SUPERLINEAR STOCHASTIC HEAT EQUATION ON $\mathbb{R}^d$

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ABSTRACT. In this paper, we study the *stochastic heat equation* (SHE) on $\mathbb{R}^d$ subject to a centered Gaussian noise that is white in time and colored in space. We establish the existence and uniqueness of the random field solution in the presence of locally Lipschitz drift and diffusion coefficients, which can have certain superlinear growth. This is a nontrivial extension of the recent work by Dalang, Khoshnevisan and Zhang [Ann. Probab. 47 (2019), pp. 519–559], where the one-dimensional SHE on $[0,1]$ subject to space-time white noise has been studied.

1. INTRODUCTION

In this paper, we study the following stochastic heat equation (SHE) on $\mathbb{R}^d$ with a drift term:

$$(1.1) \quad \begin{cases} \dfrac{\partial u(t,x)}{\partial t} = \dfrac{1}{2}\Delta u(t,x) + b\left(u(t,x)\right) + \sigma\left(u(t,x)\right)\dot{W}(t,x), & t > 0,\ x \in \mathbb{R}^d, \\ u(0,\cdot) = u_0(\cdot), \end{cases}$$

with both b and σ being locally Lipschitz continuous. The noise $\dot{W}$ is a centered Gaussian noise which is white in time and colored in space with the following covariance structure

$$(1.2) \qquad \mathbb{E}\left[\dot{W}(s,y)\dot{W}(t,x)\right] = \delta(t-s)f(x-y).$$

We assume that the correlation function f in (1.2) satisfies the improved *Dalang's condition*:

$$(1.3) \qquad \Upsilon_\alpha := (2\pi)^{-d}\int_{\mathbb{R}^d}\frac{\hat{f}(\xi)\mathrm{d}\xi}{(1+|\xi|^2)^{1-\alpha}} < \infty, \quad \text{for some } 0 < \alpha < 1,$$

where $\hat{f}(\xi)$ is the Fourier transform of f, namely, $\hat{f} = \mathcal{F}f(\xi) = \int_{\mathbb{R}^d} f(x)e^{-ix\cdot\xi}\mathrm{d}x$. Recall that condition (1.3) with $\alpha = 0$ refers to *Dalang's condition* [9]:

$$(1.4) \qquad \Upsilon(\beta) := (2\pi)^{-d} \int_{\mathbb{R}^d} \frac{\hat{f}(\mathrm{d}\xi)}{\beta + |\xi|^2} < +\infty \quad \text{for some and hence for all } \beta > 0.$$

The case when $f = \delta_0$ refers to the space-time white noise. The solution to (1.1) is understood in the mild formulation:

$$(1.5)$$
$$u(t,x) = (p_t * u_0)(x) + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y)b(u(s,y))\mathrm{d}y\mathrm{d}s$$
$$+ \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y)\sigma(u(s,y))W(\mathrm{d}s,\mathrm{d}y),$$

where the stochastic integral is the *Walsh integral* [8, 17],

$$p_t(x) = (2\pi t)^{-d/2} \exp\left(-|x|^2/2t\right)$$

is the heat kernel, and "$*$" denotes the convolution in the spatial variable.

Motivated by the work of Fernández Bonder and Groisman [11], Dalang, Khoshnevisan and Zhang [10] established the *global solution with superlinear and locally Lipschitz coefficients* for the one-dimensional SHE on $[0,1]$ subject to the space-time white noise. In particular, they assumed that

$$(1.6) \qquad |b(z)| = O\left(|z|\log|z|\right) \quad \text{and} \quad |\sigma(z)| = o\left(|z|\left(\log|z|\right)^{1/4}\right).$$

Foondun and Nualart [12] studied SHE with an additive noise, i.e., $\sigma(\cdot) \equiv \text{const.}$, and showed that the solution to (1.1) blows up in finite time if and only if b satisfies the *Osgood condition*:

$$(1.7) \qquad \int_c^\infty \frac{1}{b(u)}\mathrm{d}u < \infty \quad \text{for some } c > 0.$$

Salins [15] studied this problem for SHE on a compact domain in $\mathbb{R}^d$ under the following Osgood-type conditions, which are weaker than (1.6): There exists a positive and increasing function $h : [0,\infty) \to [0,\infty)$ that satisfies

$$\int_c^\infty \frac{1}{h(u)}\mathrm{d}u = \infty \quad \text{for all } c > 0$$

such that for some $\gamma \in (0,1/2)$ (which depends on the noise),

$$(1.8)$$
$$|b(z)| \le h\left(|z|\right) \quad \text{for all } z \in \mathbb{R}, \quad \text{and} \quad |\sigma(z)| \le |z|^{1-\gamma}\left(h\left(|z|\right)\right)^\gamma \quad \text{for all } z > 1.$$

Extending the above results to the SHE on the whole space $\mathbb{R}^d$ with both superlinear drift and diffusion coefficients is a challenging problem due to the noncompactness of the spatial domain. Indeed, for the wave equation on $\mathbb{R}^d$ ($d = 1,2,3$), the compact support of the corresponding fundamental solution can help circumvent this difficulty; see Millet and Sanz-Solé [14]. The aim of this present paper is to carry out such extension by proving the following theorem:

Theorem 1.1. *Assume the improved Dalang's condition (1.3) is satisfied for some $\alpha \in (0,1)$. Let $u(t,x)$ be the solution to (1.1) starting from $u_0 \in L^\infty(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ for some $p > (d+2)/\alpha$. Suppose that b and σ are locally Lipschitz functions such that $b(0) = \sigma(0) = 0$.*

(a) *(Global solution)* If

$$(1.9) \qquad \max\left(\frac{|b(z)|}{\log|z|}, \frac{|\sigma(z)|}{(\log|z|)^{\alpha/2}} \right) = o(|z|) \quad as \ |z| \to \infty,$$

then for any $T > 0$, there is a unique solution $u(t, x)$ to (1.1) for all $(t, x) \in (0, T] \times \mathbb{R}^d$.

(b) *(Local solution)* If

$$(1.10) \qquad \max\left(\frac{|b(z)|}{\log|z|}, \frac{|\sigma(z)|}{(\log|z|)^{\alpha/2}} \right) = O(|z|) \quad as \ |z| \to \infty,$$

then for some deterministic time $T > 0$, there exists a unique solution $u(t, x)$ to (1.1) for all $(t, x) \in (0, T] \times \mathbb{R}^d$.

