Parabolic stochastic PDEs on bounded domains with rough initial conditions: moment and correlation bounds

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Abstract

We consider nonlinear parabolic stochastic PDEs on a bounded Lipschitz domain driven by a Gaussian noise that is white in time and colored in space, with Dirichlet or Neumann boundary condition. We establish existence, uniqueness and moment bounds of the random field solution under measure-valued initial data ν. We also study the two-point correlation function of the solution and obtain explicit upper and lower bounds. For $C^{1,\alpha}$-domains with Dirichlet condition, the initial data ν is not required to be a finite measure and the moment bounds can be improved under the weaker condition that the leading eigenfunction of the differential operator is integrable with respect to $|\nu|$. As an application, we show that the solution is fully intermittent for sufficiently high level λ of noise under the Dirichlet condition, and for all $\lambda > 0$ under the Neumann condition.

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1 Introduction and main results

In this paper, we study nonlinear parabolic *stochastic partial differential equations* (SPDEs) on a bounded Lipschitz domain U in $\mathbb{R}^d$. By a domain we refer to a connected open subset of $\mathbb{R}^d$. Consider a second-order differential operator

$$\mathscr{L} = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial}{\partial x_j} \right), \tag{1.1}$$

where $[a_{ij}(x)]_{i,j}$ is a real-valued symmetric matrix that is Hölder continuous on U with some exponent $0 < \gamma \le 1$ and uniformly elliptic, i.e., there exists a positive finite constant C such that

$$C^{-1}|\xi|^2 \le \sum_{i=1}^{d} \sum_{j=1}^{d} a_{ij}(x)\xi_i\xi_j \le C|\xi|^2 \qquad \text{for all } x \in U \text{ and } \xi \in \mathbb{R}^d, \tag{1.2}$$

where $|\xi| := \sqrt{\xi_1^2 + \cdots + \xi_d^2}$. We consider operators of the form (1.1) because our approach in this paper is based on heat kernel estimates for operators in divergence form (see Sect. 3.3 below). We consider the following SPDE with (vanishing) Dirichlet boundary condition:

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) + \mathscr{L}u(t,x) = \lambda\,\sigma\,(t,x,u(t,x))\,\dot{W}(t,x), & t > 0, \ x \in U, \\ u(0,\cdot) = v(\cdot), & x \in U, \\ u(t,x) = 0, & t > 0, \ x \in \partial U, \end{cases} \tag{1.3}$$

as well as the same equation with (vanishing) Neumann boundary condition:

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) + \mathscr{L}u(t,x) = \lambda\,\sigma\,(t,x,u(t,x))\,\dot{W}(t,x), & t > 0, \ x \in U, \\ u(0,\cdot) = v(\cdot), & x \in U, \\ \frac{\partial}{\partial \mathrm{n}}u(t,x) = 0, & t > 0, \ x \in \partial U, \end{cases} \tag{1.4}$$

where n is the outward normal to the boundary ∂U of U.

We make the following assumption on the noise and correlation function:

Assumption 1.1 The noise $\dot{W}$ is a centered and spatially homogeneous Gaussian noise that is white in time with the covariance given by

$$\mathbb{E}\left[\dot{W}(t,x)\dot{W}(s,y)\right] = \delta(t-s)f(x-y), \tag{1.5}$$

where δ is the delta function and f is a nonnegative and nonnegative definite function on $\mathbb{R}^d$. We assume that there exist constants $0 < C_f < \infty$ and $0 < \beta < 2 \wedge d$ such that

$$C_f^{-1}|x-y|^{-\beta} \le f(x-y) \le C_f|x-y|^{-\beta} \qquad \text{for all } x, y \in U. \tag{1.6}$$

For example, f may be taken as the Riesz kernel $f(x - y) = |x - y|^{-\beta}$. In both Eqs. (1.3) and (1.4), $\lambda > 0$ is a constant parameter representing the level or intensity of the noise.

We need some regularity and cone conditions on the diffusion coefficient σ, which is given by the following assumption:

Assumption 1.2 We assume that $\sigma : (0, \infty) \times U \times \mathbb{R} \to \mathbb{R}$ in both Eqs. (1.3) and (1.4) is a non-random function such that $\sigma(t, x, 0) = 0$ for all $(t, x) \in (t, \infty) \times U$ and there exists a constant $L_\sigma > 0$ such that

$$|\sigma(t, x, u) - \sigma(t, x, v)| \leq L_\sigma |u - v| \quad \text{for all } t > 0, \ x \in U \text{ and } u, v \in \mathbb{R}. \quad (1.7)$$

In particular, Assumption 1.2 implies that

$$|\sigma(t, x, u)| \leq L_\sigma |u| \quad \text{for all } t > 0, \ x \in U \text{ and } u, v \in \mathbb{R}. \quad (1.8)$$

Besides, we will need the other side of condition (1.8) in order to derive some lower bounds later: there exists a constant $l_\sigma > 0$ such that

$$\sigma(t, x, u) \geq l_\sigma |u| \quad \text{for all } t > 0, \ x \in U \text{ and } u \in \mathbb{R}. \quad (1.9)$$

Our results will also cover the important case—the *parabolic Anderson model* (PAM) [5]:

$$\sigma(t, x, u) = u \quad \text{for all } t > 0, \ x \in U \text{ and } u \in \mathbb{R}. \quad (1.10)$$

In this case, our results hold with $L_\sigma = l_\sigma = 1$.

We assume that the initial condition ν is a non-random, locally finite, signed Borel measure on U. Denote $|\nu| := \nu_+ + \nu_-$, where $\mu = \mu_+ - \mu_-$ is the corresponding *Jordan decomposition* of μ with $\mu_\pm$ being two nonnegative Borel measures with disjoint support. The exact blow-up rate of the locally finite measure near the boundary will be controlled via an integrability condition by the leading eigenfunction; see (1.14). Initial conditions of this type will be called *rough initial conditions*. An important example is the *Dirac delta measure*, which plays an important role in the studying the long-time asymptotics of the solution; see, e.g., [1] and [16].

For stochastic heat equations on $\mathbb{R}^d$, the probabilistic moment bounds and the two-point correlation function, both under rough initial conditions, have been studied in [7–10, 14]. As for bounded domains, Foondun and Nualart [24] considered the stochastic heat equation on an interval $(0, L)$ with space-time white noise and either Dirichlet or Neumann boundary condition, and studied the moments and intermittency properties of the solutions. Nualart [33] and Guerngar and Nane [26] extended the results in [24] to fractional stochastic heat equations with colored noise, but only to the case when the domain is the unit ball in $\mathbb{R}^d$ plus a Dirichlet boundary condition. In all these works [24, 26, 33], the initial conditions are assumed to be a bounded function. Important initial data, such as the Dirac delta measure, have not been properly studied.

One of the main objectives/contributions of this paper is to study the moments and correlation function of the solution of parabolic SPDEs (1.3) and (1.4) under rough initial conditions with a uniformly elliptic operator $\mathscr{L}$ on a bounded domain $U \subset \mathbb{R}^d$.

| 1.1. ✓ | 1.2. ✓ | 1.3. ✓ | 1.4. ✓ | 1.5. ✗ |
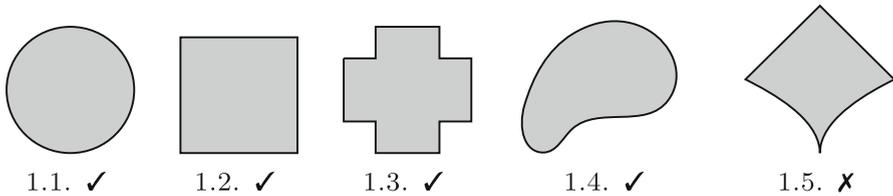
Fig. 1 Various bounded domains on $\mathbb{R}^2$: Fig. 1.1–1.4 are Lipschitz domains (either convex or not); Fig. 1.5 is a typical example of the non-Lipschitz domain where there is a cusp

For the parabolic Anderson model (i.e., $\sigma(t, x, u) = u$) on $\mathbb{R}^d$, Chen and Kim [14] have shown that the two-point correlation function can be expressed as

$$\mathbb{E}[u(t, x)u(t, x)] = \lambda^{-2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \nu(\mathrm{d}y)\, \nu\left(\mathrm{d}y'\right) \mathcal{K}\left(t, x - y, x' - y', y' - y\right)$$

(1.11)

for some kernel function $\mathcal{K}$; see also [9] for the space-time white noise case with $d = 1$. Under the conditions (1.8) and (1.9) for σ, the correlation function also admits upper and lower bounds of the same form as the right-hand side of (1.11). By establishing sharp upper and lower bounds for the kernel $\mathcal{K}$, one can then obtain sharp bounds for the correlation function. The formula (1.11) is established in [14] by using a convolution-type operator $\rhd$. It is natural to ask if one can obtain similar formula and bounds for the correlation function in the case of bounded domains. The convolution-type operator $\rhd$ on a bounded domain $U \subset \mathbb{R}^d$ has been considered by Candil in [4], where this operator is used to study the localization error between the solution of the stochastic heat equation on U and the solution of the same equation on $\mathbb{R}^d$.

In [33], it is mentioned that the extension of the moment estimates from the unit ball—a smooth, convex and bounded domain—to other bounded domains is not straightforward. One may expect that some geometric and regularity conditions on the domain would be required. The majority of our results below work for a general Lipschitz domain (see examples in Fig. 1(1.1–1.4). Convexity of the domain will be only required for the lower bounds of the moments in case of the Neumann boundary conditions. Domains with more regularity than the Lipschitz condition on the domain, such as the $C^{1,\alpha}$-domain ($\alpha > 0$), will allow us to obtain sharper upper moment bounds under Dirichlet boundary condition. See Table 2 below for a summary of our results.

For the Neumann boundary condition, there have not been many results except those in [22, 24, 29], which are concerned with the stochastic heat equation driven by the space-time white noise on an interval $(0, L)$ in one spatial dimension. In the Neumann case, they prove that full intermittency occurs for all levels $\lambda > 0$ of noise. This suggests the formation of tall peaks for the solutions even when the noise level is small. However, the precise intermittency behavior has not been well studied for general domains under the Neumann boundary condition.

Another main contribution of the paper is about the weak conditions, namely, the rough initial conditions, that we impose on the initial data ν. When $U = \mathbb{R}^d$, i.e., the boundary is at $|x| \to \infty$, and in case of $\mathcal{L} = -\frac{1}{2}\Delta$, the rough initial condition

refers to locally finite (signed) measure on $\mathbb{R}^d$ that satisfies the following integrability condition:

$$\int_{\mathbb{R}^d} e^{-a|x|^2} |v|(\mathrm{d}x) < \infty, \quad \text{for all } a > 0, \tag{1.12}$$

which is equivalent to the solution to the homogeneous heat equation exists for all time:

$$(p_t * |v|)(x) := \int_{\mathbb{R}^d} (2\pi t)^{-d/2} e^{-\frac{|x-y|^2}{2t}} |v|(\mathrm{d}y) < \infty, \quad \text{for all } t > 0 \text{ and } x \in \mathbb{R}^d. \tag{1.13}$$

Initial conditions of this type were studied in [8, 12, 13]; see also [15]. When the domain U is bounded, we will show that the integrability condition (1.12) should be replaced by

$$\int_U \Phi_1(y) |v|(\mathrm{d}y) < \infty, \tag{1.14}$$

where $\Phi_1(\cdot)$ is the eigenfunction corresponding to the leading eigenvalue of the operator $\mathcal{L}$. In particular, for the Neumann boundary condition case (see Theorems 1.4 and 1.5), since $\Phi_1(x)$ is a constant function that does not vanish at the boundary, condition (1.14) is equivalent to $|\mu|(U) < \infty$, i.e., $|\mu|$ is a finite measure on the domain U. In case of the Dirichlet boundary condition (see Theorem 1.7 and Corollary 1.9), condition (1.14) allows locally finite measure with certain growth rate near the boundary. For specific domains, condition (1.14) can be made more explicit; see examples in Table 1.

In order to obtain precise moment results and allow rough initial conditions, we start by considering in Lemmas 4.3 and 4.4 the following heat kernel integral:

$$\iint_{U^2} G(t, x, y) G(t, x', y') f(y - y') \, \mathrm{d}y \, \mathrm{d}y'.$$

By using the heat kernel estimates in Proposition 3.6, we find that the sharp bound for this integral is $e^{-2\mu_1 t}(1 \wedge t)^{-\beta/2}$. In particular, the factor of $(1 \wedge t)^{-\beta/2}$ improves the estimates in [33]. Furthermore, we consider the convolution-type integral of the heat kernel

$$\iint_{U^2} G(t - s, x, z) G(t - s, x', z') f(z - z') G(s, z, y) G(s, z', y') \, \mathrm{d}z \, \mathrm{d}z'$$

and obtain optimal bounds with a similar factor of $\left(1 \wedge \frac{(t-s)s}{t}\right)^{-\beta/2}$. Also, based on these optimal bounds and the convolution-type operator $\triangleright$ as considered in [4], we extend the two-point correlation formula (1.11) and related bounds in Proposition 6.2, and establish explicit upper and lower bounds for the kernel function in Propositions 6.5

Table 1 Simplification of Condition (1.14) for specific domains in case of Dirichlet boundary conditions and $\mathcal{L} = \Delta$

Domain U		References
	$\int_U \Phi_1(y) \, |\nu|(\mathrm{d}y) < \infty$	
Interval: $(0, 1)$	$\int_0^1 x(1-x) \, |\nu|(\mathrm{d}x) < \infty$	Example 2.1
Ball in $\mathbb{R}^d : |x| < 1$	$\int_{|x|<1} (1 - |x|) \, |\nu|(\mathrm{d}x) < \infty$	Example 2.2
Annulus in $\mathbb{R}^2 : R_1 < |x| < R_2$	$\int_{R_1 < |x| < R_2} (R_2 - |x|) \, (|x| - R_1) \, |\nu|(\mathrm{d}x) < \infty$	Example 2.3
Box: $(0, 1)^d$	$\int_{(0,1)^d} \left(\prod_{i=1}^d x_i (1 - x_i) \right) |\nu|(\mathrm{d}x) < \infty$	Example 2.4

and 6.7. In Theorem 1.5, we use these kernel bounds to obtain sharp bounds for the two-point correlation function. In case of $C^{1,\alpha}$-domains with Dirichlet boundary condition, we improve the above bounds, in Lemma 7.1 and Proposition 7.2, by including a factor which contains $\Phi_1(x)\Phi_1(x')\Phi_1(y)\Phi_1(y')$.

As some applications of our moment bounds, in Theorems 2.5 and 2.8, we establish the *full intermittency* property for sufficiently large λ under the Dirichlet boundary condition, and for all $\lambda > 0$ under the Neumann boundary condition. This extends significantly the results in [24]. We also apply our moment bounds to study the L^2-*energy* of the solution as a function of the parameter λ. This property has been studied in [22] and [29] on $(0, L)$ in the large λ regime, i.e., as $\lambda \to \infty$, under the Neumann boundary condition. In this paper, we study this property in both large and small λ regimes and for a general bounded domain with Neumann boundary condition. We find that, at a fixed time, when $\lambda > 0$ is small, the L^2-energy of the solution on U has the exponential rate $\exp(C\lambda^2)$, which is different from the rate $\exp(C\lambda^{4/(2-\beta)})$ when λ is large (see Theorem 2.8 and Corollary 2.10).

Remark 1.3 Since the domain U is bounded, it is natural to study the SPDE in (1.3) or (1.4) under the framework of infinite-dimensional stochastic differential equations as in Da Prato and Zabczyk [17]; see also Cerrai [6] and Prévôt and Röckner [37]. However, in order to obtain sharper pointwise estimates of the probabilistic moments with both $t > 0$ and $x \in U$ fixed, and in order to demonstrate how the geometric and analytic properties of the boundary ∂U affect the solution especially through the initial conditions, we adopt the random field approach in this paper. The random field approach was pioneered by Walsh [43] and extended by Dalang [18]; see [19] for a comparison of the two approaches.

Before we state our main results, let us first introduce some notations. Throughout the paper, G_D and G_N denote the Dirichlet and Neumann heat kernel, respectively. We use G to denote either G_D or G_N when we do not need to distinguish the two cases. We use $\|\cdot\|_p$ to denote the $L^p(\Omega)$-norm. Moreover, $a \wedge b = \min\{a, b\}$ for any $a, b \in \mathbb{R}$.

1.1 Main results

Our first theorem concerns the existence and uniqueness of random field solution (see Definition 3.2 below) and the p-th moment bounds of the solution. For any $c > 0$, set

$$J_c(t, x) := \int_U \frac{1}{1 \wedge t^{d/2}} e^{-c\frac{|x-y|^2}{t}} |\nu|(\mathrm{d}y). \tag{1.15}$$

It is clear that $J_c(t, x) < \infty$ for all $t > 0$ and $x \in U$ if and only if $|\nu|(U) < \infty$, i.e., $|\nu|$ is a finite Borel measure on U. Let μ_1 be the smallest positive eigenvalue of the operator $\mathscr{L}$ with Dirichlet boundary condition on U, and Φ_1 be the corresponding eigenfunction such that

$$\begin{cases} \mathscr{L}\Phi_1(x) = \mu_1\Phi_1(x), & x \in U, \\ \Phi_1(x) = 0, & x \in \partial U, \end{cases} \tag{1.16}$$

with Φ_1 chosen to be positive and usually normalized $\|\Phi_1\|_{L^2(U)} = 1$; see Sect. 3.3.

We use the following convention for the constants μ, c, and c' in Theorems 1.4, 1.5, and 1.7. In case of Dirichlet (resp. Neumann) boundary condition, we set $\mu = \mu_1$, $c = c_1$, $c' = c_2$ (resp. $\mu = 0$, $c = c_3$, $c' = c_4$) as the constants given by (3.6) (resp. (3.7) and (3.8)) in Proposition 3.6 below.

Theorem 1.4 *If U is a bounded Lipschitz domain, the noise $\dot{W}$ satisfies Assumption 1.1 and σ satisfies Assumption 1.2, then there exists a random field solution to (1.3) with Dirichlet boundary condition (and (1.4) with Neumann boundary condition, respectively). Moreover:*

(i) If ν has a bounded density, then for all $T > 0$ and all $p \geq 2$,

$$\sup_{0 < t \leq T} \sup_{x \in U} \|u(t, x)\|_p < \infty. \tag{1.17}$$

(ii) If ν is a signed Borel measure with $|\nu|(U) < \infty$, then there exists a positive finite constant C such that for all $t > 0$, $x \in U$ and all $p \geq 2$,

$$\|u(t, x)\|_p \leq C e^{t\left(Cp\lambda^2 L_\sigma^2 + Cp^{\frac{2}{2-\beta}} \lambda^{\frac{4}{2-\beta}} L_\sigma^{\frac{4}{2-\beta}} - \mu\right)} J_c(t, x). \tag{1.18}$$

In both cases, the solution is unique among all random field solutions such that for each $T > 0$, there exists $C_T < \infty$ such that

$$\|u(t, x)\|_2 \leq C_T J_c(t, x) \quad \text{for all } (t, x) \in (0, T] \times U. \tag{1.19}$$

Parts (i) and (ii) of Theorem 1.4 are proved in Sects. 4 and 5, respectively.

The next result is about the upper and lower bounds for the two-point correlation of the solution. We need a few more notations: for $x \in U$, denote

$$\text{dist}(x, \partial U) := \inf\{|x - y| : y \in \partial U\}, \quad U_\varepsilon := \{x \in U : \text{dist}(x, \partial U) > \varepsilon\}, \tag{1.20}$$

and accordingly,

$$J_{c,\varepsilon}(t, x) := \int_{U_\varepsilon} \frac{1}{1 \wedge t^{d/2}} e^{-c\frac{|x-y|^2}{t}} |\nu|(dy). \tag{1.21}$$

Theorem 1.5 (Two-point correlation) *Suppose that U is a bounded Lipschitz domain and the initial data ν is a finite nonnegative measure on U. Assume that the noise $\dot{W}$ satisfies Assumption 1.1 and σ satisfies Assumption 1.2. Let u be the solution to (1.3) with Dirichlet boundary condition or (1.4) with Neumann boundary condition.*

(i) Assume (1.10) or the nonnegativity of the solution, namely, $u(t, x) \geq 0$ a.s. for all $(t, x) \in (0, \infty) \times U$. Then there exists a positive finite constant C such that for all $t > 0$ and $x, x' \in U$,

$$\mathbb{E}\left(u(t, x)u(t, x')\right) \leq C e^{2t\left(C\lambda^2 L_\sigma^2 + C\lambda^{\frac{4}{2-\beta}} L_\sigma^{\frac{4}{2-\beta}} - \mu\right)} J_c(t, x) J_c(t, x'). \tag{1.22}$$

(ii) *Assume* (1.9) *or* (1.10). *Then, in case of Dirichlet boundary condition, there exists* $0 < \varepsilon_0 < 1$ *such that for all* $0 < \varepsilon \le \varepsilon_0$, *there exists* $\overline{C} = \overline{C}(\varepsilon) > 0$ *with* $\lim_{\varepsilon \to 0} \overline{C}(\varepsilon) = 0$ *such that for all* $t > 0$ *and* $x, x' \in U_\varepsilon$,

$$\mathbb{E}(u(t, x)u(t, x')) \ge \overline{C}e^{2t\left(\overline{C}\lambda^2 l_\sigma^2 + \overline{C}\lambda^{\frac{4}{2-\beta}} l_\sigma^{\frac{4}{2-\beta}} - \mu\right)}$$

$$e^{-16c\frac{|x-x'|^2}{t}} J_{12c',\varepsilon}(t, x)J_{12c',\varepsilon}(t, x'). \quad (1.23)$$

In case of Neumann boundary condition, if the heat kernel lower bound (3.8) *below holds (which is the case, for example, when* U *is a smooth, convex domain and* $\mathscr{L} = -\Delta$; *see Proposition 3.6 below), then there exists a constant* $\overline{C} > 0$ *such that* (1.23) *holds with* $\varepsilon = 0$ *for all* $t > 0$ *and* $x, x' \in U$.

Theorem 1.5 is proved at the end of Sect. 6.

