted, i.e., for all $(s, y) \in]0, t[\times\mathbb{R}^d$, $Z(s, y)$ is $\mathscr{F}_s$-measurable;

(2) Z is jointly measurable with respect to $\mathscr{B}\left(\mathbb{R}^{1+d}\right) \times \mathscr{F}$;

(3) $\left\|G^2(t - \cdot, x - \circ) Z(\cdot, \circ)\right\|_{M,2} < +\infty$ for all $(t, x) \in \mathbb{R}_+^* \times \mathbb{R}^d$.

Thanks to Proposition 3.3.2, for fixed $(t, x) \in \mathbb{R}_+ \times\mathbb{R}^d$, the random field $(s, y) \in [0, t] \times\mathbb{R}^d \mapsto G(t - s, x - y)\, Z(s, y)$ belongs to $\mathscr{P}_{M,2}$. Therefore, the following stochastic convolution

$$\left(G \star Z\dot{W}\right)(t, x) := \iint_{[0,t]\times\mathbb{R}^d} G(t - s, x - y)\, Z(s, y)\, \dot{W}(\mathrm{d}s, \mathrm{d}y)\,, \tag{3.3.2}$$

is a well-defined Walsh integral.

Lemma 3.3.3. *Let* Z *be the random field that satisfies the above three properties. Then the stochastic convolution in* (3.3.2) *has the following moment estimates: For all even integers* $p \geq 2$, *and all* $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$, *we have*

$$\left\|\left(G \star Z\dot{W}\right)(t, x)\right\|_p^2 \leq z_p^2 \iint_{[0,t]\times\mathbb{R}^d} G^2(t - s, x - y) \left\|Z(s, y)\right\|_p^2 \mathrm{d}s\mathrm{d}y\,, \tag{3.3.3}$$

where z_p is the constant defined in Theorem 2.3.18.

The proof can be easily adapted from the proof of [19, Lemma 2.4]. We will not repeat here.

3.3.3 A Proposition for the Picard Iteration

Proposition 3.3.4. *Suppose that for some even integer $p \geq 2$, a random field*

$$Y = \left(Y(t,x): (t,x) \in \mathbb{R}_+^* \times \mathbb{R}^d\right)$$

has the following properties

(i) Y is adapted, i.e., for all $\left(s,y\right) \in]0,t[\times\mathbb{R}^d$, $Y\left(s,y\right)$ is $\mathscr{F}_s$ measurable;

(ii) Y is jointly measurable with respect to $\mathscr{B}\left(\mathbb{R}_+^* \times \mathbb{R}^d\right) \times \mathscr{F}$;

(iii) for all $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}^d$,

$$||Y(\cdot,\circ)G(t-\cdot,x-\circ)||_{M,p}^2 < +\infty\,.$$

Then for all $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}^d$, $Y(\cdot,\circ)G(t-\cdot,x-\circ) \in \mathscr{P}_p$. So

$$w(t,x) = \iint_{]0,t[\times\mathbb{R}^d} Y\left(s,y\right) G\left(t-s,x-y\right) W(\mathrm{d}s,\mathrm{d}y), \quad \textit{for all } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$$

is well defined as a Walsh integral and the resulting random field w is adapted to $\{\mathscr{F}_s\}_{s\geq 0}$. Moreover, the random field $w = \{w(t,x): (t,x) \in \mathbb{R}_+^* \times \mathbb{R}^d\}$ is $L^p(\Omega)$)-continuous under either of the following two conditions:

(H) (Heat equation case)

(H-i) $G(\cdot,\cdot)$ is a deterministic and Borel measurable function that satisfies Assumptions 3.2.10 and 3.2.11,

(H-ii) $\sup_{(t,x)\in K} ||Y(t,x)||_p < +\infty$ for all compact sets $K \subseteq \mathbb{R}_+^* \times \mathbb{R}^d$, which is true, in particular, if Y is $L^p(\Omega)$-continuous.

(W) (1-d wave equation case) $G(t,x)$ is a deterministic and Borel measurable function that satisfies Assumptions 3.2.9.

Proof. Fix $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}^d$. By Assumption (iii) and the fact that $G(t,x)$ is Borel measurable and deterministic, the random field

$$X = \left(X\left(s,y\right): \left(s,y\right) \in]0,t[\times\mathbb{R}^d\right), \quad \text{with } X\left(s,y\right) := Y\left(s,y\right) G\left(t-s,x-y\right)$$

satisfies all conditions of Proposition 3.3.2. This implies that for all $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}^d$, $Y(\cdot,\circ)G(t-\cdot,x-\circ) \in \mathscr{P}_p$. Hence $w(t,x)$ is a well-defined Walsh integral and the resulting random field is adapted to the filtration $\{\mathscr{F}_s\}_{s\ge0}$.

Now we shall consider the two cases (H) and (W) separately. For two points $(t,x), \left(t',x'\right)\in \mathbb{R}_+\times\mathbb{R}^d$, recall (t_*,x_*) and $(\hat{t},\hat{x})$ are defined in (3.2.16).

Case (H). Choose $\beta\in\,]0,1[$ according to Assumption 3.2.10. Fix $(t,x)\in\mathbb{R}_+^*\times\mathbb{R}^d$. Let $B_{t,x,\beta} := B_{t,x,\beta,1/2,1}$ be the set defined in (3.2.14) and a be the constant used in Assumption 3.2.10. Assume that $\left(t',x'\right)\in B_{t,x,\beta}$. Set $K_a=[1/a,t+1]\times[-a,a]^d$. Let $A_a := \sup_{(s,y)\in K_a}\left|\left|Y\left(s,y\right)\right|\right|_p^2$, which is finite by (H-ii).

By Lemma 3.3.3, we have that

$$
\begin{aligned}
&\left|\left|w(t,x)-w\left(t',x'\right)\right|\right|_p^p\\
&\le 2^{p-1}\mathbb{E}\left(\left|\int_0^{t_*}\int_{\mathbb{R}^d}Y\left(s,y\right)\left(G\left(t-s,x-y\right)-G(t'-s,x'-y)\right)W(\mathrm{d}s,\mathrm{d}y)\right|^p\right)\\
&\quad+2^{p-1}\mathbb{E}\left(\left|\int_{t_*}^{\hat{t}}\int_{\mathbb{R}^d}Y\left(s,y\right)G\left(\hat{t}-s,\hat{x}-y\right)W(\mathrm{d}s,\mathrm{d}y)\right|^p\right)\\
&\le 2^{p-1}z_p^p\left(\int_0^{t_*}\int_{\mathbb{R}^d}\left|\left|Y\left(s,y\right)\right|\right|_p^2\left(G(t-s,x-y)-G(t'-s,x'-y)\right)^2\mathrm{d}s\mathrm{d}y\right)^{p/2}\\
&\quad+2^{p-1}z_p^p\left(\int_{t_*}^{\hat{t}}\int_{\mathbb{R}^d}\left|\left|Y\left(s,y\right)\right|\right|_p^2G^2\left(\hat{t}-s,\hat{x}-y\right)\mathrm{d}s\mathrm{d}y\right)^{p/2}\\
&\le 2^{p-1}z_p^p\left(L_1(t,t',x,x')\right)^{p/2}+2^{p-1}z_p^p\left(L_2(t,t',x,x')\right)^{p/2}. \qquad (3.3.4)
\end{aligned}
$$

We first consider L_1. Decompose L_1 into two parts:

$$
L_1\left(t,t',x,x'\right)=\iint_{\left([0,t_*]\times\mathbb{R}^d\right)\setminus K_a}\cdots\,\mathrm{d}s\mathrm{d}y+\iint_{\left([0,t_*]\times\mathbb{R}^d\right)\cap K_a}\cdots\,\mathrm{d}s\mathrm{d}y=L_{1,1}\left(t,t',x,x'\right)+L_{1,2}\left(t,t',x,x'\right).
$$

One can apply Lebesgue's dominated convergence theorem and Assumption 3.2.11 to show that

$$
\begin{aligned}
\lim_{(t',x')\to(t,x)}L_{1,1}\left(t,t',x,x'\right)=\lim_{(t',x')\to(t,x)}\iint_{\left([0,t_*]\times\mathbb{R}^d\right)\setminus K_a}&\left|\left|Y\left(s,y\right)\right|\right|_p^2\\
&\left(G\left(t-s,x-y\right)-G(t'-s,x'-y)\right)^2\mathrm{d}s\mathrm{d}y=0.
\end{aligned}
$$

Indeed, by Assumption 3.2.10,

$$
\sup_{(t',x')\in B_{t,x}}\left(G\left(t-s,x-y\right)-G(t'-s,x'-y)\right)^2\le 4G^2(t+1-s,x-y), \qquad (3.3.5)
$$

for all $s\in[0,t']$ and $|y|\ge a$. Moreover,

$$
\begin{aligned}
&\iint_{\left([0,t_*]\times\mathbb{R}^d\right)\setminus K_a}\left|\left|Y\left(s,y\right)\right|\right|_p^2G^2(t+1-s,x-y)\mathrm{d}s\mathrm{d}y\\
&\qquad\le\iint_{[0,t+1]\times\mathbb{R}^d}\left|\left|Y\left(s,y\right)\right|\right|_p^2G^2(t+1-s,x-y)\mathrm{d}s\mathrm{d}y<+\infty.
\end{aligned}
$$

As for $L_{1,2}$, we have that

$$
\begin{aligned}
L_{1,2}\left(t,t',x,x'\right) &\le A_a \iint_{([0,t_*]\times\mathbb{R}^d)\cap K_a} \left(G\left(t-s,x-y\right)-G(t'-s,x'-y)\right)^2 \mathrm{d}s\mathrm{d}y \\
&\le A_a \iint_{[0,\hat{t}]\times\mathbb{R}^d} \left(G\left(t-s,x-y\right)-G(t'-s,x'-y)\right)^2 \mathrm{d}s\mathrm{d}y \to 0\,,
\end{aligned}
$$

as $\left(t',x'\right)\to(t,x)$, where we have applied Assumption 3.2.11. Hence, we have proved that

$$
\lim_{(t',x')\to(t,x)} L_1\left(t',t,x,x'\right) = 0\,.
$$

Now let us consider L_2. Decompose it into two parts:

$$
L_2\left(t,t',x,x'\right) = \iint_{([t_*,\hat{t}]\times\mathbb{R}^d)\setminus K_a} \cdots \,\mathrm{d}s\mathrm{d}y + \iint_{([t_*,\hat{t}]\times\mathbb{R}^d)\cap K_a} \cdots \,\mathrm{d}s\mathrm{d}y = L_{2,1}\left(t,t',x,x'\right) + L_{2,2}\left(t,t',x,x'\right)\,.
$$

The limit $\lim_{(t',x')\to(t,x)} L_{2,1}\left(t,t',x,x'\right) = 0$ is true due to the monotone convergence theorem, thanks to the fact that

$$
\sup_{(t',x')\in B_{t,x}} G^2\left(\hat{t}-s,\hat{x}-y\right) \le G^2(t+1-s,x-y)\,.
$$

The proof for $L_{2,2}$ is similar to $L_{1,2}$:

$$
L_{2,2}\left(t,t',x,x'\right) \le A_a \int_{t_*}^{\hat{t}} \mathrm{d}s \int_{\mathbb{R}^d} G^2\left(\hat{t}-s,\hat{x}-y\right)\mathrm{d}y \to 0\,, \quad \text{as } (t',x')\to(t,x)\,,
$$

where we have applied Assumption 3.2.11 (see (3.2.15)). Hence, we have proved that

$$
\lim_{(t',x')\to(t,x)} L_2\left(t',t,x,x'\right) = 0\,.
$$

This completes the proof of (H).

Case (W). Choose $\beta\in\,]0,1[$, $\tau>0$ and $\alpha>0$ according to Assumption 3.2.9. Fix $(t,x)\in\mathbb{R}_+^*\times\mathbb{R}^d$. Let $B := B_{t,x,\beta,\tau,\alpha}$ be the set defined in (3.2.14) and C be the constant used in Assumption 3.2.9. For $\left(t',x'\right)\in B$, see (3.3.4) for the definitions of $L_1(t,t',x,x')$ and $L_2(t,t',x,x')$.

We first consider L_1. Under Assumption 3.2.9, we have that

$$
\left(G\left(t-s,x-y\right)-G\left(t'-s,x'-y\right)\right)^2 \le 4C^2G^2\left(t+1-s,x-y\right)\,.
$$

By (iii),

$$
\begin{aligned}
&\iint_{[0,t_*]\times\mathbb{R}^d} 4C^2G^2\left(t+1-s,x-y\right)\left|\left|Y\left(s,y\right)\right|\right|_p^2 \mathrm{d}s\mathrm{d}y \\
&\qquad\le 4C^2 \left|\left|Y(\cdot,\circ)G(t+1-\cdot,x-\circ)\right|\right|_{M,p}^2 < +\infty\,.
\end{aligned}
$$

By Assumption 3.2.9, for almost all (t,x), we have

$$\lim_{(t',x')\to(t,x)} \left(G\left(t-s,x-y\right)-G\left(t'-s,x'-y\right)\right)^2=0\,.$$

Hence, we can apply Lebesgue's dominated convergence theorem to conclude that

$$\lim_{(t',x')\to(t,x)} L_1(t,t',x,x')=0\,.$$

Similarly, for L_2, under Assumption 3.2.9, $G^2\left(\hat{t}-s,\hat{x}-y\right)\le C^2G^2\left(t+1-s,x-y\right)$. Then due to the fact that

$$\begin{aligned}\iint_{[t_*,\hat{t}]\times\mathbb{R}^d} & C^2G^2\left(t+1-s,x-y\right)\left|\left|Y(s,y)\right|\right|_p^2\mathrm{d}s\mathrm{d}y\\ &\le C^2\,||Y(\cdot,\circ)G(t+1-\cdot,x-\circ)||_{M,p}^2<+\infty\,,\end{aligned}$$

Lebesgue's dominated convergence theorem gives that

$$\lim_{(t',x')\to(t,x)} L_2(t,t',x,x')=0\,.$$

This completes the proof of (W). □

We still need a lemma to transform the stochastic integral equation of the form (2.2.2) to integral inequalities for its moments.

Lemma 3.3.5. *Suppose that $f(t,x)$ is an adapted and jointly measurable (with respect to $\mathscr{B}\left(\mathbb{R}_+^*\times\mathbb{R}^d\right)\times\mathscr{F}$) random field. Let $v=\left(v(t,x):(t,x)\in\mathbb{R}_+\times\mathbb{R}^d\right)$ be an adapted random field, that is, $v(t,x)$ is $\mathscr{F}_t$-measurable for all $(t,x)\in\mathbb{R}_+\times\mathbb{R}^d$. Suppose that the random field $w=\left(w(t,x):(t,x)\in\mathbb{R}_+\times\mathbb{R}^d\right)$ satisfies the following relation*

$$w(t,x)=f(t,x)+\left(G\lhd\left[\rho(v)\dot{W}\right]\right)(t,x)\,,$$

for all $(t,x)\in\mathbb{R}_+\times\mathbb{R}^d$, where

$$\left(G\lhd\left[\rho(v)\dot{W}\right]\right)(t,x):=\int_0^t\int_{\mathbb{R}^d}G\left(t-s,x-y\right)\theta\left(s,y\right)\,\rho\left(v\left(s,y\right)\right)W\left(\mathrm{d}s,\mathrm{d}y\right)\,,$$

and assume that this Walsh integral is well defined. If ρ satisfies (1.4.1), then for all even integers $p\ge2$, there is a constant $a_{p,\overline{\varsigma}}>0$ (defined in (1.4.4)) such that

$$\begin{aligned}\left|\left|\left(G\lhd\left[\rho(v)\dot{W}\right]\right)(t,x)\right|\right|_p^2 &\le z_p^2\left|\left|G\left(t-\cdot,x-\circ\right)\rho(v(\cdot,\circ))\,\theta\left(\cdot,\circ\right)\right|\right|_{M,p}^2\\ &\le\frac{1}{b_p}\left(\left(\overline{\varsigma}^2+||v||_p^2\right)\rhd\widehat{\mathscr{L}_0}\right)(t,x)\,,\end{aligned}$$

where b_p is defined in (2.4.5). Hence, for all $(t,x)\in\mathbb{R}_+\times\mathbb{R}^d$,

$$||w(t,x)||_p^2\le b_p\,\left|\left|f(t,x)\right|\right|_p^2+\left(\left(\overline{\varsigma}^2+||v||_p^2\right)\rhd\widehat{\mathscr{L}_0}\right)(t,x)\,,$$

where $\widehat{\mathscr{L}_0}(t,x) := \mathscr{L}_0\left(t,x\,;a_{p,\overline{\varsigma}}\, z_p\; L_\rho\right)$ *and the constant* $a_{p,\overline{\varsigma}}$ *is defined in* (1.4.4).

Proof. Fix an even integer $p \geq 2$. Denote

$$I(t,x) = \left(G \lhd \left[\rho(v)\dot{W}\right]\right)(t,x)\ .$$

By (3.3.3),

$$\begin{aligned}||I(t,x)||_p^2 &\leq z_p^2 \iint_{[0,t]\times\mathbb{R}^d} G^2\left(t-s,x-y\right)\left|\left|\rho\left(v\left(s,y\right)\right)\right|\right|_p^2 \theta^2\left(s,y\right)\mathrm{d}s\mathrm{d}y\\ &= z_p^2 \left|\left|G\left(t-\cdot,x-\circ\right)\rho\left(v\left(\cdot,\circ\right)\right)\theta\left(\cdot,\circ\right)\right|\right|_{M,p}^2 .\end{aligned}$$

If $\overline{\varsigma} = 0$, clearly $\left|\left|\rho(v)\right|\right|_p \leq L_\rho\, ||v||_p$. Otherwise, if $\overline{\varsigma} \neq 0$, by the linear growth condition (1.4.1), we know that

$$\mathbb{E}\left[|\rho(v)|^p\right] \leq L_\rho^p\, \mathbb{E}\left[\left(\overline{\varsigma}^2 + |v|^2\right)^{p/2}\right] \leq L_\rho^p\, 2^{(p-2)/2}\left(\overline{\varsigma}^p + \mathbb{E}[|v|^p]\right)\ .$$

By the sub-additivity of the function $|x|^{2/p}$ for $p \geq 2$ (that is, $(a+b)^{2/p} \leq a^{2/p} + b^{2/p}$ for all $a, b \geq 0$ and all $p \geq 2$), we know that

$$\left|\left|\rho\left(v\left(t,x\right)\right)\right|\right|_p^2 \leq L_\rho^2\, 2^{(p-2)/p}\left(\overline{\varsigma}^2 + ||v\left(t,x\right)||_p^2\right)\ , \quad \overline{\varsigma} > 0\ . \tag{3.3.6}$$

We have then

$$\begin{aligned}b_p\, ||I(t,x)||_p^2 &\leq z_p^2\, L_\rho^2\, a_{p,\overline{\varsigma}}^2 \iint_{[0,t]\times\mathbb{R}^d} G^2\left(t-s,x-y\right)\left(\overline{\varsigma}^2 + \left|\left|v\left(s,y\right)\right|\right|_p^2\right)\theta^2\left(s,y\right)\mathrm{d}s\mathrm{d}y\\ &= \left(\left(\overline{\varsigma}^2 + ||v||_p^2\right) \rhd \widehat{\mathscr{L}_0}\right)(t,x)\ ,\end{aligned}$$

where

$$\widehat{\mathscr{L}_0}(t,x) := \mathscr{L}_0\left(t,x\,;\, z_p L_\rho\, a_{p,\overline{\varsigma}}\right)\ ,$$

and $a_{p,\overline{\varsigma}}$ is defined in (1.4.4). We have used the facts that $a_{p,0}^2 = b_p$, and $a_{p,\overline{\varsigma}}^2 = 2^{\frac{p-2}{p}+1} = 2^{2(p-1)/p}$ for $\overline{\varsigma} \neq 0$ and $p > 2$.

Finally, by the triangle inequality, we have

$$||w(t,x)||_p \leq \left|\left|f(t,x)\right|\right|_p + \left|\left|\left(G \lhd \left[\rho(v)\dot{W}\right]\right)(t,x)\right|\right|_p\ ,$$

and so

$$||w(t,x)||_p^2 \leq b_p \left|\left|f(t,x)\right|\right|_p^2 + b_p \left|\left|\left(G \lhd \left[\rho(v)\dot{W}\right]\right)(t,x)\right|\right|_p^2\ ,$$

which then finishes the whole proof. □

3.3.4 Proof of Theorem 3.2.16

The proof is based on the standard Picard iteration. Throughout the proof, an arbitrary even integer $p \geq 2$ is fixed.

Step 1. Define $u_0(t,x) = J_0(t,x)$, which is a Borel measurable function by Assumption 3.2.12. Now we shall apply Proposition 3.3.4 with

$$Y\left(s,y\right) = \rho\left(u_0\left(s,y\right)\right)\theta\left(s,y\right)$$

by verifying the three properties that it requires. Since $\theta(\cdot,\circ)$ is deterministic, Y is clearly jointly measurable and adapted, and so Properties (i) and (ii) hold. As for Property (iii), by Lemma 3.3.5,

$$b_p z_p^2 \left|\left|\rho\left(u_0(\cdot,\circ)\right)\theta(\cdot,\circ)G(t-\cdot,x-\circ)\right|\right|_{M,p}^2 \le \left(\left[\overline{\varsigma}^2 + J_0^2\right] \rhd \widehat{\mathscr{L}_0}\right)(t,x)\,, \tag{3.3.7}$$

which is finite due to Assumption 3.2.12. Hence, by Proposition 3.3.4, we can conclude that for all $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}^d$,

$$\rho\left(u_0(\cdot,\circ)\right)\theta(\cdot,\circ)G\left(t-\cdot,x-\circ\right) \in \mathscr{P}_p\,,$$

and

$$I_1(t,x) = \iint_{[0,t]\times\mathbb{R}^d} \rho\left(u_0\left(s,y\right)\right)\theta\left(s,y\right)G\left(t-s,x-y\right)W\left(\mathrm{d}s,\mathrm{d}y\right)$$

is a well-defined Walsh integral. The random field $\left\{I_1\left(t,x\right) : (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d\right\}$ is adapted, that is, $I_1\left(t,x\right)$ is $\mathscr{F}_t$-measurable for all $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}^d$. The continuity of $(t,x) \mapsto I_1(t,x)$ from $\mathbb{R}_+^* \times \mathbb{R}^d$ into $L^p(\Omega)$ is guaranteed by Part (W) of Proposition 3.3.4 under Cond(W) and Part (H) under Cond(H). Define

$$u_1\left(t,x\right) := J_0\left(t,x\right) + I_1\left(t,x\right)\,.$$

Now we estimate its moments. We pay special attention to the second moment: The isometry property gives that

$$\left|\left|I_1\left(t,x\right)\right|\right|_2^2 = \left|\left|\rho\left(u_0(\cdot,\circ)\right)\theta\left(\cdot,\circ\right)G\left(t-\cdot,x-\circ\right)\right|\right|_{M,2}^2$$

which equals $\left(\left[\varsigma^2 + J_0^2\right] \rhd \mathscr{L}_0\right)(t,x)$ for the quasi-linear case (1.4.3), and is bounded from above (see (3.3.7) with $b_2 z_2^2 = 1$) and below (if ρ additionally satisfies (1.4.2)), in which case

$$\left(\left[\underline{\varsigma}^2 + J_0^2\right] \rhd \underline{\mathscr{L}}_0\right)(t,x) \le \left|\left|I_1\left(t,x\right)\right|\right|_2^2 \le \left(\left[\overline{\varsigma}^2 + J_0^2\right] \rhd \overline{\mathscr{L}}_0\right)(t,x)\,.$$

Because $J_0(t,x)$ is deterministic, $\mathbb{E}\left[J_0\left(t,x\right)I_1\left(t,x\right)\right] = 0$ and so

$$\left|\left|u_1\left(t,x\right)\right|\right|_2^2 = J_0^2(t,x) + \left|\left|I_1\left(t,x\right)\right|\right|_2^2\,.$$

The p-th moment is bounded as follows:

$$\left|\left|u_1\left(t,x\right)\right|\right|_p^2 \le b_p\,J_0^2(t,x) + \left(\left(\overline{\varsigma}^2 + b_p J_0^2\right) \rhd \widehat{\mathscr{L}_0}\right)(t,x)$$

according to Lemma 3.3.5, where b_p is defined in (2.4.5).

In summary, in this step we have proved that

$$\left\{u_1(t,x) = J_0(t,x) + I_1(t,x) : (t,x) \in \mathbb{R}_+^* \times \mathbb{R}^d\right\}$$

is a well-defined random field such that

(1) u_1 is adapted to the filtration $\{\mathscr{F}_t\}_{t>0}$, that is, $u_1(t,x)$ is $\mathscr{F}_t$-measurable for all $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}^d$;

(2) The function $(t,x) \mapsto I_1(t,x)$ from $\mathbb{R}_+^* \times \mathbb{R}^d$ into $L^p(\Omega)$ is continuous;

(3) $\mathbb{E}\left[u_1^2(t,x)\right] = J_0^2(t,x) + \left(\left[\overline{\varsigma}^2 + J_0^2\right] \rhd \mathscr{L}_0\right)(t,x)$ for the quasi-linear case (1.4.3) and it is bounded from above and below (if ρ additionally satisfies (1.4.2)):

$$J_0^2(t,x) + \left(\left[\underline{\varsigma}^2 + J_0^2\right] \rhd \underline{\mathscr{L}}_0\right)(t,x) \le \mathbb{E}\left[u_1^2(t,x)\right] \le J_0^2(t,x) + \left(\left[\overline{\varsigma}^2 + J_0^2\right] \rhd \overline{\mathscr{L}}_0\right)(t,x) \;;$$

(4) $||u_1(t,x)||_p^2 \le b_p\, J_0^2(t,x) + \left(\left(\overline{\varsigma}^2 + b_p J_0^2\right) \rhd \widehat{\mathscr{L}_0}\right)(t,x)$.

Step 2. We assume that for all $1 \le k \le n$, the following Walsh integral

$$I_k(t,x) = \iint_{[0,t]\times\mathbb{R}^d} \rho\left(u_{k-1}\left(s,y\right)\right)\theta\left(s,y\right) \times G\left(t-s,x-y\right) W\left(\mathrm{d}s,\mathrm{d}y\right)$$

is well defined. Hence, the random field

$$\left\{u_k(t,x) = J_0(t,x) + I_k(t,x) : (t,x) \in \mathbb{R}_+^* \times \mathbb{R}^d\right\}$$

is well defined and it satisfies the following properties:

(1) u_k is adapted to the filtration $\{\mathscr{F}_t\}_{t>0}$;

(2) The function $(t,x) \mapsto I_k(t,x)$ from $\mathbb{R}_+^* \times \mathbb{R}^d$ into $L^p(\Omega)$ is continuous;

(3) $\mathbb{E}\left[u_k^2(t,x)\right] = J_0^2(t,x) + \sum_{i=0}^{k-1}\left(\left[\overline{\varsigma}^2 + J_0^2\right] \rhd \mathscr{L}_i(t,x;\cdot,\circ)\right)(t,x)$ for the quasi-linear case (1.4.3) and it is bounded from above and below (if ρ additionally satisfies (1.4.2)):

$$J_0^2(t,x) + \sum_{i=0}^{k-1}\left(\left[\underline{\varsigma}^2 + J_0^2\right] \rhd \underline{\mathscr{L}}_i(t,x;\cdot,\circ)\right)(t,x) \le \mathbb{E}\left[u_k^2(t,x)\right]$$
$$\le J_0^2(t,x) + \sum_{i=0}^{k-1}\left(\left[\overline{\varsigma}^2 + J_0^2\right] \rhd \overline{\mathscr{L}}_i(t,x;\cdot,\circ)\right)(t,x) \;;$$

(4) $||u_k(t,x)||_p^2 \le b_p\, J_0^2(t,x) + \sum_{i=0}^{k-1}\left(\left(\overline{\varsigma}^2 + b_p\, J_0^2\right) \rhd \widehat{\mathscr{L}_i}(t,x;\cdot,\circ)\right)(t,x)$.

