

Parabolic Anderson model with rough noise in space and rough initial conditions*

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Abstract

In this note, we consider the parabolic Anderson model on $\mathbb{R}_+ \times \mathbb{R}$, driven by a Gaussian noise which is fractional in time with index $H_0 > 1/2$ and fractional in space with index $0 < H < 1/2$ such that $H_0 + H > 3/4$. Under a general condition on the initial data, we prove the existence and uniqueness of the mild solution and obtain its exponential upper bounds in time for all p -th moments with $p \geq 2$.

Keywords: stochastic partial differential equations; parabolic Anderson model; Malliavin calculus; rough initial condition; Dirac delta initial condition; rough Gaussian noise.

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1 Introduction

In this paper, we study the *parabolic Anderson model* (PAM):

$$\begin{cases} \frac{\partial u}{\partial t}(t, x) = \frac{1}{2} \frac{\partial^2 u}{\partial x^2}(t, x) + u(t, x) \dot{W}(t, x) & t > 0, x \in \mathbb{R}, \\ u(0, \cdot) = \mu_0, \end{cases} \quad (1.1)$$

with initial condition given by a non-negative Borel measure μ_0 on \mathbb{R} such that

$$\int_{\mathbb{R}} e^{-ax^2} \mu_0(dx) < \infty \quad \text{for all } a > 0. \quad (1.2)$$

Initial conditions of this type (called *rough initial conditions*) were introduced in [4] and were considered later for the *stochastic heat equation* in various settings; see, e.g., [2, 5, 6] and references therein. The noise \dot{W} is assumed to be a centered Gaussian noise that is fractional in time and space with indices H_0 , respectively H in the following range (see the gray area in Figure 1.1):

$$(H_0, H) \in (1/2, 1) \times (0, 1/2) \quad \text{and} \quad H + H_0 > 3/4. \quad (1.3)$$

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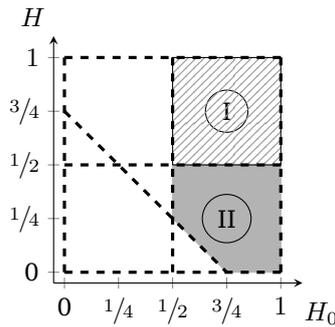


Figure 1.1: The gray area – Area II – corresponds to the range of H_0 and H in which the noise \dot{W} is rough in space. Area I corresponds to the case when the noise is non rough.

Rigorously, $\{W(\varphi); \varphi \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{R})\}$ is a zero-mean Gaussian process with covariance¹:

$$\mathbb{E}[W(\varphi)W(\psi)] = \alpha_{H_0} \int_{\mathbb{R}_+^2} \int_{\mathbb{R}} |t-s|^{2H_0-2} \mathcal{F}\varphi(t, \cdot)(\xi) \overline{\mathcal{F}\psi(s, \cdot)(\xi)} \mu(d\xi) dt ds =: \langle \varphi, \psi \rangle_{\mathcal{H}},$$

where $\alpha_{H_0} = H_0(2H_0 - 1)$ and $\mu(d\xi) = c_H |\xi|^{1-2H} d\xi$, with $c_H = \Gamma(2H + 1) \sin(\pi H)/(2\pi)$. Since $H < 1/2$, we say that W is *rough* in space.

We denote by \mathcal{H} the completion of $\mathcal{D}(\mathbb{R}_+ \times \mathbb{R})$ with respect to the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. Then $W = \{W(\varphi)\}_{\varphi \in \mathcal{H}}$ is an isonormal Gaussian process and we can use Malliavin calculus to define and analyze the solution to (1.1). We say that a process $u = \{u(t, x); t > 0, x \in \mathbb{R}\}$ is a *Skorohod solution* of (1.1) if it is adapted with respect to the filtration induced by W , and for all $t > 0$ and $x \in \mathbb{R}$,

$$u(t, x) = J_0(t, x) + \int_0^t \int_{\mathbb{R}} G(t-s, x-y) u(s, y) W(\delta s, \delta y), \tag{1.4}$$

where J_0 is the solution to the homogeneous heat equation, i.e.,

$$J_0(t, x) := \int_{\mathbb{R}} G(t, x-y) \mu_0(dy) \quad \text{with} \quad G(t, x) = (2\pi t)^{-1/2} e^{-\frac{x^2}{2t}}. \tag{1.5}$$

The stochastic integral in (1.4) is interpreted in the Skorohod sense, i.e. it is given by the divergence operator from Malliavin calculus. We refer the reader to Section 1.3 of [12] for the definition of this operator, and to [2, 10, 9] for similar developments.

The following theorem is the main result of the present article.

Theorem 1.1. *If (H_0, H) satisfy (1.3) and if μ_0 satisfies (1.2), then equation (1.1) has a unique solution u and this solution satisfies: for all $p \geq 2, t > 0$ and $x \in \mathbb{R}$,*

$$\mathbb{E}(|u(t, x)|^p) \leq C_1^p J_0^p(t, x) \exp\left(C_2 p^{\frac{H+1}{H}} t^{\frac{2H_0+H-1}{H}}\right), \tag{1.6}$$

where $C_1 > 0$ and $C_2 > 0$ are some constants which depend on H_0 and H .

The novelty of our result is the fact that we consider rough initial condition. One prominent example is the case $\mu_0 = \delta_0$ where δ_0 is the Dirac delta measure; see, e.g., [1].

Remark 1.2. When the initial condition is a bounded function or a constant, X. Chen established, in Theorem 1.2 of [7], the well-posedness of the solution to (1.1) in $L^p(\Omega)$ under conditions (1.3). Note that recently, Z.-Q. Chen and Y. Hu identified, in part (ii) of

¹In this note, we denote by $\mathcal{F}\varphi = \int_{\mathbb{R}^d} e^{-i\xi \cdot x} \varphi(x) dx$ the Fourier transform of a function $\varphi \in L^1(\mathbb{R}^d)$.