(c) *In either case* (a) *or* (b), *the solution $u(t, x)$ is Hölder continuous: $u \in C^{\alpha/2-, \alpha-}\left((0, T] \times \mathbb{R}^d \right)$ a.s., where $C^{\alpha_1-, \alpha_2-}(D)$ denotes the Hölder continuous function on the space-time domain D with exponents $\alpha_1 - \epsilon$ and $\alpha_2 - \epsilon$ in time and space, respectively, for any small $\epsilon > 0$.*

Theorem 1.1 is proved in Section 3.

Remark 1.2 (Critical vs sub-critical cases). We call the case under conditions in (1.10) the *critical case* and the one in (1.9) the *sub-critical case*. Dalang et al. [10] established the global solution for the critical case using the semigroup property of the heat equation. In this paper, we cannot restart our SHE (1.1) to pass the local solution to global solution due to the fact that it is not clear whether at time T, $u(T, \cdot)$ as the initial condition for the next step is again an element in $L^\infty(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ a.s. This issue does not present for a continuous random field on a compact spatial domain, which is the case in [10] and [15].

Remark 1.3 (Regularity of the initial conditions). In [10], the initial condition u_0 is assumed to be a Hölder continuous function on $[0, 1]$. In contrast, in this paper, we only assume that $u_0 \in L^\infty(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ for some large p. This is one example of the smoothing effect of the heat kernel in the stochastic partial differential equation context. This improvement from a Hölder continuous function to a measurable function is due to the factorization representation of the solution (see (2.10)). Similar arguments using this factorization have also been carried out by Salins [15].

Remark 1.4. The assumption $b(0) = \sigma(0) = 0$ in Theorem 1.1 is essential because when τ defined in (1.13) is nonzero, then one needs to replace the term $J_+^p(r, z)$ on the far left-hand side of (2.12) by $\tau^p + J_+^p(r, z)$. With this replacement, the double integral in (2.12) will not converge. The question of removing this restriction remains for the future research.

Example 1.5 (Examples of b and σ in Theorem 1.1). (1) The function $g(x) = x\sin(x)$ for $x \in \mathbb{R}$ is locally Lipschitz, but not globally Lipschitz, continuous with linear growth and $g(0) = 0$. Hence Theorem 1.1 holds when either b or σ takes the form of g. (2) For the function $g_{a,b}(x) := |x|^b \log^a(1 + |x|)$ for $x, a, b \in \mathbb{R}$, it is easy to see that the conditions $a + b > 0$ and $a + b \geq 1$ imply that $g_{a,b}(0) = 0$ and $g_{a,b}$ is locally Lipschitz continuous, respectively. The growth condition of either (1.9) or (1.10) makes the further restriction on the suitable choices of (a, b).

The extension given in Theorem 1.1 from a compact spatial domain to the entire space $\mathbb{R}^d$ critically relies on the sharp moment formulas obtained in Theorem 1.6. These moment formulas, as extensions of those in [2, 4, 5] to allow a Lipschitz drift term, constitute the second and independent contribution of the paper. Indeed, when the drift term is linear, i.e., $b(u) = \lambda u$, then one can work with the following heat kernel $G_d(t, x) = p_t(x)e^{\lambda t}$. However, when b is a Lipschitz nonlinear function, the situation is much more trickier, especially if one wants to allow *rough initial conditions* [2, 4, 5], namely, u_0 being a signed Borel measure such that

$$(1.11) \qquad \int_{\mathbb{R}^d} e^{-a|x|^2} |u_0|(\mathrm{d}x) < \infty \quad \text{for all } a > 0,$$

where $|u_0| = u_{0,+} + u_{0,-}$ and $u_0 = u_{0,+} - u_{0,-}$ is the Jordan decomposition of the signed measure u_0. The existence and uniqueness of the solution u is proved in [13] (the proof still works for signed Borel measure). We will prove the following theorem:

Theorem 1.6 (Moment formulas with a Lipschitz drift term). *Let $u(t, x)$ be the solution to (1.1) and suppose that b and σ are globally Lipschitz continuous functions and the correlation function f satisfies the improved Dalang's condition (1.3) for some $\alpha \in (0, 1)$. Then we have the following:*

(a) *If $u_0 \in L^\infty(\mathbb{R}^d)$, then for all $p \geq \max\left(2, 2^{-6} L_b^{-2} \Upsilon_\alpha^{-1}\right)$, $t > 0$ and $x \in \mathbb{R}^d$, it holds that*

$$(1.12) \qquad \|u(t, x)\|_p \leq \left(\frac{\tau}{2} + 2\|u_0\|_{L^\infty}\right) \exp\left(Ct \max\left(p^{1/\alpha} L_\sigma^{2/\alpha}, L_b\right)\right),$$

where $\|\cdot\|_p$ and $\|\cdot\|_{L^\infty}$ denote the $L^p(\Omega)$-norm and $L^\infty(\mathbb{R}^d)$-norm, respectively,

$$(1.13) \qquad \tau := \frac{|b(0)|}{L_b} \vee \frac{|\sigma(0)|}{L_\sigma},$$

$$C = \max\left(4, 2^{6/\alpha - 1} \Upsilon_\alpha^{1/\alpha}\right), \text{ and}$$

$$(1.14) \qquad L_b := \sup_{z \in \mathbb{R}} \frac{|b(z) - b(0)|}{|z|} \quad \text{and} \quad L_\sigma := \sup_{z \in \mathbb{R}} \frac{|\sigma(z) - \sigma(0)|}{|z|}.$$

(b) *If u_0 is a rough initial condition (see (1.11)), then for all $t > 0$, $x \in \mathbb{R}^d$ and $p \geq 2$,*

$$(1.15) \qquad \|u(t, x)\|_p \leq \sqrt{3}\left[\tau + J_+(t, x)\right] \exp\left(Ct \max\left(p^{1/\alpha} L_\sigma^{2/\alpha}, L_b\right)\right),$$

where $J_+(t, x) := (p_t * |u_0|)(x)$ and the constant C does not depend on (t, x, p, L_b, L_σ).