Now let us consider the case where $k = n+1$. Let us apply Proposition 3.3.4 with

$$Y\left(s,y\right) = \rho\left(u_n\left(s,y\right)\right)\theta\left(s,y\right)$$

by verifying the three properties that it requires. The properties (i) and (ii) are clearly satisfied by the induction assumptions (1) and (2). By Lemma 3.3.5 and the induction assumptions, we have that

$$
\begin{aligned}
& b_p z_p^2 \left|\left|\rho\left(u_n(\cdot,\circ)\right)\theta(\cdot,\circ)G(t-\cdot,x-\circ)\right|\right|_{M,p}^2 \\
&\quad \le \left(\left[\overline{\varsigma}^2 + ||u_n(\cdot,\circ)||_p^2\right] \rhd \widehat{\mathscr{L}}_0\right)(t,x) \\
&\quad \le \left(\left[\overline{\varsigma}^2 + b_p J_0^2\right] \rhd \widehat{\mathscr{L}}_0\right)(t,x) \\
&\qquad + \sum_{i=0}^{n-1} \int_0^t \int_{\mathbb{R}^d} \left(\left[\overline{\varsigma}^2 + b_p J_0^2\right] \rhd \widehat{\mathscr{L}}_i(s,y;\cdot,\circ)\right)(s,y)\ \theta^2(s,y)\ \widehat{\mathscr{L}}_0(t-s,x-y)\,\mathrm{d}s\mathrm{d}y\,.
\end{aligned}
$$

Denote the above double integral by $f_i(t,x)$. Now we apply (3.2.9) to $f_i(t,x)$ and use the definition of $\widehat{\mathscr{L}}_n(t,x)$:

$$
\begin{aligned}
f_i(t,x) &= \int_0^t \int_{\mathbb{R}^d} \left(\left[\overline{\varsigma}^2 + b_p J_0^2\right] \rhd \rhd_{i+1}\left(\widehat{\mathscr{L}}_0,\dots,\widehat{\mathscr{L}}_0\right)(s,y;\cdot,\circ)\right)(s,y) \\
&\qquad\qquad\qquad\qquad \times \theta^2(s,y)\ \widehat{\mathscr{L}}_0(t-s,x-y)\,\mathrm{d}s\mathrm{d}y \\
&= \left(\left[\overline{\varsigma}^2 + b_p J_0^2\right] \rhd \rhd_{i+2}\left(\widehat{\mathscr{L}}_0,\dots,\widehat{\mathscr{L}}_0\right)(t,x;,\cdot,\circ)\right)(t,x) \\
&= \left(\left[\overline{\varsigma}^2 + b_p J_0^2\right] \rhd \widehat{\mathscr{L}}_{i+1}(t,x;,\cdot,\circ)\right)(t,x)\,.
\end{aligned}
$$

Thus, Property (iii) is also true:

$$
\begin{aligned}
b_p z_p^2 \left|\left|\rho\left(u_n(\cdot,\circ)\right)\theta(\cdot,\circ)G(t-\cdot,x-\circ)\right|\right|_{M,p}^2 &\le \sum_{i=0}^{n} \left(\left[\overline{\varsigma}^2 + b_p J_0^2\right] \rhd \widehat{\mathscr{L}}_i(t,x;\cdot,\circ)\right)(t,x) \\
&\le \left(\left[\overline{\varsigma}^2 + b_p J_0^2\right] \rhd \widehat{\mathscr{K}}(t,x;\cdot,\circ)\right)(t,x)\ ,
\end{aligned}
\tag{3.3.8}
$$

which is finite by (3.2.17). Hence, by Proposition 3.3.4,

$$
\rho\left(u_n(\cdot,\circ)\right)\ \theta(\cdot,\circ)\ G(t-\cdot,x-\circ) \in \mathscr{P}_p\ ,
$$

and

$$
I_{n+1}(t,x) = \iint_{[0,t]\times\mathbb{R}^d} \rho\left(u_n(s,y)\right)\theta(s,y)\ G(t-s,x-y)\,W(\mathrm{d}s,\mathrm{d}y)
$$

is a well-defined Walsh integral. The random field

$$
\left\{I_{n+1}(t,x) : (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d\right\}
$$

is adapted to $\{\mathscr{F}_t\}_{t>0}$. The continuity of $(t,x) \mapsto I_{n+1}(t,x)$ from $\mathbb{R}_+^* \times \mathbb{R}^d$ into $L^p(\Omega)$ is guaranteed by Part (W) of Proposition 3.3.4 under Cond(W). Under Cond(H), we only need to show that the function

$$
(s,y) \mapsto \left|\left|\rho\left(u_n(s,y)\right)\theta(s,y)\right|\right|_p
$$

is locally bounded over $\mathbb{R}_+^* \times \mathbb{R}^d$. This is true since, by (3.3.6),

$$\begin{aligned}\left|\left|\rho\left(u_n\left(s,y\right)\right)\theta\left(s,y\right)\right|\right|_p^2 &\le L_\rho^2 \left|\theta\left(s,y\right)\right|^2 2^{(p-2)/p}\left(\overline{\varsigma}^2 + \left|\left|u_n\left(s,y\right)\right|\right|_p^2\right)\\ &\le L_\rho^2 \left|\theta\left(s,y\right)\right|^2 2^{(p-2)/p}\left(\overline{\varsigma}^2 + b_p J_0^2\left(s,y\right) + b_p \left|\left|I_n\left(s,y\right)\right|\right|_p^2\right)\end{aligned}$$

where both functions $\theta^2\left(s,y\right)$ and $J_0\left(s,y\right)$ are locally bounded by Cond(H), and the function $\left|\left|I_n\left(s,y\right)\right|\right|_p^2$ is as well thanks to the $L^p(\Omega)$-continuity of $\left(s,y\right)\mapsto I_n\left(s,y\right)$ (the induction assumption (2)). Define

$$u_{n+1}\left(t,x\right) := J_0\left(t,x\right) + I_{n+1}\left(t,x\right).$$

Now we estimate its moments. We first consider the second moment. Similar to the previous step, the isometry gives

$$\left|\left|I_{n+1}\left(t,x\right)\right|\right|_2^2 = \left|\left|\rho\left(u_n(\cdot,\circ)\right)\theta\left(\cdot,\circ\right)G\left(t-\cdot,x-\circ\right)\right|\right|_{M,2}^2$$

which equals $\sum_{i=0}^n\left(\left[\varsigma^2+J_0^2\right]\rhd\mathscr{L}_i(t,x;\cdot,\circ)\right)(t,x)$ for the quasi-linear case (1.4.3), and is bounded from above (see (3.3.8) with $b_2 z_2^2 = 1$) and below (if ρ additionally satisfies (1.4.2)), in which case

$$\sum_{i=0}^n\left(\left[\overline{\varsigma}^2+J_0^2\right]\rhd\underline{\mathscr{L}}_i(t,x;\cdot,\circ)\right)(t,x) \le \left|\left|I_{n+1}\left(t,x\right)\right|\right|_2^2 \le \sum_{i=0}^n\left(\left[\overline{\varsigma}^2+J_0^2\right]\rhd\overline{\mathscr{L}}_i(t,x;\cdot,\circ)\right)(t,x).$$

Because $J_0(t,x)$ is deterministic, $\mathbb{E}\left[J_0\left(t,x\right)I_{n+1}\left(t,x\right)\right]=0$ and so

$$\left|\left|u_{n+1}\left(t,x\right)\right|\right|_2^2 = J_0^2(t,x) + \left|\left|I_{n+1}\left(t,x\right)\right|\right|_2^2.$$

By (3.3.8), the p-th moment is bounded as follows:

$$\begin{aligned}\left|\left|u_{n+1}\left(t,x\right)\right|\right|_p^2 &\le b_p\, J_0^2(t,x) + \left(\left(\overline{\varsigma}^2 + b_p\left|\left|u_n\right|\right|_p^2\right)\rhd\widehat{\mathscr{L}_0}\right)(t,x)\\ &\le b_p\, J_0^2(t,x) + \sum_{i=0}^n\left(\left(\overline{\varsigma}^2+b_p\, J_0^2\right)\rhd\widehat{\mathscr{L}_i}(t,x;\cdot,\circ)\right)(t,x).\end{aligned}$$

Therefore, we have proved that the four properties (1) – (4) also hold for $k=n+1$. So we conclude that for all $n\in\mathbb{N}$,

$$\left\{u_{n+1}\left(t,x\right) = J_0\left(t,x\right) + I_{n+1}\left(t,x\right):\ \left(t,x\right)\in\mathbb{R}_+^*\times\mathbb{R}^d\right\}$$

is a well-defined random field such that

(1) u_{n+1} is adapted to the filtration $\{\mathscr{F}_t\}_{t>0}$;

(2) The function $(t,x)\mapsto I_{n+1}(t,x)$ from $\mathbb{R}_+^*\times\mathbb{R}^d$ into $L^p(\Omega)$ is continuous;

(3) $\mathbb{E}\left[u_{n+1}^2\left(t,x\right)\right] = J_0^2(t,x) + \sum_{i=0}^n\left(\left[\varsigma^2+J_0^2\right]\rhd\mathscr{L}_i(t,x;\cdot,\circ)\right)(t,x)$ for the quasi-linear case

(1.4.3) and is bounded from above and below (if ρ additionally satisfies (1.4.2)):

$$J_0^2(t,x) + \sum_{i=0}^{n} \left(\left[\underline{\varsigma}^2 + J_0^2 \right] \rhd \underline{\mathscr{L}}_i(t,x;\cdot,\circ) \right)(t,x) \le \mathbb{E}\left[u_{n+1}^2(t,x) \right]$$
$$\le J_0^2(t,x) + \sum_{i=0}^{n} \left(\left[\overline{\varsigma}^2 + J_0^2 \right] \rhd \overline{\mathscr{L}}_i(t,x;\cdot,\circ) \right)(t,x) \;;$$

(4) $||u_{n+1}(t,x)||_p^2 \le b_p J_0^2(t,x) + \sum_{i=0}^{n} \left(\left(\overline{\varsigma}^2 + b_p J_0^2 \right) \rhd \widehat{\mathscr{L}}_i(t,x;\cdot,\circ) \right)(t,x)$.

Step 3. We claim that for all $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}^d$, the series $\{I_n(t,x) : n \in \mathbb{N}\}$, with $I_0(t,x) := J_0(t,x)$, is a Cauchy sequence in $L^p(\Omega)$. Define

$$F_n(t,x) := ||I_{n+1}(t,x) - I_n(t,x)||_p^2 \, .$$

For $n \ge 1$, by Lemma 2.3.20, we have

$$F_n(t,x) \le z_p^2 \iint_{[0,t]\times\mathbb{R}^d} G^2(t-s,x-y)\,\theta^2(s,y) \left|\left|\rho(u_n(s,y)) - \rho(u_{n-1}(s,y))\right|\right|_p^2 \mathrm{d}s\mathrm{d}y \, .$$

Then by the Lipschitz continuity of ρ, we have

$$\begin{aligned} F_n(t,x) &\le z_p^2 \,\mathrm{Lip}_\rho^2 \iint_{[0,t]\times\mathbb{R}^d} G^2(t-s,x-y)\,\theta^2(s,y) \left|\left|u_n(s,y) - u_{n-1}(s,y)\right|\right|_p^2 \mathrm{d}s\mathrm{d}y \\ &= z_p^2 \,\mathrm{Lip}_\rho^2 \iint_{[0,t]\times\mathbb{R}^d} G^2(t-s,x-y)\,\theta^2(s,y) \left|\left|I_n(s,y) - I_{n-1}(s,y)\right|\right|_p^2 \mathrm{d}s\mathrm{d}y \\ &\le \left(F_{n-1} \rhd \widetilde{\mathscr{L}}_0 \right)(t,x) \, , \end{aligned}$$

where

$$\widetilde{\mathscr{L}}_0(t,x) := \mathscr{L}_0\left(t,x; z_p^2 \max\left(\mathrm{Lip}_\rho^2, a_{p,\overline{\varsigma}}^2 L_\rho^2 \right) \right) .$$

The following kernel functions $\widetilde{\mathscr{L}}_n(t,x;s,y)$ and $\widetilde{\mathscr{K}}(t,x;s,y)$ are defined by the same parameter. For the case $n=0$, we need to use the linear growth condition (1.4.1) instead: Apply Lemma 3.3.5 (see also (3.2.23)),

$$\begin{aligned} F_0(t,x) = ||u_1(t,x) - u_0(t,x)||_p^2 &\le \left(\left[\overline{\varsigma}^2 + J_0^2 \right] \rhd \widehat{\mathscr{L}}_0 \right)(t,x) \\ &\le \left(\left[\overline{\varsigma}^2 + J_0^2 \right] \rhd \widetilde{\mathscr{L}}_0 \right)(t,x) \, . \end{aligned}$$

Then apply the above relation recursively:

$$\begin{aligned} F_n(t,x) &\le \left(F_{n-1} \rhd \widetilde{\mathscr{L}}_0 \right)(t,x) \le \left(\left(F_{n-2} \rhd \widetilde{\mathscr{L}}_0 \right) \rhd \widetilde{\mathscr{L}}_0 \right)(t,x) \\ &\;\vdots \\ &\le \left(\left(\cdots \left(\left(F_0 \rhd \widetilde{\mathscr{L}}_0 \right) \rhd \widetilde{\mathscr{L}}_0 \right) \rhd \cdots \right) \rhd \widetilde{\mathscr{L}}_0 \right)(t,x) \, , \quad (n \text{ convolutions}) \\ &\le \left(\left(\cdots \left(\left(\left(\overline{\varsigma}^2 + J_0^2 \right) \rhd \widetilde{\mathscr{L}}_0 \right) \rhd \widetilde{\mathscr{L}}_0 \right) \rhd \cdots \right) \rhd \widetilde{\mathscr{L}}_0 \right)(t,x) \, , \quad (n+1 \text{ convolutions}) \, . \end{aligned}$$

Then use (3.2.7) to see that

$$\begin{aligned} F_n &\le \left(\left(\overline{\varsigma}^2 + J_0^2 \right) \rhd \rhd_{n+1} \left(\widetilde{\mathscr{L}}_0, \dots, \widetilde{\mathscr{L}}_0 \right) (t,x;\cdot,\circ) \right) (t,x) \\ &= \left(\left[\overline{\varsigma}^2 + J_0^2 \right] \rhd \widetilde{\mathscr{L}}_n (t,x;\cdot,\circ) \right) (t,x) \ . \end{aligned}$$

By Assumption 3.2.8, we have that $\widetilde{\mathscr{L}}_n \left(t,x;s,y\right) \le \widetilde{\mathscr{L}}_0 \left(s,y\right) B_n(t)$. Since $B_n(t)$ is nondecreasing,

$$F_n\left(t,x\right) \le \left(\left[\overline{\varsigma}^2 + J_0^2 \right] \rhd \widetilde{\mathscr{L}}_0 \right) (t,x)\, B_n(t) \ .$$

By Assumption 3.2.8 again, we have

$$\begin{aligned} \sum_{i=0}^{\infty} \left| F_i\left(t,x\right) \right|^{1/2} &\le \sum_{i=0}^{\infty} \left| \left(\left[\overline{\varsigma}^2 + J_0^2 \right] \rhd \widetilde{\mathscr{L}}_0 \right) (t,x)\, B_i(t) \right|^{1/2} \\ &= \left| \left(\left[\overline{\varsigma}^2 + J_0^2 \right] \rhd \widetilde{\mathscr{L}}_0 \right) (t,x) \right|^{1/2} \sum_{i=0}^{\infty} \left| B_i(t) \right|^{1/2} < +\infty \ . \end{aligned}$$

This proves that $\{I_n\left(t,x\right) : n \in \mathbb{N}\}$ is a Cauchy sequence in $L^p(\Omega)$. Define

$$I(t,x) := \lim_{n\to+\infty} I_n(t,x)\ , \ \text{in } L^p(\Omega) \quad \text{and} \quad u\left(t,x\right) := J_0\left(t,x\right) + I\left(t,x\right) \ .$$

The moments estimates can be obtained simply by sending n to $+\infty$ in the conclusions (3) and (4) of the previous step and using Assumption 3.2.7. For example,

$$\begin{aligned} ||u\left(t,x\right)||_p^2 &\le \lim_{n\to+\infty} \left(b_p\, J_0^2(t,x) + \sum_{i=0}^{n} \left(\left(\overline{\varsigma}^2 + b_p\, J_0^2 \right) \rhd \widehat{\mathscr{L}}_i (t,x;\cdot,\circ) \right) (t,x) \right) \\ &= b_p\, J_0^2(t,x) + \sum_{i=0}^{\infty} \left(\left(\overline{\varsigma}^2 + b_p\, J_0^2 \right) \rhd \widehat{\mathscr{L}}_i (t,x;\cdot,\circ) \right) (t,x) \\ &= b_p\, J_0^2(t,x) + \left(\left(\overline{\varsigma}^2 + b_p\, J_0^2 \right) \rhd \widehat{\mathscr{K}} (t,x;\cdot,\circ) \right) (t,x) \ , \end{aligned}$$

which is finite by (3.2.17) and Assumption 3.2.12:

$$\left(\left(\overline{\varsigma}^2 + b_p\, J_0^2 \right) \rhd \widehat{\mathscr{K}} (t,x;\cdot,\circ) \right) (t,x) \le \widehat{\Upsilon}(t) \left(\left(\overline{\varsigma}^2 + b_p\, J_0^2 \right) \rhd \widehat{\mathscr{L}}_0 (t,x;\cdot,\circ) \right) (t,x) < +\infty \ .$$

Now let us prove the $L^p(\Omega)$-continuity of the function $(t,x) \mapsto I(t,x)$ from $\mathbb{R}_+^* \times \mathbb{R}^d$ into $L^p(\Omega)$. Indeed, for all $a > 0$, denote the compact set $K_a := [1/a, a] \times [-a, a]^d$. The above $L^p(\Omega)$ limit is uniform over K_a since

$$\sum_{i=0}^{\infty} \sup_{(t,x)\in K_a} \left| F_i\left(t,x\right) \right|^{1/2} \le \left(\sum_{i=0}^{\infty} \left| B_i(a) \right|^{1/2} \right) \sup_{(t,x)\in K_a} \left| \left(\left[\overline{\varsigma}^2 + J_0^2 \right] \rhd \widetilde{\mathscr{L}}_0 \right) (t,x) \right|^{1/2}$$

from the fact that $B_n(t)$ is nondecreasing. Assumption 3.2.12 (Cond(G) (b)) implies that

$$\sup_{(t,x)\in K_a} \left| \left(\left[\overline{\varsigma}^2 + J_0^2 \right] \rhd \widetilde{\mathscr{L}}_0 \right) (t,x) \right|^{1/2} < +\infty \ .$$

Hence $\sum_{i=0}^{\infty}\sup_{(t,x)\in K_a}|F_i(t,x)|^{1/2}<+\infty$, which implies that the function in question is continuous over K_a since each $(t,x)\mapsto u_n(t,x)$ is so. As the compact set K_a can be arbitrarily close to the domain $\mathbb{R}_+^*\times\mathbb{R}^d$, we have then proved the $L^p(\Omega)$-continuity of the function $(t,x)\mapsto I(t,x)$.

Finally, we conclude that $\{I_n(t,x):\ n\in\mathbb{N}\}$ converges in $L^p(\Omega)$ to $I(t,x)$ such that

(1) u is adapted to the filtration $\{\mathscr{F}_t\}_{t>0}$;

(2) The function $(t,x)\mapsto I(t,x)$ from $\mathbb{R}_+^*\times\mathbb{R}^d$ into $L^p(\Omega)$ is continuous;

(3) for the quasi-linear case (1.4.3)

$$\begin{aligned}||u(t,x)||_2^2 &= J_0^2(t,x)+\sum_{i=0}^{\infty}\left(\left[\varsigma^2+J_0^2\right]\rhd\mathscr{L}_i(t,x;\cdot,\circ)\right)(t,x)\\ &= J_0^2(t,x)+\left(\left[\varsigma^2+J_0^2\right]\rhd\mathscr{K}(t,x;\cdot,\circ)\right)(t,x)\ ,\end{aligned}$$

and it is bounded from above and below (if ρ additionally satisfies (1.4.2)) by

$$\begin{aligned}J_0^2(t,x)+\left(\left[\underline{\varsigma}^2+J_0^2\right]\rhd\underline{\mathscr{K}}\right)(t,x) &\le ||u(t,x)||_2^2\\ &\le J_0^2(t,x)+\left(\left[\overline{\varsigma}^2+J_0^2\right]\rhd\overline{\mathscr{K}}(t,x;\cdot,\circ)\right)(t,x)\ ;\end{aligned}$$

(4) $||u(t,x)||_p^2\le b_p\,J_0^2(t,x)+\left(\left(\overline{\varsigma}^2+b_p\,J_0^2\right)\rhd\widehat{\mathscr{K}}(t,x;\cdot,\circ)\right)(t,x)$.

As a direct consequence of the above upper bound and (3.2.9), we have

$$\begin{aligned}&\left(\left[\overline{\varsigma}^2+||u||_p^2\right]\rhd\widehat{\mathscr{L}_0}\right)(t,x)\\ &\quad\le\left(\left[\overline{\varsigma}^2+J_0^2\right]\rhd\widehat{\mathscr{L}_0}\right)(t,x)\\ &\qquad+\int_0^t\int_{\mathbb{R}^d}\left(\left[\overline{\varsigma}^2+J_0^2\right]\rhd\widehat{\mathscr{K}}(s,y;\cdot,\circ)\right)(s,y)\ \theta^2(s,y)\ \widehat{\mathscr{L}_0}(t-s,x-y)\,\mathrm{d}s\mathrm{d}y\\ &\quad=\left(\left[\overline{\varsigma}^2+J_0^2\right]\rhd\widehat{\mathscr{L}_0}\right)(t,x)\\ &\qquad+\sum_{i=0}^{\infty}\int_0^t\int_{\mathbb{R}^d}\left(\left[\overline{\varsigma}^2+J_0^2\right]\rhd\widehat{\mathscr{L}_i}(s,y;\cdot,\circ)\right)(s,y)\ \theta^2(s,y)\ \widehat{\mathscr{L}_0}(t-s,x-y)\,\mathrm{d}s\mathrm{d}y\\ &\quad=\left(\left[\overline{\varsigma}^2+J_0^2\right]\rhd\widehat{\mathscr{L}_0}\right)(t,x)+\sum_{i=1}^{\infty}\left(\left[\overline{\varsigma}^2+J_0^2\right]\rhd\widehat{\mathscr{L}_i}(s,y;\cdot,\circ)\right)(t,x)\end{aligned}$$

Hence,

$$\left(\left[\overline{\varsigma}^2+||u||_p^2\right]\rhd\widehat{\mathscr{L}_0}\right)(t,x)\le\left(\left[\overline{\varsigma}^2+J_0^2\right]\rhd\widehat{\mathscr{K}}(t,x;\cdot,\circ)\right)(t,x)\ . \tag{3.3.9}$$

This inequality will be used in Step 4.

Step 4 (Verifications). Now we shall verify that $\left\{u(t,x):\ (t,x)\in\mathbb{R}_+^*\times\mathbb{R}^d\right\}$ defined in previous step is indeed a solution of the stochastic integral equation (3.1.1) in the sense of Definition 3.2.1. Clearly, u is adapted and jointly-measurable and hence it satisfies (1) and (2) of Definition 3.2.1. The continuity of the function $(t,x)\mapsto I(t,x)$ from $\mathbb{R}_+^*\times\mathbb{R}^d$

into $L^2(\Omega)$ proved in Step 3 guarantees (3) of Definition 3.2.1. So we only need to verify that $I(t,x)$ satisfies (4) of Definition 3.2.1, that is, $I(t,x)$ satisfies (3.2.1) a.s. for all $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}^d$.

Fix $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}^d$. We shall apply Proposition 3.3.4 with

$$Y(s,y) = \rho\left(u(s,y)\right)\theta(s,y)$$

by verifying the three properties that it requires. The properties (i) and (ii) are clearly satisfied. By Lemma 3.3.5 and (3.3.9),

$$\begin{aligned}\left|\left|\rho\left(u(\cdot,\circ)\right)\theta(\cdot,\circ)\,G(t-\cdot,x-\circ)\right|\right|_{M,p}^2 &\le \left(\left[\overline{\varsigma}^2 + ||u||_p^2\right] \rhd \widehat{\mathscr{L}_0}\right)(t,x) \\ &\le \left(\left[\overline{\varsigma}^2 + b_p\, J_0^2\right] \rhd \widehat{\mathscr{K}}(t,x;\cdot,\circ)\right)(t,x)\ ,\end{aligned}$$

which is finite due to Assumption 3.2.12 (see also (3.2.17)). Hence,

$$\rho\left(u(\cdot,\circ)\right)\theta(\cdot,\circ)\,G(t-\cdot,x-\circ) \in \mathscr{P}_p\,,$$

and

$$I(t,x) := \iint_{[0,t]\times\mathbb{R}^d} \rho\left(u(s,y)\right)\theta(s,y)\,G(t-s,x-y)\,W(\mathrm{d}s,\mathrm{d}y)$$

is a well-defined Walsh integral. It is adapted to $\{\mathscr{F}_t\}_{t>0}$. The $L^p(\Omega)$-continuity of the function $(t,x) \mapsto I(t,x)$ is guaranteed by Part (W) of Proposition 3.3.4 under Cond(W) and Part (H) under Cond(H).

By Step 3, we know that

$$I_n(t,x) = \iint_{[0,t]\times\mathbb{R}^d} \rho\left(I_{n-1}(s,y) + J_0(s,y)\right)\theta(s,y)\,G(t-s,x-y)\,W(\mathrm{d}s,\mathrm{d}y)$$

with the left-hand side $I_n(t,x)$ tending to $I(t,x)$ in $L^p(\Omega)$. We only need to show that the right-hand side converges in $L^p(\Omega)$ to $I(t,x)$. In fact, by Lemma 2.3.20 and the Lipschitz property of ρ,

$$\begin{aligned}&\left|\left|\iint_{[0,t]\times\mathbb{R}^d}\left[\rho\left(I(s,y)+J_0(s,y)\right)-\rho\left(I_n(s,y)+J_0(s,y)\right)\right]\theta(s,y)\,G(t-s,x-y)\,W(\mathrm{d}s,\mathrm{d}y)\right|\right|_p^2 \\ &\qquad \le \mathrm{Lip}_\rho^2 \iint_{[0,t]\times\mathbb{R}^d} \left|\left|I_n(s,y)-I(s,y)\right|\right|_p^2\,\theta^2(s,y)\,G^2(t-s,x-y)\,\mathrm{d}s\mathrm{d}y\,.\end{aligned}$$

Now apply Lebesgue's dominated convergence theorem to conclude that the above integral tends to zero as $n\to\infty$ due to:

(i) For all $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}^d$, $||I_n(t,x) - I(t,x)||_p^2 \to 0$ as $n \to +\infty$;

(ii) The integrand can be bounded in the following way:

$$\begin{aligned}||I_n(t,x)-I(t,x)||_p^2 &= ||u_n(t,x)-u(t,x)||_p^2\\ &\le 2\,||u_n(t,x)||_p^2+2\,||u(t,x)||_p^2\\ &\le 4b_p\,J_0^2(t,x)+4\left(\left[\overline{\varsigma}^2+b_p\,J_0^2\right]\rhd\widehat{\mathscr{K}}(t,x;\cdot,\circ)\right)(t,x)\ ,\end{aligned}$$

where the last inequality is true because by Step 2,

$$\begin{aligned}||u_n(t,x)||_p^2 &\le b_p\,J_0^2(t,x)+\sum_{i=0}^{n}\left(\left[\overline{\varsigma}^2+b_p\,J_0^2\right]\rhd\widehat{\mathscr{L}}_i(t,x;\cdot,\circ)\right)(t,x)\\ &\le b_p\,J_0^2(t,x)+\left(\left[\overline{\varsigma}^2+b_p\,J_0^2\right]\rhd\widehat{\mathscr{K}}(t,x;\cdot,\circ)\right)(t,x)\ ,\end{aligned}$$

and also by Step 3,

$$||u(t,x)||_p^2\le b_p\,J_0^2(t,x)+\left(\left[\overline{\varsigma}^2+b_p\,J_0^2\right]\rhd\widehat{\mathscr{K}}(t,x;\cdot,\circ)\right)(t,x)\ .$$

Because $\widehat{\Upsilon}_p(t)$ is nondecreasing in t, by (3.2.13), for $0\le s\le t$, we have that

$$\left|\left|I_n(s,y)-I(s,y)\right|\right|_p^2\le 4b_p\,J_0^2(s,y)+4\widehat{\Upsilon}_p(t)\left(\left[\overline{\varsigma}^2+b_p\,J_0^2\right]\rhd\widehat{\mathscr{L}_0}\right)(s,y)\ ,$$

where by our convention,

$$\widehat{\Upsilon}_p(t)=\Upsilon\left(t\,;\;a_{p,\overline{\varsigma}}z_p\,L_\rho\right)\ .$$

This upper bound is integrable:

$$\begin{aligned}&4\,a_{p,\overline{\varsigma}}^2z_p^2\,L_\rho^2\iint_{[0,t]\times\mathbb{R}^d}\left(b_p\,J_0^2(s,y)+\widehat{\Upsilon}_p(t)\left(\left[\overline{\varsigma}^2+b_p\,J_0^2\right]\rhd\widehat{\mathscr{L}_0}\right)(s,y)\right)\\ &\qquad\qquad\times\theta^2(s,y)\;G^2(t-s,x-y)\,\mathrm{d}s\mathrm{d}y\\ &\le 4\,\left(\widehat{\Upsilon}_p(t)\vee 1\right)\left[\left(b_p\,J_0^2\rhd\widehat{\mathscr{L}_0}\right)(t,x)+\left(\left[\left(\overline{\varsigma}^2+b_p\,J_0^2\right)\rhd\widehat{\mathscr{L}_0}\right]\rhd\widehat{\mathscr{L}_0}\right)(t,x)\right]\\ &\le 4\,\left(\widehat{\Upsilon}_p(t)\vee 1\right)\sum_{i=0}^{1}\left(\left[\overline{\varsigma}^2+b_p\,J_0^2\right]\rhd\widehat{\mathscr{L}_i}(t,x;\cdot,\circ)\right)(t,x)\\ &\le 4\,\left(\widehat{\Upsilon}_p(t)\vee 1\right)\left(\left[\overline{\varsigma}^2+b_p\,J_0^2\right]\rhd\widehat{\mathscr{K}}(t,x;\cdot,\circ)\right)(t,x)\ ,\end{aligned}$$

which is finite due to Assumption 3.2.12 (see also (3.2.17)), where we have used (3.2.8).

Therefore, we have proved that, as $n\to\infty$,

$$I_n(t,x)\overset{L^p(\Omega)}{\longrightarrow}\iint_{[0,t]\times\mathbb{R}^d}\rho\left(I(s,y)+J_0(s,y)\right)\,\theta(s,y)\;G(t-s,x-y)\,W(\mathrm{d}s,\mathrm{d}y)\ .$$

These two $L^p(\Omega)$-limits of $I_n(t,x)$ must be equal a.s., i.e., for all $(t,x)\in\mathbb{R}_+^*\times\mathbb{R}^d$,

$$I(t,x)=\iint_{[0,t]\times\mathbb{R}^d}\rho\left(I\left(s,y\right)+J_0\left(s,y\right)\right)\theta\left(s,y\right)\,G\left(t-s,x-y\right)W\left(\mathrm{d}s,\mathrm{d}y\right),\quad\text{a.s.}$$

We have therefore proved that $I(t,x)$ satisfies the integral equation (3.2.1) a.s. for all $(t,x)\in\mathbb{R}_+^*\times\mathbb{R}^d$. This finishes the proof of the existence part of Theorem 3.2.16 with the moment estimates.