Theorem 1.3 of [8], the following necessary condition for the well-posedness of (1.1), in the case when $(H_0, H) \in (1/2, 1) \times (0, 1/2)$ and $\mu_0(dx) = u_0(x)dx$ with $u_0(x) \geq c > 0$:

$$H + 2H_0 > 5/4.$$

In the present paper, we extend X. Chen’s work to the rough initial condition case. This extension is highly nontrivial. As we will see in the proof of Theorem 1.1 below, to estimate the moments of the solution, we need to compute a spatial integral and a time integral. In [7], X. Chen uses the Laplace transform to handle the time integral first and then computes the spatial integral. This method does not work for the rough initial data. In a nutshell, moving from bounded initial data to rough initial data, one essentially changes the underlying motion from the Brownian motion to the Brownian bridge. The time increments no longer take the linear difference form; see Lemma 2.1. To overcome this difficulty, we compute the spatial integral first and then use Lemma A.1 to compute the time integral. This leads to a complicated expression which contains a product γ_n of ratios of gamma functions; see (2.9). The most delicate part is to estimate γ_n . For this, we develop some novel combinatorial techniques. Unlike [7, 8] where (1.1) in \mathbb{R}^d was studied, in this note, we only study on the one-spatial-dimensional case.

Recently, Hu and Lê obtained in Theorem 3.2 of [10] both the well-posedness and the following moment asymptotics: ²

$$\mathbb{E}(|u(t, x)|^p) \leq C_1^p t^{-p(\beta + \frac{2H-1}{4})} \exp(C_2 p^{\frac{H+1}{H}} t^{\frac{2H_0+H-1}{H}}), \quad \text{for all } p \geq 2, t > 0, x \in \mathbb{R}, \quad (1.7)$$

under weaker conditions on μ_0 , namely, μ_0 is a Borel measure such that

$$\int_{\mathbb{R}} \left(1 + |\xi|^{-(H-1/2)}\right) e^{-t|\xi|^2} |\mathcal{F}\mu_0(\xi)| d\xi \leq C t^{-\beta} \quad \text{for all } t > 0, \quad (1.8)$$

for some $C > 0$ and $\beta < H_0$, where $\mathcal{F}\mu_0$ is the Fourier transform of μ_0 . Condition (1.8) is more restrictive than (1.2); see Remark 1.3 for one example. While our exponent in (1.6) recovers that of Hu and Lê in (1.7), the factor $J_0^p(t, x)$ in (1.6) looks more natural than the corresponding factor in (1.7). For example, when the initial condition is a bounded function (resp. the delta initial measure), then as $t \rightarrow 0$, the factor $J_0^p(t, x)$ will not blow up (resp. blows up at the exact rate $t^{-d/2}$ for $x = 0$ and will not blow up for $x \neq 0$). But it is not clear whether the factor $t^{-p(\beta + (2H-1)/4)}$ in (1.7) would blow up or not, which depends on the sign of the exponent of t .

Remark 1.3. Our condition on the initial data allows growing tails, for example, $\mu(dx) = x^2 dx$. In this case, $\int_{\mathbb{R}} e^{-ax^2} x^2 dx = \sqrt{\pi} 2^{-1} a^{-3/2} < \infty$ for all $a > 0$. Hence, condition (1.2) is satisfied. But this initial condition cannot satisfy condition (1.8) because $\mathcal{F}[x^2](\xi) = \delta''(\xi)$ (in the generalized sense, see, e.g., Theorem 7.4 of [14]), which is a genuine distribution and hence does not have module or absolute value.

Finally, this paper can also be viewed as a continuation of [2] where the case of rough initial conditions and $(H_0, H) \in (1/2, 1)^2$ (see Figure 1.1) was covered.

2 Proof of Theorem 1.1

We denote by $I_n : \mathcal{H}^{\otimes n} \rightarrow \mathcal{H}$ the multiple Wiener integral of order n with respect to W . It is known that the solution u exists if and only if $\sum_{n \geq 1} I_n(f_n(\cdot, t, x))$ converges in

²This is relation (3.6) of [10] with $\alpha_0 = 2 - 2H_0$ and $\alpha = 2H - 1$; see also Remark 3.5 (ii) *ibid*.

$L^2(\Omega)$, and in this case the solution has the Wiener chaos expansion:

$$u(t, x) = J_0(t, x) + \sum_{n \geq 1} I_n(f_n(\cdot, t, x)) \quad \text{with}$$

$$f_n(t_1, x_1, \dots, t_n, x_n, t, x) = \prod_{j=1}^n G(t_{j+1} - t_j, x_{j+1} - x_j) J_0(t_1, x_1) 1_{\{0 < t_1 < \dots < t_n < t\}},$$

and $t_{n+1} = t$ and $x_{n+1} = x$; see for instance [2, 9]. By the orthogonality of the terms in this series, the necessary and sufficient condition for the existence of solution is:

$$\sum_{n \geq 1} n! \|\tilde{f}_n(\cdot, t, x)\|_{\mathcal{H}^{\otimes n}}^2 < \infty, \tag{2.1}$$

where $\tilde{f}_n(\cdot, t, x)$ is the symmetrization of $f_n(\cdot, t, x)$, defined by:

$$\tilde{f}_n(t_1, x_1, \dots, t_n, x_n, t, x) = \frac{1}{n!} \sum_{\rho \in S_n} f_n(t_{\rho(1)}, x_{\rho(1)}, \dots, t_{\rho(n)}, x_{\rho(n)}, t, x),$$

where S_n is the set of permutations of $\{1, \dots, n\}$. For any $\mathbf{t} = (t_1, \dots, t_n) \in [0, t]^n$, $\mathbf{s} = (s_1, \dots, s_n) \in [0, t]^n$, we denote

$$\psi_{t,x}^{(n)}(\mathbf{t}, \mathbf{s}) = (n!)^2 \int_{\mathbb{R}^d} \mu(d\xi_1) \dots \mu(d\xi_n) \mathcal{F} \tilde{f}_n(t_1, \cdot, \dots, t_n, \cdot, t, x)(\xi_1, \dots, \xi_n) \\ \times \overline{\mathcal{F} \tilde{f}_n(s_1, \cdot, \dots, s_n, \cdot, t, x)(\xi_1, \dots, \xi_n)}.$$

We will use the following result.