(c) *If $u_0 \in L^\infty(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ for some $p \geq (2 + d)/\alpha$ and if $\sigma(0) = b(0) = 0$, then for all $t > 0$,*

$$(1.16) \qquad \left\|\sup_{(s,x) \in [0,t] \times \mathbb{R}^d} u(s, x)\right\|_p$$

$$\leq \|u_0\|_{L^\infty} + C\|u_0\|_{L^p} (L_b + L_\sigma) \exp\left(Ct \max\left(p^{1/\alpha} L_\sigma^{2/\alpha}, L_b\right)\right),$$

where $\|\cdot\|_{L^p}$ denotes the $L^p(\mathbb{R}^d)$-norm and the constant C does not depend on (t, x, p, L_b, L_σ).

Remark 1.7. Part (a) of Theorem 1.6 can be derived from part (b) by noticing that $J_+(t, x) \leq ||u_0||_{L^\infty}$. However, we still keep part (a) due to the simplicity of its proof.

Using the moment bounds in (1.15), one can extend the Hölder regularity from the SHE without drift (see [16] for the bounded initial condition case and [4] for the rough initial condition case) to the one with a Lipschitz drift.

Corollary 1.8 (Hölder regularity). *Let $u(t, x)$ be the solution to* (1.1) *starting from a rough initial condition (see* (1.11)) *and suppose that b and σ are globally Lipschitz continuous functions. If the correlation function f satisfies the improved Dalang's condition* (1.3) *for some $\alpha \in (0, 1)$. Then $u \in C^{\alpha/2-, \alpha-}\left((0, \infty) \times \mathbb{R}^d\right)$ a.s., i.e., Hölder continuous with orders $\beta/2$ and β for time and space respectively, for any $\beta < \alpha$.*

Parts (a), (b), and (c) of Theorem 1.6 are proved in Sections 2.1, 2.3, and 2.4, respectively. Corollary 1.8 is proved in Section 2.5.

Finally, we list a few open questions for future exploration: (1) Theorem 1.1 cannot handle either the constant one initial condition or the Dirac delta initial condition. It is interesting to investigate if either global or local solution exists for these two special initial conditions. (2) Can one improve Theorem 1.1 by relaxing the growth conditions in (1.9) and (1.10) to the Osgood-type conditions in (1.8) as in [15]?

In the rest of the paper, we prove Theorems 1.6 and 1.1 in Sections 2 and 3, respectively.

2. Moment bounds with a Lipschitz drift term

2.1. The bounded initial data case – proof of part (a) of Theorem 1.6.

Proof of Theorem 1.6(a). By Minkowski's inequality and the Burkholder-Davis-Gundy inequality,

$$||u(t, x)||_p$$
$$\leq (p_t * u_0)(x) + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x - y) \left(|b(0)| + L_b ||u(s, y)||_p\right) dy ds$$
$$+ z_p \left(\int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p_{t-s}(x - y) p_{t-s}(x - y') \left(|\sigma(0)| + L_\sigma ||u(s, y)||_p\right) \right.$$
$$\left. \times \left(|\sigma(0)| + L_\sigma ||u(s, y')||_p\right) f(y - y') dy dy' ds \right)^{1/2},$$

where z_p is the constant coming from the Burkholder-Davis-Gundy inequality and $z_p \sim 2\sqrt{p}$ as $p \to \infty$; see [6, Theorem 1.4] and references therein. For $\beta > 0$, consider the following norm

$$\mathcal{N}_\beta(u) := \sup_{(t, x) \in (0, \infty) \times \mathbb{R}^d} e^{-\beta t} ||u(t, x)||_p.$$

Then we see that

$$e^{-\beta t} \left\| u(t,x) \right\|_p$$

$$\leq \left\| u_0 \right\|_{L^\infty}$$

$$+ \int_0^t \int_{\mathbb{R}^d} e^{-\beta(t-s)} p_{t-s}(x-y) \left(|b(0)| + L_b \left(\sup_{(s,y)\in(0,\infty)\times\mathbb{R}^d} e^{-\beta s} \|u(s,y)\|_p \right) \right) \mathrm{d}y\mathrm{d}s$$

$$+ z_p \Bigg(\int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{-2\beta(t-s)} p_{t-s}(x-y) p_{t-s}(x-y')$$

$$\times \left(|\sigma(0)| + L_\sigma \sup_{(s,y)\in(0,\infty)\times\mathbb{R}^d} e^{-\beta s} \|u(s,y)\|_p \right)^2 f(y-y') \mathrm{d}y \mathrm{d}y' \mathrm{d}s \Bigg)^{1/2}.$$

Hence,

$$\mathcal{N}_\beta(u) \leq \left\| u_0 \right\|_{L^\infty} + \frac{1}{\beta} \left(|b(0)| + L_b \, \mathcal{N}_\beta(u) \right)$$

$$+ z_p \left((2\pi)^{-d} \int_0^\infty \int_{\mathbb{R}^d} e^{-2\beta s} e^{-s|\xi|^2} \hat{f}(\xi) \mathrm{d}\xi \mathrm{d}s \right)^{1/2} \left(|\sigma(0)| + L_\sigma \, \mathcal{N}_\beta(u) \right).$$

By the improved Dalang's condition (1.3) and by assuming that $\beta > 1/2$, we see that

$$(2\pi)^{-d} \int_0^\infty \int_{\mathbb{R}^d} e^{-2\beta s} e^{-s|\xi|^2} \hat{f}(\xi) \mathrm{d}\xi \mathrm{d}s$$

$$= (2\pi)^{-d} \int_{\mathbb{R}^d} \frac{\hat{f}(\mathrm{d}\xi)}{(2\beta + |\xi|^2)^{1-\alpha} (2\beta + |\xi|^2)^\alpha} \leq (2\beta)^{-\alpha} \Upsilon_\alpha.$$

Therefore,

$$\mathcal{N}_\beta(u) \leq \left\| u_0 \right\|_{L^\infty} + \frac{L_b}{\beta} \left(\frac{|b(0)|}{L_b} + \mathcal{N}_\beta(u) \right) + z_p \, (2\beta)^{-\alpha/2} \, \Upsilon_\alpha^{1/2} L_\sigma \left(\frac{|\sigma(0)|}{L_\sigma} + \mathcal{N}_\beta(u) \right).$$

Now by choosing β large enough, namely,

$$\beta > \frac{1}{2}, \quad \frac{L_b}{\beta} \leq \frac{1}{4},$$

$$z_p \, (2\beta)^{-\alpha/2} \, \Upsilon_\alpha^{1/2} L_\sigma \leq \frac{1}{4} \quad \Longleftrightarrow \quad \beta > \max \left(4L_b, \frac{1}{2}, \frac{1}{2} \left(16 z_p^2 L_\sigma^2 \Upsilon_\alpha \right)^{1/\alpha} \right),$$

we form a contraction map, which can be easily solved:

$$\mathcal{N}_\beta(u) \leq 2 \left\| u_0 \right\|_{L^\infty} + \frac{|b_0|}{2L_b} \vee \frac{|\sigma(0)|}{2L_\sigma}.$$