Step 5 (Uniqueness). Let $u_1(t,x)=J_0(t,x)+I_1(t,x)$ and $u_2(t,x)=J_0(t,x)+I_2(t,x)$ be two solutions to (3.1.1) in the sense of Definition 3.2.1, and denote

$$v(t,x):=u_1(t,x)-u_2(t,x)=I_1(t,x)-I_2(t,x)\ .$$

The random field $v(t,x)$ inherits the $L^2(\Omega)$-continuity from I_1 and I_2. Writing $v(t,x)$ explicitly

$$v(t,x)=\iint_{[0,t]\times\mathbb{R}^d}\left[\rho\left(u_1\left(s,y\right)\right)-\rho\left(u_2\left(s,y\right)\right)\right]\theta\left(s,y\right)\,G\left(t-s,x-y\right)W\left(\mathrm{d}s,\mathrm{d}y\right)$$

and then taking the second moment, by the isometry property and Lipschitz condition of ρ, we have

$$\mathbb{E}\left[v(t,x)^2\right]\le\left(||v||_2^2\rhd\widetilde{\mathscr{L}_0}\right)(t,x)\,,$$

with $\widetilde{\mathscr{L}_0}(t,x):=\mathscr{L}_0\left(t,x;\mathrm{Lip}_\rho\right)$. Now we convolve both sides with respect to $\widetilde{\mathscr{K}}$ and then use (3.2.8),

$$\begin{aligned}\left(||v||_2^2\rhd\widetilde{\mathscr{K}}(t,x;\cdot,\circ)\right)(t,x)&\le\left(\left[||v||_2^2\rhd\widetilde{\mathscr{L}_0}\right]\rhd\widetilde{\mathscr{K}}(t,x;\cdot,\circ)\right)(t,x)\\&=\sum_{i=0}^{\infty}\left(\left[||v||_2^2\rhd\widetilde{\mathscr{L}_0}\right]\rhd\widetilde{\mathscr{L}_i}(t,x;\cdot,\circ)\right)(t,x)\\&=\sum_{i=1}^{\infty}\left(||v||_2^2\rhd\widetilde{\mathscr{L}_i}(t,x;\cdot,\circ)\right)(t,x)\\&=\left(||v||_2^2\rhd\widetilde{\mathscr{K}}(t,x;\cdot,\circ)\right)(t,x)-\left(||v||_2^2\rhd\widetilde{\mathscr{L}_0}\right)(t,x)\ .\end{aligned}$$

So we have that

$$\left(||v||_2^2\rhd\widetilde{\mathscr{L}_0}\right)(t,x)\equiv 0\,,\quad\text{for all }(t,x)\in\mathbb{R}_+^*\times\mathbb{R}^d\,,$$

which implies $\mathbb{E}\left[v(t,x)^2\right]=0$ for all $(t,x)\in\mathbb{R}_+^*\times\mathbb{R}^d$. Now using the fact that the function $(t,x)\mapsto\mathbb{E}\left[v(t,x)^2\right]$ is non-negative and continuous as a consequence of the $L^2(\Omega)$-continuity of v, we can conclude that for all $(t,x)\in\mathbb{R}_+^*\times\mathbb{R}^d$, $u_1(t,x)=u_2(t,x)$ a.s. This proves the uniqueness.

Step 6 (Two-point correlations). We only need to prove the formula (3.2.28) for the quasi-linear case: $|\rho(u)|^2=\lambda^2\left(\varsigma^2+u^2\right)$. Let $u(t,x)$ be the solution to (3.1.1). Fix $t\in\mathbb{R}_+^*$ and $x,y\in\mathbb{R}^d$. Consider the $L^2(\Omega)$-martingale $\{U(\tau;t,x):\tau\in[0,t]\}$ defined as follows

$$U(\tau;t,x):=J_0(t,x)+\int_0^\tau\int_{\mathbb{R}^d}\rho(u(s,z))\,\theta(s,z)\,G(t-s,x-z)\,W(\mathrm{d}s,\mathrm{d}z)\ .$$

Then $\mathbb{E}[U(\tau;t,x)] = \mathbb{E}[J_0(t,x)]$. Similarly, we can define the martingale $\{U(\tau;t,y) : \tau \in [0,t]\}$. The mutual variation process of these two martingale is

$$\begin{aligned}&\left[U(\cdot;t,x), U(\cdot;t,y)\right]_\tau \\ &= \lambda^2 \int_0^\tau \mathrm{d}s \int_{\mathbb{R}^d} \left(\varsigma^2 + |u(s,z)|^2\right) \theta^2(s,z)\, G(t-s,x-z) G\left(t-s,y-z\right) \mathrm{d}z\,, \quad \text{for all } \tau \in [0,t]\,.\end{aligned}$$

Hence, by Itô's lemma, for every $\tau \in [0,t]$,

$$\begin{aligned}\mathbb{E}\left[u(t,x)u(t,y)\right] =& J_0(t,x)J_0(t,y) \\ &+ \lambda^2\varsigma^2 \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \theta^2(s,z)\, G(t-s,x-z) G\left(t-s,y-z\right) \mathrm{d}z \\ &+ \lambda^2 \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} ||u(s,z)||_2^2\, \theta^2(s,z)\, G(t-s,x-z) G\left(t-s,y-z\right) \mathrm{d}z\,.\end{aligned}$$

Then use the definition of $\Theta(t,x,y)$ in (3.2.2). This proves (3.2.28). Formulas (3.2.24) and (3.2.26) can be derived similarly.

Step 7 (Hölder continuity). In this step, we use the equivalent conditions in Lemma 3.2.15. Since $u(t,x)$ satisfies the integral equation (3.1.1), we denote the stochastic integral part by $I(t,x)$, that is, $u(t,x) = J_0(t,x) + I(t,x)$. Fix $n > 1$ and $v \in \mathbb{R}$. Let $\gamma_i \in]0,1]$, $i = 0,\dots,d$ be given by Assumption 3.2.14. Choose arbitrary two points (t,x) and $\left(t',x'\right) \in K_n$ with $t < t'$, where K_n can either be $[1/n,n] \times [-n,n]^d$ or $[0,n] \times [-n,n]^d$.

By Lemma 2.3.20 and the linear growth condition (1.4.1) of ρ, we have that for all even integers $p > 2$,

$$\begin{aligned}&\left|\left|I(t,x) - I\left(t',x'\right)\right|\right|_p^p \\ &\le 2^{p-1}\mathbb{E}\left(\left|\int_0^t \int_{\mathbb{R}^d} \rho\left(u\left(s,y\right)\right)\theta\left(s,y\right)\left(G\left(t-s,x-y\right) - G(t'-s,x'-y)\right) W(\mathrm{d}s,\mathrm{d}y)\right|^p\right) \\ &\quad + 2^{p-1}\mathbb{E}\left(\left|\int_t^{t'} \int_{\mathbb{R}^d} \rho\left(u\left(s,y\right)\right)\theta\left(s,y\right) G\left(t'-s,x'-y\right) W(\mathrm{d}s,\mathrm{d}y)\right|^p\right) \\ &\le 2^{p-1} z_p^p L_\rho^p \left(L_1(t,t',x,x')\right)^{p/2} + 2^{p-1} z_p^p L_\rho^p \left(L_2(t,t',x,x')\right)^{p/2}\,,\end{aligned}$$

where

$$L_1(t,t',x,x') = \iint_{[0,t]\times\mathbb{R}^d} \left(G\left(t-s,x-y\right) - G(t'-s,x'-y)\right)^2 \left(\overline{\varsigma}^2 + \left|\left|u\left(s,y\right)\right|\right|_p^2\right)\theta^2\left(s,y\right) \mathrm{d}s\mathrm{d}y1$$

and

$$L_2(t,t',x,x') = \iint_{[t,t']\times\mathbb{R}^d} G^2\left(t'-s,x'-y\right)\left(\overline{\varsigma}^2 + \left|\left|u\left(s,y\right)\right|\right|_p^2\right)\theta^2\left(s,y\right) \mathrm{d}s\mathrm{d}y\,.$$

Then by the subadditivity of the function $x \mapsto |x|^{2/p}$, we have

$$\left|\left|I(t,x) - I\left(t',x'\right)\right|\right|_p^2 \le 4 z_p^2 L_\rho^2 \left(L_1(t,t',x,x') + L_2(t,t',x,x')\right)\,.$$

where we have used the fact $2^{2(p-1)/p} \le 4$. We have proved in Step 3 that

$$\left|\left|u\left(s,y\right)\right|\right|_p^2 \le 2\,J_0^2\left(s,y\right) + \left(\left(\overline{\varsigma}^2 + 2\,J_0^2\right) \rhd \widehat{\mathcal{K}}\left(s,y;\cdot,\circ\right)\right)(s,y)\,.$$

We first consider the case $x = x'$. Denote $s = t' - t$. Recall the function $\Upsilon(t;\lambda)$ defined in (3.2.12)). Let

$$\Upsilon_*(t) := a_{p,\overline{\varsigma}}^2 z_p^2\, L_\rho^2\; \Upsilon\left(t\,; a_{p,\overline{\varsigma}} z_p\, L_\rho\right) < +\infty\,, \quad \text{for all } t \in \mathbb{R}_+\,.$$

Clearly, $\Upsilon_*(t) \le \Upsilon_*(n)$ for $t \le n$. By the bound on $\widehat{\mathcal{K}}(t,x)$ in (3.2.13) and Assumption 3.2.14, we have

$$\begin{aligned} L_1(t,t',x,x) \le &\left(\left(\overline{\varsigma}^2 + 2J_0^2\right) \rhd (G(\cdot,\circ) - G(\cdot+s,\circ))^2\right)(t,x) \\ &+ \Upsilon_*(n)\left(\left[\left(\overline{\varsigma}^2 + 2J_0^2\right) \rhd G^2\right] \rhd (G(\cdot,\circ) - G(\cdot+s,\circ))^2\right)(t,x) \\ \le &\left(C_{n,1} + \Upsilon_*(n) C_{n,2}\right)|s|^{\gamma_0}\,, \end{aligned}$$

and

$$\begin{aligned} L_2(t,t',x,x') \le &\iint_{[t,t']\times\mathbb{R}^d} G^2\left(t'-s,x'-y\right)\theta^2\left(s,y\right) \\ &\times \left(\overline{\varsigma}^2 + 2J_0^2\left(s,y\right) + \left(\left(\overline{\varsigma}^2 + 2\,J_0^2\right) \rhd \widehat{\mathcal{K}}\left(s,y;\cdot,\circ\right)\right)(s,y)\right)\mathrm{d}s\mathrm{d}y \\ \le &\left(C_{n,5} + \Upsilon_*(n) C_{n,6}\right)|s|^{\gamma_0}\,. \end{aligned}$$

Hence, for all $x \in [-n,n]^d$ and $1/n \le t < t' \le n$,

$$\left|\left|I(t,x) - I(t',x)\right|\right|_p^2 \le 4 z_p^2\, L_\rho^2\left(C_{n,1} + C_{n,5} + \Upsilon_*(n)\left(C_{n,2} + C_{n,6}\right)\right)\left|t'-t\right|^{\gamma_0}\,.$$

Similarly, for the case where $t = t'$, denote $h = x' - x$. By the bound on $\widehat{\mathcal{K}}(t,x)$ in (3.2.13) and Assumption 3.2.14, we only have the L_1 part and hence,

$$\begin{aligned} \left|\left|I(t,x) - I\left(t,x'\right)\right|\right|_p^2 \le &4 z_p^2\, L_\rho^2\; L_1(t,t,x,x') \\ \le &\;\; 4 z_p^2\, L_\rho^2\left(\left(\overline{\varsigma}^2 + 2J_0^2\right) \rhd (G(\cdot,\circ) - G(\cdot,\circ+h))^2\right)(t,x) \\ &+ 4 z_p^2\, L_\rho^2\, \Upsilon_*(n)\;\left(\left[\left(\overline{\varsigma}^2 + 2J_0^2\right) \rhd G^2\right] \rhd (G(\cdot,\circ) - G(\cdot,\circ+h))^2\right)(t,x) \\ \le &4 z_p^2\, L_\rho^2\,\left[C_{n,3} + \Upsilon_*(n) C_{n,4}\right]\sum_{i=1}^d |h_i|^{\gamma_i}\,. \end{aligned}$$

Finally, combing these two cases gives

$$\begin{aligned} \left|\left|I(t,x) - I\left(t',x'\right)\right|\right|_p^2 &\le 2\left|\left|I(t,x) - I\left(t,x'\right)\right|\right|_p^2 + 2\left|\left|I\left(t,x'\right) - I\left(t',x'\right)\right|\right|_p^2 \\ &\le \widetilde{C}_{p,n}\left(\left|t'-t\right|^{\gamma_0} + \sum_{i=1}^d \left|x_i' - x_i\right|^{\gamma_i}\right)\,, \end{aligned}$$

where
$$\widetilde{C}_{p,n} = 8z_p^2 L_\rho^2 \left(C_{n,1} + C_{n,3} + C_{n,5} + \Upsilon_*(n)\left(C_{n,2} + C_{n,4} + C_{n,6}\right)\right).$$

Then the Hölder continuity is proved by an application of Kolmogorov's continuity theorem (see Proposition 2.6.4). In particular, if $K_n = [1/n, n] \times [-n, n]^d$, then

$$I(t,x) \in C_{\frac{\gamma_0}{2}-,\frac{\gamma_1}{2}-\dots,\frac{\gamma_d}{2}-}\left(\mathbb{R}_+^* \times \mathbb{R}^d\right), \text{ a.s.;}$$

otherwise, if $K_n = [0, n] \times [-n, n]^d$, then

$$I(t,x) \in C_{\frac{\gamma_0}{2}-,\frac{\gamma_1}{2}-\dots,\frac{\gamma_d}{2}-}\left(\mathbb{R}_+ \times \mathbb{R}^d\right), \text{ a.s.}$$

This completes the whole proof of Theorem 3.2.16. □

3.4 Proof of the Application Theorem 3.2.17

3.4.1 A Technical Proposition on Initial Data

Proposition 3.4.1. *Suppose that $\theta(t,x) \in \Xi_r$ and $\mu \in \mathscr{D}_k'(\mathbb{R})$ with $0 \le k < r + 1/4$. Then*

$$\sup_{(t,x)\in K} \left(\left[v^2 + J_0^2\right] \rhd G_\nu^2\right)(t,x) < +\infty, \quad \textit{for all compact sets } K \subseteq \mathbb{R}_+^* \times \mathbb{R},$$

where $J_0(t,x)$ is defined in (2.6.14).

Proof. Since for some constant C, $|\theta(t,x)| \le C(1 \wedge t^r) \le C t^r$, we can simply take $\theta(t,x) = t^r$. Assume $v = 0$. So we need to prove that

$$f(t,x) := \iint_{[0,t]\times\mathbb{R}} J_0^2(s,y)\, s^{2r} G_\nu^2(t-s, x-y)\, \mathrm{d}s\mathrm{d}y < +\infty, \quad \text{for all } (t,x) \in \mathbb{R}_+^* \times \mathbb{R}.$$

From (2.6.14), we have

$$J_0^2(s,y) \le (\nu s)^{-2k} \left(|\mu_0| * \left[|\mathrm{He}|_k(\cdot;\nu s) G_\nu(s,\cdot)\right]\right)^2(y).$$

Without loss of generality, we assume from now that μ_0 is a non-negative measure. Replace the upper bound of $J_0^2(s,y)$ by the following double integral

$$(\nu s)^{-2k} \iint_{\mathbb{R}^2} G_\nu(s, y - z_1)\, G_\nu(s, y - z_2)\, |\mathrm{He}|_k(y - z_1; \nu s)\, |\mathrm{He}|_k(y - z_2; \nu s)\, \mu_0(\mathrm{d}z_1)\mu_0(\mathrm{d}z_2),$$

and then apply Lemma 2.3.7. So

$$\begin{aligned} \left|f(t,x)\right| \le \int_0^t \mathrm{d}s \frac{s^{2r}}{\nu^{2k} s^{2k} \sqrt{4\pi\nu(t-s)}} \iint_{\mathbb{R}^2} \mu_0(\mathrm{d}z_1)\mu_0(\mathrm{d}z_2)\, G_{2\nu}(s, z_1 - z_2) \\ \times \int_{\mathbb{R}} G_{\nu/2}(s, y - \bar{z})\, G_{\nu/2}(t-s, x-y)\, |\mathrm{He}|_k(y - z_1; \nu s)\, |\mathrm{He}|_k(y - z_2; \nu s)\, \mathrm{d}y, \end{aligned}$$

where $\bar{z} = \frac{z_1+z_2}{2}$. Use $\int G\,G\,|\mathrm{He}|_k\,|\mathrm{He}|_k\,\mathrm{d}y$ to denote the above $\mathrm{d}y$-integral. Notice that by Lemma 2.3.7,

$$G_{\nu/2}\left(s, y-\bar{z}\right) G_{\nu/2}\left(t-s, x-y\right) = G_{\nu/2}(t, x-\bar{z})\, G_{\nu/2}\left(\frac{s(t-s)}{t}, y - \frac{(t-s)\bar{z}+sx}{t}\right).$$

So

$$\int G\,G\,|\mathrm{He}|_k\,|\mathrm{He}|_k\,\mathrm{d}y = G_{\nu/2}(t, x-\bar{z}) \int_{\mathbb{R}} G_{\nu/2}\left(\frac{s(t-s)}{t}, y - \frac{(t-s)\bar{z}+sx}{t}\right)$$
$$\times |\mathrm{He}|_k\left(y-z_1;\nu s\right) |\mathrm{He}|_k\left(y-z_2;\nu s\right) \mathrm{d}y\,.$$

In order to integrate over y, we change the variable: $u = y - \frac{(t-s)\bar{z}+sx}{t}$ and the integrand becomes

$$G_{\nu/2}\left(\frac{s(t-s)}{t}, u\right) |\mathrm{He}|_k\left(u + \frac{t-s}{2t}z_2 - \frac{t+s}{2t}z_1 + \frac{s}{t}x\,;\nu s\right) |\mathrm{He}|_k\left(u + \frac{t-s}{2t}z_1 - \frac{t+s}{2t}z_2 + \frac{s}{t}x\,;\nu s\right).$$

Using the absolute moment of the Gaussian distribution (see, e.g., [55, p. 23])

$$\int_{\mathbb{R}} G_{\nu/2}(t,x)|x|^n \mathrm{d}x = (\nu t)^{n/2} \frac{2^{n/2}\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi}}\,,$$

we have that for some constant $C_k > 0$,

$$\int_{\mathbb{R}} G_{\nu/2}(t,x)|x|^n \mathrm{d}x \le \left(C_k\sqrt{2t}\right)^n, \quad \text{for all } 0 \le n \le 2k\,,$$

where we can choose the constant C_k to be

$$C_k = \max_{0\le n\le 2k} \sqrt{\nu} \left(\frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi}}\right)^{1/n}.$$

Hence, for any polynomial of order less than $2k$ with nonnegative coefficients, say, $f(x) = \sum_{i=0}^{2k} a_i x^i$ with $a_i \ge 0$, we have that

$$\int_{\mathbb{R}} G_{\nu/2}(t,x)\left|f(x)\right| \mathrm{d}x \le \sum_{i=0}^{2k} a_i \int_{\mathbb{R}} G_{\nu/2}(t,x)|x|^i \mathrm{d}x \le \sum_{i=0}^{2k} a_i \left(C_k\sqrt{2t}\right)^i = f\left(C_k\sqrt{2t}\right). \quad (3.4.1)$$

Notice that

$$|\mathrm{He}|_k\,(u+\ldots;\nu s)\ |\mathrm{He}|_k\,(u+\cdots;\nu s) \le |\mathrm{He}|_k\,(u+|\ldots|;\nu s)\ |\mathrm{He}|_k\,(u+|\cdots|;\nu s)\,,$$

where the highest power of the right-hand side is less than or equal to $2k$, and all its coefficients are nonnegative. Therefore, we can apply the relation (3.4.1) to obtain the follwing bound

$$\int G\,G\,|\mathrm{He}|_k\,|\mathrm{He}|_k\,\mathrm{d}y \le G_{\nu/2}(t,x-\bar{z})\,|\mathrm{He}|_k\left(C_k\sqrt{\frac{2s(t-s)}{t}}+\left|\frac{t-s}{2t}z_2-\frac{t+s}{2t}z_1+\frac{s}{t}x\right|;\nu s\right)$$
$$\times|\mathrm{He}|_k\left(C_k\sqrt{\frac{2s(t-s)}{t}}+\left|\frac{t-s}{2t}z_1-\frac{t+s}{2t}z_2+\frac{s}{t}x\right|;\nu s\right).$$

Clearly, for $s\in[0,t]$, $\frac{2s(t-s)}{t}\le t$ where the maximum is achieved at $s=t/2$. Since $|\mathrm{He}|_k\,(x;t)$ is monotone increasing in both $|x|$ and t, we have

$$\int G\,G\,|\mathrm{He}|_k\,|\mathrm{He}|_k\,\mathrm{d}y \le G_{\nu/2}(t,x-\bar{z})\,|\mathrm{He}|_k^2\left(C_k\sqrt{t}+|z_2|+|z_1|+|x|\,;\nu t\right).$$

Notice that by the inequality $a+b\le(a+1)(b+1)$ for $a,b\ge 0$, we have

$$\begin{aligned}|\mathrm{He}|_k^2\left(C_k\sqrt{t}+|z_2|+|z_1|+|x|\,;\nu t\right) &\le \sum_{i=0}^{\lfloor k/2\rfloor} a_i^2\,(\nu t)^{2i}\left(C_k\sqrt{t}+|z_2|+|z_1|+|x|\right)^{2k-4i}\\ &\le \sum_{i=0}^{\lfloor k/2\rfloor} a_i^2\,(\nu t)^{2i}\,(|z_1|+r(t,x))^{2k-4i}\,(|z_2|+r(t,x))^{2k-4i}\end{aligned}$$

where $r(t,x)=\left(C_k\sqrt{t}+|x|\right)/2+1$ and $a_i=\sqrt{2}\binom{k}{i}(2i-1)!!$, and by Lemma 2.3.8,

$$G_{2\nu}\,(s,z_1-z_2)\,G_{\nu/2}(t,x-\bar{z})\le 2\frac{\sqrt{t}}{\sqrt{s}}G_{2\nu}(t,x-z_1)G_{2\nu}(t,x-z_2).$$

Then by the non-negativity of μ_0, we have

$$\left|f(t,x)\right|\le g(t,x)\int_0^t \frac{s^{2r-2k-1/2}\sqrt{t}}{\nu^{2k}\sqrt{\pi\nu(t-s)}}\mathrm{d}s$$

where

$$g(t,x):=\sum_{i=0}^{\lfloor k/2\rfloor} a_i^2(\nu t)^{2i}\left(\mu_0 * G_{2\nu}(t,\cdot)P_{k,i}(\cdot;t,x)\right)^2(x)$$

with $P_{k,i}(z;t,x):=(|z|+|x|+r(t,x))^{2k-4i}$. Clearly, since $\mu_0\in\mathcal{M}_H(\mathbb{R})$, $g(t,x)<+\infty$ for all $(t,x)\in\mathbb{R}_+^*\times\mathbb{R}$. The integration over s is finite since $2r-2k-1/2>-1$. In particular, using the Beta integral (see (2.3.5)), we have that

$$\int_0^t \frac{s^{2r-2k-1/2}\sqrt{t}}{\nu^{2k}\sqrt{\pi\nu(t-s)}}\mathrm{d}s=\nu^{-2k-1/2}\frac{\Gamma(2r-2k+1/2)}{\Gamma(2r-2k+1)}\;t^{2r-2k+1/2},$$

where the power of t is positive: $2r-2k+1/2>0$. As for the contribution of v, we simply replace k by 0 and $\mu_0(\mathrm{d}x)$ by $v\mathrm{d}x$ in the above arguments. We will not repeat them here.

Finally, take an arbitrary compact set $K\subseteq\mathbb{R}_+^*\times\mathbb{R}$. We only need to show that

$$\sup_{(t,x)\in K} g(t,x)<+\infty.$$

By expanding each of the polynomials $P_{k,i}(z;,t,x)$ and applying the bound in (2.6.10), one can see that

$$g(t,x) \le \widehat{P}_k\left(\sqrt{t},|x|\right)\,\left(|\mu| * G_{4\nu}(t,\cdot)\right)(x)$$

for some polynomial $\widehat{P}_k(x,y)$ of two variables. The supremum of $\widehat{P}_k\left(\sqrt{t},x\right)$ over K is clearly finite. The supremum of $\left(|\mu| * G_{4\nu}(t,\cdot)\right)(x)$ over K is also finite thanks to the smoothing effect of the heat kernel; see Lemma 2.3.5. This completes the whole proof. □

3.4.2 Proof of Theorem 3.2.17

We only need to verify that the assumptions Cond(G) and Cond(H) of Theorem 3.2.16 are satisfied.

We first remark that the Lipschitz continuity of ρ implies the linear growth of the following form:

$$|\rho(u)|^2 \le L_\rho^2\left(\overline{\varsigma}^2 + u^2\right),$$

for some $\overline{\varsigma} > 0$ and $L_\rho > 0$. See Remark 1.4.1. Now fix $r \in [0,+\infty]$ and $\theta(t,x) \in \Xi_r$. By the definition of Ξ_r, for some constant $C > 0$, we have

$$\sup_{(t,x)\in\mathbb{R}_+\times\mathbb{R}} |\theta(t,x)| \le C.$$

Hence, the θ-weighted space-time convolution is bounded by C^2 times the normal space-time convolution:

$$\left(f \rhd g\right)(t,x) \le C^2\left(f \star g\right)(t,x).$$

Therefore, Assumption 3.2.3 is satisfied with

$$\Theta(t,x,x) \le C^2 \iint_{[0,t]\times\mathbb{R}} G_\nu^2(t-s,x-z)\mathrm{d}s\mathrm{d}z = C^2\frac{\sqrt{t}}{\sqrt{\pi\nu}} < +\infty, \tag{3.4.2}$$

for all $(t,x) \in \mathbb{R}_+\times\mathbb{R}$. Assumptions 3.2.7 and 3.2.8 are verified by Proposition 2.3.1 with $\lambda = C\,L_\rho$. Assumption 3.2.12 is true due to Proposition 3.4.1. Therefore, all conditions in Cond(G) are satisfied.

Both Assumptions 3.2.10 and 3.2.11 are satisfied due to Proposition 2.3.12 and Corollary 2.3.10, respectively. Assumption 3.2.13 is true by Lemma 2.6.14. Therefore, all conditions in Cond(H) are satisfied. This completes the whole proof. □

4 The One-Dimensional Nonlinear Stochastic Wave Equation

4.1 Introduction

In this chapter, we study the following nonlinear stochastic wave equation

$$\begin{cases} \left(\dfrac{\partial^2}{\partial t^2} - \kappa^2 \dfrac{\partial^2}{\partial x^2}\right) u(t,x) = \rho(u(t,x))\, \dot{W}(t,x), & x \in \mathbb{R},\ t \in \mathbb{R}_+^*\,, \\ u(0,\cdot) = g(\cdot)\,, \quad \dfrac{\partial u}{\partial t}(0,\cdot) = \mu(\cdot)\,, \end{cases} \tag{4.1.1}$$

where $\dot{W}$ is space-time white noise, ρ is Lipschitz continuous, and $g(\cdot)$ and μ are initial position and initial velocity, respectively. Our main contributions are as follows:

(1) A random field solution to (4.1.1) (in the sense of Definition 3.2.1 where (4.1.1) is recast in the integral form) exists for all initial position $g \in L^2_{loc}(\mathbb{R})$ and initial velocity $\mu \in \mathcal{M}(\mathbb{R})$ (i.e, locally finite and signed Borel measure on $\mathbb{R}$). The sample path regularity depends on the local integrability of the initial position g, not on the initial velocity μ;

(2) We derive sharp estimates for the moments $\mathbb{E}[|u(t,x)|^p]$ of the solution with both t and x fixed. For the hyperbolic Anderson model, these estimates become an explicit formula for the second moment;

(3) We obtain nontrivial bounds for the exponential growth indices.

The main results and some examples are presented in Section 4.2. Theorem 4.2.1 states the first main result about the existence, uniqueness, moment estimates, two-point correlations, and sample path regularity of the random field solution. The second result, the full intermittency of the wave equation, is stated in Theorem 4.2.8. The third one – Theorem 4.2.11 – states the estimates of the exponential growth indices. Before proving these theorems, we first prepare some results in Section 4.3. The complete proofs of these three theorems, as well as some propositions and corollaries, are given in Section 4.4

4.2 Main Results

4.2.1 Notation and Conventions

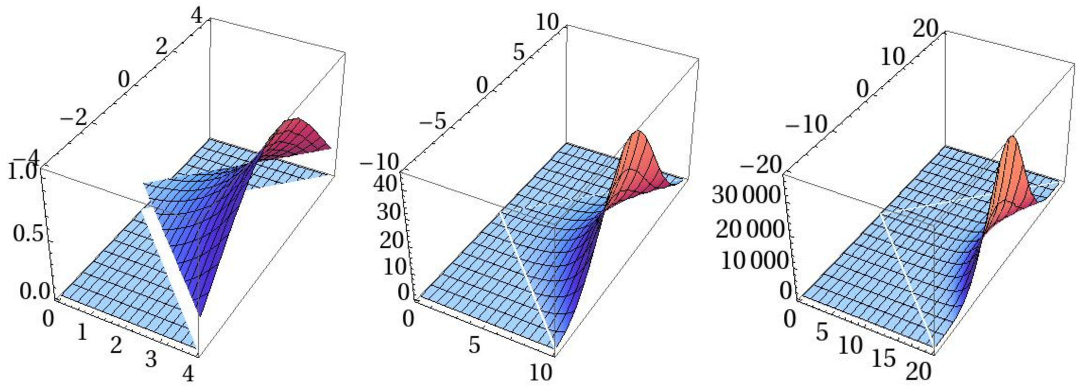
Define a kernel function

$$\mathscr{K}(t,x;\kappa,\lambda) := \begin{cases} \frac{\lambda^2}{4} I_0\left(\sqrt{\frac{\lambda^2\left((\kappa t)^2 - x^2\right)}{2\kappa}}\right) & \text{if } -\kappa t \le x \le \kappa t\,, \\ 0 & \text{otherwise}\,, \end{cases} \tag{4.2.1}$$

with two parameters $\kappa > 0$ and $\lambda > 0$, where $I_n(\cdot)$ is the modified Bessel function of the first kind of order n, or simply *hyperbolic Bessel function* ([51, 10.25.2, on p. 249])

$$I_n(x) := \left(\frac{x}{2}\right)^n \sum_{k=0}^{\infty} \frac{\left(x^2/4\right)^k}{k!\,\Gamma(n+k+1)}\,. \tag{4.2.2}$$

See [69, p. 204] and [41, Section 3.7, p. 212] for its relation with the wave equation. See Figure 4.1 for some graphs of this kernel function.