Remark 1.6 (Nonnegativity and comparison principle) The condition $u \ge 0$ a.s. in Theorem 1.5 should be interpreted as $u(t, x) \ge 0$ a.s. for all $t > 0$ and $x \in U$. It is generally believed that under condition (1.8), if the initial condition is nonnegative, then the solution to the stochastic heat equation (SHE) is nonnegative or even strictly positive, which is indeed a consequence of the well-known *sample-path comparison principle* for SHE. In particular, Mueller [31] established the sample-path comparison principle for the case of SHE on [0, 1] with Neumann boundary conditions and space-time white noise. Later, Shiga [41] proved the case of SHE on $\mathbb{R}$ with space-time white noise. The case of SHE on [0, 1] with Dirichlet boundary conditions was proved by Mueller and Nualart [32]. The case of SHE on $\mathbb{R}$ with a fractional Laplace, space-time white noise, and rough initial conditions was established in [13] and the case of SHE on $\mathbb{R}^d$ with rough initial data and with a noise that is white in time and homogeneously colored in space was proved in [12]. The sample-path comparison principle under the settings of the current paper is left as a future project.

The last set of results focus on the case of $C^{1,\alpha}$-domains with the Dirichlet boundary condition and some variations. Note that $C^{1,\alpha}$-domains are a special case of Lipschitz domains. We need to introduce some notations:

$$\Psi(t, x) := 1 \wedge \frac{\Phi_1(x)}{1 \wedge t^{1/2}} \quad \text{and} \quad J_c^*(t, x) := \int_U \Psi(t, y)\frac{e^{-c\frac{|x-y|^2}{t}}}{1 \wedge t^{d/2}}|\nu|(\mathrm{d}y), \quad (1.24)$$

where Φ_1 is the leading eigenfunction. In this case, we are able to improve the previous results by giving a new condition (1.25) below, namely $\Phi_1 \in L^1(U, |\nu|)$, which is weaker than the above condition $|\nu|(U) < \infty$ in Theorem 1.4 for existence and all moments of solutions with measure-valued initial data. This new integrability condition indicates the rate of blow-up for the initial data which is allowed near the boundary ∂U (see Examples 2.1–2.4 and Remark 1.8 below), and hence ν is not necessarily a finite measure. Moreover, because

$$\Psi(t, x)\Big|_{x \in \partial U} = 0 \quad \text{and} \quad J_c^*(t, x) \le J_c(t, x),$$

the bounds in (1.26), (1.28) and (1.29) below strengthen the previous bounds (1.18), (1.22) and (1.23), respectively, especially near the boundary of the domain. Indeed,

(1) for any $t > 0$ fixed, when x is close to the boundary of U, the term $\Psi(t, x)$ in (1.18) and (1.22) plays the dominant role in pushing the moments to zero;
(2) for any $x \in U$ fixed, since $\Psi(\cdot, \cdot) \leq 1$, when $t \to 0$, the term $J_c^*(t, x) \asymp (p_t * |\nu|)(x)$ defines the behavior of the moments. Here, $p_t(x)$ is the heat kernel on $\mathbb{R}^d$ and "$*$" refers to the spatial convolution; see (1.13).

Theorem 1.7 *Let U be a bounded $C^{1,\alpha}$-domain for some $\alpha > 0$ with the Dirichlet boundary condition at ∂U. Assume that the noise $\dot{W}$ satisfies Assumption 1.1 and σ satisfies Assumption 1.2. If the initial condition ν is any locally finite and signed measure that satisfies the following integrability condition*

$$\|\Phi_1\|_{L^1(U, |\nu|)} = \int_U \Phi_1(y) \, |\nu|(\mathrm{d}y) < \infty, \tag{1.25}$$

where $\Phi_1(\cdot)$ is the leading eigenfunction of the differential operator $\mathscr{L}$ on the domain U, then we have the following:

(i) *There exists a random field solution to (1.3). The solution has the property that for some $C < \infty$, for all $t > 0$ and $x \in U$,*

$$\|u(t, x)\|_p \leq C e^{t\left(Cp\lambda^2 L_\sigma^2 + Cp^{\frac{2}{2-\beta}} \lambda^{\frac{4}{2-\beta}} L_\sigma^{\frac{4}{2-\beta}} - \mu_1\right)} \Psi(t, x) J_{2c_1/3}^*(t, x), \tag{1.26}$$

where μ_1 and c_1 are the constants in Proposition 3.6 below. Moreover, the solution is unique among all random field solutions such that for each $T > 0$, there exists $C_T < \infty$ such that

$$\|u(t, x)\|_2 \leq C_T \Psi(t, x) J_c^*(t, x) \quad \text{for all } (t, x) \in (0, T] \times U. \tag{1.27}$$

(ii) *Assume (1.10) or the nonnegativity of the solution, namely, $u(t, x) \geq 0$ a.s. for all $(t, x) \in (0, \infty) \times U$. Then for all $t > 0$ and $x, x' \in U$,*

$$\mathbb{E}(u(t, x)u(t, x'))$$
$$\leq C e^{2t\left(C\lambda^2 L_\sigma^2 + C\lambda^{\frac{4}{2-\beta}} L_\sigma^{\frac{4}{2-\beta}} - \mu_1\right)} \Psi(t, x)\Psi(t, x') J_{2c_1/3}^*(t, x) J_{2c_1/3}^*(t, x'). \tag{1.28}$$

(iii) *Assume (1.9) or (1.10). Then, there exists $\bar{C} > 0$ such that for all $t > 0$ and $x, x' \in U$,*

$$\mathbb{E}(u(t, x)u(t, x')) \geq \bar{C} e^{2t\left(\bar{C}\lambda^2 l_\sigma^2 + \bar{C}\lambda^{\frac{4}{2-\beta}} l_\sigma^{\frac{4}{2-\beta}} - \mu_1\right)}$$
$$e^{-16c_2 \frac{|x-x'|^2}{t}} \Psi(t, x)\Psi(t, x') J_{12c_2}^*(t, x) J_{12c_2}^*(t, x'). \tag{1.29}$$

Remark 1.8 Condition (1.25) holds if and only if

$$J_c^*(t, x) < \infty \quad \text{for all } c > 0, \ t > 0 \text{ and } x \in U.$$

which is the consequence of the following bounds:

$$((c_0 D)^{-1} \wedge 1) e^{-\frac{cD^2}{t}} \|\Phi_1\|_{L^1(U, |\nu|)}$$
$$\leq J_c^*(t, x) \leq \frac{\|\Phi_1\|_{L^1(U, |\nu|)}}{1 \wedge t^{(d+1)/2}} \quad \text{for all } t > 0 \text{ and } x \in U, \qquad (1.30)$$

where $D := \sup\{|x - y|, \ x, y \in U\}$ and we have used the fact that $\Phi_1(\cdot)$ is bounded (see Remark 3.7). The proof of (1.30) is straightforward and is left as an exercise for interested readers.

In fact, Theorem 1.7 can easily be extended for Cartesian products of bounded $C^{1,\alpha}$-domains to allow some Lipschitz domains; see Example 2.1 below.

Corollary 1.9 *Let U be a bounded domain in the following Cartesian product form:*

$$U = U_1 \times U_2 \times \cdots \times U_m \subseteq \mathbb{R}^d, \quad \text{with } m \geq 1, \ U_i \subseteq \mathbb{R}^{d_i}, \ d_i \geq 1, \ \text{and} \ \sum_{i=1}^{m} d_i = d.$$

Assume that each U_i is a bounded C^{1,α_i}-domain for some $\alpha_i > 0$. Let $\mathscr{L}_i$ be a uniformly elliptic differential operator on U_i of the form (1.1) satisfying the condition (1.2). Consider the SPDE (1.3) with $\mathscr{L} = \mathscr{L}_1 + \cdots + \mathscr{L}_m$ on U with the Dirichlet boundary condition. Suppose σ satisfies Assumption 1.2 and the initial measure ν on U satisfies the following integrability condition

$$\int_U \prod_{i=1}^{m} \Phi_1^{U_i}(x_i) |\nu|(\mathrm{d}x) < \infty, \qquad (1.31)$$

where $\Phi_1^{U_i}(x_i)$ is the eigenfunction corresponding to the leading eigenvalue $\mu_i^{U_i}$ of the Dirichlet operator $\mathscr{L}_i$ on U_i. Let $J_c^*(t, x)$ be defined as in (1.24) but with Ψ replaced by

$$\Psi^*(t, x) := \prod_{i=1}^{m} \left(1 \wedge \frac{\Phi_1^{U_i}(x_i)}{1 \wedge t^{1/2}} \right). \qquad (1.32)$$

Then, the statements (i) and (ii) in Theorem 1.7 above hold with $\mu_1 = \sum_{i=1}^{m} \mu_i^{U_i}$, which is the leading eigenvalue of the Dirichlet operator $\mathscr{L}$, and with Ψ defined in (1.24) replaced by Ψ^* defined above in (1.32).

Theorem 1.7 and Corollary 1.9 are proved at the end of Sect. 7. Finally, Table 2 below summarizes the main results of this paper.

Table 2 Summary of the main results and the qualitative differences across domains of increasing regularity/conditions (progressing from the first to the third row)

Domain	Dirichlet		Neumann	
	Upper bound	Lower bound	Upper bound	Lower bound
Lipschitz	(1.18) and (1.22): †	(1.23): ‡	(1.18) and (1.22): †	
$C^{1,\alpha}$ or their products	(1.26) and (1.28): †, *	(1.29): †, *		
Smooth and convex				(1.23): †

† Hold(s) on U; ‡ Hold(s) on U_ϵ for $\epsilon > 0$; * Better estimates near ∂U

1.2 Outline of the paper

The rest of the paper is organized as follows. In Sect. 2, we first give some concrete examples and apply our moment bounds to establish the full intermittency of the solution and discuss its L^2-energy. Then in Sect. 3, we give some preliminaries which include the definition of the mild solution in Sect. 3.1, the cone condition for the domain in Sect. 3.2, and the heat kernel estimates for the Eqs. (1.3) and (1.4) in Sect. 3.3. Then in Sects. 4, resp. 5, we derive the moment bounds in case of bounded, resp. rough, initial conditions, and prove the two cases in Theorem 1.4. The two-point correlation function is studied in Sect. 6, where Theorem 1.5 is proved. The case of bounded $C^{1,\alpha}$-domains with Dirichlet condition is studied in Sect. 7, at the end of which we prove Theorem 1.7 and Corollary 1.9.

2 Examples and applications

In this Section, we give some examples of our main results and apply our moment bounds to study the intermittency property and the L^2-energy of the solutions.

2.1 Rough initial conditions under Dirichlet boundary condition

In this part, we give a few examples to illustrate Theorem 1.7 and Corollary 1.9.

Example 2.1 (Interval for $d = 1$) Consider the stochastic heat Eq. (1.3) with $\mathscr{L} = -\partial^2/\partial x^2$ on an interval $U = (0, L)$ with Dirichlet boundary condition. This is a smooth, and hence $C^{1,\alpha}$ domain. The first eigenvalue is $\mu_1 = (\pi/L)^2$ and the corresponding eigenfunction is $\Phi_1(x) = (2/L)^{1/2} \sin(\pi x/L)$. In this case, $\Psi(t, x)$ defined in (1.24) reduces to (see Fig. 2(2.1))

$$\Psi(t, x) = 1 \wedge \frac{(2/L)^{1/2} \sin(\pi x/L)}{1 \wedge \sqrt{t}}, \tag{2.1}$$

and condition (1.25) becomes

$$\int_0^L \sin(\pi x/L) |v|(\mathrm{d}x) < \infty \iff \int_0^L x(L - x) |v|(\mathrm{d}x) < \infty. \tag{2.2}$$

Due to the dissipative or cooling-down effect of the Dirichlet boundary condition, one can inject, at time zero, more heat flow into the domain from the boundary. For example, the following nonnegative measures with compact support satisfies (2.2) or equivalently (1.25):

$$v(\mathrm{d}x) = \frac{\mathbf{1}_{(0,L)}(x)}{[\sin(\pi x/L)]^\beta}\mathrm{d}x \quad \text{or} \quad v(\mathrm{d}x) = \frac{\mathbf{1}_{(0,L)}(x)}{[x(L - x)]^\beta}\mathrm{d}x, \quad \text{with } \beta < 2. \tag{2.3}$$

| 2.1. $d = 1$. | 2.2. $d \geq 2$. |
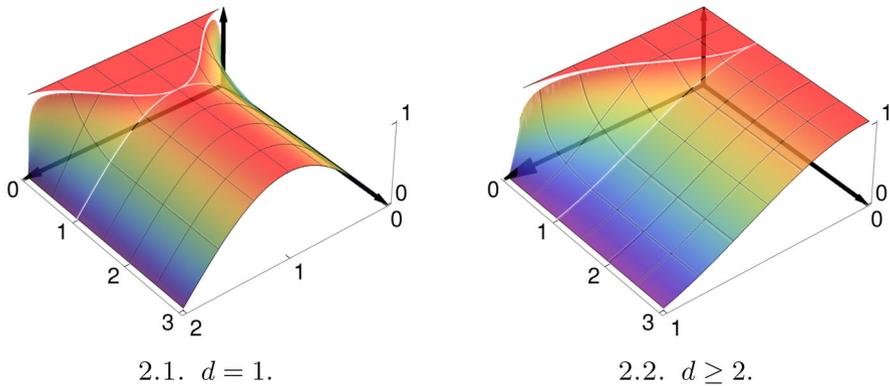
Fig. 2 Some plots of the function $\Psi(t, x)$ in case of $d = 1$ in Fig. 2.1 with $L = 2$, $x \in [0, 2]$ and $t \in [0, 3]$ and $\Psi(t, r)$ in case of $d \geq 2$ in Fig. 2.2 with $d = 4$, $r \in [0, 1]$ and $t \in [0, 3]$. The normalization constant C_d for the plot in Fig. 2.2 is chosen to be the one in (7.8) so that $\max_{(t,r)\in(0,\infty)\times(0,1)} \Psi(t, r) = 1$

But since examples in (2.3) neither have bounded densities nor are finite measures on the domain being considered, both parts of Theorem 1.4 fail to apply for such initial conditions.

Example 2.2 (Unit ball in $\mathbb{R}^d$) Consider the stochastic heat Eq. (1.3) with $\mathscr{L} = -\Delta$ on the unit disk $U = B(0, 1)$ in $\mathbb{R}^d$, $d \geq 2$, with Dirichlet boundary condition. The first eigenvalue is $\mu_1 = z_0^2$, where z_0 is the first positive zero of the Bessel function $J_{(d-2)/2}(\cdot)$, and the corresponding eigenfunction is $\Phi_1(x) = \frac{1}{C_d}|x|^{(2-d)/2} J_{(d-2)/2}(z_0|x|)$; see Remark 7.4 below for more details. In this case, $\Psi(t, x)$ defined in (1.24) reduces to (see Fig. 2(2.2) with $r = |x|$)

$$\Psi(t, x) = 1 \wedge \frac{|x|^{(2-d)/2} J_{(d-2)/2}(z_0|x|)}{C_d \left(1 \wedge \sqrt{t}\right)}. \tag{2.4}$$

Similarly to the previous example, we claim that the following locally finite nonnegative measure on $\mathbb{R}^d$ with compact support

$$\nu(dx) = \frac{\mathbf{1}_{B(0,1)}(x)}{|x|^{\beta_0} (1 - |x|)^{\beta_1}} dx, \quad \text{with } \beta_0 < 2 \text{ and } \beta_1 < 2, \tag{2.5}$$

satisfies condition (1.25). Indeed,

$$0 \leq \int_{B(0,1)} \frac{|x|^{(2-d)/2} J_{(d-2)/2}(z_0|x|)}{|x|^{\beta_0} (1 - |x|)^{\beta_1}} dx = C_d \int_0^1 \frac{r^{(2-d)/2} J_{(d-2)/2}(z_0 r)}{r^{\beta_0 - 1} (1 - r)^{\beta_1}} dr$$
$$\leq C_d \times C \times I,$$

where

$$C := \max_{r \in (0,1)} \frac{r^{(2-d)/2} J_{(d-2)/2}(z_0 r)}{1 - r} < \infty \quad \text{and} \quad I := \int_0^1 \frac{\mathrm{d}r}{r^{\beta_0 - 1}(1 - r)^{\beta_1 - 1}} < \infty.$$

Note that the above maximum is finite thanks to Lemma 7.3 below. By the same reason, condition (1.25) in this case reduces to

$$\int_{B(0,1)} (1 - |x|) \, |\nu|(\mathrm{d}x) < \infty. \tag{2.6}$$

Example 2.3 (Annular domain in $\mathbb{R}^2$) Consider the stochastic heat Eq. (1.3) with $\mathcal{L} = -\Delta$ on the following annulus with Dirichlet boundary condition.[1]:

$$U = \left\{ x \in \mathbb{R}^2 : R_1 < |x| < R_2 \right\}, \quad 0 < R_1 < R_2 < +\infty.$$

Note that U is a nonconvex, but smooth, bounded domain. The leading eigenvalue is $\mu_1 = z_0^2$, where z_0 is the first positive zero of the cross-product Bessel functions

$$J_0(R_1 z) Y_0(R_2 z) - Y_0(R_1 z) J_0(R_2 z) = 0, \tag{2.7}$$

where $J_0(\cdot)$ and $Y_0(\cdot)$ are the Bessel functions of the first and second kind of order zero, respectively. The corresponding eigenfunction is

$$\Phi_1(x) = C Z(|x|) \quad \text{with} \quad Z(r) := J_0(R_1 z_0) Y_0(r z_0) - Y_0(R_1 z_0) J_0(r z_0); \tag{2.8}$$

see Fig. 3(3.1) for a plot of $\Phi_1(x)$. Similarly to the previous example (2.2), we claim that

$$\nu(\mathrm{d}x) = \frac{|x|^{\beta_0} \mathbf{1}_U(x)}{(R_2 - |x|)^{\beta_2} (|x| - R_1)^{\beta_1}} \mathrm{d}x, \quad \text{with } \beta_i < 2, \ i = 1, 2, \text{ and } \beta_0 \in \mathbb{R}, \tag{2.9}$$
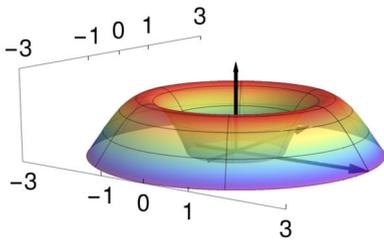
satisfies condition (1.25). Indeed,

$$0 < \int_U \frac{\Phi_1(x)|x|^{\beta_0} \mathrm{d}x}{(R_2 - |x|)^{\beta_2} (|x| - R_1)^{\beta_1}} = 2\pi \int_{R_1}^{R_2} \frac{Z(r) r^{\beta_0 + 1} \mathrm{d}r}{(R_2 - r)^{\beta_2} (r - R_1)^{\beta_1}} \leq 2\pi \times C \times I,$$
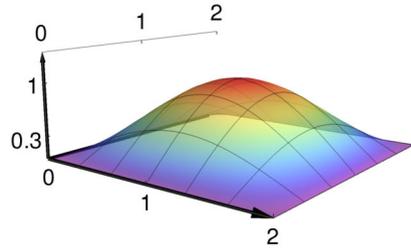
where

$$C := \max_{r \in (R_1, R_2)} \frac{Z(r)}{(R_2 - r)(r - R_1)} < \infty \quad \text{and} \quad I := \int_{R_1}^{R_2} \frac{r^{\beta_0 + 1} \mathrm{d}r}{(R_2 - r)^{\beta_2 - 1} (r - R_1)^{\beta_1 - 1}} < \infty.$$

---

[1] The explicit form of the fundamental solution can be found, e.g., in [36, Section 4.1.2 on p. 418].

3.1. Annular domain. 3.2. Rectangular domain.

Fig. 3 Some plots of the leading eigenfunction $\Phi_1(x)$ with $x \in \mathbb{R}^2$ both in case of the annular domain in Fig. 3.1 where $R_1 = 1$ and $R_2 = 3$ and of the rectangular domain in Fig. 3.2 where $L = 2$

Note that the finiteness of the above constant C is due to the fact that R_1 and R_2 are both simple zeros of $Z(r)$; see Lemma 7.5 below. By the same reason, in this case, condition (1.25) can be equivalently written as

$$\int_{R_1 < |x| < R_2} (R_2 - |x|)\,(|x| - R_1)\,\,|\upsilon|(\mathrm{d}x) < \infty.$$

Example 2.4 (Rectangular domain in $\mathbb{R}^d$ with $d \geq 2$) Consider the stochastic heat Eq. (1.3) with $\mathcal{L} = -\Delta$ on the rectangular domain $U = (0, L)^d \subseteq \mathbb{R}^d$, $d \geq 2$ and $L > 0$, with Dirichlet boundary condition. Note that for $d \geq 2$, U has corners and hence, is not a $C^{1,\alpha}$–domain, but only a Lipschitz domain. The first eigenvalue is $\mu_1 = d \times (\pi/L)^2$ and the corresponding (normalized) eigenfunction is given by (see Fig. 3(3.2) for a plot)

$$\Phi_1(x) = (2/L)^{d/2} \prod_{i=1}^{d} \sin\left(\frac{\pi x_i}{L}\right), \qquad \text{for } x = (x_1, \ldots, x_d) \in (0, L)^d.$$

Similar to Example 2.1, by Corollary 1.9, $\Psi^*(t, x)$ defined in (1.32) reduces to

$$\Psi^*(t, x) = \prod_{i=1}^{d} \left(1 \wedge \frac{(2/L)^{1/2} \sin(\pi x_i/L)}{1 \wedge \sqrt{t}}\right), \qquad (2.10)$$

and condition (1.31) becomes

$$\int_{(0,L)^d} |\upsilon|(\mathrm{d}x) \prod_{i=1}^{d} \sin\left(\frac{\pi x_i}{L}\right) < \infty \quad \Longleftrightarrow \quad \int_{(0,L)^d} |\upsilon|(\mathrm{d}x) \prod_{i=1}^{d} (x_i(L - x_i)) < \infty.$$
$$(2.11)$$

Locally finite measures similar to (2.3) can be constructed component-wise.