Lemma 2.1 (Lemma 3.2 of [2]). *If $0 < t_{\rho(1)} < \dots < t_{\rho(n)} < t =: t_{\rho(n+1)}$, then*

$$\psi_{t,x}^{(n)}(\mathbf{t}, \mathbf{s}) \leq J_0^2(t, x) \int_{\mathbb{R}^n} \prod_{k=1}^n \exp \left\{ -\frac{t_{\rho(k+1)} - t_{\rho(k)}}{t_{\rho(k+1)} t_{\rho(k)}} \left| \sum_{j=1}^k t_{\rho(j)} \xi_j \right|^2 \right\} \mu(d\xi_1) \dots \mu(d\xi_n).$$

We will also use the following estimate, which is a consequence of Hölder’s inequality with $p = 1/H$, and the Littlewood-Hardy-Sobolev inequality (see, e.g., [11]):

$$\alpha_{H_0}^n \int_{\mathbb{R}_+^{2n}} \prod_{j=1}^n |t_j - s_j|^{2H_0-2} \varphi(\mathbf{t}) \varphi(\mathbf{s}) d\mathbf{t} d\mathbf{s} \leq b_{H_0}^n \left(\int_{\mathbb{R}_+^n} |\varphi(\mathbf{t})|^{1/H_0} d\mathbf{t} \right)^{2H_0}. \tag{2.2}$$

Lemma 2.2. *For $n \geq 2$ and $x_1, \dots, x_n \in \mathbb{R}_+$, it holds that*

$$S_n := x_1 \prod_{k=2}^n (x_k + x_{k-1}) = \sum_{a \in A_n} \prod_{j=1}^n x_j^{a_j}, \tag{2.3}$$

where A_n is a set of indices $a = (a_1, \dots, a_n)$ such that $\text{card}(A_n) = 2^{n-1}$ and

$$a_1 \in \{1, 2\}, \quad a_n \in \{0, 1\}, \quad a_2, \dots, a_{n-1} \in \{0, 1, 2\}, \tag{2.4a}$$

$$\sum_{j=1}^i a_j \in \{i, i+1\} \quad \text{for } i = 1, \dots, n-1, \quad \sum_{j=1}^n a_j = n, \tag{2.4b}$$

$$a_i + a_{i+1} \in \{1, 2, 3\} \quad \text{for } i = 2, \dots, n-2, \tag{2.4c}$$

$$a_1 + a_2 \in \{2, 3\} \quad \text{and} \quad a_{n-1} + a_n \in \{1, 2\}. \tag{2.4d}$$

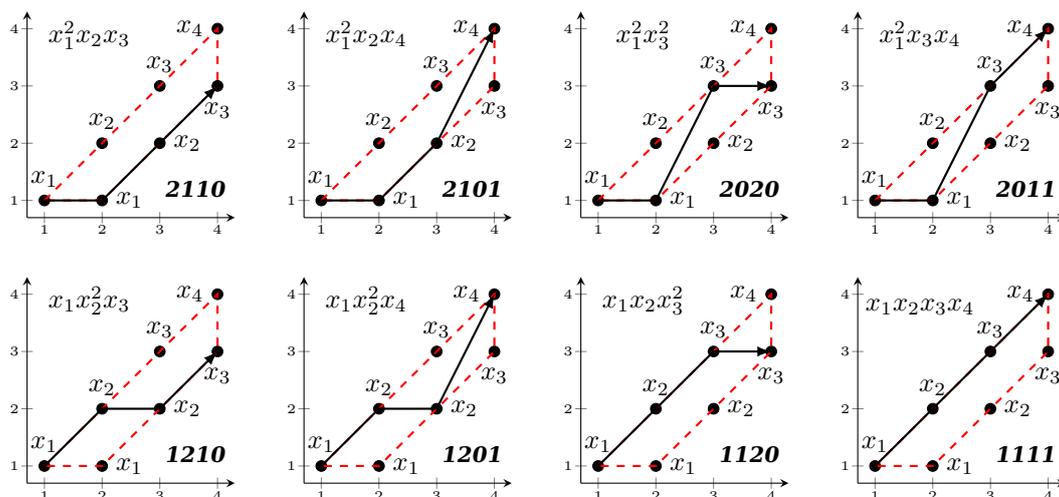


Figure 2.2: The eight paths for $n = 4$, from $(1, 1)$ to either $(4, 3)$ or $(4, 4)$, correspond to the eight monomials in the expansion of $S_4 = x_1(x_1 + x_2)(x_2 + x_3)(x_3 + x_4)$. All paths should stay in the dashed envelope. The four digits correspond to the value of (a_1, \dots, a_4) .

An expansion similar to Lemma 2.2 can be found in [9, p. 488]. Our result is slightly more precise. For example, instead of $a_1, a_n \in \{0, 1, 2\}$, we have $a_1 \in \{1, 2\}$ and $a_n \in \{0, 1\}$.