(a) t up to 4

(b) t up to 10

(c) t up to 20

Figure 4.1 – The kernel function $\mathscr{K}(t,x)$ defined in (4.2.1) with $\lambda = \kappa = 1$.

Define

$$\mathscr{H}(t;\kappa,\lambda) := (1 \star \mathscr{K})(t,x) = \cosh\left(|\lambda|\sqrt{\kappa/2}\,t\right) - 1\,, \tag{4.2.3}$$

where the second equality is proved in Lemma 4.3.3 below. We use the following conventions:

$$\begin{aligned} \mathscr{K}(t,x) &:= \mathscr{K}(t,x;\kappa,\lambda)\,, \\ \overline{\mathscr{K}}(t,x) &:= \mathscr{K}\left(t,x;\kappa,L_\rho\right)\,, \\ \underline{\mathscr{K}}(t,x) &:= \mathscr{K}\left(t,x;\kappa,l_\rho\right)\,, \\ \widehat{\mathscr{K}}_p(t,x) &:= \mathscr{K}\left(t,x;\kappa,a_{p,\bar{\varsigma}}\,z_p\,L_\rho\right)\,, \quad \text{for all } p \ge 2\,, \end{aligned}$$

where z_p is the optimal universal constant in the Burkholder-Davis-Gundy inequality (see Theorem 2.3.18) and $a_{p,\overline{\varsigma}}$ is defined in (1.4.4). Note that the kernel function $\widehat{\mathscr{K}}_p(t,x)$ depends on the parameter $\overline{\varsigma}$, which is usually clear from the context. Similarly, we define $\overline{\mathscr{H}}(t)$, $\underline{\mathscr{H}}(t)$ and $\widehat{\mathscr{H}}_p(t)$.

Define two functions:

$$T_\kappa(t,x) := \left(t - \frac{|x|}{2\kappa}\right) 1_{\{|x|\le 2\kappa t\}}\,, \tag{4.2.4}$$

$$\begin{aligned}\Theta_\kappa\left(t,x,y\right) &:= \iint_{\mathbb{R}_+\times\mathbb{R}} G_\kappa(t-s,x-z)G_\kappa\left(t-s,y-z\right)\mathrm{d}s\mathrm{d}z\\ &= \frac{\kappa}{4}T_\kappa^2\left(t,x-y\right),\end{aligned} \tag{4.2.5}$$

where the equality in (4.2.5) is proved in Lemma 4.3.4. Note that the function $\Theta_\kappa\left(t,x,y\right)$ is the realization of the function $\Theta\left(t,x,y\right)$ used in Chapter 3; see (3.2.2). It is evaluated in Lemma 4.3.4 below. We will work under the filtered probability space $\{\Omega,\mathscr{F},\{\mathscr{F}_t, t\ge 0\},P\}$ as specified in Chapter 2.

4.2.2 Existence, Uniqueness, Moments and Regularity

Recall the definition of the random field solution in Definition 3.2.1.

Theorem 4.2.1. *Suppose that*

(i) the function ρ is Lipschitz continuous with $|\rho(u)|^2 \le L_\rho^2\left(\overline{\varsigma}^2+u^2\right)$;

(ii) the initial data are such that $g(x)\in L^2_{loc}(\mathbb{R})$ and $\mu\in\mathscr{M}(\mathbb{R})$.

Then the stochastic integral equation (4.1.1) *has a random field solution, in the sense of Definition 3.2.1,*

$$\left\{u(t,x) = J_0(t,x) + I(t,x) : t>0, x\in\mathbb{R}\right\}$$

which consists of a deterministic part $J_0(t,x)$ given in (1.3.5) *and a stochastic integral part $I(t,x)$. This solution $u(t,x)$ has the following properties:*

(1) $u(t,x)$ is unique (in the sense of versions);

(2) $(t,x)\mapsto I(t,x)$ is $L^p(\Omega)$-continuous for all integers $p\ge 2$;

(3) For all even integers $p\ge 2$, the p-th moment of the solution $u(t,x)$ satisfies the upper bound

$$||u(t,x)||_p^2 \le \begin{cases} J_0^2(t,x) + \left(J_0^2\star\overline{\mathscr{K}}\right)(t,x) + \overline{\varsigma}^2\,\overline{\mathscr{H}}(t) & \text{if } p=2,\\ \\ 2J_0^2(t,x) + \left(2J_0^2\star\widehat{\mathscr{K}}_p\right)(t,x) + \overline{\varsigma}^2\,\widehat{\mathscr{H}}_p(t) & \text{if } p>2,\end{cases} \tag{4.2.6}$$

for all $t > 0$, $x \in \mathbb{R}$. *And the two-point correlation satisfies the upper bound*

$$\mathbb{E}\left[u(t,x)u(t,y)\right] \le J_0(t,x)J_0(t,y) + L_\rho^2\, \overline{\varsigma}^2\, \Theta_\kappa\left(t,x,y\right) + \frac{L_\rho^2}{2}\left(f \star G_\kappa\right)\left(T_\kappa\left(t,x-y\right), \frac{x+y}{2}\right), \tag{4.2.7}$$

for all $t > 0$, $x, y \in \mathbb{R}$, *where* $f(s,z)$ *denotes the right hand side of* (4.2.6) *for* $p = 2$*;*

(4) *If* ρ *satisfies* (1.4.2)*, then the second moment satisfies the lower bound*

$$||u(t,x)||_2^2 \ge J_0^2(t,x) + \left(J_0^2 \star \underline{\mathcal{K}}\right)(t,x) + \underline{\varsigma}^2\, \underline{\mathcal{H}}(t) \tag{4.2.8}$$

for all $t > 0$, $x \in \mathbb{R}$. *And the two-point correlation satisfies the lower bound*

$$\mathbb{E}\left[u(t,x)u(t,y)\right] \ge J_0(t,x)J_0(t,y) + l_\rho^2\, \underline{\varsigma}^2\, \Theta_\kappa\left(t,x,y\right) + \frac{l_\rho^2}{2}\left(f \star G_\kappa\right)\left(T_\kappa\left(t,x-y\right), \frac{x+y}{2}\right), \tag{4.2.9}$$

for all $t > 0$, $x, y \in \mathbb{R}$, *where* $f(s,z)$ *denotes the right hand side of* (4.2.8)*;*

(5) *In particular, for the quasi-linear case* $|\rho(u)|^2 = \lambda^2\left(\varsigma^2 + u^2\right)$*, the second moment has an explicit expression:*

$$||u(t,x)||_2^2 = J_0^2(t,x) + \left(J_0^2 \star \mathcal{K}\right)(t,x) + \varsigma^2\, \mathcal{H}(t)\,, \tag{4.2.10}$$

for all $t > 0$, $x \in \mathbb{R}$. *And the two-point correlation is given by*

$$\mathbb{E}\left[u(t,x)u(t,y)\right] = J_0(t,x)J_0(t,y) + \lambda^2 \varsigma^2\, \Theta_\kappa\left(t,x,y\right) + \frac{\lambda^2}{2}\left(f \star G_\kappa\right)\left(T_\kappa\left(t,x-y\right), \frac{x+y}{2}\right), \tag{4.2.11}$$

for all $t > 0$, $x, y \in \mathbb{R}$, *where* $f(s,z) = ||u(s,z)||_2^2$ *is defined in* (4.2.10)*;*

(6) *If* $g \in L^{2p}_{loc}(\mathbb{R})$ *with* $p \ge 1$ *and* $\mu \in \mathcal{M}(\mathbb{R})$*, then the stochastic integral part* $I(t,x)$ *is almost surely Hölder continuous:*

$$I(t,x) \in C_{\frac{1}{2p'}-,\frac{1}{2p'}-}\left(\mathbb{R}_+ \times \mathbb{R}\right), \quad a.s., \quad \frac{1}{p} + \frac{1}{p'} = 1;$$

In particular, if g *is a bounded Borel measurable function (*$p = +\infty$*), then*

$$I(t,x) \in C_{\frac{1}{2}-,\frac{1}{2}-}\left(\mathbb{R}_+ \times \mathbb{R}\right), \quad a.s.$$

The proofs of this theorem, as well as the following two corollaries, are presented in Section 4.4.1.

Corollary 4.2.2 (Constant initial data)**.** *Suppose that* $\rho^2(x) = \lambda^2(\varsigma^2 + x^2)$ *with* $\lambda \ne 0$*. If both the initial position and initial velocity are homogeneous, that is,* $g(x) \equiv w$ *and*

$\mu(\mathrm{d}x) = \widetilde{w}\mathrm{d}x$, *then we have:*

(1) The second moment has the following explicit form

$$||u(t,x)||_2^2 = w^2 + \left(w^2 + \varsigma^2 + \frac{4\kappa\widetilde{w}^2}{\lambda^2}\right)\mathscr{H}(t) + \frac{2\sqrt{2\kappa}\, w\widetilde{w}}{|\lambda|}\sinh\left(\frac{\sqrt{\kappa}|\lambda|t}{\sqrt{2}}\right).$$

for all $t \geq 0$ and $x \in \mathbb{R}$. In particular,

$$||u(t,x)||_2^2 = \begin{cases} w^2\left(\mathscr{H}(t)+1\right) & \text{if } \varsigma = \widetilde{w} = 0, \\ \dfrac{4\kappa\widetilde{w}^2}{\lambda^2}\mathscr{H}(t) & \text{if } \varsigma = w = 0. \end{cases}$$

(2) The two-point correlation function has the following explicit form

$$\begin{aligned}\mathbb{E}\left[u(t,x)u(t,y)\right] &= w^2 + \kappa\widetilde{w}\left(t - T_\kappa\left(t,x-y\right)\right)\left(2w + \kappa\widetilde{w}(t + T_\kappa\left(t,x-y\right))\right) \\ &+ \left(w^2 + \varsigma^2 + \frac{4\kappa\widetilde{w}^2}{\lambda^2}\right)\mathscr{H}\left(T_\kappa\left(t,x-y\right)\right) + \frac{2\sqrt{2\kappa}\, w\widetilde{w}}{|\lambda|}\sinh\left(\frac{\sqrt{\kappa}|\lambda|}{\sqrt{2}}T_\kappa\left(t,x-y\right)\right),\end{aligned}$$

for all $t \geq 0$ and $x, y \in \mathbb{R}$, where $T_\kappa(t,x)$ is defined in (4.2.4). *In particular,*

$$\mathbb{E}\left[u(t,x)u(t,y)\right] = \begin{cases} w^2\left(\mathscr{H}\left(T_\kappa\left(t,x-y\right)\right)+1\right) & \text{if } \varsigma = \widetilde{w} = 0, \\ \dfrac{4\kappa\widetilde{w}^2}{\lambda^2}\mathscr{H}\left(T_\kappa\left(t,x-y\right)\right) + \kappa^2\widetilde{w}^2\left(t^2 - T_\kappa^2\left(t,x-y\right)\right) & \text{if } \varsigma = w = 0. \end{cases}$$

Corollary 4.2.3 (Dirac delta initial velocity)**.** *Suppose that $\rho^2(x) = \lambda^2(\varsigma^2 + x^2)$ with $\lambda \neq 0$. If $g \equiv 0$ and $\mu = \delta_0$, then we have:*

(1) The second moment has the following explicit form

$$||u(t,x)||_2^2 = \frac{1}{\lambda^2}\mathscr{K}(t,x) + \varsigma^2\mathscr{H}(t), \quad \text{for all } t \geq 0 \text{ and } x \in \mathbb{R}.$$

(2) The two-point correlation function has the following explicit form

$$\mathbb{E}\left[u(t,x)u\left(t,y\right)\right] = \frac{1}{\lambda^2}\mathscr{K}\left(T_\kappa\left(t,x-y\right), \frac{x+y}{2}\right) + \varsigma^2\mathscr{H}\left(T_\kappa\left(t,x-y\right)\right),$$

for all $t \geq 0$ and $x, y \in \mathbb{R}$.

Example 4.2.4. Let $g(x) = |x|^{-1/4}$ and $\mu \equiv 0$. Clearly, $g \in L^2_{loc}(\mathbb{R})$. In this case,

$$J_0^2(t,x) = \frac{1}{4}\left(\frac{1}{|x+\kappa t|^{1/4}} + \frac{1}{|x-\kappa t|^{1/4}}\right)^2,$$

which is not well defined at the points when $x = \pm\kappa t$. Nevertheless, the stochastic integral part $I(t,x)$ is well defined for all $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}$ and the random field solution $u(t,x)$ in the sense of Definition 3.2.1 does exist according to Theorem 4.2.1. We have the following two comments:

(1) The argument for the heat equation in Theorem 3.2.16, which is based on Cond(H) (in particular, Assumption 3.2.13), is impossible because of the explosion of $J_0(t,x)$ at certain points. However, the wave kernel has a better property (Cond(W), or Assumption 3.2.9) than the heat case (Assumption 3.2.10).

(2) Due to the singularity of $J_0(t,x)$ along the characteristic lines $x = \pm\kappa t$, the random field solution $u(t,x)$ equals infinity along these two characteristic lines. This phenomenon is the propagation of certain singularities, which is $\mathbb{E}\left[|u(t,x)|^2\right] = +\infty$ in the current case. Note that Carmona and Nualart proved in [10] propagation of another singularity, namely, a failure of the law of the iterated logarithm.

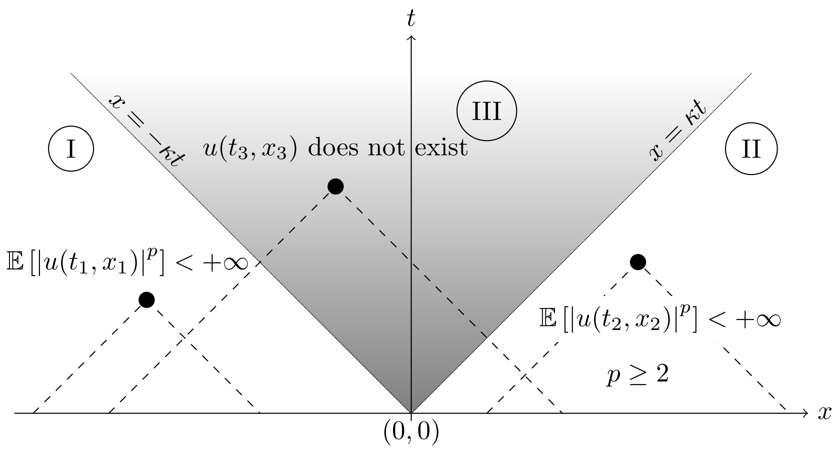


Figure 4.2 – When $g(x) = |x|^{-1/2}$ and $\mu \equiv 0$, there is a random field solution in Regions I and II, but not in Region III.

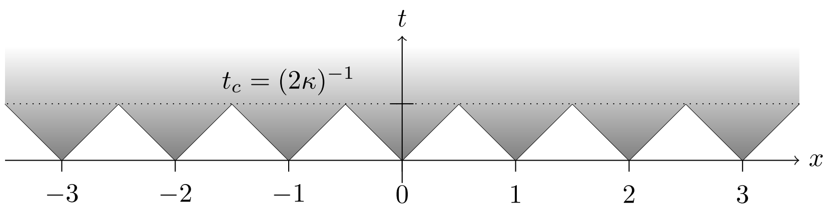


Figure 4.3 – When $g(x) = \sum_{n\in\mathbb{N}} 2^{-n}\left(|x-n|^{-1/2} + |x+n|^{-1/2}\right)$ and $\mu \equiv 0$, the random field solution $u(t,x)$ is only defined in the unshaded regions and in particular only for $t < t_c = (2\kappa)^{-1}$.

Example 4.2.5. Let $g(x) = |x|^{-1/2}$ and $\mu \equiv 0$. Clearly, $g \notin L^2_{loc}(\mathbb{R})$. So Theorem 4.2.1 does not apply. In this case, the solution $u(t,x)$ is well defined outside of the space-time cone – Regions I and II in Figure 4.2. But because

$$J_0^2(t,x) = \frac{1}{4}\left(\frac{1}{|x+\kappa t|^{1/2}} + \frac{1}{|x-\kappa t|^{1/2}}\right)^2$$

is not locally integrable when the characteristic lines $x = \pm\kappa t$ are in the integral domain (see (1.3.8)), the stochastic integral part $I(t,x)$ cannot have finite p-th moments for any $p\geq 2$. Therefore, a random field solution $u(t,x)$ in the sense of Definition 3.2.1 does not exist for all (t,x) inside the space-time cone $|x|\leq \kappa t$ — the shaded region in Figure 4.2. Although $u(t,x)$ does not exist globally, it is still well defined locally (possibly only for finite time) at places where the initial data is relatively regular; see another example in Figure 4.3.

Proposition 4.2.6. *Suppose that* $\left|\rho(u)\right|^2=\lambda^2\left(\varsigma^2+u^2\right)$. *If the initial position* $g(x)=|x|^{-a}$ *with* $a\in[0,1/2[$ *and initial velocity vanishes* $\mu\equiv 0$, *then in the neighborhood of the two characteristic lines* $|x|=\kappa t$, *the stochastic integral part* $I(t,x)$ *of the random field solution, viewed as a function from* $\mathbb{R}_+\times\mathbb{R}$ *to* $L^p(\Omega)$ *for all* $p\geq 2$, *cannot be* ρ_1*-Hölder continuous in space or* ρ_2*-Hölder continuous in time with* $\rho_i>\frac{1-2a}{2}$, $i=1,2$.

This proposition is proved in Section 4.4.2.

Remark 4.2.7 (Optimal $L^p(\Omega)$-Hölder continuity)**.** Clearly, $|x|^{-a}\in L^{2p}_{loc}(\mathbb{R})$ if and only if $2pa<1$, i.e., $p<(2a)^{-1}$. Hence, p', the dual of p, is strictly bigger than $(1-2a)^{-1}$. Therefore, in the proof of Theorem 4.2.1 (6), we show that, for all $p\geq 2$, the function

$$I:\mathbb{R}_+\times\mathbb{R}\mapsto L^p(\Omega)$$

is jointly η-Hölder continuous with $\eta=(1-2a)/2$. For example, if $a=1/4$ (see Example 4.2.4), then I is jointly $1/4$-Hölder continuous in $L^p(\Omega)$. Proposition 4.2.6 then shows that $I(t,x)$ cannot be jointly η-Hölder continuous with $\eta>1/4$. Hence, the estimates on the joint $L^p(\Omega)$-Hölder continuity are optimal. Unlike the stochastic heat equation, the wave kernel does not have a smoothing effect and the singularities propagate along the characteristics.

4.2.3 Full Intermittency

Recall that $u(t,x)$ is said to be *fully intermittent* if the lower Lyapunov exponent of order 2 is strictly positive: $\underline{\lambda}_2>0$; see Definition 1.1.1.

Theorem 4.2.8. *(Full intermittency) Suppose that for some constants* $w,\widetilde{w}\in\mathbb{R}$, *the initial data are* $g(x)\equiv w$ *and* $\mu(\mathrm{d}x)=\widetilde{w}\mathrm{d}x$. *Assume that* $|\rho(u)|^2\leq L_\rho^2(\overline{\varsigma}^2+u^2)$. *Then we have the following properties:*

(1) the upper Lyapunov exponents are bounded by

$$\frac{\overline{\lambda}_p}{p}\leq\sqrt{2\kappa}\;L_\rho\sqrt{p}\,,$$

for all even integers $p\geq 2$*;*

(2) if $|\rho(u)|^2\geq l_\rho^2(\underline{\varsigma}^2+u^2)$ *for some* $l_\rho\neq 0$ *and* $|\underline{\varsigma}|+|w|+|\widetilde{w}|\neq 0$, *then the lower Lyapunov*

exponent of order 2 *is bounded from below by*

$$\frac{\lambda_2}{2} \geq \frac{\sqrt{2\kappa}\,|\,l_\rho\,|}{4}$$

and so $u(t,x)$ *is fully intermittent.*

See Section 4.4.3 for its proof.

Remark 4.2.9. In order to get the growth rate of the Lyapunov exponents λ_p with respect to p, we still need to prove that there exists a constant C such that

$$\frac{\lambda_p}{p} \geq C\sqrt{p}\,, \quad \text{for all } p \geq 2 \text{ even integers}\,.$$

This part is not proved here because we could get the lower bound only for the second moment of the solution thanks to the Itô isometry. Upper bounds of the higher moments are derived by the Burkholder-Davis-Gundy inequality (see Theorem 2.3.18). Dalang and Mueller [30] derived the lower bound for the stochastic wave and heat equations in $\mathbb{R}_+ \times \mathbb{R}^3$ in the case where $\rho(u) = \lambda u$ and the driving noise is spatially colored. An essential tool in their paper is a Feynman-Kac-type formula that they (with Tribe) obtained in [31]. In [13], we obtain similar Feynman-Kac-type formulas for both stochastic heat and wave equations in $\mathbb{R}_+ \times \mathbb{R}$ driven by space-time white noise (with $\rho(u) = \lambda u$).

4.2.4 Exponential Growth Indices

Recall that $\mathcal{M}_G^{\beta}(\mathbb{R})$ with $\beta > 0$ is the set of locally finite Borel measures with exponential tails (see (2.2.10)).

Remark 4.2.10. Before stating the following theorem, we remark that since the kernel function $\mathcal{K}(t,x)$ has support in the same space-time cone as the fundamental solution $G_\kappa(t,x)$, it is clear that if the initial data have compact support, then the solution including the high peaks must propagate in the space-time cone with the same speed κ. Hence $\underline{\lambda}(p) \leq \overline{\lambda}(p) \leq \kappa$. Conus and Khoshnevisan showed in [19, Theorem 5.1] that with some other mild conditions on the initial data, $\underline{\lambda}(p) = \overline{\lambda}(p) = \kappa$ for all $p \geq 2$.

Theorem 4.2.11. *The following bounds hold:*

(1) Suppose that $|\rho(u)| \leq L_\rho |u|$ *with* $L_\rho \neq 0$ *and the initial data satisfy the following two conditions:*

(a) The initial position $g(x)$ *is a Borel measurable function such that* $|g(x)|$ *is bounded from above by some function* $ce^{-\beta_1|x|}$ *with* $c > 0$ *and* $\beta_1 > 0$ *for almost all* $x \in \mathbb{R}$*;*

(b) The initial velocity $\mu \in \mathcal{M}_G^{\beta_2}(\mathbb{R})$ *for some* $\beta_2 > 0$.

Then for all even integers $p \geq 2$*, the upper growth indices of order* p *satisfy the* *following upper bounds:*

$$\overline{\lambda}(p) \le \begin{cases} \dfrac{1}{2\left(\beta_1 \wedge \beta_2\right)} z_p \sqrt{\kappa}\, L_\rho + \kappa & p > 2\,, \\[2ex] \dfrac{1}{4\left(\beta_1 \wedge \beta_2\right)} \sqrt{2\kappa}\, L_\rho + \kappa & p = 2\,. \end{cases}$$

(2) *Suppose that* $|\rho(u)| \ge l_\rho\,|u|$ *with* $l_\rho \ne 0$ *and the initial data satisfy one of the following two conditions:*

(a') *The initial position* $g(x)$ *is a non-negative Borel measurable function bounded from below by some function* $c_1 e^{-\beta_1'|x|}$ *with* $c_1 > 0$ *and* $\beta_1' > 0$ *for almost all* $x \in \mathbb{R}$*;*

(b') *The initial velocity* $\mu(\mathrm{d}x)$ *is such that* $\mu(x)$ *is a non-negative Borel measurable function bounded from below by some function* $c_2 e^{-\beta_2'|x|}$ *with* $c_2 > 0$ *and* $\beta_2' > 0$ *for almost all* $x \in \mathbb{R}$.

Then for all even integers $p \ge 2$*, the lower growth indices of order* p *satisfy the following lower bound:*

$$\underline{\lambda}(p) \ge \kappa\left(1 + \frac{l_\rho^2}{8\kappa\left(\beta_1' \wedge \beta_2'\right)^2}\right)^{1/2}.$$

In particular, we have the following two special cases:

(3) *For the hyperbolic Anderson model* $\rho(u) = \lambda u$ *with* $\lambda \ne 0$*, if the initial velocity* μ *satisfies all Conditions (a), (b), (a') and (b') with* $\beta := \beta_1 \wedge \beta_2 = \beta_1' \wedge \beta_2'$*, then*

$$\kappa\left(1 + \frac{\lambda^2}{8\kappa\beta^2}\right)^{1/2} \le \underline{\lambda}(2) \le \overline{\lambda}(2) \le \kappa\left(1 + \sqrt{\frac{\lambda^2}{8\kappa\beta^2}}\right).$$

(4) *If* $l_\rho\,|u| \le |\rho(u)| \le L_\rho\,|u|$ *with* $l_\rho \ne 0$ *and* $L_\rho \ne 0$*, and both* $g(x)$ *and* μ *are non-negative Borel measurable functions with compact support, then for all even integers* $p \ge 2$*,*

$$\overline{\lambda}(p) = \underline{\lambda}(p) = \kappa\,.$$

See Section 4.4.4 for the complete proof. Note that for Conclusion (3), clearly, $\beta_i' \le \beta_i$, $i = 1, 2$. Hence, the condition $\beta_1 \wedge \beta_2 = \beta_1' \wedge \beta_2'$ has only two possible cases:

$$\beta_1' = \beta_1 \le \beta_2' \le \beta_2\,, \qquad \text{and} \qquad \beta_2' = \beta_2 \le \beta_1' \le \beta_1\,.$$

Remark 4.2.12. We notice that the behaviour of growth indices of the solution to the stochastic wave equation (4.1.1) depends not only on the size of the noise (i.e., the magnitude of κ), but also on the growth rate of the nonlinearity of ρ. But when the initial data are compactly supported, it only depends on κ.

4.3 Technical Lemmas and Propositions

Define the backward space-time cone:

$$\Lambda(t,x) := \left\{(s,y) \in \mathbb{R}_+ \times \mathbb{R} \,:\, 0 \le s \le t,\, |y-x| \le \kappa(t-s)\right\}\,, \tag{4.3.1}$$

and the wave kernel (1.3.1) can be equivalently written as

$$G_\kappa\left(t-s,x-y\right) = \frac{1}{2} 1_{\{\Lambda(t,x)\}}\left(s,y\right)\,. \tag{4.3.2}$$

The following change of variables are used many times: see Figure 4.4.

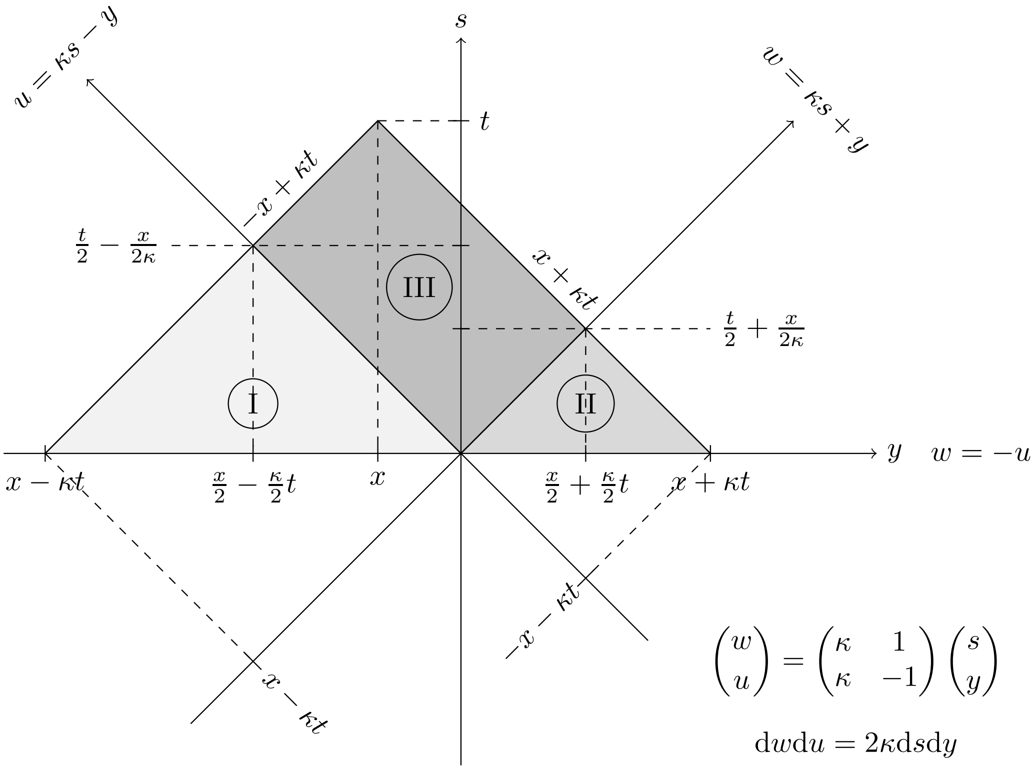


Figure 4.4 – Change of variables for the wave equation in $\mathbb{R}_+ \times \mathbb{R}$, for the case where $|x| \le \kappa t$.