Other Lipschitz domains can be considered as an application of Corollary 1.9 as well. For example, for the cylinder domain

$$U = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 < 1 \text{ and } 0 < x_3 < 1 \right\},$$

as an easy exercise (which is left for the interested readers), condition (1.31) becomes

$$\int_{\mathbb{R}^3} \left(1 - x_1^2 - x_2^2 \right) x_3 (1 - x_3) \, |\nu|(\mathrm{d}x) < \infty.$$

2.2 Intermittency

Following [23] and Definition III.1.1 of [5], we say that u is *weakly intermittent* if, for all $x \in U$,

$$\limsup_{t \to \infty} \frac{1}{t} \log \mathbb{E}(|u(t, x)|^2) > 0 \quad \text{and} \tag{2.12}$$

$$\limsup_{t \to \infty} \frac{1}{t} \log \mathbb{E}(|u(t, x)|^p) < \infty \quad \text{for all } p \geq 2, \tag{2.13}$$

and u is *fully intermittent* (or simply intermittent) if (2.12) can be strengthened to

$$\liminf_{t \to \infty} \frac{1}{t} \log \mathbb{E}(|u(t, x)|^2) > 0. \tag{2.14}$$

The following two theorems extend the corresponding results in [24] and [33]. Recall that μ_1 is the first eigenvalue of the operator $\mathscr{L}$ with Dirichlet boundary condition. Also, recall the definitions of $J_c(t, x)$ and $J_{c,\varepsilon}(t, x)$ in (1.15) and (1.21), respectively. The following theorem provides moment bounds for the solution and shows that full intermittency occurs when λ is sufficiently large, but not when λ is small.

Theorem 2.5 *Let $U \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Let u be the solution to (1.3) with Dirichlet boundary condition. Suppose (1.8) holds. Suppose $\nu \geq 0$ and $\nu(U) < \infty$. Then, there exist positive finite constants C, c and c' such that for all $p \geq 2$, $\lambda > 0$, $t > 0$, $x \in U$,*

$$\mathbb{E}(|u(t, x)|^p) \leq C^p \left(J_{c_1}(t, x) \right)^p e^{pt \left(cp\lambda^2 L_\sigma^2 + c' p^{\frac{2}{2-\beta}} \lambda^{\frac{4}{2-\beta}} L_\sigma^{\frac{4}{2-\beta}} - \mu_1 \right)}. \tag{2.15}$$

Moreover, if (1.10) or (1.9) holds, then there exists $0 < \varepsilon_0 < 1$ such that for all $0 < \varepsilon \leq \varepsilon_0$, there exist positive finite constants $\overline{C}$, $\overline{c}$ and $\widetilde{c}$ depending on ε such that for all $p \geq 2$, $\lambda > 0$, $t > 0$, $x \in U_\varepsilon$,

$$\mathbb{E}(|u(t, x)|^p) \geq \overline{C}^p \left(J_{12c_2, \varepsilon}(t, x) \right)^p e^{pt \left(\overline{c}\lambda^2 l_\sigma^2 + \widetilde{c}\lambda^{\frac{4}{2-\beta}} l_\sigma^{\frac{4}{2-\beta}} - \mu_1 \right)}. \tag{2.16}$$

Consequently, if $0 < \nu(U_\varepsilon) \leq \nu(U) < \infty$, then there exist $0 < \lambda_0 < \lambda_1 < \infty$ such that u is fully intermittent on U_ε when $\lambda > \lambda_1$, but not when $\lambda < \lambda_0$ as

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}(|u(t,x)|^2) < 0,$$

where

$$\lambda_0 := \sup \left\{ \lambda > 0 : 2c\lambda^2 L_\sigma^2 + 2^{\frac{2}{2-\beta}} c' \lambda^{\frac{4}{2-\beta}} L_\sigma^{\frac{4}{2-\beta}} \leq \mu_1 \right\},$$

$$\lambda_1 = \lambda_1(\varepsilon) := \inf \left\{ \lambda > 0 : \overline{c}\lambda^2 l_\sigma^2 + \widetilde{c}\lambda^{\frac{4}{2-\beta}} l_\sigma^{\frac{4}{2-\beta}} \geq \mu_1 \right\}.$$

Proof The upper bound (2.15) follows from Theorem 1.4. The lower bound follows from Theorem 1.5 and Jensen's inequality $\mathbb{E}(|u(t,x)|^p) \geq \mathbb{E}(|u(t,x)|^2)^{p/2}$.

It remains to prove the last statement of full intermittency. First, since $\nu(U) < \infty$,

$$\limsup_{t\to\infty} \frac{1}{t} \log \left(J_{c_1}(t,x) \right)^p \leq \limsup_{t\to\infty} \frac{p}{t} \left(-\log(1 \wedge t^{d/2}) + \log \nu(U) \right) = 0.$$

Then, (2.15) implies that for all $x \in U$ and $\lambda > 0$,

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}(|u(t,x)|^p) \leq p\left(cp\lambda^2 L_\sigma^2 + c'\, p^{\frac{2}{2-\beta}} \lambda^{\frac{4}{2-\beta}} L_\sigma^{\frac{4}{2-\beta}} - \mu_1 \right), \quad (2.17)$$

which proves (2.13). Moreover, $\nu(U_\varepsilon) > 0$ implies $\log \nu(U_\varepsilon) > -\infty$, hence

$$\liminf_{t\to\infty} \frac{1}{t} \log \left(J_{12c_2,\varepsilon}(t,x) \right)^2$$
$$\geq \liminf_{t\to\infty} \frac{2}{t} \left(-\log(1 \wedge t^{d/2}) - 4c_2 \sup_{x,y\in U} |x-y|^2 + \log \nu(U_\varepsilon) \right) = 0.$$

If $\lambda > \lambda_1$, then it follows from (2.16) that for all $x \in U_\varepsilon$,

$$\liminf_{t\to\infty} \frac{1}{t} \log \mathbb{E}(|u(t,x)|^2) \geq 2\left(\overline{c}\lambda^2 l_\sigma^2 + \widetilde{c}\lambda^{\frac{4}{2-\beta}} l_\sigma^{\frac{4}{2-\beta}} - \mu_1 \right) > 0,$$

which proves (2.14). Hence, u is fully intermittent on U_ε. On the other hand, by (2.17),

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}(|u(t,x)|^2)$$
$$\leq 2\left(2c\lambda^2 L_\sigma^2 + 2^{\frac{2}{2-\beta}} c' \lambda^{\frac{4}{2-\beta}} L_\sigma^{\frac{4}{2-\beta}} - \mu_1 \right) < 0, \quad \text{when } \lambda < \lambda_0,$$

which completes the proof of Theorem 2.5. $\qquad\square$

Similarly, we get the following result from Theorem 1.7 for $C^{1,\alpha}$-domains.

Theorem 2.6 *Let $U \subset \mathbb{R}^d$ be a bounded $C^{1,\alpha}$-domain, where $\alpha > 0$. Let u be the solution to* (1.3) *with Dirichlet boundary condition. Suppose* (1.8) *holds. Suppose $v \geq 0$ and $\Phi_1 \in L^1(U, v)$. Then, there exist positive finite constants C, c and c' such that for all $p \geq 2$, $\lambda > 0$, $t > 0$, $x \in U$,*

$$\mathbb{E}(|u(t,x)|^p) \leq C^p \Psi^p(t,x) \left(J^*_{2c_1/3}(t,x)\right)^p e^{pt\left(c p \lambda^2 L_\sigma^2 + c' p^{\frac{2}{2-\beta}} \lambda^{\frac{4}{2-\beta}} L_\sigma^{\frac{4}{2-\beta}} - \mu_1\right)}.$$
(2.18)

Moreover, if (1.10) *or* (1.9) *holds, then there exist positive finite constants $\overline{C}$, $\overline{c}$ and $\widetilde{c}$ such that for all $p \geq 2$, $\lambda > 0$, $t > 0$, $x \in U$,*

$$\mathbb{E}(|u(t,x)|^p) \geq \overline{C}^p \Psi^p(t,x) \left(J^*_{12c_2}(t,x)\right)^p e^{pt\left(\overline{c} \lambda^2 l_\sigma^2 + \widetilde{c} \lambda^{\frac{4}{2-\beta}} l_\sigma^{\frac{4}{2-\beta}} - \mu_1\right)}.$$
(2.19)

Consequently, if $0 < \|\Phi_1\|_{L^1(U,v)} < \infty$, then there exist $0 < \lambda_0 < \lambda_1 < \infty$ such that u is fully intermittent on U when $\lambda > \lambda_1$, but not when $\lambda < \lambda_0$, where

$$\lambda_0 := \sup \left\{\lambda > 0 : 2c\lambda^2 L_\sigma^2 + 2^{\frac{2}{2-\beta}} c' \lambda^{\frac{4}{2-\beta}} L_\sigma^{\frac{4}{2-\beta}} \leq \mu_1\right\},$$

$$\lambda_1 := \inf \left\{\lambda > 0 : \overline{c}\lambda^2 l_\sigma^2 + \widetilde{c} \lambda^{\frac{4}{2-\beta}} l_\sigma^{\frac{4}{2-\beta}} \geq \mu_1\right\}.$$

As in [24, 26, 33], it is not clear whether the solution is intermittent if $\lambda \in [\lambda_0, \lambda_1]$ in Theorems 2.5 and 2.6 above. Instead, we propose the following conjecture for future investigation.

Conjecture 2.7 *Under the settings of either Theorem 2.5 or Theorem 2.6, there exists $\lambda^* \in [\lambda_0, \lambda_1]$ such that when $\lambda > \lambda^*$, the solution $u(t,x)$ to* (1.3) *is fully intermittent; when $\lambda < \lambda^*$, the solution has all p-th moments ($p \geq 2$) bounded in time and is not fully intermittent.*

Theorem 2.8 *Let $U \subset \mathbb{R}^d$ be a convex bounded Lipschitz domain. Let u be the solution to* (1.4) *with Neumann boundary condition. Suppose* (1.8) *holds. Suppose $v \geq 0$ and $v(U) < \infty$. Then, there exist positive finite constants C, c and c' such that for all $p \geq 2$, $\lambda > 0$, $t > 0$, $x \in U$,*

$$\mathbb{E}(|u(t,x)|^p) \leq C^p \left(J_{c_3}(t,x)\right)^p e^{pt\left(c p \lambda^2 L_\sigma^2 + c' p^{\frac{2}{2-\beta}} \lambda^{\frac{4}{2-\beta}} L_\sigma^{\frac{4}{2-\beta}}\right)}.$$
(2.20)

Moreover, if (1.10) *or* (1.9) *holds, then there exist positive finite constants $\overline{C}$, $\overline{c}$ and $\widetilde{c}$ such that for all $p \geq 2$, $\lambda > 0$, $t > 0$, $x \in U$,*

$$\mathbb{E}(|u(t,x)|^p) \geq \overline{C}^p \left(J_{12c_4}(t,x)\right)^p e^{pt\left(\overline{c} \lambda^2 l_\sigma^2 + \widetilde{c} \lambda^{\frac{4}{2-\beta}} l_\sigma^{\frac{4}{2-\beta}}\right)}.$$
(2.21)

Consequently, if $0 < v(U) < \infty$, then u is fully intermittent on U for all $\lambda > 0$.

Proof The proof of (2.20) and (2.21) is similar to that of Theorem 2.5. Finally, (2.20) and (2.21) imply that

$$\liminf_{t\to\infty} \frac{1}{t} \log \mathbb{E}(|u(t,x)|^2) \geq 2\left(\overline{c}\lambda^2 l_\sigma^2 + \widetilde{c}\lambda^{\frac{4}{2-\beta}} l_\sigma^{\frac{4}{2-\beta}}\right) > 0$$

and

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}(|u(t,x)|^p) \leq p\left(c p \lambda^2 L_\sigma^2 + c' \, p^{\frac{2}{2-\beta}} \, \lambda^{\frac{4}{2-\beta}} L_\sigma^{\frac{4}{2-\beta}}\right) < \infty$$

for all $p \geq 2$ and all $\lambda > 0$. Hence, u is fully intermittent for all $\lambda > 0$. □

2.3 $L^2(U)$-energy of solution

Following [24, 29, 30], the L^2-*energy* of the solution u at time $t > 0$ is defined as

$$\mathscr{E}_t(\lambda) = \left(\mathbb{E}\int_U |u(t,x)|^2 \mathrm{d}x\right)^{1/2},$$

and the *excitation index* (at infinity) is defined as

$$\lim_{\lambda\to\infty} \frac{\log\log \mathscr{E}_t(\lambda)}{\log\lambda}$$

provided the limit exists. Note that, for $\varepsilon \geq 0$,

$$\mathrm{Vol}(U_\varepsilon) \times \inf_{x\in U_\varepsilon} \mathbb{E}(|u(t,x)|^2) \leq \mathscr{E}_t^2(\lambda) \leq \mathrm{Vol}(U) \times \sup_{x\in U} \mathbb{E}(|u(t,x)|^2).$$

As a result of Theorems 2.5 and 2.8, we see that the solution is intermittent for λ large under Dirichlet or Neumann boundary condition, and the energy of the solution behaves like

$$\mathscr{E}_t(\lambda) \sim Ce^{Ct\lambda^{\frac{4}{2-\beta}}} \quad \text{for } \lambda \text{ large.}$$

Under Neumann boundary condition, the solution remains intermittent even when $\lambda > 0$ is small. However, in this case, the energy of the solution has a different exponential rate in λ than the one above, namely,

$$\mathscr{E}_t(\lambda) \sim Ce^{Ct\lambda^2} \quad \text{for } \lambda > 0 \text{ small.}$$

In other words, the excitation index "at zero" is different. We obtain the following corollaries:

Corollary 2.9 *Let u be the solution of* (1.3) *with Dirichlet boundary condition.*

(1) If the conditions of Theorem 2.5 hold, then for all $t > 0$,

$$\lim_{\lambda \to \infty} \frac{\log \log \mathscr{E}_t(\lambda)}{\log \lambda} = \frac{4}{2 - \beta}.$$

(2) Moreover, if the conditions of Theorem 2.6 hold, then for all $t > 0$,

$$\mathscr{E}_t^*(\lambda) := \left(\mathbb{E} \int_U |u(t, x)|^2 \frac{\mathrm{d}x}{|\Phi_1(x)|^2} \right)^{1/2} < \infty$$

and

$$\lim_{\lambda \to \infty} \frac{\log \log \mathscr{E}_t^*(\lambda)}{\log \lambda} = \frac{4}{2 - \beta}.$$

Note that $\mathscr{E}_t(\lambda) \le C \mathscr{E}_t^*(\lambda)$ for some constant C.

Corollary 2.10 *Let u be the solution of* (1.4) *with Neumann boundary condition and the conditions in Theorem 2.8 hold. Then for any $t > 0$,*

$$\lim_{\lambda \to \infty} \frac{\log \log \mathscr{E}_t(\lambda)}{\log \lambda} = \frac{4}{2 - \beta} \quad and \quad \lim_{\lambda \to 0^+} \frac{\log \log \mathscr{E}_t(\lambda)}{\log \lambda} = 2.$$

3 Preliminaries

3.1 Mild solutions

Let $\dot{W}$ be a centered and spatially homogeneous Gaussian noise that is white in time defined on a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with covariance given in (1.5).

Remark 3.1 Let f be any spatially homogeneous correlation correlation function on $\mathbb{R}^d$; see (1.5). When restricted on the domain $\{x - y : x, y \in U\}$, $f(\cdot)$ is still a non-negative definite function. Indeed, for any test function ϕ defined on U,

$$\iint_{U^2} \phi(x) f(x - y) \phi(y) \mathrm{d}x \mathrm{d}y$$
$$= \iint_{\mathbb{R}^{2d}} (\phi(x) \mathbf{1}_U(x)) f(x - y) (\phi(y) \mathbf{1}_U(y)) \mathrm{d}x \mathrm{d}y \ge 0.$$

Let $\mathscr{B}(U)$ denote the Borel σ-algebra on $U \subseteq \mathbb{R}^d$. Let $\{W_t(A); t \ge 0, A \in \mathscr{B}(U)\}$ be the martingale measure associated to the noise $\dot{W}$ in the sense of Walsh [43]. Let $\{\mathcal{F}_t, t \ge 0\}$ be the underlying filtration generated by W and augmented by the σ-field $\mathcal{N}$ generated by all $\mathbb{P}$-null sets in $\mathcal{F}$, namely,

$$\mathscr{F}_t = \sigma \{W_s(A) : 0 \le s \le t, A \in \mathscr{B}(U)\} \vee \mathcal{N}.$$

Definition 3.2 A process $u = \{u(t, x); t > 0, x \in U\}$ is called *a random field solution* to (1.3) (or (1.4), respectively) if:

(i) u is adapted, i.e., for each $t > 0$ and $x \in U$, $u(t, x)$ is $\mathscr{F}_t$-measurable;
(ii) u is jointly measurable with respect to $\mathscr{B}((0, \infty) \times U) \times \mathscr{F}$;
(iii) for each $t > 0$ and $x \in U$,

$$\mathbb{E}\left(\int_0^t \mathrm{d}s \iint_{U^2} \mathrm{d}y\, \mathrm{d}y'\, G(t - s, x, y)\sigma(s, y, u(s, y))f(y - y')\right.$$
$$\left. G(t - s, x, y')\sigma(s, y', u(s, y'))\right) < \infty; \tag{3.1}$$

(iv) u satisfies

$$u(t, x) = J(t, x) + \lambda \int_0^t \int_U G(t - s, x, y)\sigma(s, y, u(s, y))W(\mathrm{d}s, \mathrm{d}y) \quad \text{a.s.} \tag{3.2}$$

for each $t > 0$ and $x \in U$, where $G = G_D$ (or $G = G_N$, respectively), and $J(t, x)$ is the solution to the homogeneous equation, namely,

$$J(t, x) := \int_U G(t, x, y)v(\mathrm{d}y). \tag{3.3}$$

Note that in (iii) above, the condition (3.1) ensures that the *Walsh stochastic integral*

$$\int_0^t \int_U G(t - s, x, y)\sigma(s, y, u(s, y))W(\mathrm{d}s, \mathrm{d}y)$$

is well-defined and the square of its $\|\cdot\|_2$-norm is equal to the expression in (3.1).

3.2 Regularities and geometric properties of the domain

The definition of the Lipschitz domain is standard; see e.g. Section 1.2.1 of [25].

Definition 3.3 A bounded domain $U \subset \mathbb{R}^d$ is called a *Lipschitz domain* if there exist positive constants K_U and r_0 such that for every $q \in \partial U$, there exist a Lipschitz function $F_q : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying $|F_q(x) - F_q(\overline{x})| \le K_U|x - \overline{x}|$ for all $x, \overline{x} \in \mathbb{R}^{d-1}$ and an orthonormal coordinate system with origin q such that if $z = (x, y)$, $x \in \mathbb{R}^{d-1}$, $y \in \mathbb{R}$, in this coordinate system, then

$$U \cap B(q, r_0) = B(q, r_0) \cap \{z = (x, y) : y > F_q(x)\}$$

and

$$\partial U \cap B(q, r_0) = B(q, r_0) \cap \{z = (x, y) : y = F_q(x)\},$$

where $B(q, r_0)$ is the open ball centered at q of radius r_0. We call K_U the *Lipschitz constant* of U and r_0 the *localization radius* of U.
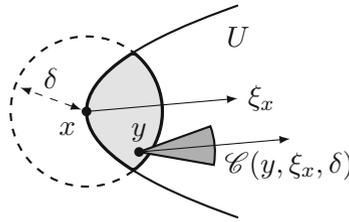
Fig. 4 Illustration for the δ-cone property in Definition 3.4

Definition 3.4 (Definition 2.4.1 of [27]) For $\delta > 0$, we say that U has the δ-*cone property* if, for every $x \in \partial U$, there exists a unit vector $\xi_x \in \mathbb{R}^d$ such that for all $y \in \overline{U} \cap B(x, \delta)$, we have $\mathscr{C}(y, \xi_x, \delta) \subset U$, where $\mathscr{C}(y, \xi, \delta)$ is the δ-*cone* defined by

$$\mathscr{C}(y, \xi, \delta) := \left\{ z \in \mathbb{R}^d : 0 < |z - y| < \delta \text{ and } (z - y) \cdot \xi \geq |z - y| \cos(\delta) \right\}. \quad (3.4)$$

See Fig. 4 for an illustration.

It is clear that if U has the δ-cone property, then it also has the δ'-cone property for $0 < \delta' \leq \delta$. It is known that for a bounded domain $U \subset \mathbb{R}^d$, it is a Lipschitz domain if and only if it has the δ-cone property for some $\delta > 0$; see, e.g., [27, Theorem 2.4.7] or [25, Theorem 1.2.2.2]. In particular, according to the above definitions, one can easily see that if U is a Lipschitz domain, then it satisfies the δ-cone condition with

$$\delta = \arctan(1/K_U) \wedge r_0. \quad (3.5)$$

The δ-cone property gives us a convenient way to handle the Lipschitz domain.

Example 3.5 Any $C^{1,\alpha}$-domain with $\alpha > 0$ is a Lipschitz domain; see, e.g., [35]. The unit ball in $\mathbb{R}^d$ is a smooth domain and also a $C^{1,\alpha}$-domain. $U = (-1, 1)^d$ is a Lipschitz domain but not a $C^{1,\alpha}$-domain for any $\alpha > 0$. Domains with cusps are not Lipschitz domain; see Fig. 1(1.5).

3.3 Heat kernel estimates

Recall that $\mathscr{L}$ is the operator defined in divergence form (1.1) satisfying the uniformly elliptic condition (1.2). It is known that the operator $\mathscr{L}$ with Dirichlet boundary condition on U has a discrete spectrum with a sequence of positive eigenvalues $0 < \mu_1 \leq \mu_2 \leq \ldots$ such that the first eigenvalue μ_1 is simple and its eigenfunction Φ_1 can be chosen to be positive and $\|\Phi_1\|_{L^2(U)} = 1$; see, e.g., [20]. Moreover, we have the following heat kernel estimates under Dirichlet and Neumann boundary conditions.

Proposition 3.6 *Let* $U \subset \mathbb{R}^d$ *be a bounded Lipschitz domain. Then the following estimates hold.*

(i) *Dirichlet heat kernel estimates: There exist positive finite constants c_1, c_2, C_1, C_2 and $0 < a_2 \leq 1 \leq a_1$ such that for all $t > 0$ and $x, y \in U$,*

$$C_2 \left(1 \wedge \frac{\Phi_1(x)}{1 \wedge t^{a_2/2}}\right) \left(1 \wedge \frac{\Phi_1(y)}{1 \wedge t^{a_2/2}}\right) \frac{e^{-\mu_1 t}}{1 \wedge t^{d/2}} e^{-c_2 \frac{|x-y|^2}{t}}$$

$$\leq G_D(t, x, y) \tag{3.6}$$

$$\leq C_1 \left(1 \wedge \frac{\Phi_1(x)}{1 \wedge t^{a_1/2}}\right) \left(1 \wedge \frac{\Phi_1(y)}{1 \wedge t^{a_1/2}}\right) \frac{e^{-\mu_1 t}}{1 \wedge t^{d/2}} e^{-c_1 \frac{|x-y|^2}{t}}.$$

Moreover, if U is a bounded $C^{1,\alpha}$-domain with $\alpha > 0$, then (3.6) holds with $a_1 = a_2 = 1$.