Proof of Lemma 2.2. Clearly it holds for $n = 2$. Assume that the statement is true for $n \geq 2$. Then

$$S_{n+1} = S_n(x_n + x_{n+1}) = \sum_{a \in A_n} \prod_{i=1}^n x_i^{a_i} x_n^{a_n+1} + \sum_{a \in A_n} \prod_{i=1}^n x_i^{a_i} x_{n+1} =: \sum_{a' \in A_{n+1}} \prod_{i=1}^{n+1} x_i^{a'_i}.$$

The statement for $n+1$ follows by considering separately two cases: (i) $a'_1 = a_1, \dots, a'_{n-1} = a_{n-1}, a'_n = a_n + 1, a'_{n+1} = 0$; and (ii) $a'_1 = a_1, \dots, a'_n = a_n, a'_{n+1} = 1$. \square

Remark 2.3. Lemma 2.2 can also be proved using a path representation. Since this representation will be used in the proof of Theorem 1.1, we explain it here. To each monomial $x_1^{a_1} \dots x_n^{a_n}$ in the expansion of S_n one can associate a path starting from $(1, 1)$ and going to (n, n) or $(n, n - 1)$, depending on whether x_n is present or absent in the monomial. This path is composed of $n - 1$ segments, which correspond to exponents a_1, \dots, a_{n-1} (in this order) and are constructed as follows:

- if the exponent is 0, the path moves 1 unit to the right and 2 units up;
- if the exponent is 1, the path moves 1 unit to the right and 1 unit up;
- if the exponent is 2, the path moves 1 unit to the right and 0 units up.

See Figure 2.2 for an illustration of this correspondence with $n = 4$. See also Figure 2.3 for the properties in (2.4).

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Denote $J_n(t, x) = I_n(f_n(\cdot, t, x))$. Let C be a constant which depends on H_0 and H and may be different from line to line. The proof consists of five steps:

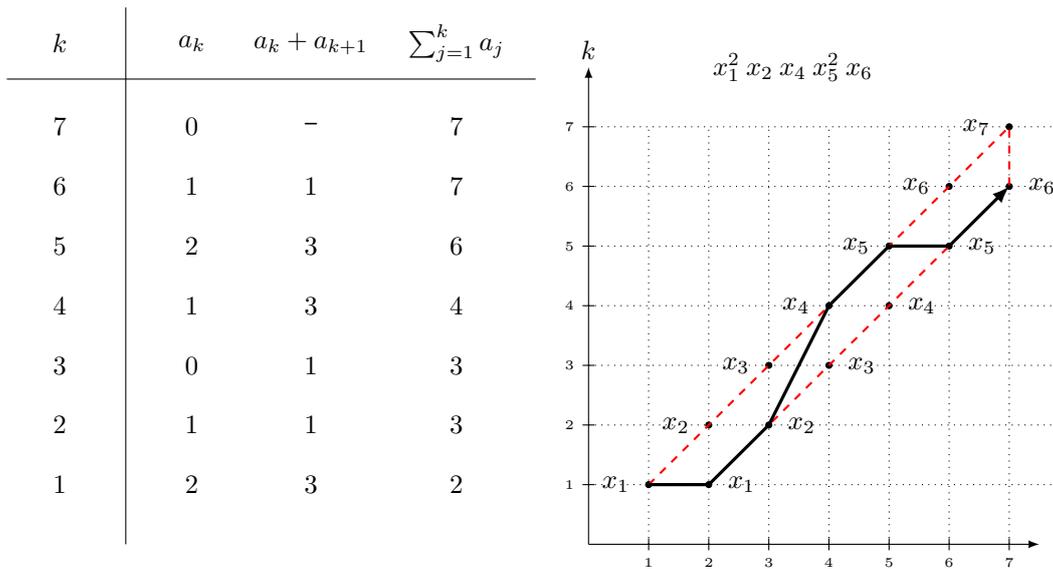


Figure 2.3: Illustrations of properties in (2.4) with $n = 7$. Each monomial in the expansion of (2.3) corresponds to a path from $(1, 1)$ to either $(7, 7)$ or $(7, 6)$. The dots indicate the possible choices for the position of the path.

Step 1. Note that $\psi_{t,x}^{(n)}(\mathbf{t}, \mathbf{s}) \leq \psi_{t,x}^{(n)}(\mathbf{t}, \mathbf{t})^{1/2} \psi_{t,x}^{(n)}(\mathbf{s}, \mathbf{s})^{1/2}$ by the Cauchy-Schwarz inequality. Combining this with inequality (2.2), we obtain:

$$\begin{aligned} \mathbb{E} (|J_n(t, x)|^2) &= n! \|\tilde{f}_n(\cdot, t, x)\|_{\mathcal{H}^{\otimes n}}^2 = \frac{1}{n!} \alpha_{H_0}^n \int_{[0,t]^{2n}} \prod_{j=1}^n |t_j - s_j|^{2H_0-2} \psi_{t,x}^{(n)}(\mathbf{t}, \mathbf{s}) d\mathbf{t} d\mathbf{s} \\ &\leq \frac{1}{n!} b_{H_0}^n \left(\sum_{\rho \in S_n} \int_{0 < t_{\rho(1)} < \dots < t_{\rho(n)} < t} \psi_{t,x}^{(n)}(\mathbf{t}, \mathbf{t})^{\frac{1}{2H_0}} d\mathbf{t} \right)^{2H_0}. \end{aligned} \tag{2.5}$$

By Lemma 2.1, for any $\rho \in S_n$ fixed, we have:

$$\begin{aligned} \int_{0 < t_{\rho(1)} < \dots < t_{\rho(n)} < t} \psi_{t,x}^{(n)}(\mathbf{t}, \mathbf{t})^{\frac{1}{2H_0}} d\mathbf{t} &\leq J_0^{1/H_0}(t, x) \int_{0 < t_{\rho(1)} < \dots < t_{\rho(n)} < t} d\mathbf{t} \\ &\times \left(\int_{\mathbb{R}^n} \prod_{k=1}^n \exp \left\{ -\frac{t_{\rho(k+1)} - t_{\rho(k)}}{t_{\rho(k+1)} t_{\rho(k)}} \left| \sum_{j=1}^k t_{\rho(j)} \xi_j \right|^2 \right\} \mu(d\xi_1) \dots \mu(d\xi_n) \right)^{\frac{1}{2H_0}}. \end{aligned}$$