4.3.1 Space-time Convolution of the Square of the Wave Kernel

Define the kernel function

$$\mathscr{L}_0(t,x;\lambda) = \lambda^2 G_\kappa^2(t,x)\,,$$

and for all $n \in \mathbb{N}^*$, define

$$\mathscr{L}_n(t,x;\lambda) \overset{\Delta}{=} (\underbrace{\mathscr{L}_0 \star \cdots \star \mathscr{L}_0}_{\substack{n+1 \text{ times of} \\ \mathscr{L}_0(t,x;\lambda)}})\,(t,x)$$

with $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}$. We use the same convention on the kernel functions $\mathscr{L}_n(t,x;\lambda)$ as $\mathscr{K}(t,x;\lambda)$ regarding the parameter λ.

Proposition 4.3.1 (Properties of the kernel functions)**.** *We have the following properties:*

(i) $\mathscr{L}_n(t,x)$ *has the following explicit form*

$$\mathscr{L}_n(t,x) = \begin{cases} \dfrac{\lambda^{2n+2}\left((\kappa t)^2 - x^2\right)^n}{2^{3n+2}(n!)^2\kappa^n} & \text{if } -\kappa t \le x \le \kappa t, \\ 0 & \text{otherwise,} \end{cases} \tag{4.3.3}$$

for any $n \in \mathbb{N}$ *and* $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}$.

(ii) The kernel functions $\mathscr{K}(t,x)$*, which is defined in* (4.2.1)*, and* $\{\mathscr{L}_n(t,x) : n \in \mathbb{N}\}$ *have the following relations*

$$\mathscr{K}(t,x) = \sum_{n=0}^{\infty} \mathscr{L}_n(t,x)\,, \tag{4.3.4}$$

and

$$(\mathscr{K} \star \mathscr{L}_0)(t,x) = \mathscr{K}(t,x) - \mathscr{L}_0(t,x)\,, \tag{4.3.5}$$

for any $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}$.

(iii) There are non-negative functions $B_n(t)$ *such that for all* $n \in \mathbb{N}$*, the function* $B_n(t)$ *is nondecreasing in* t *and*

$$\mathscr{L}_n \le \mathscr{L}_0(t,x) B_n(t), \quad \text{for all } (t,x) \in \mathbb{R}_+^* \times \mathbb{R}\,.$$

Moreover,

$$\sum_{n=1}^{\infty} (B_n(t))^{1/m} < +\infty, \quad \text{for all } m \in \mathbb{N}^*\,.$$

Proof. (i) We shall first prove (4.3.3). By induction, it holds clearly for $n = 0$. Suppose that the equation holds for n. Now we shall evaluate $\mathscr{L}_{n+1}(t,x)$ by the definition. In order to calculate the convolution, we change the variables: $u = \kappa s - y$ and $w = \kappa s + y$ (see Figure 4.4) and so

$$\begin{aligned} \mathscr{L}_{n+1}(t,x) &= (\mathscr{L}_0 \star \mathscr{L}_n)(t,x) \\ &= \frac{\lambda^{2n+4}}{2^{3n+4}(n!)^2\kappa^n} \frac{1}{2\kappa} \int_0^{x-\kappa t} \mathrm{d}u\, u^n \int_0^{x+\kappa t} w^n \mathrm{d}w \\ &= \frac{\lambda^{2(n+1)+2}\left((\kappa t)^2 - x^2\right)^{n+1}}{2^{3(n+1)+2}((n+1)!)^2\kappa^{n+1}}\,, \end{aligned}$$

for $-\kappa t \le x \le \kappa t$, and $\mathscr{L}_{n+1}(t,x) = 0$ otherwise. This proves (4.3.3).

(ii) Then the series in (4.3.4) converges to the modified Bessel function of order zero by (4.2.2). As a direct consequence, we have (4.3.5).

(iii) Take

$$B_n(t) = \frac{\lambda^{2n}(\kappa t)^{2n}}{2^{3n}(n!)^2\kappa^n},$$

which is non-negative and nondecreasing in t. Then clearly, $\mathscr{L}_n(t,x) \le \mathscr{L}_0(t,x)B_n(t)$. To show the convergence, by the ratio test, for all $m \in \mathbb{N}^*$, we have

$$\frac{(B_n(t))^{1/m}}{(B_{n-1}(t))^{1/m}} = \left(\frac{\lambda\sqrt{\kappa}\,t}{2\sqrt{2}}\right)^{\frac{2}{m}} \left(\frac{(n-1)!}{n!}\right)^{\frac{2}{m}} = \left(\frac{\lambda\sqrt{\kappa}\,t}{2\sqrt{2}}\right)^{\frac{2}{m}} \left(\frac{1}{n}\right)^{\frac{2}{m}} \to 0$$

as $n \to \infty$. This completes the proof. □

Lemma 4.3.2. *The following two statements hold:*

(1) The kernel function $\mathcal{K}(t,x)$ defined in (4.2.1) *is strictly increasing in t for $x \in \mathbb{R}$ fixed and decreasing in $|x|$ for $t > 0$ fixed.*

(2) Let $t > 0$. For all $(s,y) \in [0,t] \times \mathbb{R}$, we have that

$$\frac{\lambda^2}{4} 1_{\{|y|\le\kappa s\}} \le \mathcal{K}(s,y) \le \frac{\lambda^2}{4} I_0\left(|\lambda|\sqrt{\kappa/2}\,t\right) 1_{\{|y|\le\kappa s\}},$$

or equivalently,

$$\frac{\lambda^2}{2} G_\kappa(s,y) \le \mathcal{K}(s,y) \le \frac{\lambda^2}{2} I_0\left(|\lambda|\sqrt{\kappa/2}\,t\right) G_\kappa(s,y).$$

Proof. (1) We only need to show that the function $I_0(y)$ is increasing in $y \in \mathbb{R}$. This is clear because

$$\frac{\mathrm{d}I_0(y)}{\mathrm{d}y} = I_1(y) > 0, \quad \text{for all } y > 0,$$

by [50, (49:10:1) in p.512 and (49:6:1) on p. 511]. As for (2), The upper bound follows from (1). The lower bound is clear since $I_0(0) = 1$ by (4.2.2). □

Lemma 4.3.3. *For $t \ge 0$ and $x \in \mathbb{R}$,*

$$\int_{\mathbb{R}} \mathcal{K}(t,x)\mathrm{d}x = |\lambda|\sqrt{\kappa/2}\sinh\left(|\lambda|\sqrt{\kappa/2}\,t\right) \tag{4.3.6}$$

$$(1 \star \mathcal{K})(t,x) = \cosh\left(|\lambda|\sqrt{\kappa/2}\,t\right) - 1. \tag{4.3.7}$$

Proof. By the change of variable $y = \sqrt{\frac{\lambda^2}{2\kappa}\left[(\kappa t)^2 - x^2\right]}$, and so $x = \frac{\sqrt{2\kappa}}{|\lambda|}\sqrt{\kappa t^2\lambda^2/2 - y^2}$ the integration becomes,

$$\int_{\mathbb{R}} \mathcal{K}(t,x)\mathrm{d}x = 2\int_0^{|\lambda|\sqrt{\kappa/2}\,t} \frac{\lambda^2}{4}\frac{\sqrt{2\kappa}}{|\lambda|}\frac{y}{\sqrt{\kappa t^2\lambda^2/2 - y^2}} I_0(y)\mathrm{d}y.$$

Then use the integral [36, (6) on p. 365]:

$$\int_0^a x^{\nu+1}(a^2-x^2)^{\sigma-1} I_\nu(x)\mathrm{d}x = 2^{\sigma-1}a^{\nu+\sigma}\Gamma(\sigma)I_{\nu+\sigma}(a), \quad \Re(\nu) > -1, \quad \Re(\sigma) > 0,$$

with $\nu = 0$, $\sigma = 1/2$ and $a = |\lambda|\sqrt{\kappa/2}\, t$. So,

$$\begin{aligned}\int_{\mathbb{R}} \mathscr{K}(t,x)\mathrm{d}x &= \frac{a}{t}\int_0^a \frac{y}{\sqrt{a^2-y^2}} I_0(y)\mathrm{d}y \\ &= \frac{a^{3/2}\sqrt{\pi}}{t\sqrt{2}} I_{1/2}(a) = \frac{a^{3/2}\sqrt{\pi}}{t\sqrt{2}}\frac{\sqrt{2}}{\sqrt{\pi a}}\sinh(a)\,,\end{aligned}$$

where we have used the fact that

$$I_{1/2}(x) = \frac{\sqrt{2}}{\sqrt{\pi x}}\sinh(x)\,,$$

see [50, (28:13:3) on p. 277]. Therefore, (4.3.6) is proved by replacing a by $|\lambda|\sqrt{\kappa/2}\, t$. Finally, (4.3.7) is a simple application of (4.3.6). This finishes the whole proof. □

4.3.2 Some Continuity Properties of the Wave Kernels

Lemma 4.3.4. *For all $t \in \mathbb{R}_+$, and $x, y \in \mathbb{R}$, we have*

$$G_\kappa(t-s, x-z)G_\kappa\left(t-s, y-z\right) = \frac{1}{2}G_\kappa\left(T_\kappa\left(t, x-y\right)-s, \frac{x+y}{2}-z\right), \tag{4.3.8}$$

where $T_\kappa(t,x)$ is defined in (4.2.4). *Hence,*

$$\int_{\mathbb{R}} G_\kappa(t,x-z)G_\kappa(t,y-z)\mathrm{d}z = \frac{\kappa}{2}\, T_\kappa\left(t, x-y\right), \tag{4.3.9}$$

and

$$\iint_{\mathbb{R}_+\times\mathbb{R}} G_\kappa(t-s,x-z)G_\kappa\left(t-s,y-z\right)\mathrm{d}s\mathrm{d}z = \frac{\kappa}{4}\, T_\kappa^2\left(t,x-y\right). \tag{4.3.10}$$

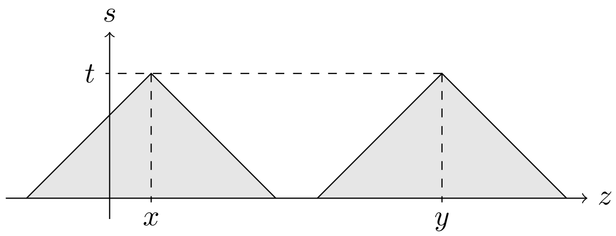
Note that we use the convention that $G_\kappa(t,\cdot) \equiv 0$ for $t \le 0$ in this lemma.

Proof. Write G_κ in the indicator form (4.3.2). Then (4.3.8) and (4.3.9) are clear from Figure 4.5. As (4.3.10), it is one quarter (due to the factor $1/2$ in each of $G_\kappa(\cdot,\circ)$) of the intersection area of the two cones $\Lambda(t,x)$ and $\Lambda(t,y)$. □

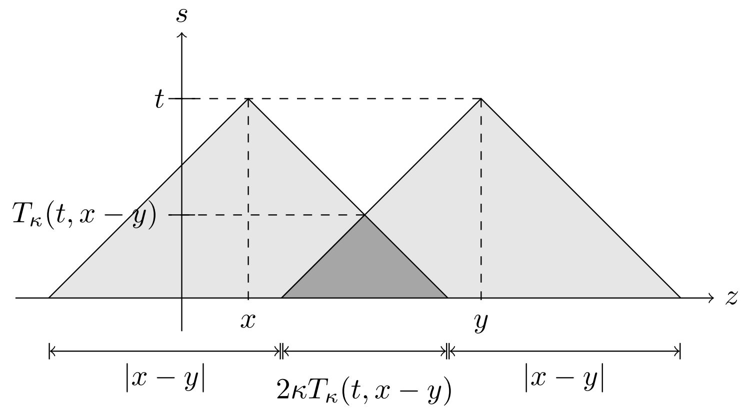
Proposition 4.3.5. *The fundamental solution $G_\kappa(t,x)$ of wave equation (see* (1.3.1)*) satisfies Assumption 3.2.11: Fix $T>0$. For all (t,x) and $\left(t',x'\right) \in [0,T]\times\mathbb{R}$ with $0 < t \le t'$,*

$$\begin{aligned}\int_0^t \mathrm{d}s\int_{\mathbb{R}}\left(G_\kappa\left(t-s,x-y\right)-G_\kappa(t'-s,x'-y)\right)^2\mathrm{d}y + \int_t^{t'}\mathrm{d}s\int_{\mathbb{R}} G_\kappa^2(t'-s,x'-y)\mathrm{d}y \\ \le C_T\left(\left|x'-x\right|+\left|t'-t\right|\right), \quad \textit{with } C_T := (\kappa\vee 1)\,T/2.\end{aligned} \tag{4.3.11}$$

Proof. Denote the left-hand side of (4.3.11) by $I\left(t,x,t',x'\right)$. We have three cases to



(a) the case where $|x-y| \geq 2\kappa t$



(b) the case where $|x-y| < 2\kappa t$

Figure 4.5 – Two lightly shaded regions denote functions $(s,z) \mapsto G_\kappa(t-s, x-z)$ and $(s,z) \mapsto G_\kappa\left(t-s, y-z\right)$ respectively.

consider as shown in Figure 4.6. In Case I where $\left|x'-x\right| \geq \kappa(t+t')$, we have

$$\begin{aligned} 4I\left(t,x,t',x'\right) &= \kappa\left(t^2+(t')^2\right) = \frac{\kappa}{2}\left((t-t')^2+(t+t')^2\right) \\ &\leq \frac{\kappa}{2}\left((t-t')^2+(t+t')\frac{\left|x'-x\right|}{\kappa}\right) \leq 2\kappa T(t'-t) + 2T\left|x'-x\right| . \end{aligned}$$

In Case III where $\left|x'-x\right| \leq \kappa(t'-t)$, we have

$$4I\left(t,x,t',x'\right) = \kappa\left((t')^2-t^2\right) = \kappa(t+t')(t'-t') \leq 2\kappa T(t'-t) .$$

As for Case II, $4I\left(t,x,t',x'\right)$ equals the area of the shaded region in Figure 4.7:

$$4I\left(t,x,t',x'\right) = \kappa t^2 + \kappa(t')^2 - 2\kappa T^2, \qquad \text{with } T = \frac{t+t'}{2} - \frac{\left|x'-x\right|}{2\kappa}.$$

After some simplifications,

$$\begin{aligned} 4I\left(t,x,t',x'\right) &= \frac{\kappa}{2}\left|t'-t\right|^2 + (t+t')\left|x'-x\right| - \frac{1}{2\kappa}\left|x'-x\right|^2 \\ &\leq \frac{\kappa}{2}\left|t'-t\right|^2 + (t+t')\left|x'-x\right| \\ &\leq 2\kappa T(t'-t) + 2T\left|x'-x\right| . \end{aligned}$$

The proposition is proved by combining all these three cases. □

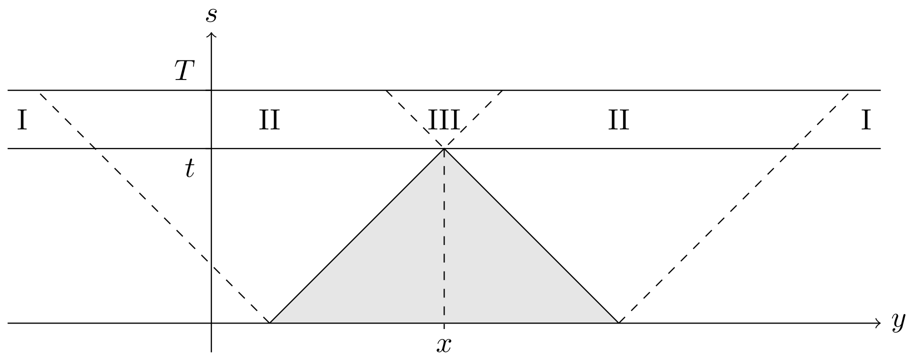


Figure 4.6 – Three cases in the proof of Proposition 4.3.5.
1. (t', x') is in the region I: $|x' - x| \geq \kappa(t + t')$;
2. (t', x') is in the region II: $\kappa(t' - t') \leq |x' - x| \leq \kappa(t + t')$;
3. (t', x') is in the region III: $|x' - x| \leq \kappa(t' - t)$.

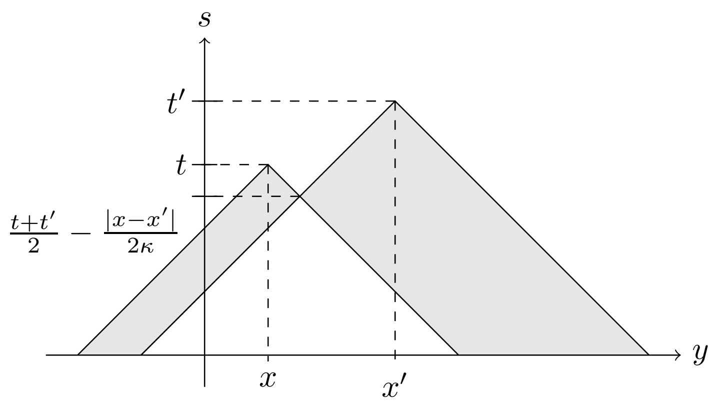


Figure 4.7 – Case II where $|x' - x| \leq \kappa(t + t')$ in the proof of Proposition 4.3.5.

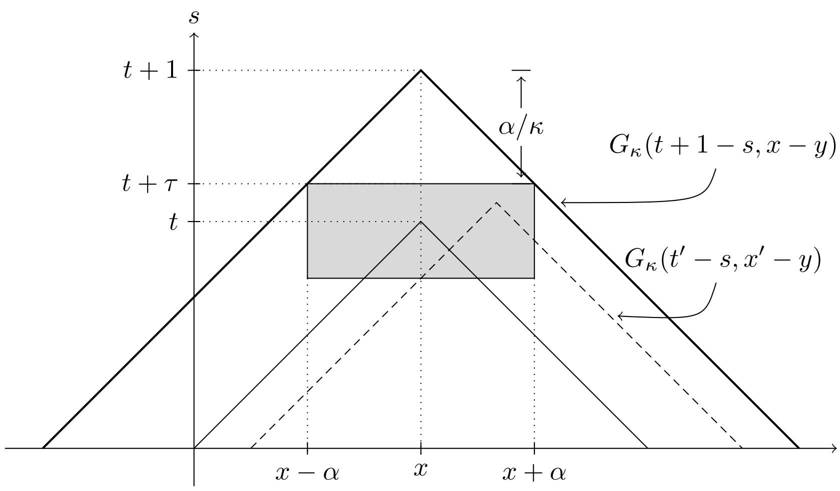


Figure 4.8 – $G_\kappa(t, x)$ verifies Assumption 3.2.9. The function, for example $G_\kappa(t' - s, x' - y)$, is understood to be a step function with value $1/2$ inside the triangle (closed set) and zero elsewhere.

Proposition 4.3.6. *The fundamental solution $G_\kappa(t,x)$ of wave equation (see* (1.3.1)*) satisfies Assumption 3.2.9 with $\tau = 1/2$, $\alpha = \kappa/2$ and all $\beta \in \,]0,1[$ and $C = 1$.*

Proof. The proof is simple: see Figure 4.8. The gray box is the set $B_{t,x,\beta,\tau,\alpha}$. Clearly, we need to find $\alpha/\kappa + \tau = 1$. By choosing $\alpha = \kappa\tau$, this relation becomes $2\tau = 1$. Therefore, we can choose $\tau = 1/2$ and $\alpha = \kappa/2$. This completes the proof. □

4.3.3 Results on Initial Data

For any $g \in L^2_{loc}(\mathbb{R})$ and $\mu \in \mathcal{M}(\mathbb{R})$, define

$$\Psi_g(x) := \int_{-x}^{x} g^2(y)\mathrm{d}y\,, \quad \text{for all } x \geq 0, \tag{4.3.12}$$

and

$$\Psi^*_\mu(x) := |\mu|^2\left([-x,x]\right)\,, \quad \text{for all } x \geq 0. \tag{4.3.13}$$

Clearly, they are nondecreasing functions in x.

Lemma 4.3.7. *For every Borel measurable function g such that $g \in L^2_{loc}(\mathbb{R})$, and for all $\mu \in \mathcal{M}(\mathbb{R})$,*

$$\left(\left[v^2 + J_0^2\right] \star G_\kappa^2\right)(t,x) \leq \frac{\kappa t^2}{4}\left(v^2 + 3\Psi^*_\mu(|x|+\kappa t)\right) + \frac{3}{16}\Psi_g(|x|+\kappa t) < +\infty$$

holds for all $v \in \mathbb{R}$ and $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$, where $J_0(t,x)$ is defined in (1.3.5). *Moreover,*

$$\sup_{(t,x)\in K} \left(\left[v^2 + J_0^2\right] \star G_\kappa^2\right)(t,x) < +\infty\,, \tag{4.3.14}$$

for all $v \in \mathbb{R}$ and all compact sets $K \subseteq \mathbb{R}_+ \times \mathbb{R}$.

Note that the conclusion of this lemma is stronger than Assumption 3.2.12 since t can be zero here.

Proof. Suppose $t > 0$. Notice that

$$\left|(\mu * G_\kappa(s,\cdot))(y)\right| \leq |\mu|\left([y-\kappa s, y+\kappa s]\right)\,,$$

and so

$$\begin{aligned}\left(\left[v^2 + J_0^2\right] \star G_\kappa^2\right)(t,x) &= \frac{1}{4}\left(v^2 \iint_{\Lambda(t,x)} \mathrm{d}s\mathrm{d}y + \iint_{\Lambda(t,x)} J_0^2(s,y)\,\mathrm{d}s\mathrm{d}y\right) \\ &\leq \frac{1}{4}\Bigg(v^2\kappa t^2 + \frac{3}{4}\int_0^t \mathrm{d}s \int_{x-\kappa(t-s)}^{x+\kappa(t-s)} \left(g^2(y+\kappa s) + g^2(y-\kappa s)\right. \\ &\qquad\qquad \left. + 4|\mu|^2\left([y-\kappa s, y+\kappa s]\right)\right)\mathrm{d}y\Bigg).\end{aligned}$$

Clearly, for all $(s,y) \in \Lambda(t,x)$, by (4.3.13),

$$|\mu|^2\left([y-\kappa s, y+\kappa s]\right) \le |\mu|^2\left([x-\kappa t, x+\kappa t]\right) \le \Psi_\mu^*(|x|+\kappa t)\ .$$

The integral for g^2 can be easily evaluated by the change of variables (see Figure 4.4):

$$\begin{aligned}\int_0^t \mathrm{d}s \int_{x-\kappa(t-s)}^{x+\kappa(t-s)} \left(g^2(y+\kappa s)+g^2(y-\kappa s)\right)\mathrm{d}y &= \frac{1}{2\kappa}\iint_{I\cup II\cup III}\left(g^2(u)+g^2(w)\right)\mathrm{d}u\mathrm{d}w\\ &\le \frac{1}{2\kappa}\int_{x-\kappa t}^{x+\kappa t}\mathrm{d}w\int_{-x-\kappa t}^{-x+\kappa t}\mathrm{d}u\left(g^2(u)+g^2(w)\right)\\ &\le \Psi_g(|x|+\kappa t)\,,\end{aligned}$$

where I, II and III denote the three regions in Figure 4.4 and Ψ_g is defined in (4.3.12). Therefore,

$$\left(\left[v^2+J_0^2\right]\star G_\kappa^2\right)(t,x) \le \frac{1}{4}\left(\left(v^2+3\Psi_\mu^*(|x|+\kappa t)\right)\kappa t^2+\frac{3}{4}\Psi_g(|x|+\kappa t)\right) < +\infty\,.$$

As for (4.3.14), let $a=\sup\left\{|x|+\kappa t : (t,x)\in K\right\}$, which is finite because K is a compact set. Then,

$$\sup_{(t,x)\in K}\left(\left[v^2+J_0^2\right]\star G_\kappa^2\right)(t,x) \le \frac{\kappa a^2}{4}\left(v^2+3\Psi_\mu^*(a)\right)+\frac{3}{16}\Psi_g(a) < +\infty\,,$$

which finishes the proof. □

4.3.4 Hölder Continuity

In this part, we will prove three propositions 4.3.8, 4.3.9 and 4.3.10, which altogether verify Assumption 3.2.14 (and hence the Hölder continuity). Among these three propositions, Propositions 4.3.9 and 4.3.10 are essentially proving the Sobolev imbedding theorem in our special case.

Proposition 4.3.8. *Let $K_n^* := [0,n]\times[-n-\kappa n, n+\kappa n]$. If for all $n>0$,*

$$\sup_{(t,x)\in K_n^*} J_0^2(t,x) < +\infty\,,$$

then Assumption 3.2.14 holds under the settings:

$$\theta(t,x)\equiv 1,\ d=1,\ \gamma_0=\gamma_1=1,\ \textit{and}\ K_n=[0,n]\times[-n,n].$$

In particular, this is the case when the initial position g vanishes and the initial velocity μ is a locally finite Borel measure:

$$\sup_{(t,x)\in K_n^*} J_0^2(t,x) \le 1/4\ \Psi_\mu^*(n+2\kappa n) < +\infty\,.$$

Proof. Fix $v \geq 0$, $n > 1$ and choose arbitrary (t,x) and $(t',x') \in K_n = [0,n] \times [-n,n]$ (note that the time variable can be zero). Notice that the support of the function

$$(s,y) \mapsto G_\kappa\left(t-s, x-y\right) - G_\kappa\left(t'-s, x'-y\right)$$

is included in the compact set

$$\left[0, t \vee t'\right] \times \left[(x-\kappa t) \wedge \left(x'-\kappa t'\right), (x+\kappa t) \vee \left(x'+\kappa t'\right)\right] ,$$

which is further included in K_n^*. Hence, the left-hand side of (3.2.18) is bounded by

$$\left(\sup_{(s,y)\in K_n^*} \left(v^2 + 2J_0^2(s,y)\right)\right) \iint_{\mathbb{R}_+\times\mathbb{R}} \left(G_\kappa\left(t-s,x-y\right) - G_\kappa\left(t'-s,x'-y\right)\right)^2 \mathrm{d}s\mathrm{d}y$$
$$\leq C_n \frac{n(\kappa \vee 1)}{2}\left(\left|x-x'\right| + \left|t-t'\right|\right), \quad \text{with } C_n := \sup_{(s,y)\in K_n^*} \left(v^2 + 2J_0^2(s,y)\right),$$

where we have applied Proposition 4.3.5.

As for (3.2.19), we have that

$$\begin{aligned}
&\iint_{\mathbb{R}_+\times\mathbb{R}} \left(\left(v^2 + J_0^2\right) \star G_k^2\right)(s,y) \left(G_\kappa\left(t-s,x-y\right) - G_\kappa\left(t'-s,x'-y\right)\right)^2 \mathrm{d}s\mathrm{d}y \\
&\quad= \iint_{\mathbb{R}_+\times\mathbb{R}} \mathrm{d}s\mathrm{d}y \iint_{\mathbb{R}_+\times\mathbb{R}} \mathrm{d}u\mathrm{d}z \left(v^2 + J_0^2(u,z)\right) \\
&\qquad\qquad \times G_\kappa^2(s-u,y-z)\left(G_\kappa\left(t-s,x-y\right) - G_\kappa\left(t'-s,x'-y\right)\right)^2 \\
&\quad\leq C_n \iint_{\mathbb{R}_+\times\mathbb{R}} \mathrm{d}s\mathrm{d}y \iint_{\mathbb{R}_+\times\mathbb{R}} \mathrm{d}u\mathrm{d}z\, G_\kappa^2(s-u,y-z)\left(G_\kappa\left(t-s,x-y\right) - G_\kappa\left(t'-s,x'-y\right)\right)^2 \\
&\quad= C_n \iint_{\mathbb{R}_+\times\mathbb{R}} \mathrm{d}s\mathrm{d}y \frac{\kappa s^2}{4}\left(G_\kappa\left(t-s,x-y\right) - G_\kappa\left(t'-s,x'-y\right)\right)^2 \\
&\quad\leq C_n \frac{\kappa n^2}{4} \iint_{\mathbb{R}_+\times\mathbb{R}} \mathrm{d}s\mathrm{d}y \left(G_\kappa\left(t-s,x-y\right) - G_\kappa\left(t'-s,x'-y\right)\right)^2 \\
&\quad\leq C_n \frac{n^2\kappa(\kappa \vee 1)}{8}\left(\left|x-x'\right| + \left|t-t'\right|\right).
\end{aligned}$$

This completes the proof. ☐

Proposition 4.3.9. *Suppose* $\mu \equiv 0$ *and* $g \in L^2_{loc}(\mathbb{R})$. *Then* (3.2.19) *holds under the settings:*

$$\theta(t,x) \equiv 1,\ d = 1,\ \gamma_0 = \gamma_1 = 1, \text{ and } K_n = [0,n] \times [-n,n].$$

Proof. We can split (3.2.19) into two parts by linearity: one is contributed by v^2 and the other by $2J_0^2$. Proposition 4.3.8 shows that the first part satisfies Assumption 3.2.14. Hence, we only need to consider the second part. Let $K_n^* = [0,n] \times [-(1+\kappa)n, (1+\kappa)n]$. By the change of variables (see Figure 4.4),

$$\left(J_0^2 \star G_\kappa^2\right)(t,x) = \frac{1}{16}\frac{1}{2\kappa} \iint_{I\cup II\cup III} \left(g(w) + g(u)\right)^2 \mathrm{d}u\mathrm{d}w$$

where I, II and III denote the three domains shown in Figure 4.4. Clearly,

$$\begin{aligned}\iint_{I\cup II\cup III}\left(g(w)+g(u)\right)^2 \mathrm{d}u\mathrm{d}w &\le \int_{x-\kappa t}^{x+\kappa t}\mathrm{d}w\int_{-x-\kappa t}^{-x+\kappa t}\mathrm{d}u\left(g(w)+g(u)\right)^2\\ &\le 2\int_{-n-\kappa n}^{n+\kappa n}\mathrm{d}w\int_{-n-\kappa n}^{n+\kappa n}\mathrm{d}u\left(g(w)^2+g(u)^2\right)\\ &= 8(1+\kappa)n\Psi_g(n+n\kappa)\,.\end{aligned}$$

Hence,

$$\left(J_0^2\star G_\kappa^2\right)(t,x)\le \frac{(1+\kappa)n}{4\kappa}\Psi_g(n+n\kappa)\,,\quad \text{for all } (t,x)\in K_n^*.$$

Therefore, this proposition is proved by applying Proposition 4.3.8. □

Proposition 4.3.10. *Suppose* $\mu\equiv 0$, $g\in L^{2p}_{loc}(\mathbb{R})$ *with* $p\ge 1$, *and* $1/p+1/p'=1$. *Then* (3.2.18) *holds under the settings:*

$$\theta(t,x)\equiv 1,\ d=1,\ \text{and}\ \gamma_0=\gamma_1=1/p'.$$

Proof. Equivalently, we shall show that (3.2.20), (3.2.21) and (3.2.22) hold under the same settings. By the same reason as that in the proof of Proposition 4.3.9, we can assume that $v=0$ in (3.2.20)–(3.2.22). Fix $n>0$, and (t,x), $(t',x')\in K_n=[0,n]\times[-n,n]$ with $t\le t'$.