(ii) *Neumann heat kernel estimates: there exist positive finite constants c_3 and C_3 such that for all $t > 0$ and $x, y \in U$,*

$$G_N(t, x, y) \leq C_3 \frac{1}{1 \wedge t^{d/2}} e^{-c_3 \frac{|x-y|^2}{t}}. \tag{3.7}$$

In addition, if U is a smooth convex domain and $\mathscr{L} = -\Delta$, then there exist positive finite constants c_4 and C_4 such that for all $t > 0$ and $x, y \in U$,

$$G_N(t, x, y) \geq C_4 \frac{1}{1 \wedge t^{d/2}} e^{-c_4 \frac{|x-y|^2}{t}}. \tag{3.8}$$

Proof The Dirichlet case is proved in [38] (see Theorem 2.1 and Remark 1 on p.123). For the Neumann case, the upper bound in (3.7) can be found in Theorem 3.2.9 of [20], where we note that the *extension property* referred in that theorem (*ibid.*) is satisfied by the Lipschitz domain (see either Proposition 1.7.9 of [20] or Theorem 1.4.3.1 of [25]). The lower bound in (3.8) follows from [40, Theorem 3.1 and Examples 3.3]; see also [39]. □

Remark 3.7 In the Dirichlet boundary condition case, by [38, (1.2)], there exists a finite constant $c_0 > 1$ such that for all $z \in U$,

$$c_0^{-1} \left[\text{dist}(z, \partial U)\right]^{a_1} \leq \Phi_1(z) \leq c_0 \left[\text{dist}(z, \partial U)\right]^{a_2}, \tag{3.9}$$

where a_1 and a_2 are constants from part (i) of Proposition 3.6.

Remark 3.8 In general, Theorem 3.1 of [40] states that the following conditions are equivalent:

- The two-sided bound (3.7) and (3.8) holds for the Neumann heat kernel on U, that is, for all $t > 0$ and $x, y \in U$,

$$C_4 \frac{1}{1 \wedge t^{d/2}} e^{-c_4 \frac{|x-y|^2}{t}} \leq G_N(t, x, y) \leq C_3 \frac{1}{1 \wedge t^{d/2}} e^{-c_3 \frac{|x-y|^2}{t}}; \tag{3.10}$$

- The parabolic Harnack inequality holds;

• The domain U has the volume doubling property and the Poincaré inequality holds.

The results of the present paper under the Neumann boundary condition, especially the lower bound results, remain valid for U satisfying any one of the above equivalent conditions.

In the rest of the paper, the lower case constants c_1, c_2, c_3, c_4 are reserved for the constants given by Proposition 3.6 above.

Before the end of this subsection, we prove that the Lipschitz domain U satisfies the lower bound in (3.11) below. The following lemma may well be buried in the literature. Since its proof is short, it will be given below. Let $\mathrm{Vol}(A)$ denote the volume (or d-dimensional Lebesgue measure) of any measurable set A in $\mathbb{R}^d$.

Lemma 3.9 *Suppose that U is a bounded Lipschitz domain in $\mathbb{R}^d$. Then, there exists positive finite constants C and C' such that for all $y \in \overline{U}$ and $r > 0$,*

$$C(1 \wedge r)^d \leq \mathrm{Vol}(U \cap B(y, r)) \leq C'(1 \wedge r)^d. \tag{3.11}$$

Proof The upper bound is trivial since $\mathrm{Vol}(U \cap B(y, r)) \leq \mathrm{Vol}(U) \wedge \mathrm{Vol}(B(y, r))$. We only need to prove the lower bound. It is known that the Lipschitz domain U satisfies the δ-cone property with δ given in (3.5).

We first consider the case of $0 < r < \delta$. If $\mathrm{dist}(y, \partial U) > r$, then $B(y, r) \subset U$ and

$$\mathrm{Vol}(U \cap B(y, r)) = \mathrm{Vol}(B(y, r)) = V_d \, r^d \geq V_d (1 \wedge r)^d,$$

where $V_d = \pi^{d/2} / \Gamma(d/2 + 1)$. If $\mathrm{dist}(y, \partial U) \leq r$, then $y \in B(x, \delta)$ for some $x \in \partial U$, and by the δ-cone property, we can find a unit vector $\xi = \xi_x \in \mathbb{R}^d$ such that $\mathscr{C}(y, \xi, \delta) \subset U$. It follows that

$$
\begin{aligned}
\mathrm{Vol}(U \cap B(y, r)) &\geq \mathrm{Vol}(\mathscr{C}(y, \xi, \delta) \cap B(y, r)) \\
&= \mathrm{Vol}\{z \in \mathbb{R}^d : 0 < |z - y| < r \text{ and } (z - y) \cdot \xi \geq |z - y| \cos \delta\} \\
&= C_{d,\delta} V_d \, r^d,
\end{aligned}
$$

where $C_{\delta,d} \in (0, 1]$. Therefore, when $r \in (0, \delta)$, $\mathrm{Vol}(U \cap B(y, r)) \geq C_{d,\delta} V_d (1 \wedge r)^d$. Finally, the case of $r \geq \delta$ follows from the previous case because

$$\mathrm{Vol}(U \cap B(y, r)) \geq \mathrm{Vol}(U \cap B(y, \delta)) \geq C_{d,\delta} V_d \, \delta^d \geq C_{d,\delta} V_d \, \delta^d (1 \wedge r)^d.$$

This implies the desired lower bound with $C = C_{d,\delta} V_d (1 \wedge \delta)^d$. □

Next, we need to replace U in (3.11) above by a subset of U with some specific properties. Take $0 < \varepsilon_1 < 1$ such that $U_{\varepsilon_1} \neq \varnothing$ (see (1.20)) and let

$$\varepsilon_0 = \varepsilon_1 \wedge (\delta/2). \tag{3.12}$$

By the compactness of U, find and fix a finite collection of open balls $\{B(y_i, \varepsilon_0)\}_{i=1}^m$ with centers $y_1, \ldots, y_m \in \partial U$ such that $\bigcup_{i=1}^m B(y_i, \varepsilon_0) \supset \overline{U} \backslash U_{\varepsilon_0/2}$. Then, for any $x \in U$, we have one of the following cases:
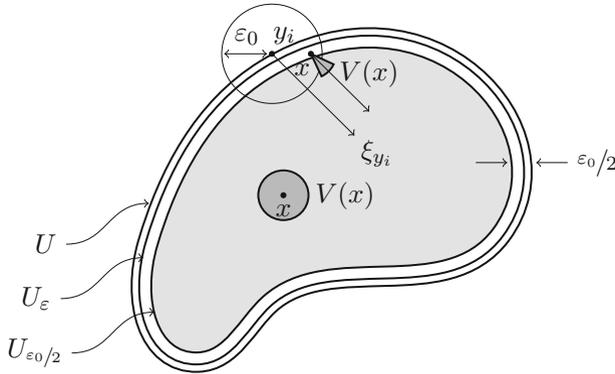
Fig. 5 Illustration for the two cases of $V(x)$ in (3.13)

(1) If $x \in \bigcup_{i=1}^{m} \overline{B(y_i, \varepsilon_0)}$, choose the smallest i such that $\overline{B(y_i, \varepsilon_0)} \ni x$. Then, by the δ-cone property of U, we have $\overline{\mathscr{C}(x, \xi_{y_i}, \varepsilon_0/2)} \subset U$.

(2) If $x \notin \bigcup_{i=1}^{m} \overline{B(y_i, \varepsilon_0)}$, then $\text{dist}(x, \partial U) > \varepsilon_0/2$ and thus $\overline{B(x, \varepsilon_0/2)} \subset U$.

Accordingly, we define (see Fig. 5 for an illustration)

$$V(x) = \begin{cases} \overline{\mathscr{C}(x, \xi_{y_i}, \varepsilon_0/2)} & \text{in case (1),} \\ \overline{B(x, \varepsilon_0/2)} & \text{in case (2).} \end{cases} \tag{3.13}$$

The next lemma will be used later together with the heat kernel estimates to derive the lower bounds in Lemmas 5.1 and 5.2 below.

Lemma 3.10 *Let $U \subset \mathbb{R}^d$ be a bounded Lipschitz domain with the δ-cone property. Let $\varepsilon_0 \in (0, 1)$ and $V(x) \subset U$, for $x \in U$, be defined by (3.12) and (3.13) above. Then, for each $\varepsilon \in (0, \varepsilon_0]$ there exists $c_\varepsilon > 0$ with $\lim_{\varepsilon \to 0} c_\varepsilon = 0$ such that for all $x \in U_\varepsilon$, we have*

$$d(z) := \text{dist}(z, \partial U) \geq c_\varepsilon \quad \text{for all } z \in V(x) \tag{3.14}$$

and

$$\text{Vol}(V(x) \cap B(x, r)) \geq C(1 \wedge r)^d \quad \text{for all } r > 0, \tag{3.15}$$

where $C > 0$ is a constant depending on d and ε_0.

Proof Let $\varepsilon \in (0, \varepsilon_0]$. We first prove (3.14). On the one hand, for each i and $x \in \overline{U_\varepsilon} \cap \overline{B(y_i, \varepsilon_0/2)}$, by the δ-cone property of U, we have $\overline{\mathscr{C}(x, \xi_{y_i}, \varepsilon_0/2)} \subset U$, which implies that

$$\text{dist}\left(\overline{\mathscr{C}(x, \xi_{y_i}, \varepsilon_0/2)}, \partial U\right) := \inf\left\{|z - y| : z \in \overline{\mathscr{C}(x, \xi_{y_i}, \varepsilon_0/2)},\ y \in \partial U\right\} > 0.$$

Since the function $x \mapsto \operatorname{dist}(\overline{\mathscr{C}(x, \xi_{y_i}, \varepsilon_0/2)}, \partial U)$ is continuous on the compact set $\overline{U_\varepsilon} \cap \overline{B(y_i, \varepsilon_0/2)}$, we can find $c_{\varepsilon,i} > 0$ such that $\operatorname{dist}(\overline{\mathscr{C}(x, \xi_{y_i}, \varepsilon_0/2)}, \partial U) \geq c_{\varepsilon,i}$ for all $x \in \overline{U_\varepsilon} \cap \overline{B(y_i, \varepsilon_0/2)}$.

On the other hand, for $x \in \overline{U_\varepsilon} \backslash \bigcup_{i=1}^m B(y_i, \varepsilon_0/2)$, we have $\overline{B(x, \varepsilon_0/2)} \subset U$, and hence

$$\operatorname{dist}(\overline{B(x, \varepsilon_0/2)}, \partial U) > 0.$$

Then, by the continuity of $x \mapsto \operatorname{dist}(\overline{B(x, \varepsilon_0/2)}, \partial U)$ and the compactness of $\overline{U_\varepsilon} \backslash \bigcup_{i=1}^m B(y_i, \varepsilon_0/2)$, we can find $c_{\varepsilon,0} > 0$ such that $\operatorname{dist}(\overline{B(x, \varepsilon_0/2)}, \partial U) \geq c_{\varepsilon,0}$ for all $x \in \overline{U_\varepsilon} \backslash \bigcup_{i=1}^m B(y_i, \varepsilon_0/2)$.

Therefore, by taking $c_\varepsilon = \min\{c_{\varepsilon,i} : 0 \leq i \leq m\}$, we get that $\operatorname{dist}(V(x), \partial U) \geq c_\varepsilon$. Also, note that for $1 \leq i \leq m$, we have $0 < c_{\varepsilon,i} \leq \varepsilon$, so $c_\varepsilon \to 0$ as $\varepsilon \to 0$. This proves (3.14).

As for (3.15), if $V(x) = \overline{\mathscr{C}(x, \xi_{y_i}, \varepsilon_0/2)}$ in case (1) of (3.13), then

$$\begin{aligned}
\operatorname{Vol}(V(x) \cap B(x, r)) &= \operatorname{Vol}(\overline{\mathscr{C}(x, \xi_{y_i}, \varepsilon_0/2)} \cap B(x, r)) \\
&= \operatorname{Vol}\{z \in \mathbb{R}^d : 0 \leq |z - x| < (\varepsilon_0/2) \wedge r \text{ and } (z - x) \\
&\quad \cdot \xi_{y_i} \geq |z - x| \cos(\varepsilon_0/2)\} \\
&= C_{d,\varepsilon_0}((\varepsilon_0/2) \wedge r)^d \geq C_{d,\varepsilon_0}(\varepsilon_0/2)^d (1 \wedge r)^d.
\end{aligned}$$

If $V(x) = \overline{B(x, \varepsilon_0/2)}$ in case (2) of (3.13), then

$$\begin{aligned}
\operatorname{Vol}(V(x) \cap B(x, r)) &= \operatorname{Vol}(B(x, (\varepsilon_0/2) \wedge r)) = C_d((\varepsilon_0/2) \wedge r)^d \\
&\geq C_d(\varepsilon_0/2)^d (1 \wedge r)^d.
\end{aligned}$$

This shows (3.15) and completes the proof of Lemma 3.10. $\qquad\square$

4 Bounded initial condition case

In this section, we give some computations to show how the noise interacts with the differential operator. Let us first give a general definition, which is not only restricted to the heat equation.

Definition 4.1 Let U be a general domain in $\mathbb{R}^d$ and let $G(t, x, y)$ be the fundamental solution to the corresponding partial differential equation. Let f be a nonnegative, nonnegative definite function. Let $h_0^U(t)$ be a locally integrable function defined on $\mathbb{R}_+ := [0, \infty)$. Define formally the following functions:

$$K_\lambda^U(t) := \sum_{n=0}^\infty \lambda^{2n} h_n^U(t) \tag{4.1a}$$

where

$$h_n^U(t) := \left(k^U * h_{n-1}^U \right)(t) \quad \text{for } n \geq 1 \text{ and,} \tag{4.1b}$$

$$k^U(t) := \sup_{x,x' \in U} \iint_{U^2} G(t, x, y) G(t, x', y') f(y - y') \, \mathrm{d}y \, \mathrm{d}y'. \tag{4.1c}$$

These functions depend on the fundamental solution G. When it is clear from the context, the superscript U will be omitted.

In the above, "$*$" is the standard convolution in the time variable:

$$h * k(t) = \int_0^t h(t - s) k(s) \mathrm{d}s.$$

For $n \geq 1$, we will also denote by k^{*n} the n-th convolution power of k, i.e., $k^{*1} = k$ and

$$k^{*(n+1)}(t) = \int_0^t k^{*n}(t - s) k(s) \mathrm{d}s.$$

Remark 4.2 In [14] and [12], the kernel function $k(t)$ is defined as

$$k(t) = \int_{\mathbb{R}^d} f(z) G(t, z) \mathrm{d}z,$$

where $G(t, x)$ is the heat kernel on $\mathbb{R}^d$. This is consistent to (4.1c) (up to a factor of 2):

$$\begin{aligned}
k^{\mathbb{R}^d}(t) &= \sup_{x,x' \in \mathbb{R}^d} \iint_{\mathbb{R}^{2d}} G(t, x - y) G(t, x' - y') f(y - y') \mathrm{d}y \mathrm{d}y' \\
&= \sup_{x,x' \in \mathbb{R}^d} \iint_{\mathbb{R}^{2d}} G(t, x - y) G(t, x' - y + z) f(z) \mathrm{d}z \mathrm{d}y \\
&= \sup_{x,x' \in \mathbb{R}^d} \int_{\mathbb{R}^d} G(2t, x - x' - z) f(z) \mathrm{d}z = \int_{\mathbb{R}^d} G(2t, z) f(z) \mathrm{d}z.
\end{aligned}$$

The following two lemmas provide estimates for $k^U(t)$ in the case of the Dirichlet and Neumann heat kernel, respectively. In particular, for the case of $\mathscr{L} = -\Delta$, our lower and upper bounds generalize (3.3) and (3.5) of [33] from U being an open ball to more general domains.

Lemma 4.3 *If U is a bounded Lipschitz domain, then we have the following integral estimates for the Dirichlet heat kernel:*

(i) There exists a positive finite constant C such that for all $t > 0$, for all $x, x' \in U$,

$$\iint_{U \times U} G_D(t, x, y) G_D(t, x', y') f(y - y') \, \mathrm{d}y \, \mathrm{d}y' \leq C e^{-2\mu_1 t} (1 \wedge t)^{-\beta/2}. \tag{4.2}$$

(ii) There exists $0 < \varepsilon_0 < 1$ such that for any $0 < \varepsilon \leq \varepsilon_0$, there exists a positive finite constant C_ε such that for all $t > 0$, for all $x, x' \in U_\varepsilon$ with $|x - x'| \leq \sqrt{t}$,

$$\iint_{U \times U} G_D(t, x, y) G_D(t, x', y') f(y - y') \, dy \, dy' \geq C_\varepsilon e^{-2\mu_1 t}(1 \wedge t)^{-\beta/2}. \quad (4.3)$$

Proof

(i) Let I be the left-hand side of (4.2), By (1.6) and the upper bound in (3.6),

$$I \leq C e^{-2\mu_1 t}\left(\iint_{U^2} |y - y'|^{-\beta} \, dy \, dy' \, \mathbf{1}_{\{t \geq 1\}}\right.$$
$$\left. + \iint_{U^2} \frac{1}{t^{d/2}} e^{-c_1 \frac{|x-y|^2}{t}} \frac{1}{t^{d/2}} e^{-c_1 \frac{|x'-y'|^2}{t}} |y - y'|^{-\beta} \, dy \, dy' \, \mathbf{1}_{\{t<1\}}\right).$$

The first integral is a finite constant since $\beta < d$. For the second integral, we can enlarge the domain of integration to $\mathbb{R}^d \times \mathbb{R}^d$ and use the Plancherel theorem to get the upper bound

$$C \int_{\mathbb{R}^d} e^{-i(x-x')\cdot\xi} \, e^{-\frac{2}{c_1} t|\xi|^2} |\xi|^{\beta-d} d\xi \leq C \int_{\mathbb{R}^d} e^{-\frac{2}{c_1} t|\xi|^2} |\xi|^{\beta-d} d\xi = C' t^{-\beta/2}.$$

The last equality can be obtained by scaling. This proves the upper bound (4.2).

(ii) We now turn to the proof of the lower bound (4.3). Let $t > 0$ and $x, x' \in U_\varepsilon$ be such that $|x - x'| \leq \sqrt{t}$. By (1.6), we have

$$I \geq C e^{-2\mu_1 t}\left(\iint_{U_\varepsilon \times U_\varepsilon} G_D(t, x, y) G_D(t, x', y') |y - y'|^{-\beta} \, dy \, dy' \, \mathbf{1}_{\{t \geq 1\}}\right.$$
$$\left. + \iint_{U \times U} G_D(t, x, y) G_D(t, x', y') |y - y'|^{-\beta} \, dy \, dy' \, \mathbf{1}_{\{t<1\}}\right).$$
$$\quad (4.4)$$

We estimate the two integrals separately. Since U is bounded, $\sup_{y, y' \in U} |y - y'| \leq M < \infty$. By the lower bounds in (3.6) and (3.9), the first integral in (4.4) is bounded below by

$$C_2^2(1 \wedge (c_0^{-1}\varepsilon^{a_1}))^4 e^{-2c_2 M^2} M^{-\beta}\big[\operatorname{Vol}(U_\varepsilon)\big]^2 \mathbf{1}_{\{t \geq 1\}} = C_\varepsilon \mathbf{1}_{\{t \geq 1\}}.$$

By the lower bound in (3.6), the second integral in (4.4) is bounded below by

$$\iint_{V(x) \times V(x')} C_2^2(1 \wedge \Phi_1(x))(1 \wedge \Phi_1(y))(1 \wedge \Phi_1(x'))(1 \wedge \Phi_1(y'))$$
$$\times \frac{1}{t^d} e^{-2c_2} \mathbf{1}_{\{|x-y| \leq \sqrt{t}, \, |x'-y'| \leq \sqrt{t}\}} |y - y'|^{-\beta} \, dy \, dy' \, \mathbf{1}_{\{t<1\}}.$$
$$\quad (4.5)$$

Since $|x - x'| \leq \sqrt{t}$, we have $|y - y'| \leq 3\sqrt{t}$ on the set $\{|x - y| \leq \sqrt{t}, |x' - y'| \leq \sqrt{t}\}$. By (3.9) and Lemma 3.10, for $0 < \varepsilon \leq \varepsilon_0$, (4.5) is bounded below by

$$C_2^2 (1 \wedge (c_0^{-1} \varepsilon^{a_1}))^2 (1 \wedge (c_0^{-1} c_\varepsilon^{a_1}))^2 t^{-d} e^{-2c_2} (3\sqrt{t})^{-\beta}$$
$$\times \mathrm{Vol}(V(x) \cap B(x, \sqrt{t})) \times \mathrm{Vol}(V(x') \cap B(x', \sqrt{t})) \mathbf{1}_{\{t < 1\}}$$
$$\geq C_\varepsilon t^{-\beta/2} \mathbf{1}_{\{t < 1\}}.$$

With this, we complete the proof of Lemma 4.3. □

Lemma 4.4 *If U is a bounded Lipschitz domain, then we have the following integral estimates for the Neumann heat kernel:*

(i) There exists a positive finite constant C_1 such that for all $t > 0$, for all $x, x' \in U$,

$$\iint_{U^2} G_N(t, x, y) G_N(t, x', y') f(y - y') \, dy \, dy' \leq C_1 (1 \wedge t)^{-\beta/2}. \qquad (4.6)$$

(ii) If (3.8) holds, then there exists a positive finite constant C_2 such that for all $t > 0$, for all $x, x' \in U$ with $|x - x'| \leq \sqrt{t}$,

$$\iint_{U^2} G_N(t, x, y) G_N(t, x', y') f(y - y') \, dy \, dy' \geq C_2 (1 \wedge t)^{-\beta/2}. \qquad (4.7)$$

The proof of Lemma 4.4 follows the same strategy as that of Lemma 4.3. We will leave it to the interested readers.

Lemmas 4.3 and 4.4 suggest the study of the following functions.