Using the change of variables $t'_k = t_{\rho(k)}$ for $k = 1, \dots, n$, we see that the integral on the right-hand side above does not depend on ρ . Hence,

$$\int_{0 < t_{\rho(1)} < \dots < t_{\rho(n)} < t} \psi_{t,x}^{(n)}(\mathbf{t}, \mathbf{t})^{\frac{1}{2H_0}} d\mathbf{t} \leq J_0^{1/H_0}(t, x) \int_{\{0 < t_1 < \dots < t_n < t\}} I_t^{(n)}(t_1, \dots, t_n)^{\frac{1}{2H_0}} d\mathbf{t},$$

where

$$I_t^{(n)}(t_1, \dots, t_n) = \int_{\mathbb{R}^n} \prod_{k=1}^n \exp \left\{ -\frac{t_{k+1} - t_k}{t_{k+1} t_k} \left| \sum_{j=1}^k t_j \xi_j \right|^2 \right\} \mu(d\xi_1) \dots \mu(d\xi_n)$$

and $t_{n+1} = t$. Taking the sum over all $\rho \in S_n$ and coming back to (2.5), we obtain:

$$\mathbb{E} (|J_n(t, x)|^2) \leq J_0^2(t, x) b_{H_0}^n (n!)^{2H_0-1} \left(\int_{\{0 < t_1 < \dots < t_n < t\}} I_t^{(n)}(t_1, \dots, t_n)^{\frac{1}{2H_0}} dt \right)^{2H_0}. \quad (2.6)$$

This inequality is similar to Lemma 3.3 of [2].

Step 2. We now estimate $I_t^{(n)}(t_1, \dots, t_n)$. We use the change of variables $z_j = t_j \xi_j$ for $j = 1, \dots, n$, followed by $\eta_k = \sum_{j=1}^k z_j$ for $k = 1, \dots, n$. We obtain:

$$\begin{aligned} I_t^{(n)}(t_1, \dots, t_n) &= c_H^n \left(\prod_{i=1}^n t_i \right)^{2H-2} \int_{\mathbb{R}^n} d\vec{z} \prod_{k=1}^n \exp \left\{ -\frac{t_{k+1} - t_k}{t_{k+1} t_k} \left| \sum_{j=1}^k z_j \right|^2 \right\} |z_k|^{1-2H} \\ &= c_H^n \left(\prod_{i=1}^n t_i \right)^{2H-2} \int_{\mathbb{R}^n} d\vec{\eta} \left(\prod_{k=1}^n \exp \left\{ -\frac{t_{k+1} - t_k}{t_{k+1} t_k} |\eta_k|^2 \right\} \right) |\eta_1|^{1-2H} \prod_{k=2}^n |\eta_k - \eta_{k-1}|^{1-2H} \\ &\leq c_H^n \left(\prod_{i=1}^n t_i \right)^{2H-2} \int_{\mathbb{R}^n} d\vec{\eta} \left(\prod_{k=1}^n \exp \left\{ -\frac{t_{k+1} - t_k}{t_{k+1} t_k} |\eta_k|^2 \right\} \right) |\eta_1|^{1-2H} \prod_{k=2}^n (|\eta_k|^{1-2H} + |\eta_{k-1}|^{1-2H}), \end{aligned}$$

where $d\vec{z} = dz_1 \dots dz_n$ and similarly $d\vec{\eta} = d\eta_1 \dots d\eta_n$.

By Lemma 2.2 (see also Remark 2.3 and Figure 2.2 for more explanations),

$$|\eta_1|^{1-2H} \prod_{k=2}^n (|\eta_k|^{1-2H} + |\eta_{k-1}|^{1-2H}) = \sum_{a \in A_n} \prod_{j=1}^n |\eta_j|^{(1-2H)a_j} = \sum_{\alpha \in D_n} \prod_{j=1}^n |\eta_j|^{\alpha_j},$$

where D_n is the set of all multi-indices $\alpha = (\alpha_1, \dots, \alpha_n)$ with $\alpha_j = (1 - 2H)a_j$ for all $j = 1, \dots, n$, and $a = (a_1, \dots, a_n) \in A_n$. Therefore,

$$I_t^{(n)}(t_1, \dots, t_n) \leq c_H^n \left(\prod_{j=1}^n t_j \right)^{2H-2} \sum_{\alpha \in D_n} \prod_{j=1}^n \left\{ \int_{\mathbb{R}} \exp \left(-\frac{t_{j+1} - t_j}{t_{j+1} t_j} |\eta_j|^2 \right) |\eta_j|^{\alpha_j} d\eta_j \right\}.$$

Each of the integrals above can be computed explicitly. By Lemma 3.1 of [3],

$$\int_{\mathbb{R}} e^{-t|\xi|^2} |\xi|^\alpha d\xi = \Gamma \left(\frac{1 + \alpha}{2} \right) t^{-\frac{1+\alpha}{2}} \quad \text{for any } t > 0 \text{ and } \alpha > -1.$$

Hence,

$$\begin{aligned} I_t^{(n)}(t_1, \dots, t_n) &\leq C^n \left(\prod_{j=1}^n t_j \right)^{2H-2} \sum_{\alpha \in D_n} \prod_{j=1}^n \left(\frac{t_{j+1} - t_j}{t_j t_{j+1}} \right)^{-\frac{1+\alpha_j}{2}} \\ &= C^n \sum_{\alpha \in D_n} t_1^{\frac{4H-3+\alpha_1}{2}} \left(\prod_{j=2}^n t_j^{\frac{4H-2+\alpha_{j-1}+\alpha_j}{2}} \right) t^{\frac{\alpha_n+1}{2}} \prod_{j=1}^n (t_{j+1} - t_j)^{-\frac{\alpha_j+1}{2}}. \end{aligned}$$