Notice that $z_p \leq 2\sqrt{p}$, we have that

$$\frac{1}{2} < \frac{1}{2} \left(16 z_p^2 L_\sigma^2 \Upsilon_\alpha \right)^{1/\alpha} \quad \Longleftrightarrow \quad 1/p < 64 L_b^2 \Upsilon_\alpha.$$

Therefore, for all $t > 0$, when $1/p < \min \left(64 L_b^2 \Upsilon_\alpha, 1/2 \right)$, we can take

$$\max \left(4L_b, \frac{1}{2} \left(16(2\sqrt{p})^2 L_\sigma^2 \Upsilon_\alpha \right)^{1/\alpha} \right)$$

$$\leq \max \left(4, 2^{6/\alpha-1} \Upsilon_\alpha^{1/\alpha} \right) \max \left(L_b, \, p^{1/\alpha} L_\sigma^{2/\alpha} \right) =: \beta$$

to have that

$$\|u(t,x)\|_p \leq \left(2\|u_0\|_{L^\infty} + \frac{|b_0|}{2L_b} \vee \frac{|\sigma(0)|}{2L_\sigma}\right) \exp(\beta t), \quad \text{for all } t > 0.$$

This proves part (a) of Theorem 1.6. □

2.2. **A Gronwall-type lemma.** Let us introduce some functions. For $a, b \geq 0$, denote

$$(2.1) \qquad k_{a,b}(t) := \int_{\mathbb{R}^d} (af(z) + bt)\, p_t(z)\mathrm{d}z = ak_{1,0}(t) + bt.$$

By the Fourier transform, this function can be written in the following form

$$(2.2) \qquad k_{1,0}(t) := (2\pi)^{-d} \int_{\mathbb{R}^d} \hat{f}(\mathrm{d}\xi) \exp\left(-\frac{t|\xi|^2}{2}\right).$$

Define $h_0^{a,b}(t) := 1$ and for $n \geq 1$,

$$(2.3) \qquad h_n^{a,b}(t) = \int_0^t \mathrm{d}s\, h_{n-1}^{a,b}(s)k_{a,b}(t-s).$$

Let

$$(2.4) \qquad H_{a,b}(t;\gamma) := \sum_{n=0}^{\infty} \gamma^n h_n^{a,b}(t), \qquad \text{for all } \gamma \geq 0.$$

When we have $a = 1$ and $b = 0$, we will use $k(t)$, $h_n(t)$ and $H(t;\gamma)$ to denote $k_{1,0}(t)$, $h_n^{1,0}(t)$ and $H_{1,0}(t;\gamma)$, respectively. Note that this convention makes our notation in case of $a = 1$ and $b = 0$ consistent with those in [4], [5] or [1]. The following lemma generalizes Lemma 2.5 in [5] or Lemma 3.8 in [1] from the case $a = 1$ and $b = 0$ to the case with general parameters a and b.

Lemma 2.1. *Suppose that the correlation function f satisfies Dalang's condition (1.4). Then for all $a \geq 0$, $b \geq 0$, and $\gamma \geq 0$, it holds that*

$$(2.5) \qquad \limsup_{t\to\infty} \frac{1}{t} \log H_{a,b}(t;\gamma) \leq \inf\left\{\beta > 0 : a\Upsilon(2\beta) + \frac{b}{2\beta^2} < \frac{1}{2\gamma}\right\},$$

where $\Upsilon(\beta)$ is defined in (1.4).

Proof. Here we follow the arguments in the proof of Lemma 3.8 of [1]. In particular,

$$\limsup_{t\to\infty} \frac{1}{t} \log H_{a,b}(t;\gamma) \leq \inf\left\{\beta > 0; \int_0^\infty e^{-\beta t} H_{a,b}(t;\gamma)dt < \infty\right\}.$$

Notice that

$$
\begin{aligned}
\int_0^\infty e^{-\beta t} H_{a,b}(t;\gamma)\mathrm{d}t &= \sum_{n\geq 0} \gamma^n \int_0^\infty e^{-\beta t} h_n^{a,b}(t)\mathrm{d}t \\
&= \sum_{n\geq 0} \gamma^n \left[\int_0^\infty e^{-\beta t} k_{a,b}(t)\mathrm{d}t\right]^n \left[\int_0^\infty e^{-\beta t} h_0^{a,b}(t)\mathrm{d}t\right] \\
&= \frac{1}{\beta} \sum_{n\geq 0} \gamma^n \left[a\int_0^\infty e^{-\beta t} k(t)\mathrm{d}t + \frac{b}{\beta^2}\right]^n \\
&= \frac{1}{\beta} \sum_{n\geq 0} \gamma^n \left[a\,(2\pi)^{-d}\int_{\mathbb{R}^d} \frac{\widehat{f}(\mathrm{d}\xi)}{\beta + \frac{|\xi|^2}{2}} + \frac{b}{\beta^2}\right]^n \\
&= \frac{1}{\beta} \sum_{n\geq 0} \gamma^n \left[2a\Upsilon(2\beta) + \frac{b}{\beta^2}\right]^n,
\end{aligned}
$$

where in the second equality we have used the Laplace transform to factorize the time convolutions as product and in the fourth equality we have used (2.2). The lemma is proved by noticing that

$$
\int_0^\infty e^{-\beta t} H_{a,b}(t;\gamma)\mathrm{d}t < \infty \quad \Longleftrightarrow \quad 2a\Upsilon(2\beta) + \frac{b}{\beta^2} < \frac{1}{\gamma}.
$$

One may check the proof of Lemma 3.8 of [1] for more details. This proves the lemma. $\qquad\square$

Corollary 2.2. *Suppose that the correlation function f satisfies the improved Dalang's condition (1.3) for some $\alpha \in (0,1)$. Then for all $a \geq 0$ and $b \geq 0$, when $\gamma > 0$ is large enough, it holds that*

$$
(2.6) \qquad \limsup_{t\to\infty} \frac{1}{t} \log H_{a,b}(t;\gamma) \leq \max\left(2^{3/\alpha}\,(aC\gamma)^{1/\alpha},\sqrt{2b\gamma}\right),
$$

where the constant C can be chosen to be

$$
(2.7) \qquad C = (2\pi)^{-d}\,2^{-\alpha}\max\left(\int_{|\xi|\leq 1} \widehat{f}(\mathrm{d}\xi), \int_{|\xi|>1} \frac{\widehat{f}(\mathrm{d}\xi)}{|\xi|^{2(1-\alpha)}}\right).
$$