We first prove (3.2.20). Notice that the support of the function $G_\kappa - G_\kappa$ is in $K_n^*=[0,n]\times[-(1+\kappa)n,(1+\kappa)n]$ (see the proof of Proposition 4.3.8). By Hölder's inequality,

$$\begin{aligned}I&:=\int_0^t\mathrm{d}s\int_{\mathbb{R}}J_0^2(s,y)\left(G_\kappa(t-s,x-y)-G_\kappa(t'-s,x-y)\right)^2\mathrm{d}y\\ &\le\int_0^t\mathrm{d}s\left(\int_{-(1+\kappa)n}^{(1+\kappa)n}J_0^{2p}(s,y)\,\mathrm{d}y\right)^{1/p}\left(\int_{\mathbb{R}}\left(G_\kappa(t-s,x-y)-G_\kappa(t'-s,x-y)\right)^{2p'}\mathrm{d}y\right)^{1/p'}.\end{aligned}$$

By convexity of the function $x\mapsto |x|^{2p}$,

$$J_0^{2p}(s,y)=\left(\frac{g(y+\kappa s)+g(y-\kappa s)}{2}\right)^{2p}\le\frac{g^{2p}(y+\kappa s)+g^{2p}(y-\kappa s)}{2}.$$

Hence,

$$\begin{aligned}\int_{-(1+\kappa)n}^{(1+\kappa)n}J_0^{2p}(s,y)\,\mathrm{d}y&\le\frac{1}{2}\int_{-(1+\kappa)n}^{(1+\kappa)n}\left(g^{2p}(y+\kappa s)+g^{2p}(y-\kappa s)\right)\mathrm{d}y\\ &\le\int_{-(1+2\kappa)n}^{(1+2\kappa)n}g^{2p}(u)\mathrm{d}u=\Psi_{g^p}(n+2\kappa n)\,,\end{aligned}$$

which is independent of s. Therefore,

$$I\le\Psi_{g^p}(n+2\kappa n)\int_0^t\mathrm{d}s\left(\int_{\mathbb{R}}\left(G_\kappa(t-s,x-y)-G_\kappa(t'-s,x-y)\right)^{2p'}\mathrm{d}y\right)^{1/p'}.$$

Clearly, by writing $G_\kappa(t-\cdot, x-\circ)$ in the indicator form (see (4.3.1)),

$$\int_{\mathbb{R}} \left(G_\kappa\left(t-s,x-y\right)-G_\kappa(t'-s,x-y)\right)^{2p'} \mathrm{d}y = 2^{-2p'}\int_{\mathbb{R}} \left(1_{\{\Lambda(t,x)\}}\left(s,y\right)-1_{\{\Lambda(t',x)\}}\left(s,y\right)\right)\mathrm{d}y$$
$$= 2^{-2p'}\kappa\left|t'-t\right| .$$

Therefore,

$$I \le \frac{\kappa^{1/p'} n}{4} \Psi_{g^p}(n+2\kappa n)\left|t'-t\right|^{1/p'} ,$$

which finishes the proof of (3.2.20).

Now let us consider (3.2.21). Similar to the previous case, we have

$$I := \int_0^t \mathrm{d}s \int_{\mathbb{R}} J_0^2\left(s,y\right)\left(G_\kappa\left(t-s,x-y\right)-G_\kappa(t-s,x'-y)\right)^2 \mathrm{d}y$$
$$\le \Psi_{g^p}(n+2\kappa n)\int_0^t \mathrm{d}s \left(\int_{\mathbb{R}} \left(G_\kappa\left(t-s,x-y\right)-G_\kappa(t-s,x'-y)\right)^{2p'}\mathrm{d}y\right)^{1/p'} .$$

Clearly, by writing G_κ functions in indicator forms,

$$\int_{\mathbb{R}} \left(G_\kappa\left(t-s,x-y\right)-G_\kappa(t-s,x'-y)\right)^{2p'} \mathrm{d}y = 2^{-2p'}\int_{\mathbb{R}} \left(1_{\{\Lambda(t,x)\}}\left(s,y\right)-1_{\{\Lambda(t,x')\}}\left(s,y\right)\right)\mathrm{d}y$$
$$= 2^{1-2p'}\left|x'-x\right| 1_{\{|x'-x|\le 2\kappa(t-s)\}} + 2^{1-2p'}\kappa(t-s)\, 1_{\{|x'-x|>2\kappa(t-s)\}} \le 2^{1-2p'}\left|x'-x\right| ,$$

see Figure 4.5. Therefore,

$$I \le 2^{-2+1/p'} n\, \Psi_{g^p}(n+2\kappa n)\left|x'-x\right|^{1/p'} ,$$

which finishes the proof of (3.2.21).

Now let us consider (3.2.22). By the same arguments as above,

$$I := \int_t^{t'} \mathrm{d}s \int_{\mathbb{R}} J_0^2\left(s,y\right) G_\kappa^2(t'-s,x'-y)\mathrm{d}y$$
$$\le \Psi_{g^p}(n+2\kappa n)\int_t^{t'} \mathrm{d}s\left(\int_{\mathbb{R}} G_\kappa^{2p'}(t'-s,x'-y)\mathrm{d}y\right)^{1/p'} ,$$

and

$$\int_{\mathbb{R}} G_\kappa^{2p'}(t'-s,x'-y)\mathrm{d}y = 2^{-2p'}2\kappa(t'-s) \le 2^{-2p'}2\kappa n .$$

Therefore,

$$I \le 2^{-2+1/p'} n\kappa\, \Psi_{g^p}(n+2\kappa n)\left|t'-t\right| .$$

Finally, (3.2.22) is proved by the fact that

$$\left|t'-t\right| = \left|t'-t\right|^{1-1/p'}\left|t'-t\right|^{1/p'} \le (2n)^{1/p}\left|t'-t\right|^{1/p'} .$$

This completes the proof. □

4.4 Proof of the Main Results

4.4.1 Proof of the Existence Theorem (Theorem 4.2.1) and Its Corollaries

The conclusions of Theorem 4.2.1 for the stochastic wave equation are similar to those of Theorems 2.2.2 and 3.2.16/3.2.17 for the stochastic heat equation. The proof of Theorem 4.2.1, given below, has the same general structure as the proofs of those two other theorems. See Table 4.1 for a comparison of how the various assumptions of Chapter 3 are checked.

Proof of Theorem 4.2.1. We need to verify Cond(G), Cond(W) and Assumption 3.2.14 of Theorem 3.2.16 with $\theta(t,x)\equiv 1$. Let us first check Cond(G): (a) is satisfied by (1.3.2) and Proposition 4.3.1; (b) is verified by Lemma 4.3.7; (c) is part of our assumption on ρ. Cond(W) is true due to Proposition 4.3.6.

As for the sample path regularity, Assumption 3.2.14 holds for $K_n=[0,n]\times[-n,n]$ thanks to Propositions 4.3.8, 4.3.9 and 4.3.10. More precisely, let $J_{0,1}(t,x)$ and $J_{0,2}(t,x)$ be the homogeneous solutions contributed respectively by the initial position g and initial velocity μ. Clearly, when both g and μ are nonvanishing,

$$J_0(t,x) = J_{0,1}(t,x) + J_{0,2}(t,x)\,.$$

Since

$$J_0^2(t,x) \le 2J_{0,1}^2(t,x) + 2J_{0,2}^2(t,x)\,,$$

we can consider the contributions by initial position g and initial velocity μ separately when verifying Assumption 3.2.14. In particular, Proposition 4.3.8 shows that the contribution by $J_{0,2}(t,x)$ satisfies Assumption 3.2.14, and Propositions 4.3.10 and 4.3.9 guarantee that the contribution by $J_{0,1}(t,x)$ satisfies Assumption 3.2.14.

We still need to show that the two-point correlation function (3.2.28) can reduce to (4.2.11). By comparing these two expressions, we need to show that

$$\int_0^t \mathrm{d}s \int_{\mathbb{R}} f(s,z)G_\kappa(t-s,x-z)G_\kappa\left(t-s,y-z\right)\mathrm{d}z = \frac{1}{2}\left(f\star G_\kappa\right)\left(T_\kappa\left(t,x-y\right),\frac{x+y}{2}\right),$$

which is true by (4.3.8). This completes the proof. □

The following three integrals will be used in the following proof:

$$\int_0^t \cosh(as)(t-s)\mathrm{d}s = \frac{1}{a^2}\left(\cosh(at)-1\right), \tag{4.4.1}$$

$$\int_0^t \sinh(as)(t-s)\mathrm{d}s = \frac{1}{a^2}\left(\sinh(at)-at\right), \tag{4.4.2}$$

$$\int_0^t \sinh(as)(t-s)^2\mathrm{d}s = \frac{1}{a^3}\left(2\cosh(at)-a^2t^2-2\right). \tag{4.4.3}$$

	Stochastic heat equation (1.1.3) with $\theta(t,x) \equiv 1$	Stochastic heat equation (1.2.2) with a general $\theta(t,x)$	Stochastic wave equation (1.3.3) with $\theta(t,x) \equiv 1$
Cond(G) (a)	(1.1.2) and Proposition 2.3.1	(3.4.2) and Proposition 2.3.1	(1.3.2) and Proposition 4.3.1
Cond(G) (b)	Lemma 2.3.6	Proposition 3.4.1	Lemma 4.3.7
Cond(W)	—	—	Proposition 4.3.6
Cond(H) (a)	Proposition 2.3.12, Corollary 2.3.10	Proposition 2.3.12, Corollary 2.3.10	—
Cond(H) (b)	True since $\theta\left(s,y\right) \equiv 1$	Assumption on $\theta(t,x)$	—
Cond(H) (c)	Lemma 2.3.5	Lemma 2.3.5	—
Assumption 3.2.14	Propositions 2.6.16 and 2.6.17 with $K_n = [1/n, n] \times [-n, n]$	—	Propositions 4.3.8, 4.3.9 and 4.3.10 with $K_n = [0, n] \times [-n, n]$

Table 4.1 – A comparison/summary of the proofs of Theorems 2.2.2, 3.2.16/3.2.17 and 4.2.1.

Proof of Corollary 4.2.2. (1) In this case, $J_0(t,x) = w + \kappa\widetilde{w}t$. Then by the moment formula (4.2.10), (4.3.6), (4.4.2) and (4.4.3), we have

$$\begin{aligned}
||u(t,x)||_2^2 &= (w+\kappa\widetilde{w}t)^2 + \int_0^t \mathrm{d}s\, \left[\varsigma^2 + (w+\kappa\widetilde{w}s)^2\right] \int_{\mathbb{R}} \mathscr{K}\left(t-s, x-y\right)\mathrm{d}y \\
&= (w+\kappa\widetilde{w}t)^2 + |\lambda|\sqrt{\kappa/2}\int_0^t \left[\varsigma^2 + (w+\kappa\widetilde{w}s)^2\right] \sinh\left(|\lambda|\sqrt{\kappa/2}\,(t-s)\right)\mathrm{d}s \\
&= C_1 + C_2\cosh\left(\frac{\sqrt{\kappa}|\lambda|t}{\sqrt{2}}\right) + C_3\sinh\left(\frac{\sqrt{\kappa}|\lambda|t}{\sqrt{2}}\right),
\end{aligned}$$

with the three constants

$$C_1 = -\varsigma^2 - \frac{4\kappa\widetilde{w}^2}{\lambda^2}, \qquad C_2 = w^2 + \varsigma^2 + \frac{4\kappa\widetilde{w}^2}{\lambda^2}, \qquad C_3 = \frac{2\sqrt{2\kappa}\,w\widetilde{w}}{|\lambda|}.$$

Then the formula follows by replacing $\cosh\left(\sqrt{\kappa/2}|\lambda|t\right)$ by $\mathscr{H}(t)+1$. The special cases, $\varsigma = \widetilde{w} = 0$ and $\varsigma = w = 0$, are clear.

(2) Now let us consider the two-point correlation function. Denote $T := T_\kappa\left(t, x-y\right)$. By the two-point correlation formula (4.2.11), the second moment formula obtained in (1), and (4.2.5), we have that for all $t>0$ and $x,y\in\mathbb{R}$,

$$\begin{aligned}
\mathbb{E}\left[u(t,x)u(t,y)\right] &= (w+\kappa\widetilde{w}t)^2 + \lambda^2\varsigma^2\,\Theta_\kappa\left(t,x,y\right) \\
&\quad + \frac{\lambda^2}{2}\int_0^T \mathrm{d}s \left(C_1 + C_2\cosh\left(\frac{\sqrt{\kappa}|\lambda|s}{\sqrt{2}}\right) + C_3\sinh\left(\frac{\sqrt{\kappa}|\lambda|s}{\sqrt{2}}\right)\right) \\
&\qquad\qquad \times \int_{\mathbb{R}} G_\kappa\left(T-s, \frac{x+y}{2} - z\right)\mathrm{d}z \\
&= (w+\kappa\widetilde{w}t)^2 + \frac{\lambda^2\kappa}{4}\varsigma^2\,T^2 \\
&\quad + \frac{\lambda^2\kappa}{2}\int_0^T \left(C_1 + C_2\cosh\left(\frac{\sqrt{\kappa}|\lambda|s}{\sqrt{2}}\right) + C_3\sinh\left(\frac{\sqrt{\kappa}|\lambda|s}{\sqrt{2}}\right)\right)(T-s)\,\mathrm{d}s.
\end{aligned}$$

Now apply the two integrals in (4.4.2) and (4.4.1) to evaluate the above integral:

$$\begin{aligned}
&\frac{\lambda^2\kappa}{2}\int_0^T \left(C_1 + C_2\cosh\left(\frac{\sqrt{\kappa}|\lambda|s}{\sqrt{2}}\right) + C_3\sinh\left(\frac{\sqrt{\kappa}|\lambda|s}{\sqrt{2}}\right)\right)(T-s)\,\mathrm{d}s \\
&\qquad = \frac{\lambda^2\kappa}{2}TC_1 + C_2\underbrace{\left(\cosh\left(\sqrt{\kappa/2}|\lambda|T\right)-1\right)}_{=\mathscr{H}(T)} + C_3\left(\sinh\left(\sqrt{\kappa/2}|\lambda|T\right) - \sqrt{\kappa/2}|\lambda|T\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathbb{E}\left[u(t,x)u(t,y)\right] = (w+\kappa\widetilde{w}t)^2 &+ \frac{\lambda^2\kappa}{4}\varsigma^2\,T^2 - C_3\sqrt{\kappa/2}|\lambda|T + \frac{\lambda^2\kappa}{2}TC_1 \\
&+ C_2\mathscr{H}(T) + C_3\sinh\left(\sqrt{\kappa/2}|\lambda|T\right).
\end{aligned}$$

The formula follows after some simplifications. □

Proof of Corollary 4.2.3. In this case, $J_0(t,x) = G_\kappa(t,x)$. Notice that $\lambda^2 J_0^2(t,x) = \mathscr{L}_0(t,x)$.

So, by (4.2.10) and (4.3.5), we have

$$\mathbb{E}\left[|u(t,x)|^2\right] = J_0^2(t,x) + \frac{1}{\lambda^2}\left(\mathscr{L}_0 \star \mathscr{K}\right)(t,x) + \varsigma^2\,\mathscr{H}(t) = \frac{1}{\lambda^2}\mathscr{K}(t,x) + \varsigma^2\,\mathscr{H}(t)\,.$$

Then, by the two-point correlation function (4.2.11), we have

$$\begin{aligned}\mathbb{E}\left[u(t,x)u\left(t,y\right)\right] &= J_0(t,x)J_0\left(t,y\right) + \lambda^2\varsigma^2\,\Theta_\kappa\left(t,x,y\right)\\ &+ \frac{\lambda^2}{2}\int_0^{T_\kappa(t,x-y)}\mathrm{d}s\int_{\mathbb{R}}\left(\frac{1}{\lambda^2}\mathscr{K}(s,z) + \varsigma^2\,\mathscr{H}(s)\right)G_\kappa\left(T_\kappa\left(t,x-y\right)-s,\frac{x+y}{2}-z\right)\mathrm{d}z\,. \end{aligned} \tag{4.4.4}$$

By (4.3.8),

$$J_0(t,x)J_0\left(t,y\right) = \frac{1}{2}G_\kappa\left(T_\kappa\left(t,x-y\right),\frac{x+y}{2}\right).$$

By (4.3.5), the double integral with $\lambda^2/2$ in (4.4.4) is equal to

$$\begin{aligned}&\frac{1}{\lambda^2}\mathscr{K}\left(T_\kappa\left(t,x-y\right),\frac{x+y}{2}\right) - \frac{1}{2}G_\kappa\left(T_\kappa\left(t,x-y\right),\frac{x+y}{2}\right)\\ &\qquad + \underbrace{\frac{\lambda^2\varsigma^2}{2}\int_0^t \mathrm{d}s\,\mathscr{H}(s)\int_{\mathbb{R}}G_\kappa\left(T_\kappa\left(t,x-y\right)-s,\frac{x+y}{2}-z\right)\mathrm{d}z}_{:=I}\,.\end{aligned}$$

Now let us evaluate the integral I in the above expression: The dz–integral is clear; Noticing that $\mathscr{H}(s)$ is related to $\cosh(\cdot)$ (see (4.2.3)), by (4.4.1), we have that

$$\begin{aligned} I &= \frac{\lambda^2\varsigma^2}{2}\int_0^{T_\kappa(t,x-y)}\mathscr{H}(s)\,\kappa\left(T_\kappa\left(t,x-y\right)-s\right)\mathrm{d}s\\ &= \varsigma^2\,\mathscr{H}\left(T_\kappa\left(t,x-y\right)\right) - \frac{\kappa\lambda^2\varsigma^2}{4}T_\kappa^2\left(t,x-y\right)\\ &= \varsigma^2\,\mathscr{H}\left(T_\kappa\left(t,x-y\right)\right) - \lambda^2\varsigma^2\,\Theta_\kappa\left(t,x,y\right).\end{aligned}$$

Finally, combing these terms, we have then

$$\mathbb{E}\left[u(t,x)u\left(t,y\right)\right] = \frac{1}{\lambda^2}\mathscr{K}\left(T_\kappa\left(t,x-y\right),\frac{x+y}{2}\right) + \varsigma^2\,\mathscr{H}\left(T_\kappa\left(t,x-y\right)\right),$$

which finishes the whole proof. □

4.4.2 Optimality of the Hölder Exponents (Proof of Proposition 4.2.6)

To prove Proposition 4.2.6, a key ingredient is the following lemma.

Lemma 4.4.1. *If the initial position is* $g(x) = |x|^{-a}$ *with* $a \in [0,1/2[$ *and the initial velocity* *is* $\mu \equiv 0$, *then*

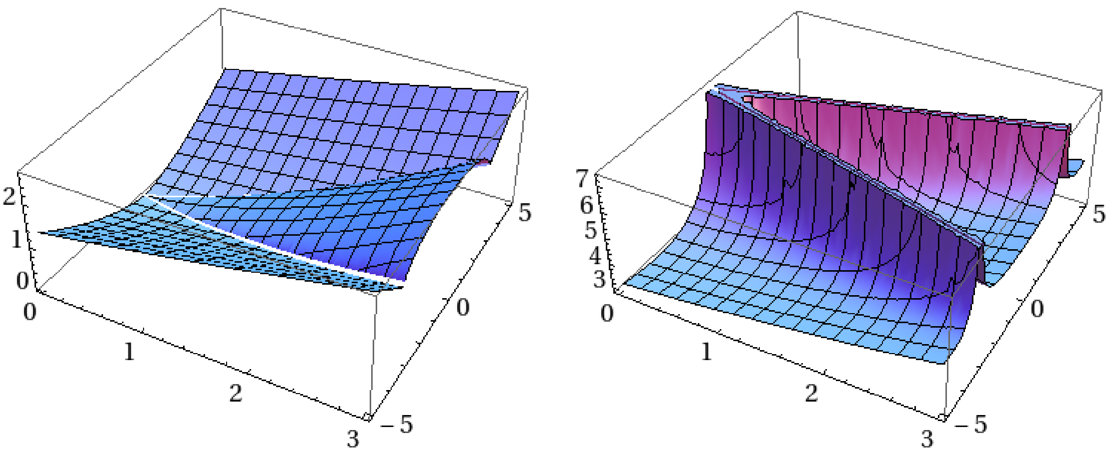
$$\left(J_0^2 \star G_\kappa^2\right)(t,x) = \begin{cases} \frac{a^2-4a+2}{32\kappa(1-2a)(1-a)^2}\,|\kappa t - x|^{2(1-a)}, & \text{if } x < -\kappa t,\\ \frac{1}{32\kappa(1-a)^2}\left[(\kappa t-x)^{1-a}+(\kappa t+x)^{1-a}\right]^2 & \\ \qquad +\frac{t}{16(1-2a)}\left[(\kappa t-x)^{1-2a}+(\kappa t+x)^{1-2a}\right], & \text{if } |x|\le \kappa t,\\ \frac{a^2-4a+2}{32\kappa(1-2a)(1-a)^2}\,|\kappa t + x|^{2(1-a)}, & \text{if } x > \kappa t, \end{cases}$$

where $J_0(t,x) = \left(g\left(x-\kappa t\right)+g\left(x+\kappa t\right)\right)/2$.

As a special case, for $a = 1/4$, we have

$$\left(J_0^2 \star G_\kappa^2\right)(t,x) = \begin{cases} \frac{17}{144\kappa}\,|\kappa t - x|^{3/2}, & \text{if } x < -\kappa t,\\ \frac{1}{18\kappa}\left[(\kappa t-x)^{3/4}+(\kappa t+x)^{3/4}\right]^2 & \\ \qquad +\frac{t}{8}\left[(\kappa t-x)^{1/2}+(\kappa t+x)^{1/2}\right], & \text{if } |x|\le \kappa t,\\ \frac{17}{144\kappa}\,|\kappa t + x|^{3/2}, & \text{if } x > \kappa t. \end{cases}$$

This function is plotted in Figure 4.9 (a).



(a) $\left(J_0^2 \star G_\kappa^2\right)(t,x)$ (b) $||u_1(t,x)||_2^2 = J_0^2(t,x) + \left(J_0^2 \star G_\kappa^2\right)(t,x)$

Figure 4.9 – $\kappa = 1$, $g(x) = |x|^{-1/4}$, $\mu \equiv 0$. The plot ranges are $0 \le t \le 3$ and $|x| \le 5$. $J_0(t,x) = \frac{1}{2}\left(|x-t|^{-1/4} + |x+t|^{-1/4}\right)$. $u_1(t,x)$ is the random field in the first Picard iteration.

Proof of Lemma 4.4.1. We first assume that $|x| \le \kappa t$. Then

$$\left(J_0^2 \star G_\kappa^2\right)(t,x) = \frac{1}{16}\int_0^t \mathrm{d}s \int_{x-\kappa(t-s)}^{x+\kappa(t-s)} \left(g(y-\kappa s)+g(y+\kappa s)\right)^2 \mathrm{d}y = \frac{1}{16}\left(S_1+S_2+S_3\right),$$

where S_1, S_2 and S_3 correspond to the integrations in the regions I, II and III shown in Figure 4.4. To evaluate these three integrals, we change the variables: $w = \kappa s + y$ and $u = \kappa s - y$ (see Figure 4.4). Then

$$S_1 = \frac{1}{2\kappa}\int_{x-\kappa t}^{0} \mathrm{d}w \int_{-w}^{-x+\kappa t} \left(|u|^{-a}+|w|^{-a}\right)^2 \mathrm{d}u = \frac{a^2-4a+2}{2\kappa(1-2a)(1-a)^2}\,(\kappa t - x)^{2(1-a)}.$$

Similarly,

$$S_2 = \frac{1}{2\kappa}\int_0^{x+\kappa t} \mathrm{d}w \int_{-w}^{0} \left(|u|^{-a}+|w|^{-a}\right)^2 \mathrm{d}u = \frac{a^2-4a+2}{2\kappa(1-2a)(1-a)^2}\,(\kappa t+x)^{2(1-a)}\,.$$

As for S_3, we have that

$$\begin{aligned} S_3 &= \frac{1}{2\kappa}\int_0^{x+\kappa t} \mathrm{d}w \int_0^{-x+\kappa t} \left(|u|^{-a}+|w|^{-a}\right)^2 \mathrm{d}u \\ &= \frac{1}{\kappa(1-a)^2}\left(\kappa^2 t^2 - x^2\right)^{1-a} + \frac{1}{2\kappa(1-2a)}\left((\kappa t-x)^{1-2a}(\kappa t+x)+(\kappa t+x)^{1-2a}(\kappa t-x)\right). \end{aligned}$$

Use the fact that

$$\frac{a^2-4a+2}{2\kappa(1-2a)(1-a)^2} = \frac{(1-2a)+(1-a)^2}{2\kappa(1-2a)(1-a)^2} = \frac{1}{2\kappa(1-a)^2}+\frac{1}{2\kappa(1-2a)}$$

to sum up these S_i. The other two cases, $x<-\kappa t$ and $x>\kappa t$, can be calculated similarly to S_1 and S_2 respectively. This completes the proof. □

Proof of Proposition 4.2.6. Let $I(t,x)$ be the stochastic integral part of random field solution, i.e., $u(t,x) = J_0(t,x)+I(t,x)$. For (t,x) and $(t',x')\in\mathbb{R}_+\times\mathbb{R}$,

$$\begin{aligned} \left|\left|I(t,x)-I(t',x')\right|\right|_p^2 &\ge \left|\left|I(t,x)-I(t',x')\right|\right|_2^2 \\ &= \lambda^2 \iint_{\mathbb{R}_+\times\mathbb{R}} \left(G_\kappa\left(t-s,x-y\right)-G_\kappa(t'-s,x'-y)\right)^2 \left(\varsigma^2+\left|\left|u\left(s,y\right)\right|\right|_2^2\right)\mathrm{d}s\mathrm{d}y \\ &\ge \lambda^2 \iint_{\mathbb{R}_+\times\mathbb{R}} \left(G_\kappa\left(t-s,x-y\right)-G_\kappa(t'-s,x'-y)\right)^2 J_0^2\left(s,y\right))\mathrm{d}s\mathrm{d}y\,, \end{aligned} \tag{4.4.5}$$

since $\varsigma^2+\left|\left|u\left(s,y\right)\right|\right|_2^2 \ge J_0^2\left(s,y\right)$.