Step 3. Taking power $1/(2H_0)$ and returning to (2.6), we obtain:

$$J_n(t, x) \leq J_0^2(t, x) C^n (n!)^{2H_0-1} \left(\sum_{\alpha \in D_n} t^{\frac{\alpha_n+1}{4H_0}} \int_{\{0 < t_1 < \dots < t_n < t\}} \prod_{i=1}^n t_i^{\bar{\alpha}_i} (t_{i+1} - t_i)^{\bar{\beta}_i} dt \right)^{2H_0} \quad (2.7)$$

where

$$\tilde{\alpha}_j = \begin{cases} \frac{4H - 3 + \alpha_1}{4H_0} & j = 1, \\ \frac{4H - 2 + \alpha_{j-1} + \alpha_j}{4H_0} & j = 2, \dots, n, \end{cases} \quad \text{and} \quad \tilde{\beta}_j = -\frac{\alpha_j + 1}{4H_0}, \quad j = 1, \dots, n.$$

Now we verify that the conditions of Lemma A.1 hold for the integrals in (2.7). Clearly, $\tilde{\alpha}_1 > -1$. When $\alpha_j = 2(1 - 2H)$, condition $\tilde{\beta}_j > -1$ becomes $H + H_0 > 3/4$; see (1.3). Now we verify that

$$\sum_{i=1}^k (\tilde{\alpha}_i + \tilde{\beta}_i) + k + 1 + \alpha_{k+1} > 0 \quad \text{for all } k = 1, \dots, n - 1, \tag{2.8}$$

using induction on k . For $k = 1$, using again the condition $H_0 + H > 3/4$, we see that

$$\tilde{\alpha}_1 + \tilde{\beta}_1 + \tilde{\alpha}_2 + 2 = \frac{8H_0 + 8H - 6 + \alpha_1 + \alpha_2}{4H_0} > 0.$$

Suppose that (2.8) holds for $k - 1$. We write

$$\sum_{i=1}^k (\tilde{\alpha}_i + \tilde{\beta}_i) + k + 1 + \alpha_{k+1} = \left(\sum_{i=1}^{k-1} (\tilde{\alpha}_i + \tilde{\beta}_i) + \tilde{\alpha}_k + k \right) + (\alpha_{k+1} + \tilde{\beta}_k + 1),$$

and we notice that $\alpha_{k+1} + \tilde{\beta}_k + 1 = (4H_0 + 4H - 3 + \alpha_{k+1}) / (4H_0) > 0$. Therefore, we can apply Lemma A.1 to see that

$$\int_{\{0 < t_1 < \dots < t_n < t\}} \prod_{i=1}^n t_i^{\tilde{\alpha}_i} (t_{i+1} - t_i)^{\tilde{\beta}_i} dt = \frac{\Gamma(\tilde{\alpha}_1 + 1) \prod_{i=1}^n \Gamma(\tilde{\beta}_i + 1)}{\Gamma(|\tilde{\alpha}| + |\tilde{\beta}| + n + 1)} \gamma_n t^{|\tilde{\alpha}| + |\tilde{\beta}| + n}, \tag{2.9}$$

with $\gamma_n := \prod_{k=1}^{n-1} \frac{\Gamma(\sum_{i=1}^k (\tilde{\alpha}_i + \tilde{\beta}_i) + k + 1 + \tilde{\alpha}_{k+1})}{\Gamma(\sum_{i=1}^k (\tilde{\alpha}_i + \tilde{\beta}_i) + k + 1)}$.

Step 4. In this step, we will show that $\gamma_n \leq 1$. Note that

$$\tilde{\alpha}_1 + \tilde{\beta}_1 = \frac{H - 1}{H_0} \quad \text{and} \quad \tilde{\alpha}_k + \tilde{\beta}_k = \frac{4H - 3 + \alpha_{k-1}}{4H_0}, \quad k = 2, \dots, n.$$

Denote $\theta_k := \sum_{i=1}^k (\tilde{\alpha}_i + \tilde{\beta}_i) + k + 1$. Hence,

$$\theta_k = \begin{cases} \frac{H - 1}{H_0} + 2 & \text{if } k = 1, \\ 1 - \frac{1}{4H_0} + k \frac{4H_0 + 4H - 3}{4H_0} + \frac{1 - 2H}{4H_0} \sum_{i=1}^{k-1} a_i & \text{if } k = 2, \dots, n. \end{cases} \tag{2.10}$$

Note that

$$\theta_k - \theta_{k-1} = \frac{4H_0 + 4H - 3}{4H_0} + \frac{1 - 2H}{4H_0} a_{k-1}, \quad \text{for } k = 2, \dots, n. \tag{2.11}$$

We see that γ_n is a function of a_i :

$$\gamma_n(a_1, \dots, a_n) = \prod_{k=1}^{n-1} \frac{\Gamma\left(\theta_k + \frac{1-2H}{4H_0} (a_k + a_{k+1} - 2)\right)}{\Gamma(\theta_k)}.$$

Recall that any choice of a_i corresponds to a path as shown in Figure 2.3. We claim that

$$\text{when we move the path downwards, the value of } \gamma_n \text{ decreases.} \tag{2.12}$$