Proof. Notice that for $\beta > 0$,

$$
\begin{aligned}
\Upsilon(2\beta) &= (2\pi)^{-d}\int_{\mathbb{R}^d} \frac{1}{(2\beta + |\xi|^2)^\alpha}\frac{\hat{f}(\mathrm{d}\xi)}{(2\beta + |\xi|^2)^{1-\alpha}} \\
&\leq \frac{(2\pi)^{-d}}{(2\beta)^\alpha}\left(\int_{|\xi|\leq 1} \frac{\hat{f}(\mathrm{d}\xi)}{(2\beta)^{1-\alpha}} + \int_{|\xi|>1} \frac{\hat{f}(\mathrm{d}\xi)}{|\xi|^{2(1-\alpha)}}\right) \leq C\left(\frac{1}{\beta} + \frac{1}{\beta^\alpha}\right),
\end{aligned}
$$

where the constant C can be chosen as in (2.7). When γ is large enough, we may assume that $\beta > 1$. Hence, in light of (2.6),

$$
a\Upsilon(2\beta) + \frac{b}{2\beta^2} \leq \frac{2aC}{\beta^\alpha} + \frac{b}{2\beta^2}.
$$

Therefore,

$$
\begin{aligned}
a\Upsilon(2\beta) + \frac{b}{2\beta^2} < \frac{1}{2\gamma} \quad &\Longleftarrow \quad \frac{2aC}{\beta^\alpha} + \frac{b}{2\beta^2} < \frac{1}{2\gamma} \\
&\Longleftarrow \quad \frac{2aC}{\beta^\alpha} < \frac{1}{4\gamma} \quad \text{and} \quad \frac{b}{2\beta^2} < \frac{1}{4\gamma} \\
&\Longleftrightarrow \quad \beta > 2^{3/\alpha}(aC\gamma)^{1/\alpha} \quad \text{and} \quad \beta > \sqrt{2b\gamma}.
\end{aligned}
$$

This proves the corollary. $\qquad\square$

2.3. Moment bounds for rough initial data – proof of part (b) of Theorem 1.6. In this part, we extend the moment bounds obtained in [4] to allow a Lipschitz drift term.

Proof of Theorem 1.6(b). Taking the p-th norm on both sides of the mild form (1.5) with $p \geq 2$ and applying the Minkowski inequality, we see that

(2.8)
$$
\|u(t,x)\|_p \leq J_+(t,x) + L_b \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} p_{t-s}(x-y) \left(\frac{|b(0)|}{L_b} + \|u(s,y)\|_p \right) \mathrm{d}y + \|I(t,x)\|_p,
$$

where we use $I(t,x)$ to denote the stochastic integral in (1.5).

By the Burkholder-Davis-Gundy inequality (see also a similar argument in the step 1 of the proof of Theorem 1.7 of [4] on p. 1000), we see that

$$
\begin{aligned}
\|I(s,y)\|_p^2 \leq 4p\, L_\sigma^2 \int_0^s \iint_{\mathbb{R}^{2d}} & p_{s-r}(y-z_1)p_{s-r}(y-z_2)f(z_1-z_2) \\
& \times \sqrt{2\left(\frac{\sigma(0)^2}{L_\sigma^2} + \|u(r,z_1)\|_p^2 \right)} \sqrt{2\left(\frac{\sigma(0)^2}{L_\sigma^2} + \|u(r,z_2)\|_p^2 \right)} \,\mathrm{d}r\mathrm{d}z_1\mathrm{d}z_2.
\end{aligned}
$$

Then by the sub-additivity of square root,

(2.9)
$$
\begin{aligned}
\|I(t,x)\|_p^2 \leq 8p\, L_\sigma^2 \int_0^t \mathrm{d}s \iint_{\mathbb{R}^{2d}} & \mathrm{d}y_1\mathrm{d}y_2\, p_{t-s}(x-y_1)p_{t-s}(x-y_2)f(y_1-y_2) \\
& \times \left(\frac{|\sigma(0)|}{L_\sigma} + \|u(s,y_1)\|_p \right) \left(\frac{|\sigma(0)|}{L_\sigma} + \|u(s,y_2)\|_p \right).
\end{aligned}
$$

By the Cauchy-Schwartz inequality applied to the dt integral, the square of second term on the right-hand side of (2.8) is bounded by

$$
L_b^2\, t \int_0^t \mathrm{d}s \left(\int_{\mathbb{R}^d} p_{t-s}(x-y) \left(\frac{|b(0)|}{L_b} + \|u(s,y)\|_p \right) \mathrm{d}y \right)^2.
$$

Hence, by raising both sides of (2.8) by a power two and recalling that the constant τ is defined in (1.13), we obtain that

$$
\begin{aligned}
||u(t,x)||_p^2 &\le 3J_+^2(t,x) + 3\,L_b^2\,t \int_0^t \mathrm{d}s \left(\int_{\mathbb{R}^d} p_{t-s}(x-y)\left(\frac{|b(0)|}{L_b} + ||u(s,y)||_p\right)\mathrm{d}y\right)^2 \\
&\quad + 24p\,L_\sigma^2 \int_0^t \mathrm{d}s \iint_{\mathbb{R}^{2d}} \mathrm{d}y_1\mathrm{d}y_2\,p_{t-s}(x-y_1)p_{t-s}(x-y_2)f(y_1-y_2) \\
&\quad \times \left(\frac{|\sigma(0)|}{L_\sigma} + ||u(s,y_1)||_p\right)\left(\frac{|\sigma(0)|}{L_\sigma} + ||u(s,y_2)||_p\right) \\
&\le 3J_+^2(t,x) + 3\int_0^t \mathrm{d}s \iint_{\mathbb{R}^{2d}} \mathrm{d}y_1\mathrm{d}y_2\,p_{t-s}(x-y_1)p_{t-s}(x-y_2) \\
&\quad \times \left(8p\,L_\sigma^2\,f(y_1-y_2) + L_b^2\,t\right)\left(\tau + ||u(s,y_1)||_p\right)\left(\tau + ||u(s,y_2)||_p\right).
\end{aligned}
$$