Definition 4.5 Let $\widehat{K}_\lambda(t)$ and $\widehat{h}_n(t)$ be defined as (4.1a) and (4.1b), respectively, except that $h_0^U(t)$ and $k^U(t)$ in (4.1c) be replaced, respectively, by

$$\widehat{h}_0(t) \equiv 1 \quad \text{and} \quad \widehat{k}(t) = (1 \wedge t)^{-\rho} \quad \text{with } \rho \in (0, 1), \qquad (4.8)$$

namely,

$$\widehat{K}_\lambda(t) := \sum_{n=0}^\infty \lambda^{2n} \widehat{h}_n(t) \quad \text{and} \quad \widehat{h}_n(t) := (\widehat{k}^{*n} * 1)(t) \quad \text{for } n \geq 1.$$

Before proceeding to the next lemma, we recall some useful formulas and inequalities:

- From the Beta integral (see, e.g., [34]), we have that for all $t > 0$, $n \geq 2$, and $r_0, r_1, \ldots, r_n > -1$,

$$\int_0^t ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{n-1}} ds_n (t - s_1)^{r_0} (s_1 - s_2)^{r_1} \ldots (s_{n-1} - s_n)^{r_{n-1}} s_n^{r_n}$$
$$= \frac{\prod_{i=0}^n \Gamma(1 + r_i)}{\Gamma(n + \sum_{i=0}^n r_i + 1)} t^{n + \sum_{i=0}^n r_i}. \qquad (4.9)$$

- For any $a > 0$, there exist positive finite constants C and c depending on a such that

$$c^n (n!)^a \leq \Gamma(an+1) \leq C^n (n!)^a, \qquad \text{for all integers } n \geq 0. \tag{4.10}$$

This inequality can be easily verified using Stirling's formula; see (68) of [2].
- The following result is proved in [2, Lemma A.1] and [3, Lemma 5.2]: For any $a > 0$, there exist positive finite constants C_1, C_2, C_3, C_4 depending on a such that for all $x > 0$,

$$C_1 \exp(C_2 x^{1/a}) \leq \sum_{n=0}^{\infty} \frac{x^n}{(n!)^a} \leq C_3 \exp(C_4 x^{1/a}). \tag{4.11}$$

Lemma 4.6 *Let $\lambda > 0$. Let $\widehat{K}_\lambda(t)$ and $\widehat{h}_n(t)$ be defined as in Definition 4.5. Then:*

(i) *The function $t \mapsto \widehat{h}_n(t)$ is nondecreasing for each $n \geq 0$.*
(ii) *There exist positive finite constants C_1 and C_2 depending only on ρ such that for all $t > 0$, for all integers $n \geq 1$,*

$$((1-\rho)/2)^n t^n \sum_{k=0}^{n} \frac{C_1^k t^{-k\rho}}{(n-k)!(k!)^{1-\rho}} \leq \widehat{h}_n(t) \leq t^n \sum_{k=0}^{n} \frac{C_2^k t^{-k\rho}}{(n-k)!(k!)^{1-\rho}}. \tag{4.12}$$

(iii) *There exist positive finite constants $C_3, \ldots, C_8$ depending only on ρ such that for all $t > 0$,*

$$C_3 \exp\left(t\left(C_4\lambda^2 + C_5\lambda^{\frac{2}{1-\rho}}\right)\right) \leq \widehat{K}_\lambda(t) \leq C_6 \exp\left(t\left(C_7\lambda^2 + C_8\lambda^{\frac{2}{1-\rho}}\right)\right). \tag{4.13}$$

(iv) *For any $p \geq 1$ and $t > 0$, it holds that $\sum_{n=0}^{\infty} \left[\lambda^{2n}\widehat{h}_n(t)\right]^{1/p} < \infty$.*

Proof

(i) Obviously, $\widehat{h}_0$ is nondecreasing. Suppose $\widehat{h}_n$ is nondecreasing. Then

$$\widehat{h}_{n+1}(t) = \int_0^t (1 \wedge (t-s))^{-\rho} \widehat{h}_n(s) \mathrm{d}s = \int_0^t (1 \wedge s)^{-\rho} \widehat{h}_n(t-s)\mathrm{d}s,$$

which is also nondecreasing (see also the proof of Lemma 2.6 of [14]).
(ii) By expanding $\widehat{h}_n(t)$ recursively, we see that

$$\widehat{h}_n(t) = \int_0^t \mathrm{d}s_1 \int_0^{s_1} \mathrm{d}s_2 \cdots \int_0^{s_{n-1}} \mathrm{d}s_n \prod_{i=1}^{n} (1 \wedge (s_{i-1} - s_i))^{-\rho},$$

where $s_0 = t$. Note that

$$\frac{1}{2}(1 + x^{-\rho}) \leq (1 \wedge x)^{-\rho} \leq 1 + x^{-\rho} \quad \text{for all } x \geq 0. \tag{4.14}$$

So, in order to prove (4.12), it suffices to prove that

$$(1-\rho)^n t^n \sum_{k=0}^{n} \frac{C_1^k t^{-k\rho}}{(n-k)!(k!)^{1-\rho}} \leq h_n^*(t) \leq t^n \sum_{k=0}^{n} \frac{C_2^k t^{-k\rho}}{(n-k)!(k!)^{1-\rho}}, \qquad (4.15)$$

where

$$h_n^*(t) = \int_0^t ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{n-1}} ds_n \prod_{i=1}^{n} \left(1 + (s_{i-1} - s_i)^{-\rho}\right).$$

We observe that

$$\prod_{i=1}^{n}(1 + (s_{i-1} - s_i)^{-\rho}) = \sum_{k=0}^{n} \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \prod_{j=1}^{k}(s_{i_j-1} - s_{i_j})^{-\rho},$$

where we have used the convention that when $k = 0$ the summation and product inside gives one. Then, by (4.9), we have

$$\begin{aligned} h_n^*(t) &= \sum_{k=0}^{n} \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \int_0^t ds_1 \int_0^{s_1} ds_2 \ldots \int_0^{s_{n-1}} ds_n \prod_{j=1}^{k} \left(s_{i_j-1} - s_{i_j}\right)^{-\rho} \\ &= \sum_{k=0}^{n} \binom{n}{k} \frac{\Gamma(1-\rho)^k}{\Gamma(n-k\rho+1)} t^{n-k\rho}. \end{aligned}$$

$$(4.16)$$

By the recursion identity of the Gamma function $\Gamma(z+1) = z\Gamma(z)$, we have

$$\Gamma(n - k\rho + 1) = \Gamma\big(k(1-\rho) + 1\big) \prod_{j=1}^{n-k} \big(k(1-\rho) + j\big).$$

Because $0 < \rho < 1$, for all $0 \leq j \leq n$, it holds that $(1-\rho)(k+j) < k(1-\rho)+j < k+j$ and hence,

$$(1-\rho)^{n-k} \frac{n!}{k!} \Gamma\left(k(1-\rho) + 1\right) \leq \Gamma\left(n - k\rho + 1\right) \leq \frac{n!}{k!} \Gamma\left(k(1-\rho) + 1\right).$$

Also, by (4.10), $c^k(k!)^{1-\rho} \leq \Gamma(k(1-\rho)+1) \leq C^k(k!)^{1-\rho}$. Hence, it follows that

$$(1-\rho)^{n-k} c^k \frac{n!}{(k!)^\rho} \leq \Gamma(n-k\rho+1) \leq C^k \frac{n!}{(k!)^\rho}. \qquad (4.17)$$

Putting this back into (4.16), we get (4.15) with $C_1 = C^{-1}\Gamma(2-\rho)$ and $C_2 = c^{-1}(1-\rho)^{-1}\Gamma(1-\rho)$. Hence, (4.12) follows.

As for part (iii), using (4.12), interchanging the order of summation and applying (4.11) yield (4.13). Finally, for part (iv), after an application of the sub-additivity

of the function $x \mapsto x^{1/p}$ to the far right-hand side of (4.12), one can carry out the same arguments as the proof of the upper bound of (4.13) to show that the series in question converges. This completes the proof of Lemma 4.6. □

The above lemma plays the same role as, e.g., Lemma A.2 of [11] where the kernel function $k(t)$ takes the form of $(t - s)^{-\rho}$. In that case, the computations can be made explicit by using the Mittag-Leffler function. Indeed, Lemma 4.6 can be rephrased as the Gronwall-type lemma below.

Lemma 4.7 (Gronwall-type Lemma) *Let $\lambda \in \mathbb{R}_+$, $\rho \in (0, 1)$, $\widehat{k}(t) = (1 \wedge t)^{-\rho}$ and $b : \mathbb{R}_+ \to \mathbb{R}_+$ be a nonnegative function. Suppose that $H : \mathbb{R}_+ \to \mathbb{R}_+$ is a locally integrable nonnegative function such that for all $t \geq 0$,*

$$H(t) \leq b(t) + \lambda^2 \int_0^t \widehat{k}(t - s)H(s)\mathrm{d}s. \tag{4.18}$$

Then

$$H(t) \leq b(t) + \sum_{n=1}^{\infty} \lambda^{2n} \left(\widehat{k}^{*n} * b\right)(t). \tag{4.19}$$

Remark 4.8 The following variation will not be used in this paper but it is worth noting that if $b(t) \geq 0$ and "$\leq$" in (4.18) is replaced by "$\geq$" (resp. "$=$"), then we will obtain the conclusion (4.19) with "$\leq$" replaced by "$\geq$" (resp. "$=$"). This can be shown by the same proof below.

Proof of Lemma 4.7 Using (4.18) inductively, we get that for any $t \geq 0$, for any $N \geq 1$,

$$H(t) \leq b(t) + \sum_{n=1}^{N-1} \lambda^{2n} \left(\widehat{k}^{*n} * b\right)(t) + \lambda^{2N} \left(\widehat{k}^{*N} * H\right)(t).$$

Similarly to the proof of Lemma 4.6(ii), we can use (4.9) to deduce that

$$\widehat{k}^{*N}(t) \leq \sum_{k=0}^{N} \binom{N}{k} \frac{\Gamma(1 - \rho)^k}{\Gamma(N - k\rho)} t^{N-1-k\rho}.$$

Then, by (4.17) and the bound $\binom{N}{k} \leq 2^N$, we have

$$\left(\widehat{k}^{*N} * H\right)(t) \leq \int_0^t H(t - s) \sum_{k=0}^{N} \frac{\widetilde{C}^N}{(N!)^{1-\rho}} s^{N-1-k\rho}\mathrm{d}s.$$

Since $0 < \rho < 1$, if N is large enough, then $N - 1 - k\rho \geq N - 1 - N\rho = N(1 - \rho) - 1 > 0$ for all $k \leq N$, and hence

$$\lambda^{2N} \left(\widehat{k}^{*N} * H\right)(t) \leq \frac{(\lambda^2 \widetilde{C})^N}{(N!)^{1-\rho}} N\left(t^{N-1} + t^{N(1-\rho)-1}\right) \int_0^t H(t - s)\mathrm{d}s.$$

Since H is locally integrable, $\lambda^{2N}(\widehat{k}^{*N} * H)(t) \to 0$ as $N \to \infty$ and (4.19) follows. □

4.1 Proof of part (i) of Theorem 1.4

Now we are ready to prove Theorem 1.4 for the case where the initial data is bounded.

Proof of Theorem 1.4 (the bounded initial data case) Here we assume that the initial condition ν is absolutely continuous with respect to the Lebesgue measure with a bounded density g, namely, $\nu(\mathrm{d}x) = g(x)\mathrm{d}x$ and $g \in L^\infty(U)$. The proof follows a standard Picard iteration scheme. Let $u_0(t, x) = J(t, x)$ (see (3.3)) and for $n \geq 1$,

$$u_n(t, x) = J(t, x) + \lambda \int_0^t \int_U G(t - s, x, y)\sigma(s, y, u_{n-1}(s, y))W(\mathrm{d}s, \mathrm{d}y).$$

By Burkholder's inequality and Minkowski's inequality, for all $p \geq 2, n \geq 2$,

$$\sup_{0 < s \leq t} ||u_n(s, x) - u_{n-1}(s, x)||_p^2$$
$$\leq C_p L_\sigma^2 \lambda^2 \int_0^t \mathrm{d}s \iint_{U^2} G(t - s, x, y)f(y - y')G(t - s, x, y')$$
$$\times \sup_{z \in U} ||u_{n-1}(s, z) - u_{n-2}(s, z)||_p^2 \, \mathrm{d}y \, \mathrm{d}y',$$

where C_p is a constant depending only on p. By Lemma 4.3 (for the Dirichlet case) or Lemma 4.4 (for the Neumann case),

$$\sup_{x \in U} \iint_{U^2} G(t - s, x, y)f(y - y')G(t - s, x, y') \, \mathrm{d}y \, \mathrm{d}y'$$
$$\leq Ce^{-2\mu(t-s)}(1 \wedge (t - s))^{-\beta/2},$$

where $\mu = \mu_1$ in the Dirichlet case and $\mu = 0$ in the Neumann case. Let

$$H_n(t) := e^{2\mu t} \sup_{(s,x) \in (0,t] \times U} ||u_n(s, x) - u_{n-1}(s, x)||_p^2.$$

It follows that for all $n \geq 2$,

$$H_n(t) \leq a \int_0^t H_{n-1}(s)(1 \wedge (t - s))^{-\beta/2}\mathrm{d}s, \tag{4.20}$$

where $a = C_p C L_\sigma^2 \lambda^2$, and for $n = 1$,

$$H_1(t) \leq a \, ||g||_{L^\infty(U)}^2 \int_0^t e^{2\mu s}(1 \wedge (t - s))^{-\beta/2}\mathrm{d}s.$$

For any $t \in [0, T]$ with T fixed, using the notation in Definition 4.5 with $\rho = \beta/2$, the above bound can be written as $H_1(t) \leq C_T a(\widehat{k} * \widehat{h}_0)(t)$ for some constant C_T, and we have $H_n(t) \leq C_T a^n \widehat{h}_n(t)$ for all $n \geq 0$. Hence, we can apply Lemma 4.6(iv) with $p = 2$ to see that $\sum_{n=1}^{\infty} H_n^{1/2}(t) < \infty$. This implies that $u_n(t, x)$ converges to some $u(t, x)$ in $L^p(\Omega)$ which satisfies (3.2) and (1.17).

For the uniqueness, suppose that u and $\widetilde{u}$ are two mild solutions satisfying (1.17) with $p = 2$. By Burkholder's inequality with $p = 2$ and similar calculations to those above, we get that

$$H(t) \leq C\lambda^2 L_\sigma^2 \int_0^t H(s)(1 \wedge (t-s))^{-\beta/2} \mathrm{d}s, \quad \text{for all } t > 0,$$

where $H(t) := \sup_{(s,x) \in (0,t] \times U} ||u(s, x) - \widetilde{u}(s, x)||_2^2$. The condition (1.17) implies that H is locally bounded. Then we can apply the Gronwall-type Lemma 4.7 with $b(t) \equiv 0$ to see that $H(t) \equiv 0$, i.e., $u(t, x) = \widetilde{u}(t, x)$ a.s. This completes the proof of Theorem 1.4 in the case where the initial measure has a bounded density. $\qquad \square$

5 The p-th moment bounds and rough initial data

Our next task is to establish more bounds for the convolution-type integrals of the heat kernels. Specifically, we seek optimal upper and lower bounds for the integral:

$$\iint_{U \times U} G(t-s, x, z)G(t-s, x', z')f(z-z')G(s, z, y)G(s, z', y') \, \mathrm{d}z \, \mathrm{d}z'.$$

The following identity plays a key role in the estimation of the above integral: for all $0 < s < t$ and $v, w \in \mathbb{R}^d$,

$$\exp\left(-C\frac{|v|^2}{t-s}\right) \exp\left(-C\frac{|v-w|^2}{s}\right) = \exp\left(-C\frac{|w|^2}{t}\right) \exp\left(-C\frac{|v - \frac{t-s}{t}w|^2}{(t-s)s/t}\right).$$
(5.1)

This identity can be verified by direct calculations; see [14, p.657]. It can also be interpreted as an expression for the density of the Brownian bridge in terms of a conditional density; see [4, Chapter 6].

Lemma 5.1 *If U is a bounded Lipschitz domain, then we have the following integral estimates:*

(i) There exists a finite constant C such that for all $0 < s < t$, for all $x, x', y, y' \in U$,

$$\iint_{U \times U} \frac{e^{-2\mu_1(t-s)}}{1 \wedge (t-s)^d} e^{-c_1 \frac{|x-z|^2 + |x'-z'|^2}{t-s}} \frac{e^{-2\mu_1 s}}{1 \wedge s^d} e^{-c_1 \frac{|z-y|^2 + |z'-y'|^2}{s}} f(z-z') \, \mathrm{d}z \, \mathrm{d}z'$$

$$\leq C \frac{e^{-2\mu_1 t}}{1 \wedge t^d} e^{-c_1 \frac{|x-y|^2 + |x'-y'|^2}{t}} \left(1 \wedge \frac{(t-s)s}{t}\right)^{-\beta/2}.$$
(5.2)

(ii) *If U is convex, then there exists $0 < \varepsilon_0 < 1$ such that for all $0 < \varepsilon \le \varepsilon_0$, there exists a positive finite constant C_ε such that for all $0 < s < t$, for all $x, x', y, y' \in U_\varepsilon$ with $|x - x'| \le \sqrt{(t-s)s/t}$ and $|y - y'| \le \sqrt{(t-s)s/t}$,*

$$\iint_{U \times U} \widetilde{G}_D(t-s, x, x', z, z') \widetilde{G}_D(s, z, z', y, y') f(z - z') \, dz \, dz'$$

$$\ge C_\varepsilon \frac{e^{-2\mu_1 t}}{1 \wedge t^d} e^{-c_2 \frac{|x-y|^2 + |x'-y'|^2}{t}} \left(1 \wedge \frac{(t-s)s}{t}\right)^{-\beta/2}. \tag{5.3}$$

Proof

(i) Denote the left-hand side of (5.2) by I. By (1.6),

$$I \le C_f C_1^4 e^{-2\mu_1 t} \frac{1}{(1 \wedge (t-s)^d)(1 \wedge s^d)}$$

$$\iint_{U^2} e^{-c_1 \left(\frac{|x-z|^2}{t-s} + \frac{|x'-z'|^2}{t-s} + \frac{|z-y|^2}{s} + \frac{|z'-y'|^2}{s} \right)} |z - z'|^{-\beta} \, dz \, dz'$$

$$= C_f C_1^4 e^{-2\mu_1 t} e^{-c_1 \left(\frac{|y-x|^2}{t} + \frac{|y'-x'|^2}{t} \right)}$$

$$\times \frac{1}{(1 \wedge (t-s)^d)(1 \wedge s^d)}$$

$$\iint_{U^2} e^{-c_1 \left(\frac{|(z-x) - \frac{t-s}{t}(y-x)|^2}{(t-s)s/t} + \frac{|(z'-x') - \frac{t-s}{t}(y'-x')|^2}{(t-s)s/t} \right)} |z - z'|^{-\beta} \, dz \, dz'.$$

For the last equality, we have applied the identity (5.1) with $v = z - x$ and $w = y - x$, and also with $v' = z' - x'$ and $w' = y' - x'$. Set $\tau = (t-s)s/t$. We claim that there exists a finite constant C such that for all $a, a' \in \mathbb{R}^d$, for all $\tau > 0$,

$$I' := \iint_{U^2} e^{-c_1 \left(\frac{|z-a|^2}{\tau} + \frac{|z'-a'|^2}{\tau} \right)} |z - z'|^{-\beta} \, dz \, dz' \le C(1 \wedge \tau)^{d - \beta/2}. \tag{5.4}$$

Indeed, if $\tau \ge 1$, then (5.4) holds since $\beta < d$. If $\tau < 1$, then by the Plancherel theorem,

$$I' \le \iint_{\mathbb{R}^d \times \mathbb{R}^d} e^{-c_1 \left(\frac{|z-a|^2}{\tau} + \frac{|z'-a'|^2}{\tau} \right)} |z - z'|^{-\beta} \, dz \, dz'$$

$$= C \tau^d \int_{\mathbb{R}^d} e^{-i(a-a') \cdot \xi} e^{-\frac{2}{c_1} \tau |\xi|^2} |\xi|^{\beta - d} \, d\xi$$

$$\le C \tau^d \int_{\mathbb{R}^d} e^{-\frac{2}{c_1} \tau |\xi|^2} |\xi|^{\beta - d} \, d\xi = C' \tau^{d - \beta/2},$$

where the last equality can be obtained by a scaling argument. This verifies (5.4) and hence the following:

$$I \leq C e^{-2\mu_1 t} e^{-c_1 \left(\frac{|y-x|^2}{t} + \frac{|y'-x'|^2}{t} \right)} h(s,t) \left(1 \wedge \frac{(t-s)s}{t} \right)^{-\beta/2},$$

where

$$h(s,t) := \frac{1}{(1 \wedge (t-s)^d)(1 \wedge s^d)} \left(1 \wedge \frac{(t-s)^d s^d}{t^d} \right).$$

It can be verified by straightforward calculations that if $t - s < 1$ and $s < 1$, then $h(s,t) \leq t^{-d}$; otherwise, $h(s,t) \leq 1$. Therefore, we obtain the upper bound (5.2).

(ii) To prove the lower bound (4.3), let $0 < s < t$ and $x, x', y, y' \in U_\varepsilon$ be such that $|x - x'| \leq \sqrt{(t-s)s/t}$ and $|y - y'| \leq \sqrt{(t-s)s/t}$. Denote the left-hand side of (4.3) by $\tilde{I}$. By (1.6), (3.6), (3.9) and (5.1), we have that

$$\tilde{I} \geq C_\varepsilon \frac{e^{-2\mu_1 t}}{(1 \wedge (t-s)^d)(1 \wedge s^d)} e^{-c_2 \left(\frac{|y-x|^2}{t} + \frac{|y'-x'|^2}{t} \right)} \iint_{U \times U} \mathrm{d}z \mathrm{d}z'$$
$$\times (1 \wedge \Phi_1(z))^2 (1 \wedge \Phi_1(z'))^2 e^{-c_2 \left(\frac{|(z-x) - \frac{t-s}{t}(y-x)|^2}{1 \wedge \tau} + \frac{|(z'-x') - \frac{t-s}{t}(y'-x')|^2}{1 \wedge \tau} \right)} |z - z'|^{-\beta}$$
$$=: A, \quad \text{with } \tau = (t-s)s/t.$$

Now, we consider the following two cases: $\tau \geq 1$ and $\tau < 1$. In the rest of the proof, the constant C_ε above will be used to denote a generic constant that depends on ε, whose value may change at each appearance.