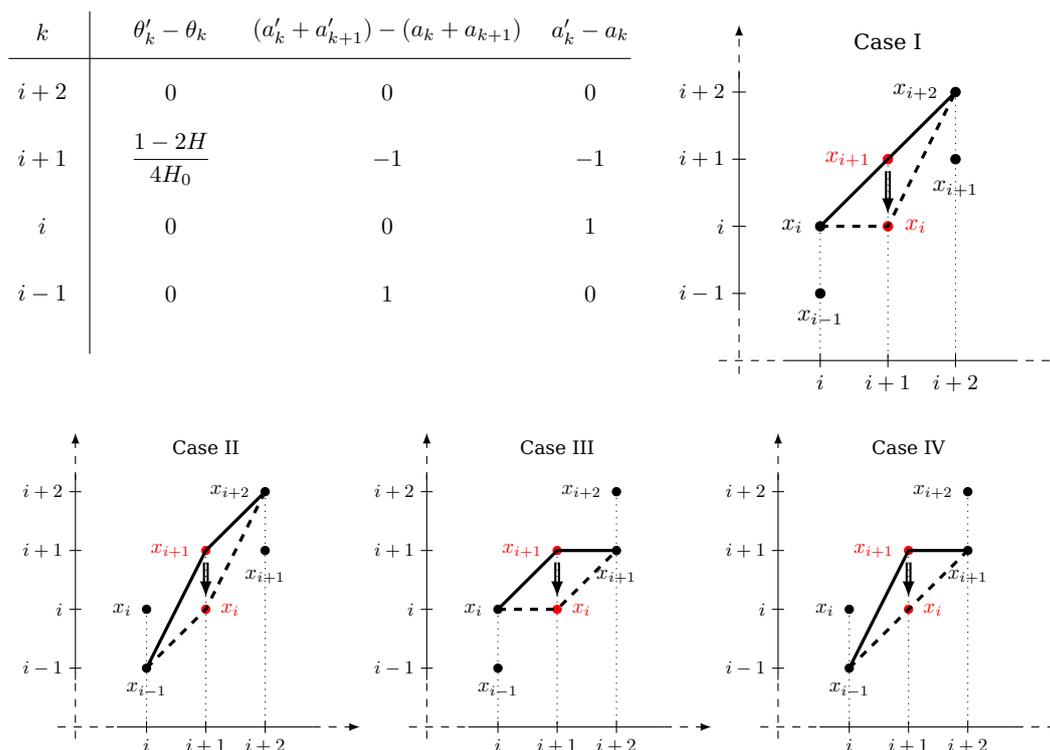


Figure 2.4: Four cases for the positions of the paths (a_1, \dots, a_n) and (a'_1, \dots, a'_n) around point $(i + 1, i + 1)$. The values of $a'_k - a_k$, $(a'_k + a'_{k+1}) - (a_k + a_{k+1})$, and $\theta'_k - \theta_k$ are the same for all four cases.

As a consequence, the path that achieves the maximum for γ_n is the one going through (i, i) for $i = 1, \dots, n$, i.e., the straight diagonal line — the topmost line. In this case, we have all a_i are equal to one and hence $a_i + a_{i+1} = 2$ for all $i = 1, \dots, n - 1$. Therefore,

$$\gamma_n(a_1, \dots, a_n) \leq \gamma(1, \dots, 1) = 1. \tag{2.13}$$

It remains to prove the claim (2.12). Note that all paths stay between the diagonal and the line parallel to the diagonal, one unit down. If the path does not touch the diagonal, then no action is taken (the argument below will show that the value of γ_n is minimal for this path).

Let $(a_1, \dots, a_n) \in A_n$ be a path which touches the diagonal on at least one point. Say this point is $(i + 1, i + 1)$ with $i + 1 < n$. (The case $i + 1 = n$ is similar.) We compare the value $\gamma_n(a_1, \dots, a_n)$ with the value $\gamma_n(a'_1, \dots, a'_n)$ corresponding to another path $(a'_1, \dots, a'_n) \in A_n$ which is obtained by moving the point $(i + 1, i + 1)$ 1 unit down. There are 4 possible cases for the shapes of the two paths around the point $(i + 1, i + 1)$, which are illustrated in Figure 2.4. Since a_k gives the number of points that the path (a_1, \dots, a_n) has on line k , it follows that in all 4 cases, $a'_i = a_i + 1$ (since line i received one point), $a'_{i+1} = a_{i+1} - 1$ (since line $i + 1$ lost one point) and $a'_k = a_k$ for all $k \notin \{i, i + 1\}$ (since the rest of the path remains unchanged). Hence,

$$(a'_k + a'_{k+1}) - (a_k + a_{k+1}) = \begin{cases} 1 & \text{if } k = i - 1 \\ -1 & \text{if } k = i + 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \theta'_k - \theta_k = \begin{cases} \frac{1-2H}{4H_0} & \text{if } k = i + 1 \\ 0 & \text{otherwise} \end{cases}.$$

By direct calculation, we see that

$$\begin{aligned} \theta'_{i-1} + \frac{1-2H}{4H_0}(a'_{i-1} + a'_i - 2) &= \theta_{i-1} + \frac{1-2H}{4H_0}(a_{i-1} + a_i - 2) + \frac{1-2H}{4H_0} \quad \text{and} \\ \theta'_k + \frac{1-2H}{4H_0}(a'_k + a'_{k+1} - 2) &= \theta_k + \frac{1-2H}{4H_0}(a_k + a_{k+1} - 2) \quad \text{for all } k \neq i-1. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\gamma(a'_1, \dots, a'_n)}{\gamma(a_1, \dots, a_n)} &= \prod_{k=1}^{n-1} \frac{\Gamma(\theta'_k + \frac{1-2H}{4H_0}(a'_k + a'_{k+1} - 2))}{\Gamma(\theta_k + \frac{1-2H}{4H_0}(a_k + a_{k+1} - 2))} \times \prod_{k=1}^{n-1} \frac{\Gamma(\theta_k)}{\Gamma(\theta'_k)} \\ &= \frac{\Gamma\left(\theta_{i-1} + \frac{1-2H}{4H_0}(a_{i-1} + a_i - 2) + \frac{1-2H}{4H_0}\right)}{\Gamma\left(\theta_{i-1} + \frac{1-2H}{4H_0}(a_{i-1} + a_i - 2)\right)} \times \frac{\Gamma(\theta_{i+1})}{\Gamma\left(\theta_{i+1} + \frac{1-2H}{4H_0}\right)}. \end{aligned}$$

By applying Lemma A.2, we see that the above ratio is always less than or equal to one. For this, we need to check that

$$z_1 := \theta_{i-1} + \frac{1-2H}{4H_0}(a_{i-1} + a_i - 2) \leq z_2 := \theta_{i+1}.$$

This is clear, since by (2.11), $\theta_{i+1} = \theta_{i-1} + 2\frac{4H_0+4H-3}{4H_0} + \frac{1-2H}{4H_0}(a_{i-1} + a_i - 2)$. Here we use again condition (1.3). This proves the claim in (2.12).