Now apply the same arguments as those in the proof of Theorem 1.7 of [4] with $k(t)$ replaced by $k_{8p\,L_\sigma^2,L_b^2}(t)$ to see that

$$
||u(t,x)||_p \le \left[\tau + \sqrt{3}\,J_+(t,x)\right]H_{8p\,L_\sigma^2,L_b^2}(t;1)^{1/2}.
$$

In particular, if f satisfies the improved Dalang's condition (1.3) for some $\alpha \in (0,1)$, then by Corollary 2.2, for all $t > 0$ and $x \in \mathbb{R}^d$,

$$
||u(t,x)||_p \le \sqrt{3}\left[\frac{|b(0)|}{L_b} \vee \frac{|\sigma(0)|}{L_\sigma} + J_+(t,x)\right]\exp\left(Ct\max\left(p^{1/\alpha}\,L_\sigma^{2/\alpha}, L_b\right)\right).
$$

This proves part (b) of Theorem 1.6. $\qquad\square$

2.4. Uniform moment bounds – Proof of part (c) of Theorem 1.6.

Proof of Theorem 1.6(c). Fix arbitrary $T > 0$ and recall that $\alpha \in (0,1)$ as in (1.3). The proof relies on the factorization lemma (see, e.g., Section 5.3.1 of [7] or Section 3 of [3]), which says that under the improved Dalang's condition (1.3), the solution to (1.1) can be factorized as follows:

$$
(2.10) \qquad u(t,x) = (p_t * u_0)(x) + \Psi(t,x) + \Phi(t,x),
$$

where

$$
\Phi(t,x) = \frac{\sin(\pi\alpha/2)}{\pi}\int_0^t\int_{\mathbb{R}^d}(t-r)^{-1+\alpha/2}p_{t-r}(x-z)Y(r,z)\mathrm{d}z\mathrm{d}r \quad \text{with}
$$

$$
Y(r,z) = \int_0^r\int_{\mathbb{R}^d}(r-s)^{-\alpha/2}p_{r-s}(z-y)\sigma(u(s,y))W(\mathrm{d}s,\mathrm{d}y)
$$

and

$$
\Psi(t,x) = \frac{\sin(\pi\alpha/2)}{\pi}\int_0^t\int_{\mathbb{R}^d}(t-r)^{-1+\alpha/2}p_{t-r}(x-z)B(r,z)\mathrm{d}z\mathrm{d}r \quad \text{with}
$$

$$
B(r,z) = \int_0^r\int_{\mathbb{R}^d}(r-s)^{-\alpha/2}p_{r-s}(z-y)b(u(s,y))\mathrm{d}s\mathrm{d}y.
$$

It is clear that

$$
\sup_{(t,x)\in[0,T]\times\mathbb{R}^d}|(p_t*u_0)(x)|^p \le ||u_0||_{L^\infty(\mathbb{R}^d)}^p.
$$

Step 1. In this step, we will show that

(2.11)
$$\mathbb{E}\left(\sup_{(t,x)\in[0,T]\times\mathbb{R}^d}|\Phi(t,x)|^p\right) \le C\,||u_0||_{L^p(\mathbb{R}^d)}^p\,L_\sigma^p\exp\left(CTp\max\left(L_b,\,p^{1/\alpha}L_\sigma^{2/\alpha}\right)\right).$$

Let p and q be a conjugate pair on positive numbers, i.e., $1/p + 1/q = 1$, whose values will be determined below. By Hölder's inequality, we see that

$$|\Phi(t,x)| \le \frac{\sin(\pi\alpha/2)}{\pi}\int_0^t (t-r)^{-1+\alpha/2}\,||p_{t-r}(x-\cdot)||_{L^q(\mathbb{R}^d)}\,||Y(r,\cdot)||_{L^p(\mathbb{R}^d)}\,\mathrm{d}r$$

$$\le C\int_0^t (t-r)^{-1+\alpha/2-(1-1/q)d/2}\,||Y(r,\cdot)||_{L_\rho^p(\mathbb{R}^d)}\,\mathrm{d}r$$

$$\le C\left(\int_0^t (t-r)^{(-1+\alpha/2)q-(q-1)d/2}\mathrm{d}r\right)^{1/q}\left(\int_0^t ||Y(r,\cdot)||_{L^p(\mathbb{R}^d)}^p\,\mathrm{d}r\right)^{1/p},$$

where we have used the fact that $||p_{t-r}(x-\cdot)||_{L^q(\mathbb{R}^d)}^q \le C(t-r)^{-d(q-1)/2}$ in the second inequality. Hence, since

$$(-1+\alpha/2)\,q - (q-1)\,d/2 > -1 \quad \Longleftrightarrow \quad p > (2+d)/\alpha,$$

we have

$$\mathbb{E}\left(\sup_{(t,x)\in[0,T]\times\mathbb{R}^d}|\Phi(t,x)|^p\right) \le C_T\int_0^t \mathbb{E}\left(||Y(r,\cdot)||_{L^p(\mathbb{R}^d)}^p\right)\,\mathrm{d}r$$

$$= C_T\int_0^t \mathrm{d}r\int_{\mathbb{R}^d}\mathrm{d}z\,\mathbb{E}\left(|Y(r,z)|^p\right).$$

Because $b(0) = \sigma(0) = 0$, we have

$$||Y(r,z)||_p^2 \le L_\sigma^2\int_0^r \mathrm{d}s\iint_{\mathbb{R}^{2d}}\mathrm{d}y\mathrm{d}y'\,(r-s)^{-\alpha}f(y-y')p_{r-s}(z-y)\,||u(s,y)||_p$$
$$\times p_{r-s}(z-y')\,||u(s,y')||_p,$$

and by (1.15),

$$||u(s,y)||_p \le C\exp\left(CT\max\left(L_b,\,p^{1/\alpha}L_\sigma^{2/\alpha}\right)\right)J_+(s,y).$$

Combining the above three bounds shows that

$$\mathbb{E}\left(\sup_{(t,x)\in[0,T]\times\mathbb{R}^d}|\Phi(t,x)|^p\right)$$
$$\le C\exp\left(CTp\max\left(L_b,\,p^{1/\alpha}L_\sigma^{2/\alpha}\right)\right)\int_0^t \mathrm{d}r\int_{\mathbb{R}^d}\mathrm{d}z\,I^{p/2}(r,z)$$

with

$$I(r,z) := \int_0^r \mathrm{d}s\iint_{\mathbb{R}^{2d}}\mathrm{d}y\mathrm{d}y'\,(r-s)^{-\alpha}p_{r-s}(z-y)J_+(s,y)f(y-y')p_{r-s}(z-y')J_+(s,y').$$