Spatial increments. Let us first consider the spatial increments. Fix $t>0$. For x and $x'\in\mathbb{R}$, by (4.3.8), the inequality (4.4.5) reduces to

$$\begin{aligned} \left|\left|I(t,x)-I(t,x')\right|\right|_p^2 \ge \lambda^2 &\iint_{\mathbb{R}_+\times\mathbb{R}} \mathrm{d}s\mathrm{d}y \\ \times J_0^2\left(s,y\right)&\left(G_\kappa^2\left(t-s,x-y\right)-2G_\kappa^2\left(T_\kappa(t,x-x')-s,\frac{x+x'}{2}-y\right)+G_\kappa^2(t-s,x'-y)\right). \end{aligned}$$

Denote this lower bound by $\lambda^2 L(t,x,x')$. Then

$$L(t,x,x') = \left(J_0^2\star G_\kappa^2\right)(t,x)+\left(J_0^2\star G_\kappa^2\right)(t,x')-2\left(J_0^2\star G_\kappa^2\right)\left(T_\kappa(t,x-x'),\frac{x+x'}{2}\right).$$

Let $x=\kappa t$ and $x'<x$ such that $|x'-x|\le 2\kappa t$. Hence, $T_\kappa(t,x-x') = t-|x-x'|/(2\kappa)$. By Lemma 4.4.1, we know that

$$\left(J_0^2\star G_\kappa^2\right)(t,\kappa t) = \frac{a^2-4a+2}{32\kappa(1-2a)(1-a)^2}(2\kappa t)^{2(1-a)}$$

$$= \frac{1}{32\kappa(1-a)^2}(2\kappa t)^{2(1-a)} + \frac{t}{16(1-2a)}(2\kappa t)^{1-2a},$$

and

$$\left(J_0^2 \star G_\kappa^2\right)(t,x') = \frac{1}{32\kappa(1-a)^2}\left[\left(\kappa t - x'\right)^{1-a} + \left(\kappa t + x'\right)^{1-a}\right]^2 + \frac{t}{16(1-2a)}\left[\left(\kappa t - x'\right)^{1-2a} + \left(\kappa t + x'\right)^{1-2a}\right],$$

and

$$\left(J_0^2 \star G_\kappa^2\right)\left(T_\kappa(t,x-x'), \frac{x+x'}{2}\right) = \frac{1}{32\kappa(1-a)^2}\left[(\kappa t - x)^{1-a} + \left(\kappa t + x'\right)^{1-a}\right]^2 + \frac{t}{16(1-2a)}\left[(\kappa t - x)^{1-2a} + \left(\kappa t + x'\right)^{1-2a}\right],$$

where in the last equality we have used the fact that

$$\kappa\left(t - \frac{x-x'}{2\kappa}\right) + \frac{x+x'}{2} = \kappa t + x', \quad \text{and} \quad \kappa\left(t - \frac{x-x'}{2\kappa}\right) - \frac{x+x'}{2} = \kappa t - x.$$

Hence,

$$L(t,\kappa t,x') = \frac{1}{32\kappa(1-a)^2}L_1(t,x') + \frac{t}{16(1-2a)}L_2(t,x'),$$

where

$$L_1(t,x') := (2\kappa t)^{2(1-a)} + \left[\left(\kappa t - x'\right)^{1-a} + \left(\kappa t + x'\right)^{1-a}\right]^2 - 2\left(\kappa t + x'\right)^{2(1-a)},$$

and

$$L_2(t,x') := (2\kappa t)^{1-2a} + (\kappa t - x')^{1-2a} - (\kappa t + x')^{1-2a}.$$

Let $h = \kappa t - x'$. Then

$$L_1(t,x') = (2\kappa t)^{2(1-a)} + \left[h^{1-a} + (2\kappa t - h)^{1-a}\right]^2 - 2(2\kappa t - h)^{2(1-a)} \geq h^{2(1-a)},$$

and

$$L_2(t,x') = (2\kappa t)^{1-2a} + h^{1-2a} - (2\kappa t - h)^{1-2a} \geq h^{1-2a}.$$

Since $1-2a \in\ ]0,1]$ and $2(1-a) \in\ ]1,2]$, by discarding $L_1(t,x')$, we have

$$||I(t,\kappa t) - I(t,\kappa t - h)||_p^2 = \lambda^2 L(t,\kappa t,x') \geq \frac{\lambda^2 t}{16(1-2a)}h^{1-2a}.$$

Time increments. Now fix $x \in \mathbb{R}$. By symmetry, we assume that $x > 0$. For $t' \geq t \geq 0$, the inequality (4.4.5) reduces to

$$\left|\left|I(t,x) - I(t',x)\right|\right|_p^2 \geq \lambda^2 \iint_{\mathbb{R}_+\times\mathbb{R}} J_0^2(s,y)\left(G_\kappa^2(t'-s,x-y) - G_\kappa^2(t-s,x-y)\right)\mathrm{d}s\mathrm{d}y$$
$$= \lambda^2\left(\left(J_0^2 \star G_\kappa^2\right)(t',x) - \left(J_0^2 \star G_\kappa^2\right)(t,x)\right),$$

since $G_\kappa(t,x)G_\kappa(t',x) = G_\kappa^2(t,x)$. Then take $t = x/\kappa$ and $h = t'-t = t'-x/\kappa$ and so, similarly to the previous case,

$$\left(J_0^2 \star G_\kappa^2\right)(x/\kappa,x) = \frac{1}{32\kappa(1-a)^2}(2x)^{2(1-a)} + \frac{x}{16\kappa(1-2a)}(2x)^{1-2a}$$

and

$$\begin{aligned}\left(J_0^2 \star G_\kappa^2\right)\left(t',x\right) =& \frac{1}{32\kappa(1-a)^2}\left[\left(\kappa t'-x\right)^{1-a} + \left(\kappa t'+x\right)^{1-a}\right]^2 \\ &+ \frac{x}{16\kappa(1-2a)}\left[\left(\kappa t'-x\right)^{1-2a} + \left(\kappa t'+x\right)^{1-2a}\right] \\ =& \frac{1}{32\kappa(1-a)^2}\left[(\kappa h)^{1-a} + (\kappa h+2x)^{1-a}\right]^2 \\ &+ \frac{x}{16\kappa(1-2a)}\left[(\kappa h)^{1-2a} + (\kappa h+2x)^{1-2a}\right].\end{aligned}$$

Hence, by symmetry, for all $x \in \mathbb{R}$, and $h = t' - |x|/\kappa > 0$,

$$\left|\left|I\left(|x|/\kappa,x\right) - I(t',x)\right|\right|_p^2 \geq \frac{\lambda^2|x|}{16\kappa^{2a}(1-2a)}h^{1-2a}.$$

Finally, we conclude that in the neighborhood of the characteristic lines $|x| = \kappa t$, the the function $(t,x) \mapsto I(t,x)$ from $\mathbb{R}_+ \times \mathbb{R}$ to $L^p(\Omega)$ cannot be ρ_1-Hölder continuous in space and ρ_2-Hölder continuous in time with $\rho_i = \frac{1-2a}{2}$, $i = 1,2$. This completes the proof. □

4.4.3 Proof of Full Intermittency (Theorem 4.2.8)

Proof of Theorem 4.2.8. In this case, $J_0(t,x) = w + \kappa\widetilde{w}t$.

(1) If $|\overline{\varsigma}| + |w| + |\widetilde{w}| = 0$, then $J_0(t,x) \equiv 0$ and $\rho(0) = 0$, so $u(t,x) \equiv 0$ and the bound is trivially true. Now suppose that $|\overline{\varsigma}| + |w| + |\widetilde{w}| \neq 0$. By (4.2.6), for all even integers $p \geq 2$,

$$\begin{aligned}||u(t,x)||_p^2 &\leq 2\left(w+\kappa\widetilde{w}t\right)^2 + \int_0^t \mathrm{d}s\left[\overline{\varsigma}^2 + 2\left(w+\kappa\widetilde{w}s\right)^2\right]\int_{\mathbb{R}}\mathrm{d}x\,\widehat{\mathscr{K}_p}(t-s,x) \\ &\leq 2\left(w+\kappa\widetilde{w}t\right)^2 + \left[\overline{\varsigma}^2 + 2\left(w+\kappa\widetilde{w}t\right)^2\right]\widehat{\mathscr{H}_p}(t) \\ &\leq \left[\overline{\varsigma}^2 + 2\left(w+\kappa\widetilde{w}t\right)^2\right]\cosh\left(a_{p,\overline{\varsigma}}\, z_p\, L_\rho\sqrt{\kappa/2}\, t\right).\end{aligned}$$

Note that the second term on the right-hand side of the above inequality is non vanishing since $|\overline{\varsigma}| + |w| + |\widetilde{w}| \neq 0$. Hence,

$$\overline{\lambda}_p \leq a_{p,\overline{\varsigma}}\, z_p\, L_\rho\sqrt{\kappa/2}\,\frac{p}{2}.$$

Then using the fact that $a_{p,\overline{\varsigma}} \leq 2$ and $z_p \leq 2\sqrt{p}$, we have that $\overline{\lambda}_p \leq \sqrt{2\kappa}\, L_\rho\, p^{3/2}$.

(2) By (4.2.8) with $p = 2$ and Corollary 4.2.2,

$$||u(t,x)||_2^2 \geq -\underline{\varsigma}^2 - \frac{4\kappa\,\widetilde{w}^2}{l_\rho^2} + \left(w^2 + \underline{\varsigma}^2 + \frac{4\kappa\,\widetilde{w}^2}{l_\rho^2}\right)\cosh\left(|\,l_\rho\,|\sqrt{\kappa/2}\;t\right)$$

Clearly, $|\underline{\varsigma}| + |w| + |\widetilde{w}| \neq 0$ implies that $\underline{\lambda}_2 \geq |\,l_\rho\,|\sqrt{\kappa/2}$. This completes the proof. □

4.4.4 Proof of Exponential Growth Indices (Theorem 4.2.11)

Proof of Theorem 4.2.11. The statements of (1) and (2) are a consequence of the two propositions 4.4.4 and 4.4.7 below. More precisely, let $J_{0,1}(t,x)$ and $J_{0,2}(t,x)$ be the homogeneous solution contributed by the initial position g and initial velocity μ, respectively. Clearly, when both g and μ are nonvanishing,

$$J_0(t,x) = J_{0,1}(t,x) + J_{0,2}(t,x)\,.$$

For the upper bounds, we use the fact that

$$J_0^2(t,x) \leq 2J_{0,1}^2(t,x) + 2J_{0,2}^2(t,x)\,.$$

Using the upper bounds of the p-th moment in (4.2.6), we simply choose the larger of the upper bounds between Proposition 4.4.4 (1) and Proposition 4.4.7 (1).

As for the lower bounds, notice that both g and μ are nonnegative, so

$$J_0^2(t,x) \geq J_{0,1}^2(t,x) + J_{0,2}^2(t,x)\,.$$

Hence, using the lower bound of the second moment in (4.2.8), we only need to take the larger of the lower bounds between Proposition 4.4.4 (2) and Proposition 4.4.7 (2). This proves both (1) and (2). Part (3) is a direct consequence of (1) and (2). When the initial data have compact support, both (1) and (2) hold for all $\beta_i > 0$ in Part (1) with $i = 1,2$ and all $\beta > 0$ in Part (2). Then letting these β's tend to $+\infty$ proves Part (4). □

Contributions of the initial position

We first consider the case where $\mu \equiv 0$. Suppose $|g(x)| \leq Ce^{-\beta|x|}$ for some constants $C > 0$ and $\beta > 0$.

Lemma 4.4.2. *Suppose that* $a \neq c$, $t > 0$ *and* $b \in [0,1]$. *Then*

$$\int_{bt}^{t} \cosh\left(a(t-s)\right)\sinh\left(cs\right)\mathrm{d}s = \frac{1}{a^2-c^2}\Big(c\cosh(bct)\cosh\left(a(1-b)t\right) - c\cosh(ct) + a\sinh(bct)\sinh\left(a(1-b)t\right)\Big)\,. \quad (4.4.6)$$

Proof. Denote the integral by I. Apply integration by parts twice,

$$
\begin{aligned}
I &= \frac{1}{c}\cosh(cs)\cosh(a(t-s))\Big|_{s=bt}^{s=t} + \frac{a}{c}\int_{bt}^{t}\cosh(cs)\sinh(a(t-s))\mathrm{d}s \\
&= \frac{1}{c}\left[\cosh(ct)-\cosh(bct)\cosh(a(1-b)t)\right] + \frac{a}{c^2}\sinh(cs)\sinh(a(t-s))\Big|_{s=bt}^{s=t} + \frac{a^2}{c^2}I\,.
\end{aligned}
$$

Therefore,

$$
\left(c^2-a^2\right)I = c\left[\cosh(ct)-\cosh(bct)\cosh(a(1-b)t)\right] - a\sinh(bct)\sinh(a(1-b)t)\,,
$$

which finishes the proof. ☐

Lemma 4.4.3. *Let $f(t,x)=\frac{1}{2}\left(e^{-\beta|x-\kappa t|}+e^{-\beta|x+\kappa t|}\right)H(t)$, where $H(t)$ is the Heaviside function. We have*

(1) for $(t,x)\in\mathbb{R}_+^*\times\mathbb{R}$ and $\beta>0$,

$$
\left(f\star G_\kappa\right)(t,x) = \begin{cases} \dfrac{t}{2\beta}\left(1-e^{-\beta\kappa t}\cosh(\beta x)\right) & \text{if } |x|\le\kappa t\,,\\[2ex] \dfrac{t}{2\beta}e^{-\beta|x|}\sinh(\beta\kappa t) & \text{if } |x|>\kappa t\,.\end{cases}
$$

Moreover, for fixed $t>0$, the above convolution decreases as $|x|$ increases.

(2) for $(t,x)\in\mathbb{R}_+^*\times\mathbb{R}$, $\beta>0$ and $a,b\in\,]0,1[$,

$$
\left(f\star\mathcal{K}\right)(t,x) \ge \begin{cases} \dfrac{1}{2}e^{-\beta\kappa t}\cosh(\beta|x|)\left(I_0\left(\sqrt{\dfrac{\lambda^2(\kappa^2t^2-x^2)}{2\kappa}}\right)-1\right) & \text{if } |x|\le\kappa t,\\[3ex] \dfrac{\lambda^2e^{-\beta|x|}}{2(1-a^2)\beta^2\kappa}I_0\left(\sqrt{\dfrac{\lambda^2(1-a^2)}{2\kappa}}\,b\kappa t\right)g(t\,;a,b,\beta,\kappa) & \text{if } |x|>\kappa t\,,\end{cases}
$$

where

$$
\begin{aligned}
g\left(t\,;a,b,\beta,\kappa\right) :=\, & a\cosh\left(abt\beta\kappa\right)\cosh\left((1-b)t\beta\kappa\right) - a\cosh\left(at\beta\kappa\right)\\
& + \sinh\left((1-b)t\beta\kappa\right)\sinh\left(abt\beta\kappa\right)\,.
\end{aligned}
$$

Proof. (1) We consider three cases: I ($x<-\kappa t$), II ($x>\kappa t$) and III ($|x|\le\kappa t$); See Figure 4.4.

We first consider Case I: $x<-\kappa t$. In this case, $f(t,x)=\frac{1}{2}\left(e^{\beta(x-\kappa t)}+e^{\beta(x+\kappa t)}\right)H(t)$. Hence,

$$
\left(f\star G_\kappa\right)(t,x) = \frac{1}{2}\int_0^t \mathrm{d}s\int_{-\kappa s}^{\kappa s}\mathrm{d}y\,\frac{1}{2}\left(e^{\beta(x-y-\kappa(t-s))}+e^{\beta(x-y+\kappa(t-s))}\right)
$$

$$
\begin{aligned}
&= \frac{1}{4\beta}\int_0^t \left(e^{\beta(x-\kappa t+2\kappa s)} - e^{\beta(x-\kappa t)} + e^{\beta(x+\kappa t)} - e^{\beta(x+\kappa t-2\kappa s)}\right) \mathrm{d}s \\
&= \frac{t}{4\beta}\left(e^{\beta(x+\kappa t)} - e^{\beta(x-\kappa t)}\right) + \frac{1}{8\kappa\beta^2}\left(e^{\beta(x+\kappa t)} - e^{\beta(x-\kappa t)} + e^{\beta(x-\kappa t)} - e^{\beta(x+\kappa t)}\right) \\
&= \frac{te^{-\beta|x|}\sinh(\beta\kappa t)}{2\beta}\,.
\end{aligned}
$$

Similarly, for Case II, we have the same formula.

Now let us consider Case III: $|x| \le \kappa t$. As shown in Figure 4.4, we decompose the space-time convolution into three parts S_i, $i = 1,2,3$. Denote the corresponding three regions D_i, $i = 1,2,3$. Clearly,

$$
\left(f \star G_\kappa\right)(t,x) = \sum_{i=1}^{3} S_i = \sum_{i=1}^{3} \frac{1}{2}\iint_{D_i} f\left(s,y\right)\mathrm{d}s\mathrm{d}y\,,
$$

where the factor $1/2$ comes from the kernel function $G_\kappa(t,x)$. Now S_1 and S_2 can be calculated by the formula in Case I and II with (t,x) replaced by $\left(\frac{1}{2\kappa}(\kappa t - x), \frac{1}{2}(x-\kappa t)\right)$ and $\left(\frac{1}{2\kappa}(x+\kappa t), \frac{1}{2}(x+\kappa t)\right)$ respectively. In particular, after some simplifications,

$$
S_1 + S_2 = \frac{e^{-\beta\kappa t}(x\sinh(\beta x) - \kappa t\cosh(\beta x)) + \kappa t}{4\kappa\beta}\,.
$$

To calculate S_3, we change the variable $w = \kappa s + y$ and $u = \kappa s - y$; see Figure 4.4. Hence,

$$
S_3 = \frac{1}{8\kappa}\int_0^{x+\kappa t}\mathrm{d}w\int_0^{\kappa t - x}\left(e^{-\beta w} + e^{-\beta u}\right)\mathrm{d}u
$$

where we have used the fact that $w, u \ge 0$ in D_3. This integral can be easily calculated

$$
S_3 = \frac{\kappa t - e^{-\beta\kappa t}(\kappa t\cosh(\beta x) + x\sinh(\beta x))}{4\kappa\beta}\,.
$$

Hence, the space-time convolution is proved by summing up these three integrals. Finally, it is clear that for fixed $t > 0$, the convolution decreases as $|x|$ increases. This completes the proof of (1).

(2) Similarly, we consider three cases. Let us first consider Case III: $|x| \le \kappa t$. Let S_i, $i = 1,2,3$ be the integral of $f\left(s,y\right)\mathscr{K}\left(t-s,x-y\right)$ over the three regions as shown in Figure 4.4. Clearly,

$$
\left(f \star \mathscr{K}\right)(t,x) \ge S_3\,.
$$

Notice that in this case, $f\left(s,y\right) \ge \frac{1}{2}\left(e^{-\beta(\kappa t - x)} + e^{-\beta(\kappa t + x)}\right)$ for all $\left(s,y\right)$ in Region III of Figure 4.4. Hence,

$$
\begin{aligned}
S_3 &\ge \left(e^{-\beta(\kappa t - x)} + e^{-\beta(\kappa t + x)}\right)\left(G_\kappa^2 \star \mathscr{K}\right)(t,x) \\
&= \frac{2}{\lambda^2}e^{-\beta\kappa t}\cosh\left(\beta|x|\right)\left(\mathscr{L}_0 \star \mathscr{K}\right)(t,x)\,.
\end{aligned}
$$

This case is then proved by (4.3.5).

Now let us consider Case I: $x \le -\kappa t$. Fix $a, b \in]0,1[$. Then

$$\begin{aligned}
\left(f \star \mathcal{K}\right)(t,x) &= \frac{\lambda^2}{8}\int_0^t \mathrm{d}s \int_{-\kappa s}^{\kappa s} \mathrm{d}y\; I_0\left(\sqrt{\frac{\lambda^2(\kappa^2 s^2 - y^2)}{2\kappa}}\right)\left(e^{\beta(x-y-\kappa(t-s))} + e^{\beta(x-y+\kappa(t-s))}\right)\\
&\ge \frac{\lambda^2}{8}\int_{bt}^t \mathrm{d}s \int_{-a\kappa s}^{a\kappa s} \mathrm{d}y\; I_0\left(\sqrt{\frac{\lambda^2(\kappa^2 s^2 - y^2)}{2\kappa}}\right)\left(e^{\beta(x-y-\kappa(t-s))} + e^{\beta(x-y+\kappa(t-s))}\right)\\
&\ge \frac{\lambda^2}{8}\int_{bt}^t \mathrm{d}s\; I_0\left(\sqrt{\frac{\lambda^2(1-a^2)}{2\kappa}}\,\kappa s\right)\int_{-a\kappa s}^{a\kappa s} \mathrm{d}y\;\left(e^{\beta(x-y-\kappa(t-s))} + e^{\beta(x-y+\kappa(t-s))}\right)\\
&\ge \frac{\lambda^2}{8} I_0\left(\sqrt{\frac{\lambda^2(1-a^2)}{2\kappa}}\,b\kappa t\right)\int_{bt}^t \mathrm{d}s \int_{-a\kappa s}^{a\kappa s} \mathrm{d}y\;\left(e^{\beta(x-y-\kappa(t-s))} + e^{\beta(x-y+\kappa(t-s))}\right).
\end{aligned} \tag{4.4.7}$$

Notice that

$$e^{\beta(x-y-\kappa(t-s))} + e^{\beta(x-y+\kappa(t-s))} = 2e^{\beta x}\cosh(\kappa(t-s))e^{-\beta y}.$$

So by using the integral in (4.4.6), we have

$$\begin{aligned}
&\int_{bt}^t \mathrm{d}s \int_{-a\kappa s}^{a\kappa s} \mathrm{d}y\;\left(e^{\beta(x-y-\kappa(t-s))} + e^{\beta(x-y+\kappa(t-s))}\right)\\
&\qquad = 2e^{\beta x}\int_{bt}^t \mathrm{d}s\; \cosh(\kappa\beta(t-s)) \int_{-a\kappa s}^{a\kappa s} \mathrm{d}y\; e^{-\beta y}\\
&\qquad = \frac{4e^{\beta x}}{\beta}\int_{bt}^t \cosh(\kappa\beta(t-s))\sinh(a\beta\kappa s)\mathrm{d}s\\
&\qquad = \frac{4e^{\beta x}}{\beta^3\kappa^2(1-a^2)}\Big[a\beta\kappa\cosh(ba\beta\kappa t)\cosh(\kappa\beta(1-b)t)\\
&\qquad\quad - a\beta\kappa\cosh(a\beta\kappa t) + \kappa\beta\sinh(ba\beta\kappa t)\sinh(\kappa\beta(1-b)t)\Big]\\
&\qquad = \frac{4e^{-\beta|x|}}{(1-a^2)\beta^2\kappa} g(t\,; a, b, \beta, \kappa),
\end{aligned}$$

where the function $g(t\,; a, b, \beta, \kappa)$ is defined in the statement of the lemma. This completes the proof of (2). □

Now let us calculate the upper growth indices. One useful asymptotic formula is that the hyperbolic Bessel function of order n has the following asymptotic behavior (see, e.g., [51, (10.30.4) on p. 252]):

$$I_n(x) \sim \frac{e^x}{\sqrt{2\pi x}}, \quad \text{as } x \to \infty, \text{ for all } n \ne -1, -2, \dots. \tag{4.4.8}$$

Proposition 4.4.4. *Suppose that the initial velocity measure $\mu \equiv 0$ vanishes.*

(1) If $|\rho(u)| \le L_\rho |u|$ with $L_\rho \ne 0$ and the initial position $g(x)$ is a Borel measurable

function such that for some constants $C>0$ and $\beta>0$,

$$|g(x)|\le C\,e^{-\beta|x|}\quad \text{for almost all } x\in\mathbb{R}\,,$$

then for all $p\ge 2$ even integers, the upper growth indices of order p satisfy the upper bounds:

$$\overline{\lambda}(p)\le\begin{cases}(2\beta)^{-1}\,z_p\sqrt{\kappa}\,L_\rho+\kappa & p>2\,,\\(4\beta)^{-1}\,\sqrt{2\kappa}\,L_\rho+\kappa & p=2\,.\end{cases}$$

(2) *If $|\rho(u)|\ge l_\rho\,|u|$ with $l_\rho\ne 0$ and the initial position $g(x)$ is a Borel measurable function such that for some constants $c>0$ and $\beta>0$,*

$$|g(x)|\ge c\,e^{-\beta|x|}\,,\quad \text{for almost all } x\in\mathbb{R}\,,$$

then for all even integers $p\ge 2$, the lower growth indices of order p satisfy the lower bound:

$$\underline{\lambda}(p)\ge\kappa\left(1+\frac{l_\rho^2}{8\kappa\,\beta^2}\right)^{1/2}.$$

In particular, for the hyperbolic Anderson model $\rho(u)=\lambda u$ with $\lambda\ne 0$, if the initial position $g(x)$ satisfies both Conditions (1) and (2), then

$$\kappa\left(1+\frac{\lambda^2}{8\kappa\,\beta^2}\right)^{1/2}\le\underline{\lambda}(2)\le\overline{\lambda}(2)\le\kappa\left(1+\sqrt{\frac{\lambda^2}{8\kappa\,\beta^2}}\right).$$

Proof. (1) Let $J_0(t,x)=\frac{1}{2}\left(g(x-\kappa t)+g(x+\kappa t)\right)H(t)$. By the assumptions on $g(x)$,

$$|J_0(t,x)|^2\le\frac{C^2}{2}\left(e^{-2\beta|x-\kappa t|}+e^{-2\beta|x+\kappa t|}\right)H(t)\,,$$

for almost all $(t,x)\in\mathbb{R}_+\times\mathbb{R}$. We first consider the case $p>2$. Using the moment formula (4.2.6) and the upper bound of $\widehat{\mathcal{K}_p}(t,x)$ in Lemma 4.3.2, we have that

$$||u(t,x)||_p^2\le 2J_0^2(t,x)+\frac{a_{p,\overline{\varsigma}}^2\,z_p^2\,L_\rho^2}{2}I_0\left(a_{p,\overline{\varsigma}}\,z_p\,L_\rho\,\sqrt{\kappa/2}\,t\right)\left(2|J_0|^2\star G_\kappa\right)(t,x)\,.$$

Then by Lemma 4.4.3 and the asymptotic formula (4.4.8) for $I_0(x)$, we have that

$$\lim_{t\to\infty}\frac{1}{t}\sup_{|x|\ge\alpha t}\log||u(t,x)||_p^p\le\begin{cases}4^{-1}a_{p,\overline{\varsigma}}\,z_p\sqrt{2\kappa}\,L_\rho\,p & \text{if } \alpha\in[0,\kappa]\,,\\4^{-1}a_{p,\overline{\varsigma}}\,z_p\sqrt{2\kappa}\,L_\rho\,p-\beta\,p(\alpha-\kappa) & \text{if } \alpha>\kappa\,,\end{cases}$$

where we have used the fact that the upper bound is decreasing in $|x|$ so that the supremum over $|x|\ge\alpha t$ is attained at $|x|=\alpha t$ (see Lemma 4.3.2). Therefore,

$$\overline{\lambda}(p)\le\frac{1}{4\beta}a_{p,\overline{\varsigma}}\,z_p\sqrt{2\kappa}\,L_\rho+\kappa\,.$$

Then use the fact that $\overline{\varsigma} = 0$ and $a_{p,0} = \sqrt{2}$ (see (1.4.4)). Similarly, for the case $p = 2$, we simply replace both of the above z_p and $a_{p,\overline{\varsigma}}$ by 1.

(2) We only need to consider $p = 2$. Without loss of generality, assume $\rho(u) = \lambda u$. For such initial data, we have that

$$J_0^2(t,x) \ge \frac{c^2}{4}\left(e^{-\beta|x-\kappa t|} + e^{-\beta|x+\kappa t|}\right)^2 \ge \frac{c^2}{4}\left(e^{-2\beta|x-\kappa t|} + e^{-2\beta|x+\kappa t|}\right).$$

If $|x| \le \kappa t$, by the lower bound of the second moment in (4.2.8) and Lemma 4.4.3,

$$||u(t,x)||_2^2 \ge \left(J_0^2 \star \mathscr{K}\right)(t,x) \ge \frac{c^2}{4} e^{-2\beta\kappa t}\cosh(2\beta|x|)\left(I_0\left(\sqrt{\frac{\lambda^2(\kappa^2t^2-x^2)}{2\kappa}}\right)-1\right).$$

Hence, for $0 \le \alpha \le \kappa$, by (4.4.8),

$$\lim_{t\to+\infty}\sup_{|x|\ge\alpha t}\log||u(t,x)||_2^2 \ge -2\beta\kappa + 2\beta\alpha + |\lambda|\sqrt{\frac{\kappa^2-\alpha^2}{2\kappa}}.$$

The equation

$$h(\alpha) := -2\beta\kappa + 2\beta\alpha + \frac{|\lambda|}{\sqrt{2\kappa}}\sqrt{\kappa^2-\alpha^2} = 0$$

has two solutions,

$$\alpha_1 = \kappa, \qquad \alpha_2 = \kappa\,\frac{8\kappa\beta^2-\lambda^2}{8\kappa\beta^2+\lambda^2} \le \kappa.$$

As α tends to κ from left side, $h(\alpha)$ remains positive, which can be seen by $h''(\alpha) = -\frac{|\lambda|}{\sqrt{2\kappa}}\left(\frac{\alpha^2}{(\kappa^2-\alpha^2)^{3/2}} + \frac{1}{(\kappa^2-\alpha^2)^{1/2}}\right) \le 0$ for $0 \le \alpha \le \kappa$. Therefore, we can conclude that $\underline{\lambda}(2) \ge \kappa$.

Now let us consider Case II: $x \le -\kappa t$. Again, by Lemma 4.4.3, for all $a, b \in\,]0,1[$,

$$||u(t,x)||_2^2 \ge \left(J_0^2 \star \mathscr{K}\right)(t,x) \ge \frac{c^2\lambda^2 e^{-2\beta|x|}}{4(1-a^2)\beta^2\kappa} I_0\left(\sqrt{\frac{\lambda^2(1-a^2)}{2\kappa}}b\kappa t\right) g(t\,;a,b,2\beta,\kappa),$$

where $g(t,x;a,b,\beta,\kappa)$ is defined in Lemma 4.4.3. For large t, by replacing both $\cosh(Ct)$ and $\sinh(Ct)$ by $\exp(Ct)/2$ with $C \ge 0$, and using the fact that

$$\cosh(Ct) - 1 \approx \frac{e^{Ct}}{2}, \quad C > 0,$$

we know that

$$g(t\,;a,b,2\beta,\kappa) \approx \frac{1+a}{4}\exp\left(2(1+(a-1)b)t\beta\kappa\right).$$

Hence, for $\alpha > \kappa$, by (4.4.8),

$$\lim_{t\to\infty}\frac{1}{t}\sup_{|x|\ge\alpha t}\log||u(t,x)||_2^2 \ge \sqrt{\frac{\lambda^2(1-a^2)}{2\kappa}}b\kappa - 2\beta\alpha + 2(1-(1-a)b)\beta\kappa.$$

The inequality

$$h(\alpha) := \sqrt{\frac{\lambda^2(1-a^2)}{2\kappa}}\, b\kappa - 2\beta\alpha + 2(1-(1-a)b)\beta\kappa > 0$$

is equivalent to

$$\alpha < \left(\sqrt{\frac{\lambda^2(1-a^2)}{2\kappa}}\,\frac{b}{2\beta} + 1 - (1-a)b\right)\kappa\,.$$

Since $a \in \,]0,1[$ is arbitrary, we can choose

$$a := \underset{a\in\,]0,1[}{\arg\max}\left(\sqrt{\frac{\lambda^2(1-a^2)}{2\kappa}}\,\frac{b}{2\beta} + 1 - (1-a)b\right) = \left(1 + \frac{\lambda^2}{8\kappa\beta^2}\right)^{-1/2}.$$

In this case, the critical growth rate is $\alpha = b\kappa\sqrt{1+\frac{\lambda^2}{8\kappa\beta^2}} + (1-b)\kappa$. Finally, since b can be arbitrarily close to 1, we have then

$$\underline{\lambda}(2) \geq \kappa\sqrt{1+\frac{\lambda^2}{8\kappa\beta^2}}\,,$$

and for the general case $|\rho(u)| \geq l_\rho\,|u|$, we have

$$\underline{\lambda}(p) \geq \underline{\lambda}(2) \geq \kappa\sqrt{1+\frac{l_\rho^2}{8\kappa\beta^2}}\,.$$

This completes the proof. □

Contributions of the initial velocity

Now, let us consider the case where the initial position $g(x) \equiv 0$ vanishes. We shall first study the case where the initial velocity $\mu(\mathrm{d}x)$ equals $e^{-\beta|x|}\mathrm{d}x$ with $\beta > 0$. In this case, the homogeneous solution $J_0(t,x)$ is given by the following lemma.