Step 5. We claim that for n large enough,

$$\Gamma\left(|\tilde{\alpha}| + |\tilde{\beta}| + n + 1\right) \geq C^n (n!)^{\frac{2H_0+H-1}{2H_0}}. \tag{2.14}$$

Indeed, by (2.10),

$$|\tilde{\alpha}| + |\tilde{\beta}| + n + 1 = n \frac{2H_0 + H - 1}{2H_0} - \frac{1 + \alpha_n}{4H_0} + 1 \geq n \frac{2H_0 + H - 1}{2H_0} - \frac{1 - H}{2H_0} + 1.$$

We use the fact that for any $a > 0, b \in \mathbb{R}$, there exists $N_{a,b} \in \mathbb{N}$ depending on a and b such that $\Gamma(an + 1 + b) \geq C_{a,b}^n (n!)^a$ for all $n \geq N_{a,b}$. Since Γ is increasing on $(2, \infty)$, we see that (2.14) holds true. Therefore, thanks to (2.14), we see that for n large enough,

$$\int_{\{0 < t_1 < \dots < t_n < t\}} \prod_{i=1}^n t_i^{\tilde{\alpha}_i} (t_{i+1} - t_i)^{\tilde{\beta}_i} dt \leq C^n (n!)^{-\frac{2H_0+H-1}{2H_0}} t^{n \frac{2H_0+H-1}{2H_0} - \frac{1+\alpha_n}{4H_0}}.$$

Returning to (2.7), it follows that $\mathbb{E}(|J_n(t, x)|^2) \leq J_0^2(t, x) C^n (n!)^{-H} t^{n(2H_0+H-1)}$. Finally, by hypercontractivity, the $\|\cdot\|_p$ -norm on $L^p(\Omega)$ is equivalent to the $\|\cdot\|_2$ -norm (see e.g. page 62 of [12]), and hence

$$\begin{aligned} \|u(t, x)\|_p &\leq \sum_{n \geq 0} (p-1)^{n/2} \|J_n(t, x)\|_2 \leq J_0(t, x) \sum_{n \geq 0} (p-1)^{n/2} C^{n/2} \frac{1}{(n!)^{H/2}} t^{n \frac{2H_0+H-1}{2}} \\ &\leq C \exp\left(C p^{1/H} t^{\frac{2H_0+H-1}{H}}\right), \end{aligned}$$

where for the last line we used the fact that $\sum_{n \geq 0} \frac{x^n}{(n!)^a} \leq C \exp(cx^{1/a})$ for any $x > 0$ and $a > 0$. We conclude the proof of Theorem 1.1 by taking power p . \square

A Some auxiliary results

Lemma A.1. Suppose that $\alpha_1 > -1$, $\beta_i > -1$ for any $i = 1, \dots, n$, and

$$\sum_{i=1}^k (\alpha_i + \beta_i) + k + 1 + \alpha_{k+1} > 0 \quad \text{for all } k = 1, \dots, n-1. \quad (\text{A.1})$$

Then by setting $t_{n+1} = t$, $|\alpha| = \sum_{i=1}^n \alpha_i$ and $|\beta| = \sum_{i=1}^n \beta_i$, we have that

$$\begin{aligned} I_n(t, \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n) &:= \int_{\{0 < t_1 < \dots < t_n < t\}} \prod_{i=1}^n t_i^{\alpha_i} (t_{i+1} - t_i)^{\beta_i} dt \\ &= \frac{\Gamma(\alpha_1 + 1) \prod_{i=1}^n \Gamma(\beta_i + 1)}{\Gamma(|\alpha| + |\beta| + n + 1)} \prod_{k=1}^{n-1} \frac{\Gamma(\sum_{i=1}^k (\alpha_i + \beta_i) + k + 1 + \alpha_{k+1})}{\Gamma(\sum_{i=1}^k (\alpha_i + \beta_i) + k + 1)} t^{|\alpha| + |\beta| + n}. \end{aligned} \quad (\text{A.2})$$

Proof. The lemma is proved by induction. For $n = 1$, we have:

$$I_1(t, \alpha, \beta) = \int_0^t t_1^{\alpha_1} (t - t_1)^{\beta_1} dt_1 = \frac{\Gamma(\alpha_1 + 1) \Gamma(\beta_1 + 1)}{\Gamma(\alpha_1 + \beta_1 + 2)} t^{\alpha_1 + \beta_1 + 1}.$$

For the induction step, we use the fact that

$$I_n(t, \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n) = \int_0^t t_n^{\alpha_n} (t - t_n)^{\beta_n} I_{n-1}(t_n, \alpha_1, \dots, \alpha_{n-1}, \beta_1, \dots, \beta_{n-1}) dt_n.$$

This proves the lemma. □

Lemma A.2. For any $a > 0$, the function $z \mapsto \Gamma(z + a)/\Gamma(z)$ is non-decreasing on $(0, \infty)$.

Proof. Let $f(z) = \Gamma(z + a)/\Gamma(z)$. Note that $f'(z) = f(z) (\psi(z + a) - \psi(z))$, where $\psi(z) := \Gamma'(z)/\Gamma(z)$ is the psi function; see 5.2.2 on p. 136 of [13]. By the following expression

$$\psi(z) = -\gamma + \sum_{k \geq 1} \left(\frac{1}{k+1} - \frac{1}{k+z} \right) \quad \text{for any } z > 0,$$

with $\gamma = \lim_{n \rightarrow \infty} (\sum_{k=1}^n k^{-1} - \ln n) \approx 0.5772$ being the Euler's constant (see, e.g., 5.7.6 on p. 139 *ibid.*), we see that ψ is a nondecreasing function on $(0, \infty)$. Hence, $f'(z) \geq 0$ for all $z > 0$, which implies the desired result. □

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