Case 1. Suppose $\tau \geq 1$. Observe that $(1 \wedge (t-s)^d)(1 \wedge s^d) \leq 1 \wedge t^d$ and $(1 \wedge \tau)^{-\beta/2} = 1$. Also, by $\sup_{y,y' \in D} |y - y'| \leq M < \infty$ and (3.9), we see that

$$A \geq C_\varepsilon \frac{e^{-2\mu_1 t}}{1 \wedge t^d} e^{-c_2 \left(\frac{|y-x|^2}{t} + \frac{|y'-x'|^2}{t} \right)} \iint_{U_\varepsilon \times U_\varepsilon} (1 \wedge (c_0^{-1} \varepsilon^{a_1}))^4 e^{-2c_2 M^2} M^{-\beta} \mathrm{d}z \, \mathrm{d}z'$$
$$= C_\varepsilon \frac{e^{-2\mu_1 t}}{1 \wedge t^d} e^{-c_2 \left(\frac{|y-x|^2}{t} + \frac{|y'-x'|^2}{t} \right)} (1 \wedge \tau)^{-\beta/2},$$

which proves (5.3).

Case 2. Suppose $\tau < 1$. In this case, $1 \wedge \tau = \tau$. Notice that

$$(z - x) - \frac{t-s}{t}(y - x) = z - a, \quad \text{where } a := \frac{s}{t} x + \frac{t-s}{t} y,$$

and a' is defined similarly. The convexity assumption on U ensures that both a and a' are in U_ε. By Lemma 3.10, for $0 < \varepsilon \leq \varepsilon_0$,

$$A \geq C_\varepsilon \frac{e^{-2\mu_1 t}}{(1 \wedge (t-s)^d)(1 \wedge s^d)} e^{-c_2 \left(\frac{|y-x|^2}{t} + \frac{|y'-x'|^2}{t} \right)}$$

$$\times \iint_{E(a) \times E(a')} e^{-2c_2} \mathbf{1}_{\{|z-a| \le \sqrt{\tau}, \, |z'-a'| \le \sqrt{\tau}\}} |z - z'|^{-\beta} \, dz \, dz'.$$

Since $|x - x'| \le \sqrt{\tau}$ and $|y - y'| \le \sqrt{\tau}$, we have $|a - a'| \le \sqrt{\tau}$. Hence, $|z - z'| \le 3\sqrt{\tau}$ on the set $\{|z - a| \le \sqrt{\tau}, |z' - a'| \le \sqrt{\tau}\}$. Also, by (3.15), it follows that

$$A \ge C_\varepsilon \frac{e^{-2\mu_1 t}}{(1 \wedge (t-s)^d)(1 \wedge s^d)} e^{-c_2 \left(\frac{|y-x|^2}{t} + \frac{|y'-x'|^2}{t} \right)} \tau^{d - \beta/2}.$$

To finish the proof, it remains to show that the following bound holds for some constant $c > 0$:

$$\frac{\tau^d}{(1 \wedge (t-s)^d)(1 \wedge s^d)} \ge \frac{c}{1 \wedge t^d}. \tag{5.5}$$

Indeed, for $t \le 2$, we can use the bound

$$\frac{\tau^d}{(t-s)^d s^d} = \frac{1}{t^d} = \frac{1}{2 \wedge t^d} \ge \frac{1}{2(1 \wedge t^d)}.$$

For $t > 2$, consider the following two cases. If $s < t/2$, then $t - s > t/2 > 1$ and hence

$$\frac{\tau^d}{(1 \wedge (t-s)^d)(1 \wedge s^d)} = \frac{\tau^d}{s^d} = \frac{(t-s)^d}{t^d} \ge \frac{1}{2^d} = \frac{1}{2^d(1 \wedge t^d)}.$$

If $s \ge t/2$, which is > 1, then

$$\frac{\tau^d}{(1 \wedge (t-s)^d)(1 \wedge s^d)} = \frac{\tau^d}{(t-s)^d} = \frac{s^d}{t^d} \ge \frac{1}{2^d} = \frac{1}{2^d(1 \wedge t^d)}.$$

This proves (5.5) and completes the proof of the lower bound (5.3). □

Lemma 5.2 *If U is a bounded Lipschitz domain, then we have the following integral estimates:*

(i) *There exists a positive finite constant C such that for all $0 < s < t$, for all $x, x', y, y' \in U$,*

$$\iint_{U \times U} \widetilde{G}_N(t-s, x, x', z, z') \widetilde{G}_N(s, z, z', y, y') f(z - z') \, dz \, dz'$$
$$\le \frac{C}{1 \wedge t^d} e^{-c_3 \frac{|x-y|^2 + |x'-y'|^2}{t}} \left(1 \wedge \frac{(t-s)s}{t} \right)^{-\beta/2}. \tag{5.6}$$

(ii) *If U is convex and (3.8) holds, then there exists a positive finite constant C such that for all $0 < s < t$, for all $x, x', y, y' \in U$ with $|x - x'| \le \sqrt{(t-s)s/t}$ and*

$$|y - y'| \leq \sqrt{(t-s)s/t},$$

$$
\iint_{U \times U} \widetilde{G}_N(t-s, x, x', z, z') \widetilde{G}_N(s, z, z', y, y') f(z - z') \, dz \, dz'
$$
$$
\geq \frac{C}{1 \wedge t^d} e^{-c_4 \frac{|x-y|^2 + |x'-y'|^2}{t}} \left(1 \wedge \frac{(t-s)s}{t} \right)^{-\beta/2}. \tag{5.7}
$$

The proof of Lemma 5.2 is similar to that of Lemma 5.1, which will be left to interested readers.

Lemma 5.3 *Let $0 < \rho < 1$. Define $\widetilde{h}_0(t) \equiv 1$ and for $n \geq 1$*

$$
\widetilde{h}_n(t) := \int_0^t ds_1 \int_0^{s_1} ds_2 \ldots \int_0^{s_{n-1}} ds_n \prod_{j=1}^n \left(1 \wedge \frac{(s_{j-1} - s_j)s_j}{s_{j-1}} \right)^{-\rho}, \tag{5.8}
$$

where we use the convention that $s_0 = t$. Then, for all $t > 0$, for all integers $n \geq 0$,

$$
2^{-n} \widehat{h}_n(t) \leq \widetilde{h}_n(t) \leq \left(2^{1+\rho} \right)^n \widehat{h}_n(t), \tag{5.9}
$$

where $\widehat{h}_n(t)$ is as defined in Lemma 4.6. Moreover, for any $\lambda > 0$, there exists positive finite constants $C_3, \ldots, C_8$ such that

$$
C_3 \exp \left(t \left(C_4 \lambda^2 + C_5 \lambda^{\frac{2}{1-\rho}} \right) \right) \leq \sum_{n=0}^{\infty} \lambda^{2n} \widetilde{h}_n(t) \leq C_6 \exp \left(t \left(C_7 \lambda^2 + C_8 \lambda^{\frac{2}{1-\rho}} \right) \right). \tag{5.10}
$$

Proof The estimate (5.9) can be deduced using Lemma 4.17 of [4]. For completeness, we provide a direct proof. We prove (5.9) by induction. Clearly, it holds for $n = 0$. Suppose it holds for some $n \geq 0$. Since $(t-s)s/t \geq s/2$ for $s \in [0, t/2]$ and $(t-s)s/t \geq (t-s)/2$ for $s \in [t/2, t]$, we have

$$
\widetilde{h}_{n+1}(t) = \int_0^t \left(1 \wedge \frac{(t-s)s}{t} \right)^{-\rho} \widetilde{h}_n(s) \, ds
$$
$$
\leq 2^{\rho} (2^{1+\rho})^n \int_0^{t/2} (1 \wedge s)^{-\rho} \widehat{h}_n(s) \, ds
$$
$$
+ 2^{\rho} (2^{1+\rho})^n \int_{t/2}^t (1 \wedge (t-s))^{-\rho} \widehat{h}_n(s) \, ds
$$
$$
= 2^{\rho} \int_{t/2}^t (1 \wedge (t-s))^{-\rho} \widehat{h}_n(t-s) \, ds + 2^{\rho} \int_{t/2}^t (1 \wedge (t-s))^{-\rho} \widehat{h}_n(s) \, ds.
$$

Since $t - s \le s$ for $s \in [t/2, t]$, and $\widehat{h}_n(\cdot)$ is nondecreasing by Lemma 4.6, we see that

$$
\begin{aligned}
\widetilde{h}_{n+1}(t) &\le 2^\rho (2^{1+\rho})^n \int_{t/2}^t (1 \wedge (t - s))^{-\rho} \widehat{h}_n(s) \, ds \\
&\quad + 2^\rho (2^{1+\rho})^n \int_{t/2}^t (1 \wedge (t - s))^{-\rho} \widehat{h}_n(s) \, ds \\
&\le 2^{1+\rho} (2^{1+\rho})^n \int_0^t (1 \wedge (t - s))^{-\rho} \widehat{h}_n(s) \, ds \\
&\le (2^{1+\rho})^{n+1} \widehat{h}_{n+1}(t),
\end{aligned}
$$

where we have applied (4.14) in the last step. This proves the upper bound of (5.9). On the other hand, for the lower bound, it holds clearly for $n = 0$. For general $n \ge 1$, by the bound $(t - s)s/t \le t - s$ for all $s \in [0, t]$, the induction hypothesis and (4.14), we have

$$
\begin{aligned}
\widetilde{h}_{n+1}(t) &\ge 2^{-n} \int_0^t (1 \wedge (t - s))^{-\rho} \widehat{h}_n(s) \, ds \\
&\ge 2^{-n-1} \int_0^t \left(1 + (t - s)^{-\rho}\right) \widehat{h}_n(s) \, ds \\
&= 2^{-(n+1)} \widehat{h}_{n+1}(t).
\end{aligned}
$$

Finally, (5.10) follows clearly from (5.9) and (4.13). This proves Lemma 5.3. $\qquad \square$

Recall the following form of Burkholder's inequality [28, Theorem B.1]: For any $p \in [2, \infty)$ and any continuous L^2-martingale $\{M_t, t \ge 0\}$,

$$
\mathbb{E}(|M_t|^p) \le (4p)^{p/2} \mathbb{E}(\langle M \rangle_t^{p/2}), \tag{5.11}
$$

where $\langle M \rangle_t$ denotes the quadratic variation of M_t.

5.1 Proof of part (ii) of Theorem 1.4

Now we are ready to prove the second part of Theorem 1.4.

Proof of Theorem 1.4 (the rough initial data case) We first assume the Dirichlet boundary condition. Let $u_0(t, x) = J(t, x)$ and

$$
u_n(t, x) = J(t, x) + \lambda \int_0^t \int_U G(t - s, x, y) \sigma(s, y, u_{n-1}(s, y)) W(ds, dy) \quad \text{for } n \ge 1.
$$

Let $a = 8p\lambda^2 L_\sigma^2 C$, where C is the constant in (5.2). We claim that, for all $n \ge 0$,

$$
\|u_n(t, x)\|_p \le \sqrt{2} e^{-\mu_1 t} J_{c_1}(t, x) \left(\sum_{i=0}^n a^i \widetilde{h}_i(t) \right)^{1/2} \quad \text{for all } t > 0, x \in U, \tag{5.12}
$$

where $\widetilde{h}_n(t)$ is defined in (5.8). By Proposition 3.6, (5.12) holds for $n = 0$. Suppose that (5.12) is true for some $n \geq 0$. By Burkholder's inequality (5.11) and Minkowski's inequality, we get that

$$\|u_{n+1}(t, x)\|_p^2 \leq 2J^2(t, x) + 8p\lambda^2 L_\sigma^2 I_n(t, x),$$

where

$$I_n(t, x) := \int_0^t ds \iint_{U^2} dy\, dy'\, G_D(t - s, x, y) G_D(t - s, x, y') f(y - y')$$
$$\|u_n(s, y)\|_p \|u_n(s, y')\|_p.$$

By the induction hypothesis,

$$I_n(t, x) \leq 2 \int_0^t ds \iint_{U^2} dy\, dy'\, f(y - y') G_D(t - s, x, y) G_D(t - s, x, y')$$
$$\times |J(s, y)|\, |J(s, y')| \left(\sum_{i=0}^n a^i\, \widetilde{h}_i(s) \right)$$
$$\leq 2 \sum_{i=0}^n a^i \int_0^t ds\, \widetilde{h}_i(s) \iint_{U^2} dy\, dy'\, f(y - y') G_D(t - s, x, y) G_D(t - s, x, y')$$
$$\times \iint_{U^2} |\nu|(dz)|\nu|(dz') G_D(s, y, z) G_D(s, y, z').$$

Now interchange the order of the two double integrals and apply Lemma 5.1(i) to get that

$$I_n(t, x) \leq 2 \sum_{i=0}^n a^i \int_0^t ds\, \widetilde{h}_i(s) \iint_{U^2} |\nu|(dz)|\nu|(dz')$$
$$\times C \frac{e^{-2\mu_1 t}}{1 \wedge t^d} e^{-c_1 \frac{|x-z|^2 + |x-z'|^2}{t}} \left(1 \wedge \frac{(t - s)s}{t} \right)^{-\beta/2}$$
$$= 2C e^{-2\mu_1 t} J_{c_1}^2(t, x) \sum_{i=0}^n a^i \int_0^t ds \left(1 \wedge \frac{(t - s)s}{t} \right)^{-\beta/2} \widetilde{h}_i(s)$$
$$= 2C e^{-2\mu_1 t} J_{c_1}^2(t, x) \sum_{i=0}^n a^i\, \widetilde{h}_{i+1}(t).$$

Recall that $a = 8p\lambda^2 L_\sigma^2 C$. Hence,

$$\|u_{n+1}(t, x)\|_p^2 \leq 2e^{-2\mu_1 t} J_{c_1}^2(t, x) + 16p\lambda^2 L_\sigma^2 C e^{-2\mu_1 t} J_{c_1}^2(t, x) \sum_{i=0}^n a^i\, \widetilde{h}_{i+1}(t)$$

$$= 2e^{-2\mu_1 t} J_{c_1}^2(t, x) \sum_{i=0}^{n+1} a^i \, \widetilde{h}_i(t).$$

This proves (5.12).

Next, we prove that $u_n(t, x)$ is a Cauchy sequence in $L^p(\Omega)$. Let $u_{-1}(t, x) = 0$. By Burkholder's inequality (5.11) and Minkowski's inequality,

$$\|u_m(t, x) - u_n(t, x)\|_p^2$$
$$\leq 4p\lambda^2 L_\sigma^2 \int_0^t \mathrm{d}s \iint_{U^2} \mathrm{d}y \, \mathrm{d}y' G_D(t - s, x, y) G_D(t - s, x, y') f(y - y')$$
$$\times \|u_{m-1}(s, y) - u_{n-1}(s, y)\|_p \|u_{m-1}(s, y') - u_{n-1}(s, y')\|_p.$$

Then, similarly to the above, we can show by induction that, for all $0 \leq n < m$,

$$\|u_m(t, x) - u_n(t, x)\|_p \leq \sqrt{2} e^{-\mu_1 t} J_{c_1}(t, x) \left(\sum_{i=n+1}^m a^i \, \widetilde{h}_i(t) \right)^{1/2} \quad \text{for all } t > 0 \text{ and } x \in U.$$

By Lemma 5.3,

$$\sum_{i=0}^\infty a^i \, \widetilde{h}_i(t) \leq C_6 e^{t\left(C_7 a + C_8 a^{\frac{2}{2-\beta}} \right)} < \infty.$$

This implies that $u_n(t, x)$ is a Cauchy sequence in $L^p(\Omega)$, and hence converges in $L^p(\Omega)$ to some $u(t, x)$ which satisfies (3.2). In particular, by (5.12) and Lemma 5.3, we have

$$\|u(t, x)\|_p \leq \sqrt{2} e^{-\mu_1 t} J_{c_1}(t, x) \left(\sum_{i=0}^\infty (8p\lambda^2 L_\sigma^2 C)^i \, \widetilde{h}_i(t) \right)^{1/2}$$
$$\leq \sqrt{2} C_6 J_{c_1}(t, x) e^{\frac{1}{2}t\left(8C_7 C p\lambda^2 L_\sigma^2 + C_8(8C)^{\frac{2}{2-\beta}} p^{\frac{2}{2-\beta}} \lambda^{\frac{4}{2-\beta}} L_\sigma^{\frac{4}{2-\beta}} - \mu_1 \right)}.$$

For uniqueness, suppose that $u(t, x)$ and $\widetilde{u}(t, x)$ are mild solutions of (1.3) satisfying (1.19). Let $T > 0$. Then, for all $t \in (0, T]$, for all $x \in U$,

$$\|u(t, x) - \widetilde{u}(t, x)\|_2^2 \leq \lambda^2 L_\sigma^2 \int_0^t \mathrm{d}s \iint_{U^2} \mathrm{d}y \, \mathrm{d}y' G_D(t - s, x, x, y_1, y_1') f(y - y')$$
$$\times \|u(s, y) - \widetilde{u}(s, y)\|_2 \|u(s, y') - \widetilde{u}(s, y')\|_2.$$
$$(5.13)$$

We claim that there exists $C_T < \infty$ such that for all $n \geq 1$, for all $t \in (0, T]$ and all $x \in U$,

$$\|u(t, x) - \widetilde{u}(t, x)\|_2^2 \leq 4C_T^2 J_c^2(t, x)(\lambda L_\sigma)^{2n} \widetilde{h}_n(t), \qquad (5.14)$$

where $\widetilde{h}_n$ is defined in Lemma 5.3. We prove this claim by induction. First, consider $n = 1$. Using the condition that u and $\widetilde{u}$ both satisfy (1.19), we get that

$$
\begin{aligned}
&\|u(t, x) - \widetilde{u}(t, x)\|_2^2 \\
&\quad \leq 4C_T^2 \lambda^2 L_\sigma^2 \int_0^t ds \iint_{U^2} dy\, dy'\, \widetilde{G}(t - s_1, x, x, y_1, y_1') f(y - y') J_c(s, y) J_c(s, y') \\
&\quad = 4C_T^2 \lambda^2 L_\sigma^2 \iint_{U^2} |\nu|(dz) |\nu|(dz') \\
&\qquad \times \int_0^t ds \iint_{U^2} dy\, dy'\, \widetilde{G}(t - s_1, x, x, y_1, y_1') f(y - y') \frac{1}{1 \wedge s^d} e^{-c\frac{|y-z|^2+|y'-z'|^2}{s}}.
\end{aligned}
$$

Then, by Lemma 5.1(i),

$$
\begin{aligned}
\|u(t, x) - \widetilde{u}(t, x)\|_2^2 &\leq 4C_T^2 \lambda^2 L_\sigma^2 J_c^2(t, x) \int_0^t e^{-2\mu_1(t-s)} \left(1 \wedge \frac{(t - s)s}{t}\right)^{-\beta/2} ds \\
&\leq 4C_T^2 J_c^2(t, x)(\lambda L_\sigma)^2 \int_0^t \left(1 \wedge \frac{(t - s)s}{t}\right)^{-\beta/2} ds.
\end{aligned}
$$

This proves (5.14) for $n = 1$. Assume that (5.14) holds for some $n \geq 1$. We apply the induction hyposthesis to $\|u(s, y) - \widetilde{u}(s, y)\|_2$ and $\|u(s, y') - \widetilde{u}(s, y')\|_2$ on the right-hand side of (5.13) to get

$$
\begin{aligned}
\|u(t, x) - \widetilde{u}(t, x)\|_2^2 &\leq 4C_T^2 (\lambda L_\sigma)^{2(n+1)} \int_0^t ds\, \widetilde{h}_n(s) \\
&\quad \times \iint_{U^2} dy\, dy'\, \widetilde{G}(t - s_1, x, x, y_1, y_1') f(y - y') J_c(s, y) J_c(s, y').
\end{aligned}
$$

Then, by similar calculations to those in the $n = 1$ case, we obtain

$$
\begin{aligned}
\|u(t, x) - \widetilde{u}(t, x)\|_2^2 &\leq 4C_T^2 J_c^2(t, x)(\lambda L_\sigma)^{2(n+1)} \int_0^t \widetilde{h}_n(s) \left(1 \wedge \frac{(t - s)s}{t}\right)^{-\beta/2} ds \\
&= 4C_T^2 J_c^2(t, x)(\lambda L_\sigma)^{2(n+1)} \widetilde{h}_{n+1}(t).
\end{aligned}
$$

This proves the claim (5.14). Finally, by Lemma 5.3, we have $\sum_{n=0}^{\infty}(\lambda L_\sigma)^{2n} \widetilde{h}_n(t) < \infty$, which implies that $(\lambda L_\sigma)^{2n} \widetilde{h}_n(t) \to 0$ as $n \to \infty$. Hence, by (5.14), $u(t, x) = \widetilde{u}(t, x)$ a.s.

The case of the Neumann boundary condition can be proved in the same way except that Lemma 5.2 (i) is applied in place of Lemma 5.1 (i). This completes the proof of part (ii) of Theorem 1.4. □

6 The two-point correlation function

6.1 A general formula for the two-point correlation function

The authors in [14] have defined and studied the space-time convolution-type operator
"$\triangleright$" for the heat kernel in the whole space $\mathbb{R}^d$, where the heat kernel $G(t, x, y)$ can
be written as $G(t, x - y)$. However, when the domain is not the whole space $\mathbb{R}^d$, for
example when it is a bounded domain, this translation invariant property is no longer
true and one has to keep the fundamental solution in the form of three parameters. This
natural generalization of the operator "$\triangleright$" for fundamental solutions from those with
two parameters to those with three has been carried out by the first author's thesis; see
[4, Chapter 5]. Here, let us first briefly recall this generalization.

For two measurable functions $k_1, k_2 : \mathbb{R}_+ \times U^4 \to \mathbb{R}$, we define

$$
(k_1 \triangleright k_2)(t, x, x', y, y')
$$
$$
= \int_0^t ds \iint_{U^2} dz\, dz'\, k_1(t - s, x, x', z, z')k_2(s, z, z', y, y')f(z - z') \quad (6.1)
$$

whenever the multiple integral is well-defined. For $h : \mathbb{R}_+ \times U^2 \to \mathbb{R}$, we define
$k_1 \triangleright h = k_1 \triangleright \overline{h}$, where $\overline{h}(t, x, x', y, y') := h(t, x, x')$.