Lemma 4.4.5. *Suppose that* $\mu(\mathrm{d}x) = e^{-\beta|x|}\mathrm{d}x$ *with* $\beta > 0$. *For all* $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$ *and* $z > 0$, *we have*

$$\left(\mu * 1_{\{|\cdot|\leq z\}}\right)(x) = \begin{cases} 2\beta^{-1} e^{-\beta|x|}\sinh(\beta z) & |x| \geq z\,, \\ 2\beta^{-1}\left(1 - e^{-\beta z}\cosh(\beta x)\right) & |x| \leq z\,. \end{cases}$$

In particular, by letting $z = \kappa t$, *we have*

$$J_0(t,x) = \left(\mu * G_\kappa(t,\cdot)\right)(x) = \begin{cases} \beta^{-1} e^{-\beta|x|}\sinh(\beta\kappa t) & |x| \geq \kappa t\,, \\ \beta^{-1}\left(1 - e^{-\beta\kappa t}\cosh(\beta x)\right) & |x| \leq \kappa t\,. \end{cases} \tag{4.4.9}$$

Proof. Similar to the proof of Lemma 4.4.3, we shall consider three cases: I ($x < -z$), II ($x > z$) and III ($|x| \leq z$); see Figure 4.4.

Let us consider Case III first:

$$\left(\mu * 1_{\{|\cdot|\le z\}}\right)(x) = \int_{x-z}^{0} e^{\beta y}\mathrm{d}y + \int_{0}^{x+z} e^{-\beta y}\mathrm{d}y$$
$$= \frac{1}{\beta}\left(1 - e^{\beta(x-z)} + 1 - e^{-\beta(x+z)}\right) = \frac{2}{\beta}\left(1 - e^{-\beta z}\cosh(\beta x)\right).$$

As for Case I, we have

$$\left(\mu * 1_{\{|\cdot|\le z\}}\right)(x) = \int_{x-z}^{x+z} e^{\beta y}\mathrm{d}y = \frac{1}{\beta}\left(e^{\beta(x+z)} - e^{\beta(x-z)}\right) = \frac{2}{\beta}e^{-\beta|x|}\sinh(z).$$

The same is true for Case II. This completes the proof. □

Lemma 4.4.6. *Suppose that* $\mu \in \mathcal{M}_G^{\beta}(\mathbb{R})$ *with* $\beta > 0$. *Denote*

$$h(t,x) := \left(\mu * G_\kappa(t,\cdot)\right)(x).$$

Then for all $t \ge 0$ *and* $x \in \mathbb{R}$,

$$|h(t,x)| \le C\exp\left(\beta\kappa t - \beta|x|\right), \tag{4.4.10}$$

where $C = 1/2\int_{\mathbb{R}} e^{\beta|x|}|\mu|(\mathrm{d}x)$. *Moreover, for all* $t \ge 0$ *and* $x \in \mathbb{R}$,

$$(|h| \star G_\kappa)(t,x) \le \frac{2Ct}{\beta}\exp\left(\beta\kappa t - \beta|x|\right). \tag{4.4.11}$$

Proof. The proof of (4.4.10) is straightforward:

$$e^{\beta|x|}\left|\left(\mu * G_\kappa(t,\cdot)\right)(x)\right| \le \frac{1}{2}\int_{x-\kappa t}^{x+\kappa t} e^{\beta|x|}|\mu|(\mathrm{d}y) \le \frac{1}{2}\int_{x-\kappa t}^{x+\kappa t} e^{\beta|x-y|}e^{\beta|y|}|\mu|(\mathrm{d}y)$$
$$\le \frac{1}{2}e^{\beta\kappa t}\int_{x-\kappa t}^{x+\kappa t} e^{\beta|y|}|\mu|(\mathrm{d}y) \le \frac{1}{2}e^{\beta\kappa t}\int_{\mathbb{R}} e^{\beta|y|}|\mu|(\mathrm{d}y).$$

As for (4.4.11), denote $f(t,x) = \exp\left(\beta\kappa t - \beta|x|\right)$. Then

$$\left(f \star G_\kappa\right)(t,x) = \int_0^t e^{\beta\kappa(t-s)}\left(e^{-\beta|\cdot|} * G_\kappa(s,\cdot)\right)(x)\mathrm{d}s.$$

If $|x| \ge \kappa t$, then $|x| \ge \kappa s$ and by (4.4.9),

$$\left(e^{-\beta|\cdot|} * G_\kappa(s,\cdot)\right)(x) \le \frac{1}{\beta}e^{-\beta|x|}\sinh\left(\beta\kappa s\right) \le \frac{1}{\beta}e^{-\beta|x|}\exp\left(\beta\kappa s\right),$$

and hence,

$$\left(f \star G_\kappa\right)(t,x) \le \int_0^t \frac{1}{\beta}\exp\left(\beta\kappa(t-s) - \beta|x|\right)\exp(\beta\kappa s)\mathrm{d}s$$
$$= \frac{t}{\beta}\exp\left(\beta\kappa t - \beta|x|\right).$$

If $|x| \le \kappa t$, then by (4.4.9),

$$\left(e^{-\beta|\cdot|} * G_\kappa(s,\cdot)\right)(x) \le \begin{cases} \dfrac{1}{\beta} e^{-\beta|x|} \sinh\left(\beta\kappa s\right) & \text{if } 0 \le s \le |x|/\kappa\,, \\ 1/\beta & \text{if } |x|/\kappa \le s \le t\,, \end{cases}$$

and hence

$$\begin{aligned}\left(f \star G_\kappa\right)(t,x) &\le \int_{|x|/\kappa}^{t} \frac{1}{\beta} e^{\beta\kappa(t-s)} \mathrm{d}s + \int_0^{|x|/\kappa} \frac{1}{\beta} \exp\left(\beta\kappa(t-s) - \beta|x|\right) \sinh(\beta\kappa s) \mathrm{d}s \\ &\le \int_0^t \frac{1}{\beta} e^{\beta\kappa t - \beta|x|} \mathrm{d}s + \int_0^t \frac{1}{\beta} \exp\left(\beta\kappa(t-s) - \beta|x|\right) \sinh(\beta\kappa s) \mathrm{d}s\,.\end{aligned}$$

Then using the fact that $\sinh\left(\beta\kappa s\right) \le \exp\left(\beta\kappa s\right)$, we have that

$$\left(f \star G_\kappa\right)(t,x) \le \frac{2t}{\beta} \exp\left(\beta\kappa t - \beta|x|\right)\,,$$

which finishes the whole proof. □

Proposition 4.4.7. *Suppose that the initial position $g \equiv 0$ vanishes.*

(1) If $|\rho(u)| \le L_\rho |u|$ with $L_\rho \ne 0$ and the initial velocity $\mu \in \mathcal{M}_G^\beta(\mathbb{R})$, then for all even integers $p \ge 2$, the upper growth indices of order p satisfy the upper bounds:

$$\overline{\lambda}(p) \le \begin{cases} \left(2\beta\right)^{-1} z_p \sqrt{\kappa}\, L_\rho + \kappa & p > 2\,, \\ \left(4\beta\right)^{-1} \sqrt{2\kappa}\, L_\rho + \kappa & p = 2\,. \end{cases}$$

(2) If $|\rho(u)| \ge l_\rho |u|$ with $l_\rho \ne 0$ and the initial velocity $\mu(\mathrm{d}x) = \mu(x)\mathrm{d}x$ is such that $\mu(x)$ is a Borel measurable function satisfying the following bound

$$\mu(x) \ge c e^{-\beta|x|}\,, \quad \textit{for all almost all } x \in \mathbb{R}\,,$$

for some constants $c > 0$ and $\beta > 0$, then for all even integers $p \ge 2$, the lower growth indices of order p satisfy the lower bound:

$$\underline{\lambda}(p) \ge \kappa \left(1 + \frac{l_\rho^2}{8\kappa\beta^2}\right)^{1/2}\,.$$

In particular, for the hyperbolic Anderson model $\rho(u) = \lambda u$ with $\lambda \ne 0$, if the initial velocity μ satisfies both Conditions (1) and (2), then

$$\kappa \left(1 + \frac{\lambda^2}{8\kappa\beta^2}\right)^{1/2} \le \underline{\lambda}(2) \le \overline{\lambda}(2) \le \kappa \left(1 + \sqrt{\frac{\lambda^2}{8\kappa\beta^2}}\right)\,.$$

Proof. (1) Let $p > 2$ be an even integer. Using the higher moment formula (4.2.6), the upper bound for the kernel function $\widehat{\mathcal{K}}_p(t,x)$ in Lemma 4.3.2, and $h(t,x)$ defined in

Lemma 4.4.6, we have that

$$||u(t,x)||_p^2 \leq 2h^2(t,x) + \frac{a_{p,\overline{\varsigma}}^2 \, z_p^2 \, L_\rho^2}{2} I_0\left(a_{p,\overline{\varsigma}} \, z_p \, L_\rho \sqrt{\kappa/2}\, t\right) \left(|h|^2 \star G_\kappa\right)(t,x)\,.$$

Then by Lemma 4.4.6 (since the bound for h^2 has the same form as the bound for h) and the asymptotic formula (4.4.8) for $I_0(x)$, we have that

$$\lim_{t\to\infty} \frac{1}{t} \sup_{|x|\geq \alpha t} \log ||u(t,x)||_p^p \leq \frac{1}{4} a_{p,\overline{\varsigma}} \, z_p \sqrt{2\kappa}\, L_\rho \, p - \beta \, p(\alpha-\kappa)\,,$$

where we have used the fact that the upper bound is decreasing in $|x|$ and so the supremum over $|x| \geq \alpha t$ is attained at $|x| = \alpha t$. Therefore,

$$\overline{\lambda}(p) \leq \frac{1}{4\beta} a_{p,\overline{\varsigma}} \, z_p \sqrt{2\kappa}\, L_\rho + \kappa\,.$$

Then use the fact that $a_{p,0} = \sqrt{2}$. Similarly, for the case where $p = 2$, we simply replace both of the above z_p and $a_{p,\overline{\varsigma}}$ by 1.

(2) Now without loss of generality, suppose that $\mu(x) \geq e^{-\beta|x|}$ and $\rho(u) = \lambda u$. Denote $J_0(t,x) = (e^{-\beta|\cdot|} * G_\kappa(t,\cdot))(x)$.

We first consider the case where $|x| \leq \kappa t$. As shown in Figure 4.4, split the integral that defines $\left(J_0^2 \star \mathscr{K}\right)(t,x)$ over the three regions I, II, and III, so that

$$||u(t,x)||_2^2 \geq \left(J_0^2 \star \mathscr{K}\right)(t,x) = S_1 + S_2 + S_3 \geq S_3\,.$$

As in (4.4.7), for arbitary $a, b \in \,]0,1[$, we have that

$$\begin{aligned} S_3 &\geq \frac{\lambda^2}{4} \int_{bt}^{t} \mathrm{d}s \int_{-a\kappa s}^{a\kappa s} \mathrm{d}y \, J_0^2\left(t-s, x-y\right) I_0\left(\sqrt{\frac{\lambda^2\left((\kappa s)^2 - y^2\right)}{2\kappa}}\right) \\ &\geq \frac{\lambda^2}{4} \int_{bt}^{t} \mathrm{d}s \, I_0\left(\sqrt{\frac{\lambda^2\left(1-a^2\right)}{2\kappa}}\,\kappa s\right) \int_{-a\kappa s}^{a\kappa s} \mathrm{d}y \, J_0^2\left(t-s, x-y\right) \\ &\geq \frac{\lambda^2}{4} I_0\left(\sqrt{\frac{\lambda^2\left(1-a^2\right)}{2\kappa}}\,\kappa bt\right) \int_{bt}^{t} \mathrm{d}s \int_{-ab\kappa t}^{ab\kappa t} \mathrm{d}y \, J_0^2\left(t-s, x-y\right)\,. \end{aligned}$$

Clearly, for (s,y) in Region III of Figure 4.4, we have that $|x-y| \leq \kappa(t-s)$ and so by Lemma 4.4.5,

$$J_0\left(t-s, x-y\right) = \frac{1}{\beta}\left(1 - e^{-\beta\kappa(t-s)} \cosh\left(\beta(x-y)\right)\right)\,.$$

Then by expanding $J_0^2\left(t-s,x-y\right)$ and integrating term-by-term, we have

$$\begin{aligned}\int_{bt}^{t}\mathrm{d}s\int_{-ab\kappa t}^{ab\kappa t}J_0^2\left(t-s,x-y\right)\mathrm{d}y =& \frac{abt\left(4(1-b)\,\beta\kappa t+1-e^{-2(1-b)\,\beta\kappa t}\right)}{2\,\beta^3}\\ &+\frac{1-e^{-2(1-b)\,\beta\,t\kappa}}{4\,\beta^4\,\kappa}\cosh\left(2\,\beta\,x\right)\sinh\left(2ab\,\beta\kappa\,t\right)\\ &-\frac{4\left(1-e^{-(1-b)\,\beta\,t\kappa}\right)}{\beta^4\,\kappa}\cosh\left(\beta\,x\right)\sinh\left(ab\,\beta\kappa\,t\right)\\ \approx& \frac{1}{4\,\beta^4\,\kappa}\cosh\left(2\,\beta\,x\right)\sinh\left(2ab\,\beta\kappa\,t\right), \quad \text{as } t\to+\infty.\end{aligned}$$

Therefore, by (4.4.8), for $\alpha\le\kappa$

$$\lim_{t\to+\infty}\frac{1}{t}\sup_{|x|\ge\alpha t}\log||u(t,x)||_2^2\ge 2\,\beta\,\alpha+2ab\,\beta\kappa+b|\lambda|\sqrt{\kappa/2}\,\sqrt{1-a^2}\,>0\,,$$

for all $a,b\in\,]0,1[$. This implies that $\underline{\lambda}(2)\ge\kappa$.

Now let us consider the case where $|x|\ge\kappa t$. For arbitrary $a,b\in\,]0,1[$, we have, by Lemma 4.4.5,

$$\begin{aligned}||u(t,x)||_2^2 &\ge \left(J_0^2\star\mathscr{K}\right)(t,x)\\ &=\frac{\lambda^2}{16\,\beta^2}\int_0^t\mathrm{d}s\,\sinh^2(\beta\kappa(t-s))\int_{-\kappa s}^{\kappa s}\mathrm{d}y\,e^{-2\,\beta\,|x-y|}I_0\left(\sqrt{\frac{\lambda^2(\kappa^2s^2-y^2)}{2\kappa}}\right)\\ &\ge\frac{\lambda^2}{16\,\beta^2}\int_{bt}^t\mathrm{d}s\,\sinh^2(\beta\kappa(t-s))\int_{-a\kappa s}^{a\kappa s}\mathrm{d}y\,e^{-2\,\beta\,|x-y|}I_0\left(\sqrt{\frac{\lambda^2(\kappa^2s^2-y^2)}{2\kappa}}\right)\\ &\ge\frac{\lambda^2}{16\,\beta^2}\int_{bt}^t\mathrm{d}s\,\sinh^2(\beta\kappa(t-s))\,I_0\left(\sqrt{\frac{\lambda^2(1-a^2)}{2\kappa}}\kappa s\right)\int_{-a\kappa s}^{a\kappa s}\mathrm{d}y\,e^{-2\,\beta\,|x-y|}\\ &\ge\frac{\lambda^2}{16\,\beta^2}I_0\left(\sqrt{\frac{\lambda^2(1-a^2)}{2\kappa}}\,b\kappa t\right)\int_{bt}^t\mathrm{d}s\,\sinh^2(\beta\kappa(t-s))\int_{-a\kappa s}^{a\kappa s}\mathrm{d}y\,e^{-2\,\beta\,|x-y|}.\end{aligned}$$

After some elementary calculations, we have

$$\int_{-a\kappa s}^{a\kappa s}e^{-2\,\beta\,|x-y|}\mathrm{d}y=\frac{e^{-2\,\beta\,|x|}}{\beta}\cosh(2a\kappa s\,\beta)\ge\frac{e^{-2\,\beta\,|x|}}{2\,\beta}\exp\left(2a\kappa bt\,\beta\right), \quad \text{for all } s\in[bt,t]\,.$$

Thus,

$$\begin{aligned}\int_{bt}^t\mathrm{d}s\,\sinh^2(\beta\kappa(t-s))\int_{-a\kappa s}^{a\kappa s}e^{-2\,\beta\,|x-y|}\mathrm{d}y&\ge\frac{e^{-2\,\beta\,|x|}}{2\,\beta}\exp\left(2a\kappa bt\,\beta\right)\int_{bt}^t\sinh^2(\beta\kappa(t-s))\mathrm{d}s\\ &=\frac{e^{-2\,\beta\,|x|}}{2\,\beta}\exp\left(2a\kappa bt\,\beta\right)\left(\frac{\sinh(2(1-b)\,\beta\kappa t)}{4\,\beta\kappa}-\frac{1}{2}(1-b)t\right),\end{aligned}$$

and so

$$||u(t,x)||_2^2 \geq \frac{\lambda^2 \exp\left(-2\beta|x| + 2a\kappa b t\beta\right)}{32\beta^3} \times \left(\frac{\sinh(2(1-b)\beta\kappa t)}{4\beta\kappa} - \frac{1}{2}(1-b)t\right) I_0\left(\sqrt{\frac{\lambda^2(1-a^2)}{2\kappa}}\, b\kappa t\right).$$

Therefore, for $\alpha > \kappa$,

$$\lim_{t\to+\infty} \frac{1}{t} \sup_{|x|\geq\alpha t} \log ||u(t,x)||_2^2 \geq -2\beta\alpha + 2a\kappa b\beta + 2(1-b)\beta\kappa + \sqrt{\frac{\lambda^2(1-a^2)}{2\kappa}} b\kappa = \sqrt{\frac{\lambda^2(1-a^2)}{2\kappa}} b\kappa - 2\beta\alpha + 2(1-(1-a)b)\beta\kappa\,.$$

Then the rest argument is exactly the same as the proof of the second part of Proposition 4.4.4. We do not repeat here. This completes the proof. □

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List of figures and tables

Nomenclature

Calligraphic Symbols

$\mathscr{D}'(\mathbb{R})$ the set of distributions over $\mathbb{R}$, page 84

$\mathscr{D}'_k(\mathbb{R})$ $:= \left\{\mu \in \mathscr{D}'(\mathbb{R}) : \exists \mu_0 \in \mathscr{M}_H(\mathbb{R}), \text{ s.t.}, \mu = \mu_0^{(k)}\right\}$ and define $\mathscr{D}'_{+\infty}(\mathbb{R}) := \bigcup_{k\in\mathbb{N}} \mathscr{D}'_k(\mathbb{R})$, page 84

$\mathscr{F}[f](\xi)$ the Fourier transform of f, $\mathscr{F}[f](\xi) := \int_{\mathbb{R}} e^{-i\xi x} f(x)\mathrm{d}x$, page 32

$\mathscr{M}(\mathbb{R})$ the set of locally finite (signed) Borel measures on $\mathbb{R}$, page 15

$\mathscr{M}_G^{\beta}(\mathbb{R})$ the set of signed Borel measures on $\mathbb{R}$ that have exponential decay at infinity, see (2.2.10), page 16

$\mathscr{M}_H(\mathbb{R})$ the set of signed Borel measures μ over $\mathbb{R}$ such that $\left(|\mu| * G_\nu(t,\cdot)\right)(x) < +\infty$ for all $t > 0$ and $x \in \mathbb{R}$, page 16

$\mathscr{P}$ the predictable σ-field, page 38

$\mathscr{P}_p$ predictable and $L^p(\Omega \times \mathbb{R}_+ \times \mathbb{R})$ (resp. $L^p\left(\Omega \times \mathbb{R}_+ \times \mathbb{R}^d\right)$) integrable random fields, page 38 (resp. page 120)

$\mathscr{S}'(\mathbb{R})$ the set of Schwartz (or tempered) distributions over $\mathbb{R}$, page 84

Greek Symbols

$\Gamma(x)$ the Gamma function, page 25

κ the speed of wave propagation, $\kappa > 0$, page 7

$\Lambda(t,x)$ the backward space-time cone: $\left\{(s,y) \in \mathbb{R}_+ \times \mathbb{R} : 0 \le s \le t,\ |y-x| \le \kappa(t-s)\right\}$, page 152

ν the diffusion parameter of the heat equation, $\nu > 0$, page 2

$\Phi(x)$ the probability distribution function of the standard normal distribution, page 14

$\Psi_\mu^*(x)$ $:= |\mu|^2([-x,x])$ for $\mu \in \mathscr{M}(\mathbb{R})$ and $x \ge 0$, page 158

$\Psi_g(x)$ $:= \int_{-x}^{x} f^2(y)\mathrm{d}y$ for $g \in L^2_{loc}(\mathbb{R})$ and $x \ge 0$, page 158

$\Theta(t,x,y)$ $:= \iint_{[0,t]\times\mathbb{R}^d} G(t-s,x-z)G\left(t-s,y-z\right)\theta^2(s,z)\mathrm{d}s\mathrm{d}z$, see (3.2.2), page 109

$\Theta(t,x,y)$ $:= \iint_{[0,t]\times\mathbb{R}} G(t-s,x-z)G\left(t-s,y-z\right)\mathrm{d}s\mathrm{d}z$, see (1.0.3), page 1

$\Theta_\kappa(t,x,y) := \frac{\kappa}{4} T_\kappa^2(t, x-y)$, page 145

$\overline{\varsigma}$ a constant related to the linear growth (upper bound), see (1.4.3), page 11

$\underline{\varsigma}$ a constant related to the linear growth (lower bound), see (1.4.3), page 11

ς a constant related to the parabolic Anderson model, see (1.4.3), page 11

$\overline{\lambda}_p$ the upper Lyapunov exponent of order p, see (1.1.6), page 3

$\underline{\lambda}_p$ the lower Lyapunov exponent of order p, see (1.1.6), page 3

$\overline{\lambda}(p)$ the upper exponential growth index of order p, see (1.1.9), page 4

$\underline{\lambda}(p)$ the lower exponential growth index of order p, see (1.1.9), page 4

Math Operations

$*$ convolution in the spatial variable, see (1.1.5), page 3

$\star$ the space-time convolution, page 15

$\rhd$ the θ-weighted space-time convolution, see (3.2.2), page 109

$\rhd_n$ the θ-weighted space-time convolutions with multiple functions, see (3.2.5), page 111

$||\cdot||_p$ the $L^p(\Omega)$-norm, page 15

$||\cdot||_{M,p}$ a norm on the predictable and $L^p(\Omega\times\mathbb{R}_+\times\mathbb{R})$ (resp. $L^p(\Omega\times\mathbb{R}_+\times\mathbb{R}^d)$) integrable random fields, page 38 (resp. page 120)

$\lceil p \rceil_2$ the smallest even integer greater than or equal to p, page 15

Roman Symbols

$a_{p,\overline{\varsigma}}$ a constant defined to be $2^{(p-1)/p}$ if $\overline{\varsigma} \neq 0$ and $\sqrt{2}$ otherwise, see (1.4.4), page 11

b_p $b_p = 1$ if $p = 2$ and $b_p = 2$ if $p > 2$, see (2.4.5), page 47

$C_c^\infty(\mathbb{R}^n)$ $C^\infty(\mathbb{R}^n)$ functions with compact support, page 40

$C_{\beta_1,\beta_2}(D)$ the set of trajectories that are β_1-Hölder continuous in time and β_2-Hölder continuous in space over the domain $(t,x)\in D\subseteq\mathbb{R}_+\times\mathbb{R}$, page 4

$\mathrm{erf}(x)$ the error function $\mathrm{erf}(x) := \frac{2}{\sqrt{\pi}}\int_0^x e^{-x^2}\mathrm{d}x$, page 14

$\mathrm{erfc}(x)$ the complementary error function $\mathrm{erfc}(x) = 1-\mathrm{erf}(x)$, page 14

$E_{a,\beta}(x)$ a smooth version of the continuous function $e^{\beta|x|}$ defined by $e^{-\beta x}\Phi\left(\frac{a\beta-x}{\sqrt{a}}\right) + e^{\beta x}\Phi\left(\frac{a\beta+x}{\sqrt{a}}\right)$, see (2.5.0), page 67

$G_\kappa(t,x)$ the one-dimensional wave kernel function, $\kappa > 0$, see (1.3.1), page 7

$G_\nu(t,x)$ the heat kernel function, $\nu > 0$, see (1.1.1), page 2

$H(t)$ the Heaviside function, page 7

$\mathrm{He}_n(x;t)$ the Hermite polynomials, page 84

$|\mathrm{He}|_n(x;t)$ polynomials with each entry in $\mathrm{He}(x;t)$ replaced by its absolute value, page 84

$I_n(x)$ the modified Bessel function of the first kind of order n, or hyperbolic Bessel function, page 144

L_ρ the linear growth constant (upper bound), see (1.4.3), page 11

l_ρ the linear growth constant (lower bound), see (1.4.3), page 11

$T_\kappa(t,x)$ $:= \left(t - \frac{|x|}{2\kappa}\right) 1_{\{|x|\le 2\kappa t\}}$, page 145

z_p the universal constants in the Burkholder-Davis-Gundy inequality, see Theorem 2.3.18, page 41

Curriculum Vitae

Le Chen was born on January 10, 1979 in Taiyuan, Shanxi province, China. From 1992 to 1998, he studied at the No. 12 (key) middle school. In 1995, he won a second prize in the National Physics Competition for Junior Middle School Students. In 1997, he obtained a first prize in the National Physics Olympiad (ranked 7th in Shanxi Province).

From 1998 to 2002, he was a student in computer science at Dalian Jiaotong University (the former Dalian Railway Institute). He obtained five first-class and one second-class fellowships out of six evaluations based mainly on course performance. In 2002, he scored full mark for mathematics in the competitive national entrance exam for graduate study, which earned him a Wendeng Chen scholarship.

From the academic year 2002 to 2005, he was a graduate student in computer science at Tsinghua University. After obtaining a master's degree in computer science and engineering in June 2005, he joined the Web Search and Mining Group, Microsoft Research Asia, as a visiting scholar. He was involved in a research project for developing a web search engine for large-scale high-quality photos.

In the summer of 2006, he moved to Switzerland to do a Ph.D. in computer science at the Swiss Federal Institute of Technology in Lausanne (EPFL). After a year, he decided to switch to do his Ph.D. in mathematics. He was hired as a research and teaching assistant in Professor Robert C. Dalang's chair in January 2008.

As an assistant, he gave tutorial sessions for many courses such as analysis, probability, stochastic calculus, etc. He helped to supervise seven semester projects and one master project. He received a reward from EPFL for his teaching. He provided some IT support for both the probability chair (Professor Robert C. Dalang) and the stochastic process chair (Professor Thomas Mountford). He helped Professor Dalang *et al* organize two Ascona conferences and a conference in Lausanne in June 2012 (see below).

His first research topic concerned the Feynman-Kac type formulas for some stochastic PDE's. Then he studied the parabolic Anderson model with Dirac initial data. The initial results were obtained in spring 2011. These results constitute the main material for his thesis. The author reported these results along with some extensions in several conferences (see below).

Presentations delivered

- Sixth Ph.D. Student Conference in Stochastics (Zurich, Switzerland, September 30 – October 2, 2010). *A Feynman-Kac type formula for the deterministic wave equation on a domain with boundary conditions.*
- Evolution Equations: Randomness and Asymptotics (Bad Herrenalb, Germany, October 10-14, 2011). *Growth indices in a parabolic Anderson model.*
- Stochastic Analysis and Stochastic Partial Differential Equations (Banff, Canada, April 1-6, 2012). *Intermittency and exponential growth indices for some parabolic and hyperbolic Anderson models.*
- Stochastic Analysis and Applications (Lausanne, Switzerland, June 4-8, 2012). *Intermittency for some parabolic and hyperbolic Anderson models* (poster).
- Stochastic Partial Differential Equations (SPDEs) (Cambridge, UK, September 10-14, 2012). *Some properties of the parabolic Anderson model* (poster).

Conferences attended

- Seventh Seminar on Stochastic Analysis, Random Fields and Applications (Ascona, Switzerland, May 19-23, 2008).
- Sixth Seminar on Stochastic Analysis, Random Fields and Applications (Ascona, Switzerland, May 23-27, 2011).
- Stochastic Partial Differential Equations: Analysis, Numerics, Geometry and Modeling (Zurich, Switzerland, September 12-16, 2011).
- Recent Developments in Stochastic Analysis (Lausanne, Switzerland, January 30 – February 3, 2012).

Articles in preparation (with Professor Robert C. Dalang)

- Feynman-Kac type formula for deterministic wave equations in a domain with boundaries.
- Moment formulae for one-dimensional SPDEs with space-time white noise.
- Moment formulae for SPDEs in higher spatial dimensions.
- Exponential growth indices for one-dimensional stochastic heat equations with measure-valued initial conditions.
- Exponential growth indices for one-dimensional stochastic wave equations.
- Moments and intermittency in one-dimensional stochastic space-fractional heat equations.