By the same arguments as the proof of Theorem 1.8 of [4] (see, in particular, the bound for $I_{1,1}(t, x, x')$ on p. 1006 *ibid.*), we see that

$$
I(r, z) \leq J_+^2(r, z) \int_0^r \int_{\mathbb{R}^d} e^{-\frac{(r-s)s|\xi|^2}{r}} (r-s)^{-\alpha} \hat{f}(\xi) \mathrm{d}\xi \mathrm{d}s
$$

$$
\leq C J_+^2(r, z) \int_{\mathbb{R}^d} \frac{\hat{f}(\xi)\mathrm{d}\xi}{(1+|\xi|^2)^{1-\alpha}}.
$$

By Hölder's inequality, we see that

(2.12)
$$
\int_0^T \mathrm{d}r \int_{\mathbb{R}^d} \mathrm{d}z\, J_+^p(r, z) \leq \int_0^T \mathrm{d}r \int_{\mathbb{R}^d} \mathrm{d}x\, p_{2r}(x - z) \int_{\mathbb{R}^d} \mathrm{d}z |u_0(z)|^p = T \left\| u_0 \right\|_{L^p(\mathbb{R}^d)}^p.
$$

Therefore,

$$
\mathbb{E}\left(\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} |\Phi(t, x)|^p \right) \leq C L_\sigma^p e^{CTp \max\left(L_b,\, p^{1/\alpha} L_\sigma^{2/\alpha}\right)} \int_0^T \mathrm{d}r \int_{\mathbb{R}^d} \mathrm{d}z\, J_+^p(r, z).
$$

Combining the last two inequalities proves (2.11).

Step 2. In this step, we will show that

(2.13)
$$
\mathbb{E}\left(\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} |\Psi(t, x)|^p \right) \leq C \left\| u_0 \right\|_{L^p(\mathbb{R}^d)}^p L_b^p \exp\left(CTp \max\left(L_b,\, p^{1/\alpha} L_\sigma^{2/\alpha} \right) \right).
$$

By the same arguments as in Step 1, we see that

$$
\mathbb{E}\left(\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} |\Phi(t, x)|^p \right) = C_T \int_0^t \mathrm{d}r \int_{\mathbb{R}^d} \mathrm{d}z\, \mathbb{E}\left(|B(r, z)|^p \right).
$$

Notice that

$$
\| B(r, z) \|_p
$$
$$
\leq L_b \int_0^r \int_{\mathbb{R}^d} (r-s)^{-\alpha/2} p_{r-s}(z - y) \| u(s, y) \|_p \,\mathrm{d}s\mathrm{d}y
$$
$$
\leq C L_b \exp\left(CT \max\left(L_b,\, p^{1/\alpha} L_\sigma^{2/\alpha} \right) \right) \int_0^r \int_{\mathbb{R}^d} (r-s)^{-\alpha/2} p_{r-s}(z-y) J_+(s, y)\mathrm{d}s\mathrm{d}y
$$
$$
\leq C L_b \exp\left(CT \max\left(L_b,\, p^{1/\alpha} L_\sigma^{2/\alpha} \right) \right) J_+(r, z) \int_0^r (r-s)^{-\alpha/2}\mathrm{d}s,
$$

from which we deduce (2.13). This proves part (c) of Theorem 1.6. □

2.5. Hölder regularity – proof of Corollary 1.8.

Proof of Corollary 1.8. Denote the last two parts of right-hand side of (1.5) by $B(t, x)$ and $I(t, x)$. One can use the same arguments as those in the proof of Theorem 1.8 of [4], but with the slightly different moment formula (1.15), to show that $I \in C^{\alpha/2-,\, \alpha-}\left((0, \infty) \times \mathbb{R}^d \right)$. It remains to show that $B \in C^{\alpha/2-,\, \alpha-}\left((0, \infty) \times \mathbb{R}^d \right)$.

Now choose and fix arbitrary $n > 1$ and $p > 2$. For any $(t, x), (t', x') \in [1/n, n] \times \mathbb{R}^d$ with $t' > t$, an application of the Minkowski inequality shows that

$$\|B(t, x) - B(t', x')\|_p \le CL_b \left(I_1(t, x, x') + I_2(t, t', x') + I_3(t, t', x')\right), \quad \text{with}$$

$$I_1(t, x, x') = \int_0^t \int_{\mathbb{R}^d} |p_{t-s}(x - y) - p_{t-s}(x' - y)| \, \|u(s, y)\|_p \, \mathrm{d}s \mathrm{d}y,$$

$$I_2(t, t', x') = \int_0^t \int_{\mathbb{R}^d} |p_{t-s}(x' - y) - p_{t'-s}(x' - y)| \, \|u(s, y)\|_p \, \mathrm{d}s \mathrm{d}y,$$

$$I_3(t, t', x') = \int_t^{t'} \int_{\mathbb{R}^d} p_{t'-s}(x' - y) \, \|u(s, y)\|_p \, \mathrm{d}s \mathrm{d}y.$$

By the moment formula (1.15) and by setting $\mu(\mathrm{d}z) := |u_0|(\mathrm{d}z) + \tau \mathrm{d}z$, we see that

$$I_1(t, x, x') \le C \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}y \int_{\mathbb{R}^d} \mu(\mathrm{d}z) \, |p_{t-s}(x - y) - p_{t-s}(x' - y)| \, p_s(y - z),$$

$$I_2(t, t', x') \le C \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}y \int_{\mathbb{R}^d} \mu(\mathrm{d}z) \, |p_{t-s}(x' - y) - p_{t'-s}(x' - y)| \, p_s(y - z),$$

$$I_3(t, t', x') \le C \int_t^{t'} \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}y \int_{\mathbb{R}^d} \mu(\mathrm{d}z) \, p_{t'-s}(x' - y) p_s(y - z).$$

It is clear that μ is a rough initial condition, i.e., condition (1.11) is satisfied for μ. Denote $J_0(t, x) = (p_t * \mu)(x)$. It is straightforward to see that $I_3(t, t', x') \le C(t' - t)J_0(t', x')$. As for I_1 and I_2, for any $\alpha \in (0, 1)$, by Lemma 3.1 of [4], we have that