Remark 6.1 In [14], where $U = \mathbb{R}^d$, the operator "$\triangleright$" is defined by

$$
(k_1 \triangleright k_2)(t, x, x'; y) = \int_0^t ds \iint_{\mathbb{R}^d \times \mathbb{R}^d} dz\, dz'\, k_1\big(t - s, x - z, x' - z'; y - (z - z')\big)
$$
$$
\times k_2(s, z, z'; y)f\big(y - (z - z')\big)
$$

for measurable functions $k_i : \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}$, $i = 1, 2$. The prototype of the function
k is $k(t, x, x'; y) = G(t, x - y)G(t, x' - y)$, which corresponds to (and actually
equals to) $J(t, x)J(t, x')$ when the initial condition is $v = \delta_y$. But when there is
a lack of space invariance, one has to write this function k as $k(t, x, x', y, y') =
G(t, x, y)G(t, x', y')$. Indeed, introducing one more parameter in the definition of the
convolution operator "$\triangleright$" makes the associative property much more straightforward:
Provided that one can apply Fubini's theorem to interchange of the order of integration,
one gets that for any measurable functions $k_1, k_2, k_3 : \mathbb{R}_+ \times U^4 \to \mathbb{R}$,

$$
(k_1 \triangleright (k_2 \triangleright k_3))(t, x, x', y, y') = ((k_1 \triangleright k_2) \triangleright k_3)(t, x, x', y, y'). \quad (6.2)
$$

Therefore, it makes sense to write $k_1 \triangleright k_2 \triangleright k_3$ without parentheses to specify the order.
Moreover, for any $n \geq 1$, we define $k^{\triangleright n}$ by $k^{\triangleright n} := \underbrace{k \triangleright \cdots \triangleright k}_{n\,\text{times}}$.

Now we apply this operator to the function $\widetilde{G}(t, x, x', y, y') = G(t, x, y)G(t, x', y')$. For any $\lambda > 0$ and $x, x', y, y' \in U$, define formally the function $\mathcal{K}^\lambda$ by

$$\mathcal{K}^\lambda(t, x, x', y, y') := \sum_{n=1}^{\infty} \lambda^{2n} \widetilde{G}^{\rhd n}(t, x, x', y, y'). \tag{6.3}$$

Specifically, we will write $\mathcal{K}_D^\lambda$ and $\widetilde{G}_D$ when $G = G_D$ is the Dirichlet heat kernel, and write $\mathcal{K}_N^\lambda$ and $\widetilde{G}_N$ when $G = G_N$ is the Neumann heat kernel. Upper and lower bounds for $\mathcal{K}_D^\lambda$ and $\mathcal{K}_N^\lambda$ will be proved later in Propositions 6.5 and 6.7. In particular, the upper bounds there imply the convergence of the series in (6.3).

The connection between the two-point correlation function and the function $\mathcal{K}^\lambda$ is given by the next proposition.

Proposition 6.2 *Let U be a bounded Lipschitz domain and $u(t, x)$ be the solution of (1.3) or (1.4). Recall that $J(t, x)$ is the solution to the homogeneous equation; see (3.3). Let $\widetilde{J}(t, x, x') = J(t, x)J(t, x')$.*

(i) If $\sigma(t, x, u) = u$ for all $t > 0$, $x \in U$ and $u \in \mathbb{R}$, then

$$\mathbb{E}(u(t, x)u(t, x')) = \widetilde{J}(t, x, x') + \sum_{n=1}^{\infty} \lambda^{2n} (\widetilde{G}^{\rhd n} \rhd \widetilde{J})(t, x, x', 0, 0). \tag{6.4}$$

(ii) If there exists a constant $l_\sigma > 0$ such that $\sigma(t, x, u) \geq l_\sigma |u|$ for all $t > 0$, $x \in U$, $u \in \mathbb{R}$, then

$$\mathbb{E}(u(t, x)u(t, x')) \geq \widetilde{J}(t, x, x') + \sum_{n=1}^{\infty} (\lambda l_\sigma)^{2n} (\widetilde{G}^{\rhd n} \rhd \widetilde{J})(t, x, x', 0, 0). \tag{6.5}$$

(iii) If $\sigma(t, x, u) \leq L_\sigma u$ for all $t > 0$, $x \in U$, $u \in [0, \infty)$ and $u(t, x) \geq 0$ a.s. for all $t > 0$ and $x \in U$, then

$$\mathbb{E}(u(t, x)u(t, x')) \leq \widetilde{J}(t, x, x') + \sum_{n=1}^{\infty} (\lambda L_\sigma)^{2n} (\widetilde{G}^{\rhd n} \rhd \widetilde{J})(t, x, x', 0, 0). \tag{6.6}$$

Moreover, for any $a > 0$, we have

$$\widetilde{J}(t, x, x') + \sum_{n=1}^{\infty} a^{2n} (\widetilde{G}^{\rhd n} \rhd \widetilde{J})(t, x, x', 0, 0) = a^{-2} \iint_{U^2} \mathcal{K}^a(t, x, x', y, y')\nu(dy)\nu(dy'). \tag{6.7}$$

Remark 6.3 Note that case (i) in the above proposition covers the important special case—the Anderson model. In this case, the two-point correlation function enjoys an explicit formula, having an equality in (6.4). For the nonlinear case, one needs to either introduce a cone condition, $\sigma(t, x, u) \geq l_\sigma |u|$, for the lower bounds as in case (ii) or assume the nonnegativity of solution for the upper bounds as in case (iii).

Proof of Proposition 6.2 Let $u_0(t, x) = J(t, x)$. For $n \geq 1$, let

$$u_n(t, x) = J(t, x) + \lambda \int_0^t \int_U G(t - s, x, y)\sigma(s, y, u_{n-1}(s, y))W(ds, dy).$$

Let $\rho_n(t, x, x') := \mathbb{E}(u_n(t, x)u_n(t, x'))$. From the proof of Theorem 1.4, we have $u_n(t, x) \to u(t, x)$ in $L^2(\Omega)$ as $n \to \infty$, and $u(t, x)$ satisfies (3.2). It follows that $\rho_n(t, x, x') \to \rho(t, x, x')$ as $n \to \infty$, where $\rho(t, x, x') = \mathbb{E}(u(t, x)u(t, x'))$.

(i) Suppose that $\sigma(t, x, u) = u$ for all $u \in \mathbb{R}$. Then, we have

$$\mathbb{E}(u_n(t, x)u_n(t, x'))$$
$$= \widetilde{J}(t, x, x') + \lambda^2 \int_0^t ds$$
$$\iint_{U^2} dy\, dy'\, G(t - s, x, y)f(y - y')G(t - s, x', y')\mathbb{E}[u_{n-1}(s, y)u_{n-1}(s, y')]$$
$$= \widetilde{J}(t, x, x') + \lambda^2(\widetilde{G} \triangleright \rho_{n-1})(t, x, x', 0, 0).$$

Iterating this, we get

$$\rho_n(t, x, x') = \widetilde{J}(t, x, x') + \lambda^2(\widetilde{G} \triangleright \rho_{n-1})(t, x, x', 0, 0)$$
$$= \widetilde{J}(t, x, x') + \sum_{m=1}^n \lambda^{2m}(\widetilde{G}^{\triangleright m} \triangleright \widetilde{J})(t, x, x', 0, 0).$$

Then, we let $n \to \infty$ to get (6.4). Note that this also implies that the series

$$\sum_{n=1}^\infty \lambda^{2n}\left(\widetilde{G}^{\triangleright n} \triangleright \widetilde{J}\right)(t, x, x', 0, 0)$$

is convergent for any $\lambda > 0$, $t > 0$ and $x, x' \in U$.

(ii) Suppose that $\sigma(t, x, u) \geq l_\sigma|u|$ for all $t > 0$, $x \in U$ and $u \in \mathbb{R}$. We have

$$\rho_n(t, x, x') = \widetilde{J}(t, x, x')$$
$$+ \lambda^2 \int_0^t ds \iint_{U^2} dy\, dy'\, G(t - s, x, y)f(y - y')G(t - s, x', y')$$
$$\times \mathbb{E}[\sigma(s, y, u_{n-1}(s, y))\sigma(s, y', u_{n-1}(s, y'))].$$

For any $s \geq 0$ and $y, y' \in U$,

$$\mathbb{E}[\sigma(s, y, u_{n-1}(s, y))\sigma(s, y', u_{n-1}(s, y'))] \geq l_\sigma^2\, \mathbb{E}(|u_{n-1}(s, y)u_{n-1}(s, y')|)$$
$$\geq l_\sigma^2\, \mathbb{E}(u_{n-1}(s, y)u_{n-1}(s, y')).$$

It follows that

$$
\begin{aligned}
\rho_n & (t, x, x') \\
& \geq \widetilde{J}(t, x, x') + (\lambda l_\sigma)^2 \\
& \int_0^t ds \iint_{U^2} dy\, dy'\, G(t - s, x, y) f(y - y') G(t - s, x', y') \rho_{n-1}(s, y, y') \\
& = \widetilde{J}(t, x, x') + (\lambda l_\sigma)^2 \left(\widetilde{G}^{\triangleright n} \triangleright \rho_{n-1} \right)(t, x, x', 0, 0).
\end{aligned}
$$

By induction,

$$
\rho_n(t, x, x') \geq \widetilde{J}(t, x, x') + \sum_{m=1}^{n} (\lambda l_\sigma)^{2m} \left(\widetilde{G}^{\triangleright m} \triangleright \widetilde{J} \right)(t, x, x', 0, 0).
$$

Then, we let $n \to \infty$ to get (6.5).

(iii) Suppose $\sigma(t, x, u) \leq L_\sigma u$ for all $t > 0$, $x \in U$ and $u \in [0, \infty)$. By the nonnegativity assumption of the solution, we see that

$$
\begin{aligned}
\rho & (t, x, x') \\
& = \mathbb{E}(u(t, x) u(t, x')) \\
& = \widetilde{J}(t, x, x') + \lambda^2 \int_0^t ds \iint_{U^2} dy\, dy'\, G(t - s, x, y) f(y - y') G(t - s, x', y') \\
& \quad \times \mathbb{E}[\sigma(s, y, u(s, y)) \sigma(s, y', u(s, y'))] \\
& \leq \widetilde{J}(t, x, x') + (\lambda L_\sigma)^2 \int_0^t ds \\
& \iint_{U^2} dy\, dy'\, G(t - s, x, y) f(y - y') G(t - s, x', y') \mathbb{E}[u(s, y) u(s, y')] \\
& = \widetilde{J}(t, x, x') + (\lambda L_\sigma)^2 \left(\widetilde{G} \triangleright \rho \right)(t, x, x', 0, 0).
\end{aligned}
$$

Iterating this, we get,

$$
\begin{aligned}
\rho(t, x, x') \leq {} & \widetilde{J}(t, x, x') + \sum_{m=1}^{n-1} (\lambda L_\sigma)^{2m} \left(\widetilde{G}^{\triangleright m} \triangleright \widetilde{J} \right)(t, x, x', 0, 0) \\
& + (\lambda L_\sigma)^{2n} \left(\widetilde{G}^{\triangleright n} \triangleright \rho \right)(t, x, x', 0, 0).
\end{aligned}
$$

Finally, we let $n \to \infty$ to complete the proof of part (iii).

It remains to prove (6.7). Recall that $|v|(U) < \infty$. By Proposition 6.5(i) or 6.7(i), for any $t > 0$ and $x, x' \in U$,

$$
\iint_{U^2} \mathcal{K}^a(t, x, x', y, y') |v|(dy) |v|(dy') < \infty.
$$

Then, by the definition (6.3) of $\mathcal{K}^a$ and Fubini's theorem,

$$a^{-2} \iint_{U^2} \mathcal{K}^a(t, x, x', y, y') v(dy) v(dy')$$
$$= \sum_{n=1}^{\infty} a^{2n-2} \iint_{U^2} \widetilde{G}^{\triangleright n}(t, x, x', y, y') v(dy) v(dy').$$

We compare the right-hand side above with (6.7). For $n = 1$,

$$\iint_{U^2} \widetilde{G}(t, x, x', y, y') v(dy) v(dy') = \widetilde{J}(t, x, x').$$

For $n \geq 2$, by expressing $\widetilde{G}^{\triangleright n} = \widetilde{G}^{\triangleright (n-1)} \triangleright \widetilde{G}$ and interchanging the order of integration, we have

$$\iint_{U^2} \widetilde{G}^{\triangleright n}(t, x, x', y, y') v(dy) v(dy')$$
$$= \iint_{U^2} v(dy) v(dy') \int_0^t ds$$
$$\iint_{U^2} dz\, dz'\, \widetilde{G}^{\triangleright (n-1)}(t - s, x, x', z, z') f(z - z') \widetilde{G}(s, z, z', y, y')$$
$$= \int_0^t ds \iint_{U^2} dz\, dz'\, \widetilde{G}^{\triangleright (n-1)}(t - s, x, x', z, z') f(z - z') \widetilde{J}(s, z, z')$$
$$= \left(\widetilde{G}^{\triangleright (n-1)} \triangleright \widetilde{J} \right)(t, x, x', 0, 0).$$

This proves (6.7) and completes the proof of Proposition 6.2. □

6.2 Estimation of the resolvent kernel functions $\mathcal{K}$

From Proposition 6.2, we see that it is possible to estimate the two-point correlation function once we have some useful bounds for $\mathcal{K}^\lambda$. Therefore, our goal is to establish explicit upper and lower bounds for $\mathcal{K}^\lambda$. To this end, we first prove a lemma which strengthens Lemma 3.10.

Lemma 6.4 *Let $U \subset \mathbb{R}^d$ be a bounded Lipschitz domain with the δ-cone property. Let $0 < \varepsilon_0 < 1$ be defined by (3.12), and recall that, for each $x \in U$, $V(x)$ is the subset of U defined in (3.13). Then, there is a positive constant C such that the following property holds: For any $\varepsilon \in (0, \varepsilon_0]$, for any $x \in U_\varepsilon$, for any r, s such that $0 < r \leq s \leq \varepsilon_0/2$, for any $z \in V(x) \cap B(x, s)$, we have*

$$\mathrm{Vol}(V(x) \cap B(x, s) \cap B(z, r)) \geq C(1 \wedge r)^d. \tag{6.8}$$

Proof First, suppose that case (2) in (3.13) holds. Then for all $z \in B(x, s)$ and $0 < r \leq s \leq \varepsilon_0/2$, we have

$$V(x) \cap B(x, s) \cap B(z, r) = B(x, s) \cap B(z, r).$$

This intersection clearly contains a ball of radius $r/3$, and hence has volume which is bounded from below by $C(r/3)^d$.

Suppose that case (1) in (3.13) holds. Let

$$\mathscr{C}_1 = \{y \in \mathbb{R}^d : 0 < |y| < 1 \text{ and } y \cdot \xi > |y| \cos(\varepsilon_0/2)\}.$$

Note that $\mathscr{C}_1$ is a bounded convex domain, so it satisfies the $\bar{\delta}$-cone property for some $\bar{\delta} > 0$ by Proposition 2.4.4 of [27]. By scaling and translation, we see that

$$\text{Vol}(V(x) \cap B(x, s) \cap B(z, r)) = s^d \, \text{Vol}\left(\overline{\mathscr{C}_1} \cap B((z - x)/s, r/s)\right)$$

with $(z - x)/s \in \overline{\mathscr{C}_1}$ and $0 < r/s \leq 1$. So, it suffices to prove the existence of a constant $C > 0$ such that

$$\text{Vol}(\overline{\mathscr{C}_1} \cap B(w, r)) \geq Cr^d, \quad \text{for all } w \in \overline{\mathscr{C}_1} \text{ and } r \in (0, 1]. \tag{6.9}$$

To prove (6.9), we first consider the case that $0 < r \leq \bar{\delta}$. If $\text{dist}(w, \partial \mathscr{C}_1) \geq \bar{\delta}$, then $B(w, r) \subset \overline{\mathscr{C}_1}$ and hence

$$\text{Vol}(\overline{\mathscr{C}_1} \cap B(w, r)) = C_1 r^d.$$

If $\text{dist}(w, \partial \mathscr{C}_1) < \bar{\delta}$, then $w \in \overline{\mathscr{C}_1} \cap B(v, \bar{\delta})$ for some $v \in \partial \mathscr{C}_1$. By the $\bar{\delta}$-cone property of $\mathscr{C}_1$, we can find a unit vector $\eta = \eta_v \in \mathbb{R}^d$ such that $\mathscr{C}(w, \eta, \bar{\delta}) \subset \mathscr{C}_1$. Then,

$$\begin{aligned}
\text{Vol}(\overline{\mathscr{C}_1} \cap B(w, r)) \\
\geq \text{Vol}(\mathscr{C}(w, \eta, \bar{\delta}) \cap B(w, r)) \\
= \text{Vol}\{y \in \mathbb{R}^d : 0 < |y - w| < r \text{ and } (y - w) \cdot \eta \geq |y - w| \cos(\varepsilon_0/2)\} \\
\geq C_2 r^d.
\end{aligned}$$

Finally, consider the case that $\bar{\delta} < r \leq 1$. We have $\overline{\mathscr{C}_1} \cap B(w, r) \supset \overline{\mathscr{C}_1} \cap B(w, \bar{\delta})$. Then, by the first case that we just proved,

$$\text{Vol}(\overline{\mathscr{C}_1} \cap B(w, r)) \geq (C_1 \wedge C_2)\bar{\delta}^d \geq (C_1 \wedge C_2)\bar{\delta}^d r^d.$$

This completes the proof of Lemma 6.4. $\qquad\square$

Now, we are ready to establish upper and lower bounds for $\mathcal{K}_D^\lambda$ and $\mathcal{K}_N^\lambda$.

Proposition 6.5 *Let U be a bounded Lipschitz domain. Then:*

(i) *There exist positive finite constants C, c, c' such that for all $t > 0$, for all $x, x', y, y' \in U$,*

$$\mathcal{K}_D^\lambda(t, x, x', y, y') \leq \frac{C\lambda^2}{1 \wedge t^d} e^{-c_1\left(\frac{|x-y|^2}{t} + \frac{|x'-y'|^2}{t}\right)} e^{2t\left(c\lambda^2 + c'\lambda^{\frac{4}{2-\beta}} - \mu_1\right)}. \quad (6.10)$$

(ii) *For all $\varepsilon > 0$ small, there exist positive finite constants $\overline{C}$, $\overline{c}$, $\widetilde{c}$ depending on ε such that for all $t > 0$, for all $x, x', y, y' \in U_\varepsilon$,*

$$\mathcal{K}_D^\lambda(t, x, x', y, y') \geq \frac{\overline{C}\lambda^2}{1 \wedge t^d} e^{-16c_2\frac{|x-x'|^2}{t}} e^{-12c_2\left(\frac{|x-y|^2}{t} + \frac{|x'-y'|^2}{t}\right)} e^{2t\left(\overline{c}\lambda^2 + \widetilde{c}\lambda^{\frac{4}{2-\beta}} - \mu_1\right)}$$

(6.11)

and $\overline{C} \to 0$, $\overline{c} \to 0$ and $\widetilde{c} \to 0$ as $\varepsilon \to 0$.

We first make a few remarks:

Remark 6.6

(1) Under the conditions of Proposition 6.2, for the delta initial condition $\nu = \delta_y$, we have

$$(\lambda l_\sigma)^{-2} \mathcal{K}_D^\lambda(t, x, x', y, y) \leq \mathbb{E}(u(t, x)u(t, x')) \leq (\lambda L_\sigma)^{-2} \mathcal{K}_D^\lambda(t, x, x', y, y).$$

(2) Proposition 6.5 applies to all bounded Lipschitz domains. But in case of $C^{1,\alpha}$-domains, we will improve the bounds (6.10) and (6.11) in Proposition 7.2 so that the moment estimates will be consistent with Dirichlet condition at the boundary, namely,

$$\lim_{x \text{ or } x' \to \partial U} \mathbb{E}(u(t, x)u(t, x')) = 0.$$

(3) In case of $U = \mathbb{R}^d$, the factor $\exp(-c|x - x'|^2/t)$ in (6.11) also appears in the lower bound in Lemma 2.7 of [14]. We think that a sharp upper bound for $\mathcal{K}_D^\lambda$ would have this extra exponential factor as well.

Proof of Proposition 6.5

(i). We claim that, for all $n \geq 1$, $t > 0$ and $x, x', y, y' \in U$,

$$\widetilde{G}_D^{\triangleright n}(t, x, x', y, y') \leq C^n \frac{e^{-2\mu_1 t}}{1 \wedge t^d} e^{-c_1\left(\frac{|x-y|^2}{t} + \frac{|x'-y'|^2}{t}\right)} \widetilde{h}_{n-1}(t), \quad (6.12)$$

where $\widetilde{h}_n(t)$ is the iterated integral defined in Lemma 5.3. We prove this by induction. For $n = 1$, this follows from the upper bound in (3.6). Assume that (6.12) holds for some $n \geq 1$. Then, by the induction hypothesis, the upper bound in (3.6) and Lemma 5.1(i),

$$\widetilde{G}_D^{\rhd(n+1)}(t, x, x', y, y')$$

$$= \int_0^t ds \iint_{U^2} \widetilde{G}_D(t - s, x, x', z, z') \widetilde{G}_D^{\rhd n}(s, z, z', y, y') f(z - z') \, dz \, dz'$$

$$\leq C^n \int_0^t ds \, \widetilde{h}_{n-1}(s)$$

$$\iint_{U^2} dz \, dz' \, \widetilde{G}_D(t - s, x, x', z, z') \frac{e^{-2\mu_1 s}}{1 \wedge s^d} e^{-c_1 \frac{|z-y|^2 + |z'-y'|^2}{s}} f(z - z') \, dz \, dz'$$

$$\leq C^{n+1} \frac{e^{-2\mu_1 t}}{1 \wedge t^d} e^{-c_1 \left(\frac{|x-y|^2}{t} + \frac{|x'-y'|^2}{t} \right)} \int_0^t \widetilde{h}_{n-1}(s) \left(1 \wedge \frac{(t-s)s}{t} \right)^{-\beta/2} ds$$

$$= C^{n+1} \frac{e^{-2\mu_1 t}}{1 \wedge t^d} e^{-c_1 \left(\frac{|x-y|^2}{t} + \frac{|x'-y'|^2}{t} \right)} \widetilde{h}_n(t).$$

This proves the claim (6.12). Then, by Lemmas 5.3 and 4.6,

$$\mathcal{K}_D^\lambda(t, x, x', y, y') = \sum_{n=1}^\infty \lambda^{2n} \widetilde{G}_D^{\rhd n}(t, x, x', y, y')$$

$$\leq C\lambda^2 \frac{e^{-2\mu_1 t}}{1 \wedge t^d} e^{-c_1 \left(\frac{|x-y|^2}{t} + \frac{|x'-y'|^2}{t} \right)} \sum_{n=1}^\infty (C\lambda^2)^{n-1} \widetilde{h}_{n-1}(t)$$

$$\leq C'\lambda^2 \frac{e^{-2\mu_1 t}}{1 \wedge t^d} e^{-c_1 \left(\frac{|x-y|^2}{t} + \frac{|x'-y'|^2}{t} \right)} e^{t \left(c\lambda^2 + c'\lambda^{\frac{4}{2-\beta}} \right)}.$$

This proves the upper bound (6.10).