$I_1(t, x, x')$

$$\le C|x - x'|^\alpha \int_0^t \frac{\mathrm{d}s}{(t-s)^{\alpha/2}} \int_{\mathbb{R}^d} \mathrm{d}y \int_{\mathbb{R}^d} \mu(\mathrm{d}z) \left[p_{2(t-s)}(x - y) + p_{2(t-s)}(x' - y)\right] p_{2s}(y - z)$$

$$= C|x - x'|^\alpha t^{1-\alpha/2} \left(J_0(2t, x) + J_0(2t, x')\right),$$

and similarly,

$$I_2(t, t', x') \le C(t' - t)^{\alpha/2} \int_0^t \frac{\mathrm{d}s}{(t'-s)^{\alpha/2}} \int_{\mathbb{R}^d} \mathrm{d}y \int_{\mathbb{R}^d} \mu(\mathrm{d}z) \, p_{4(t'-s)}(x' - y) p_{4s}(y - z)$$

$$\le C(t' - t)^{\alpha/2} J_0(4t, x').$$

Combining the above bounds proves Corollary 1.8. $\qquad \square$

3. Proof of Theorem 1.1

Proof of Theorem 1.1. For $N \ge 1$, let us consider the truncated stochastic heat equation:

$$u_N(t, x) = (p_t * u_0)(x) + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x - y)b_N(u_N(s, y))\mathrm{d}y\mathrm{d}s$$

(3.1)

$$+ \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x - y)\sigma_N(u_N(s, y))W(\mathrm{d}s, \mathrm{d}y),$$

where

(3.2) $\qquad \sigma_N(x) = \sigma\left(\left(1 \wedge \frac{N}{|x|}\right) x\right) \quad \text{and} \quad b_N(x) = b\left(\left(1 \wedge \frac{N}{|x|}\right) x\right).$

Recall that L_{b_N} and L_{σ_N} denote the growth rate; see (1.14). According to Theorem 1.1 of [13], there exists a unique solution $\{u_N(t,x) : t > 0, x \in \mathbb{R}^d\}$ to (3.1). In the following, we will use C to denote a generic constant that may change its value at each appearance, does not depend on (N, t, x, ϵ), but may depend on (p, α).

Step 1. In this step, we will prove (a). For any $T > 0$ fixed, consider the following stopping time

$$\tau_N := \inf \left\{ t > 0 : \sup_{x \in \mathbb{R}^d} |u_N(t,x)| \geq N \right\} \wedge T.$$

Noticing that for all $M \geq N$, we have that $\tau_N \leq \tau_M$ and

$$u_N(t,x) = u_M(t,x) \quad \text{a.s. on } (t,x) \in [0, \tau_N) \times \mathbb{R}^d,$$

we can construct the solution $u(t,x)$ via

$$(3.3) \qquad u(t,x) = u_N(t,x), \quad \text{for all } N \geq 1 \text{ and } (t,x) \in [0, \tau_N) \times \mathbb{R}^d.$$

From the definition, it is clear that on $0 \leq t \leq \tau_N$,

$$b_N(u_N(t,x)) = b(u_N(t,x)) = b(u(t,x)) \quad \text{and} \quad \sigma_N(u_N(t,x)) = \sigma(u_N(t,x)) = \sigma(u(t,x)).$$

By the Chebyshev inequality and the moment formula (1.16),

$$\mathbb{P}(0 \leq \tau_N < T) = \mathbb{P}\left(\sup_{(t,x) \in [0,T] \times \mathbb{R}^d} |u_N(t,x)| \geq N \right) \leq \frac{1}{N^p} \mathbb{E}\left(\sup_{(t,x) \in [0,T] \times \mathbb{R}^d} |u_N(t,x)|^p \right)$$

$$(3.4)$$
$$\leq \frac{C}{N^p} \left(||u_0||_{L^\infty}^p + C ||u_0||_{L^p}^p (L_{b_N} + L_{\sigma_N})^p \exp\left(CpT \max\left(L_{b_N}, \, p^{1/\alpha} L_{\sigma_N}^{2/\alpha} \right) \right) \right).$$

The sub-critical conditions in (1.9) imply that

$$L_{b_N} = o(\log N) \quad \text{and} \quad L_{\sigma_N} = o\left((\log N)^{\alpha/2} \right),$$

which ensure that above probability in (3.4) goes to zero as $N \to \infty$. Therefore, by sending $N \to \infty$, we see that $u(t,x)$ is well defined on $(0, T] \times \mathbb{R}^d$. The uniqueness is inherited from the uniqueness of $u_N(t,x)$ in (3.1).

Step 2. Now we prove part (b), the proof of which is similar to that of part (a). Fix an arbitrary $T_0 > 0$. Denote

$$\tau_N := \inf \left\{ t > 0 : \sup_{x \in \mathbb{R}^d} |u_N(t,x)| \geq N \right\} \wedge T_0.$$

We claim that

$$(3.5) \qquad \lim_{N \to \infty} \mathbb{P}(0 \leq \tau_N < T) = 0, \quad \text{for some nonrandom constant } T > 0.$$

Indeed, for all $\epsilon > 0$, by replacing T by ϵ in (3.4), we see that

$$(3.6) \quad \mathbb{P}(0 \leq \tau_N < \epsilon)$$
$$\leq \frac{C}{N^p} \left(||u_0||_{L^\infty}^p + C ||u_0||_{L^p}^p (L_{b_N} + L_{\sigma_N})^p \exp\left(Cp\epsilon \max\left(L_{b_N}, \, p^{1/\alpha} L_{\sigma_N}^{2/\alpha} \right) \right) \right).$$

By the critical conditions in (1.10), for some $C > 0$,

$$L_{b_N} \leq C \log N \quad \text{and} \quad L_{\sigma_N} \leq C(\log N)^{\alpha/2}.$$

Hence, when ϵ is small enough, by plugging the above constants into (3.6), we see that the probability in (3.6) goes to zero as $N \to \infty$. Therefore, by choosing any positive constant $T \in (0, \epsilon)$, we prove the claim (3.5). The uniqueness is proved in the same way as the proof of part (a).

Step 3. Finally, the Hölder continuity of the solution of u inherits that of u_N thanks to their relation given in (3.3), where the Hölder regularity of u_N with given exponents is proved in Corollary 1.8. This completes the proof of Theorem 1.1.

$\square$

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