(ii). To prove the lower bound (6.11), we need to derive lower bounds for $\widetilde{G}_D^{\rhd n}(t, x, x', y, y')$ for each $n \geq 1$. Let $t > 0$ and $x, x', y, y' \in U_\varepsilon$. First, by the lower bounds in (3.6) and (3.9),

$$\widetilde{G}_D(t, x, x', y, y') \geq C_\varepsilon^2 \frac{e^{-2\mu_1 t}}{1 \wedge t^d} e^{-c_2 \left(\frac{|x-y|^2}{t} + \frac{|x'-y'|^2}{t} \right)}.$$

For $n = 2$, we have

$$\widetilde{G}_D^{\rhd 2}(t, x, x', y, y') \geq \int_{t/4}^{t/2} ds \iint_{[V(x) \cap B(x, \sqrt{t})]^2} dz \, dz'$$
$$\times \widetilde{G}_D(t - s, x, x', z, z') f(z - z') \widetilde{G}_D(s, z, z', y, y'),$$

where $V(x) \subset U$ is defined in (3.13). By (3.6), for $t/4 < s < t/2$, we have

$$\widetilde{G}_D(t - s, x, x', z, z') \geq C_2^2 \left(1 \wedge \Phi_1(x) \right) \left(1 \wedge \Phi_1(x') \right) \left(1 \wedge \Phi_1(z) \right) \left(1 \wedge \Phi_1(z') \right)$$
$$\times \frac{e^{-2\mu_1(t-s)}}{1 \wedge (t-s)^d} e^{-c_2 \frac{|x-z|^2 + |x'-z'|^2}{t/2}}$$

and

$$\widetilde{G}_D(s, z, z', y, y') \geq C_2^2 \ (1 \wedge \Phi_1(z)) \left(1 \wedge \Phi_1(z')\right) (1 \wedge \Phi_1(y)) \left(1 \wedge \Phi_1(y')\right)$$
$$\times \frac{e^{-2\mu_1 s}}{1 \wedge s^d} e^{-c_2 \frac{|z-y|^2 + |z'-y'|^2}{t/4}}.$$

For $z, z' \in V(x) \cap B(x, \sqrt{t})$, by the triangle inequality and Cauchy–Schwarz inequality, we have

$$|x - z|^2 + |x' - z'|^2 \leq |x - z|^2 + 2(|x' - x|^2 + |x - z'|^2)$$
$$\leq 2|x - x'|^2 + 3t$$

and

$$|z - y|^2 + |z' - y'|^2 \leq 2(|z - x|^2 + |x - y|^2)$$
$$+ 3(|z' - x|^2 + |x - x'|^2 + |x' - y'|^2)$$
$$\leq 5t + 2|x - y|^2 + 3|x' - y'|^2 + 3|x - x'|^2.$$

By applying the bounds above, (3.9) and Lemma 3.10, we get that

$$\widetilde{G}_D^{\rhd 2}(t, x, x', y, y') \geq C_\varepsilon^4 \frac{e^{-2\mu_1 t}}{1 \wedge t^d} e^{-16c_2 \frac{|x-x'|^2}{t}} e^{-12c_2 \frac{|x-y|^2 + |x'-y'|^2}{t}} t^{1-\beta/2}.$$

For $n \geq 2$, expanding $\widetilde{G}_D^{\rhd(n+1)}$ in its full integral form, we have

$$\widetilde{G}_D^{\rhd(n+1)}(t, x, x', y, y')$$
$$\geq \int_0^t ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{n-2}} ds_{n-1} \int_0^{s_{n-1}} ds_n$$
$$\times \iint_{U^2} dz_1 \, dz_1' \, \widetilde{G}_D(t - s_1, x, x', z_1, z_1') f(z_1 - z_1')$$
$$\times \iint_{U^2} dz_2 \, dz_2' \, \widetilde{G}_D(s_1 - s_2, z_1, z_1', z_2, z_2') f(z_2 - z_2')$$
$$\times \cdots \times \iint_{U^2} dz_{n-1} \, dz_{n-1}' \, \widetilde{G}_D(s_{n-2} - s_{n-1}, z_{n-2}, z_{n-2}', z_{n-1}, z_{n-1}') f(z_{n-1} - z_{n-1}')$$
$$\times \iint_{U^2} dz_n \, dz_n' \, \widetilde{G}_D(s_{n-1} - s_n, z_{n-1}, z_{n-1}', z_n, z_n') f(z_n - z_n') \widetilde{G}_D(s_n, z_n, z_n', y, y').$$

Then, we derive a lower bound by integrating on the smaller intervals

$$
s_1 \in \left[\left(1 - \frac{1}{4n}\right)\frac{t}{2}, \frac{t}{2}\right],
$$
$$
s_2 \in \left[\left(1 - \frac{1}{2n}\right)\frac{t}{2}, s_1 - \frac{t}{8n}\right],
$$
$$
s_3 \in \left[\left(1 - \frac{2}{2n}\right)\frac{t}{2}, s_2 - (s_1 - s_2)\right], \tag{6.13}
$$
$$
\vdots
$$
$$
s_n \in \left[\left(1 - \frac{n-1}{2n}\right)\frac{t}{2}, s_{n-1} - (s_{n-2} - s_{n-1})\right].
$$

Let $0 < \varepsilon_0 < 1$ be given by (3.12). For each $\varepsilon \in (0, \varepsilon_0)$ and $x \in U_\varepsilon$, recall the subset $V(x) \subset U$ defined by (3.13). Consider

$$
\begin{aligned}
z_1, z_1' \in A_1 \quad &:= V(x) \cap B(x, \sqrt{(t - s_1)/(5n)} \wedge (\varepsilon_0/2)), \\
z_2 \in A_2 \quad &:= V(x) \cap B(x, \sqrt{s_1 - s_2} \wedge (\varepsilon_0/2)) \cap B(z_1, \sqrt{s_1 - s_2} \wedge (\varepsilon_0/2)), \\
z_2' \in A_2' \quad &:= V(x) \cap B(x, \sqrt{s_1 - s_2} \wedge (\varepsilon_0/2)) \cap B(z_1', \sqrt{s_1 - s_2} \wedge (\varepsilon_0/2)), \\
&\vdots \\
z_n \in A_n \quad &:= V(x) \cap B(x, \sqrt{s_{n-1} - s_n} \wedge (\varepsilon_0/2)) \cap B(z_{n-1}, \sqrt{s_{n-1} - s_n} \wedge (\varepsilon_0/2)), \\
z_n' \in A_n' \quad &:= V(x) \cap B(x, \sqrt{s_{n-1} - s_n} \wedge (\varepsilon_0/2)) \cap B(z_{n-1}', \sqrt{s_{n-1} - s_n} \wedge (\varepsilon_0/2)).
\end{aligned}
$$

Note that by (6.13), we have $(t - s_1)/(5n) \leq t/(8n) \leq s_1 - s_2 \leq s_2 - s_3 \leq \cdots \leq s_{n-1} - s_n$, which ensures that for $2 \leq i \leq n$, both z_{i-1} and z_{i-1}' lie in $V(x) \cap B(x, \sqrt{s_{i-1} - s_i} \wedge (\varepsilon_0/2))$. Then, by Lemma 6.4, for $2 \leq i \leq n$,

$$
\mathrm{Vol}(A_i) \wedge \mathrm{Vol}(A_i') \geq C(\sqrt{s_{i-1} - s_i} \wedge (\varepsilon_0/2))^d \geq C(\varepsilon_0/2)^d((s_{i-1} - s_i) \wedge 1)^{d/2}.
$$

By Lemma 3.10 and (3.9), we have

$$
\mathrm{Vol}(A_1) \geq C\left(\sqrt{\frac{t - s_1}{5n}} \wedge (\varepsilon_0/2)\right)^d \geq C(\varepsilon_0/2)^d(5n)^{-d/2}((t - s_1) \wedge 1)^{d/2},
$$
$$
1 \wedge \Phi_1(z) \geq 1 \wedge (c_0^{-1}\varepsilon^{a_1}) \quad \text{for all } z \in V(x).
$$

Also, on $A_1 \times A_1$, we have $f(z_1 - z_1') \geq C(((t - s_1)/n) \wedge 1)^{-\beta/2}$, and on $A_i \times A_i'$, where $2 \leq i \leq n$, we have $f(z_i - z_i') \geq C((s_{i-1} - s_i) \wedge 1)^{-\beta/2}$. Then, by the lower bound in (3.6) and the inequalities

$$
|z_n - z_{n-1}|^2 + |z_n' - z_{n-1}'|^2 \leq 2(s_{n-1} - s_n),
$$
$$
|z_n - y|^2 + |z_n' - y'|^2 \leq 5(s_{n-1} - s_n) + 2|x - y|^2 + 3|x' - y'|^2 + 3|x - x'|^2,
$$

we have

$$\iint_{A_n \times A'_n} dz_n\, dz'_n\, \widetilde{G}_D(s_{n-1} - s_n, z_{n-1}, z'_{n-1}, z_n, z'_n) f(z_n - z'_n) \widetilde{G}_D(s_n, z_n, z'_n, y, y')$$

$$\geq C_\varepsilon^4 e^{-2\mu_1 s_{n-1}} (1 \wedge (s_{n-1} - s_n))^{-\frac{\beta}{2}} \frac{1}{1 \wedge t^d} e^{-c_2 \frac{3|x-x'|^2 + 3|x-y|^2 + 3|x'-y'|^2 + 5s_{n-1}}{s_n}}.$$

For $i = 2, \ldots, n-1$, by $|z_i - z_{i-1}|^2 + |z'_i - z'_{i-1}|^2 \leq 2(s_{i-1} - s_i)$, we have

$$\iint_{A_i \times A'_i} dz_i\, dz'_i\, \widetilde{G}_D(s_{i-1} - s_i, z_{i-1}, z'_{i-1}, z_i, z'_i) f(z_i - z'_i)$$

$$\geq C_\varepsilon^2 e^{-2\mu_1(s_{i-1} - s_i)} (1 \wedge (s_{i-1} - s_i))^{-\frac{\beta}{2}}.$$

For $i = 1$, by the inequality

$$|z_1 - x|^2 + |z'_1 - x'|^2 \leq 3(t - s_1) + 2|x - x'|^2,$$

we have

$$\iint_{A_1 \times A_1} dz_1\, dz'_1\, \widetilde{G}_D(t - s_1, x, x', z_1, z'_1) f(z_1 - z'_1)$$

$$\geq \frac{C_\varepsilon^2}{n^{d/2}} e^{-2\mu_1(t-s_1)} e^{-2c_2 \frac{|x-x'|^2}{t-s_1}} \left(1 \wedge \frac{t - s_1}{n}\right)^{-\frac{\beta}{2}}.$$

Combining these estimates and using (4.14), we get that

$$\widetilde{G}_D^{\triangleright(n+1)}(t, x, x', y, y')$$

$$\geq C_\varepsilon^{2(n+1)} \frac{e^{-2\mu_1 t}}{1 \wedge t^d} \int_{(1-\frac{1}{4n})\frac{t}{2}}^{\frac{t}{2}} ds_1 \left(1 + \left(\frac{t - s_1}{n}\right)^{-\frac{\beta}{2}}\right) e^{-2c_2 \frac{|x-x'|^2}{t-s_1}}$$

$$\times \int_{(1-\frac{1}{2n})\frac{t}{2}}^{s_1 - \frac{t}{8n}} ds_2 \left(1 + (s_1 - s_2)^{-\frac{\beta}{2}}\right) \times \int_{(1-\frac{2}{2n})\frac{t}{2}}^{s_2 - (s_1 - s_2)} ds_3 \left(1 + (s_2 - s_3)^{-\frac{\beta}{2}}\right) \times \cdots$$

$$\cdots \times \int_{(1-\frac{n-2}{2n})\frac{t}{2}}^{s_{n-2} - (s_{n-3} - s_{n-2})} ds_{n-1} \left(1 + (s_{n-2} - s_{n-1})^{-\frac{\beta}{2}}\right)$$

$$\times \int_{(1-\frac{n-1}{2n})\frac{t}{2}}^{s_{n-1} - (s_{n-2} - s_{n-1})} ds_n \left(1 + (s_{n-1} - s_n)^{-\frac{\beta}{2}}\right) e^{-c_2 \frac{3|x-x'|^2 + 3|x-y|^2 + 3|x'-y'|^2 + 5t}{t/4}}.$$

Let I denote the above multiple integral for $s_1, \ldots, s_n$. By (6.13), we have $s_{i-1} - s_i \geq \frac{t}{8n}$ and $s_{i-1} - (1 - \frac{i}{2n})\frac{t}{2} \geq (1 - \frac{i-1}{2n})\frac{t}{2} - (1 - \frac{i}{2n})\frac{t}{2} = \frac{t}{4n}$ for $2 \leq i \leq n$. Fixing $s_1 \in \left[(1 - \frac{1}{4n})\frac{t}{2}, \frac{t}{2}\right]$, by the change of variables $s_n \mapsto s_{n-1} - s_n, s_{n-1} \mapsto s_{n-2} - s_{n-1}$,

$\ldots$, $s_2 \mapsto s_1 - s_2$, we have

$$s_n \in \left[s_{n-2} - s_{n-1}, s_{n-1} - \left(1 - \frac{n-1}{2n} \right) \frac{t}{2} \right] \supset \left[\frac{t}{8n}, \frac{t}{4n} \right],$$

$$\vdots$$

$$s_3 \in \left[s_1 - s_2, s_2 - \left(1 - \frac{2}{2n} \right) \frac{t}{2} \right] \supset \left[\frac{t}{8n}, \frac{t}{4n} \right],$$

$$s_2 \in \left[\frac{t}{8n}, s_1 - \left(1 - \frac{1}{2n} \right) \frac{t}{2} \right] \supset \left[\frac{t}{8n}, \frac{t}{4n} \right].$$

It follows that

$$I \geq e^{-10c_2} e^{-16c_2 \frac{|x-x'|^2}{t}} e^{-12c_2 \frac{|x-y|^2 + |x'-y'|^2}{t}}$$

$$\times \int_{(1-\frac{1}{4n})\frac{t}{2}}^{\frac{t}{2}} ds_1 \left(1 + \left(\frac{t - s_1}{n} \right)^{-\frac{\beta}{2}} \right) \times \int_{\frac{t}{8n}}^{\frac{t}{4n}} ds_2 \left(1 + s_2^{-\frac{\beta}{2}} \right) \times \cdots$$

$$\times \int_{\frac{t}{8n}}^{\frac{t}{4n}} ds_{n-1} \left(1 + s_{n-1}^{-\frac{\beta}{2}} \right) \times \int_{\frac{t}{8n}}^{\frac{t}{4n}} ds_n \left(1 + s_n^{-\frac{\beta}{2}} \right)$$

$$=: e^{-10c_2} e^{-16c_2 \frac{|x-x'|^2}{t}} e^{-12c_2 \frac{|x-y|^2 + |x'-y'|^2}{t}} \prod_{i=1}^{n} I_i.$$

The above $ds_2 \cdots ds_n$ integrals can be evaluated explicitly, which is equal to

$$\prod_{i=2}^{n} I_i = \left(\frac{t}{8n} + \frac{\left(\frac{t}{4n} \right)^{1-\frac{\beta}{2}} - \left(\frac{t}{8n} \right)^{1-\frac{\beta}{2}}}{1 - \frac{\beta}{2}} \right)^{n-1} \geq c^{n-1} \left(\frac{t}{n} \right)^{n-1} \left(1 + \left(\frac{t}{n} \right)^{-\frac{\beta}{2}} \right)^{n-1}.$$

As for the ds_1 integral, we have that

$$I_1 \geq \int_{(1-\frac{1}{4n})\frac{t}{2}}^{\frac{t}{2}} ds_1 \left(1 + \left(\frac{t - t/2}{n} \right)^{-\frac{\beta}{2}} \right) \geq c \frac{t}{n} \left(1 + \left(\frac{t}{n} \right)^{-\frac{\beta}{2}} \right).$$

Therefore, we see that

$$I \geq c^n n^{-\beta/2} e^{-10c_2} e^{-16c_2 \frac{|x-x'|^2}{t}} e^{-12c_2 \frac{|x-y|^2 + |x'-y'|^2}{t}} \left(\frac{t}{n} \right)^n \left(1 + \left(\frac{t}{n} \right)^{-\beta/2} \right)^n$$

$$= c^n n^{-\beta/2} e^{-10c_2} e^{-16c_2 \frac{|x-x'|^2}{t}} e^{-12c_2 \frac{|x-y|^2 + |x'-y'|^2}{t}} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \left(\frac{t}{n} \right)^{n-k\beta/2}.$$

By Stirling's formula, $\frac{n^{k\beta/2}}{n^n} \geq C^n \frac{(k!)^{\beta/2}}{n!}$ for all $k = 0, 1, \ldots, n$ and $n \geq 2$. It follows that

$$\widetilde{G}_D^{\triangleright(n+1)}(t, x, x', y, y') \geq C_\varepsilon^{2(n+1)} c^n \frac{e^{-2\mu_1 t}}{1 \wedge t^d} e^{-16c_2 \frac{|x-x'|^2}{t}} e^{-12c_2 \frac{|x-y|^2+|x'-y'|^2}{t}}$$
$$\sum_{k=0}^{n} \frac{t^{n-k\beta/2}}{(n-k)!(k!)^{1-\beta/2}}.$$

Finally, recalling the definition of $\mathcal{K}_D^\lambda$, interchanging the order of summation, and using (4.11), we get that

$$\mathcal{K}_D^\lambda(t, x, x', y, y')$$
$$= \lambda^2 \sum_{n=0}^{\infty} \lambda^{2n} \widetilde{G}_D^{\triangleright(n+1)}(t, x, x', y, y')$$
$$\geq C_\varepsilon^2 \lambda^2 \frac{e^{-2\mu_1 t}}{1 \wedge t^d} e^{-16c_2 \frac{|x-x'|^2}{t}} e^{-12c_2 \frac{|x-y|^2+|x'-y'|^2}{t}} \sum_{n=0}^{\infty} (C_\varepsilon^2 c \lambda^2 t)^n \sum_{k=0}^{n} \frac{t^{-k\beta/2}}{(n-k)!(k!)^{1-\beta/2}}$$
$$\geq C_\varepsilon^2 C \lambda^2 \frac{e^{-2\mu_1 t}}{1 \wedge t^d} e^{-16c_2 \frac{|x-x'|^2}{t}} e^{-12c_2 \frac{|x-y|^2+|x'-y'|^2}{t}} e^{t\left(K\lambda^2 + K'\lambda^{\frac{4}{2-\beta}}\right)}.$$

The proof of Proposition 6.5 is complete. $\qquad\square$

Proposition 6.7 *Let U be a bounded Lipschitz domain. Then:*

(i) There exist positive finite constants C, c, c' such that for all $t > 0$, for all $x, x', y, y' \in U$,

$$\mathcal{K}_N^\lambda(t, x, x', y, y') \leq \frac{C\lambda^2}{1 \wedge t^d} e^{-c_3\left(\frac{|x-y|^2}{t} + \frac{|x'-y'|^2}{t}\right)} e^{t\left(c\lambda^2 + c'\lambda^{\frac{4}{2-\beta}}\right)}. \qquad (6.14)$$

(ii) If U is convex (or if (3.8) holds), then there exist positive finite constants $\overline{C}$, $\overline{c}$, $\widetilde{c}$ such that for all $t > 0$, for all $x, x', y, y' \in U$,

$$\mathcal{K}_N^\lambda(t, x, x', y, y') \geq \frac{\overline{C}\lambda^2}{1 \wedge t^d} e^{-16c_4 \frac{|x-x'|^2}{t}} e^{-12c_4\left(\frac{|x-y|^2}{t} + \frac{|x'-y'|^2}{t}\right)} e^{t\left(\overline{c}\lambda^2 + \widetilde{c}\lambda^{\frac{4}{2-\beta}}\right)}. \qquad (6.15)$$

Proof The proof is similar to that of Lemma 6.5. $\qquad\square$

6.3 Proof of Theorem 1.5

Proof of Theorem 1.5 The correlation bounds (1.22) and (1.23) under Dirichlet condition (or Neumann condition, respectively) follows immediately from Propositions 6.2 and 6.5 (or 6.2 and 6.7, respectively). $\qquad\square$

7 The case of bounded $C^{1,\alpha}$-domains with Dirichlet condition and some variations

In the case of $C^{1,\alpha}$-domains, one can expect better moment estimates under Dirichlet boundary condition because the heat kernel estimates in (3.6) hold with $a_1 = a_2 = 1$. This implies that the Dirichlet heat kernel estimates in (3.6) are sharp, yielding matching upper and lower bounds with the same factor $\left(1 \wedge \frac{\Phi_1(x)}{1 \wedge t^{1/2}}\right)\left(1 \wedge \frac{\Phi_1(y)}{1 \wedge t^{1/2}}\right)$.

7.1 Better estimates for the resolvent kernel $\mathcal{K}$

In this part, we study the case of bounded $C^{1,\alpha}$-domains with Dirichlet boundary condition. For the points x, x', y, y' that are close to ∂U, the lemma below provides more precise estimate than what one gets from the upper bound in Lemma 5.1.

Lemma 7.1 *If U is a bounded $C^{1,\alpha}$-domain for some $\alpha > 0$, then there exists a positive finite constant C such that for all $t > 0$, for all $x, x', y, y' \in U$,*

$$
\int_0^t ds \iint_{U^2} dz\, dz'\; \Psi(t-s, x)\Psi(t-s, x')\Psi(t-s, z)\Psi(t-s, z')
$$

$$
\frac{e^{-2\mu_1(t-s)}}{1 \wedge (t-s)^d} e^{-c_1 \frac{|x-z|^2 + |x'-z'|^2}{t-s}}
$$

$$
\times f(z-z')\Psi(s, z)\Psi(s, z')\Psi(s, y)\Psi(s, y') \frac{e^{-2\mu_1 s}}{1 \wedge s^d} e^{-\frac{2c_1}{3} \frac{|z-y|^2 + |z'-y'|^2}{s}}
$$

$$
\leq C\Psi(t, x)\Psi(t, x')\Psi(t, y)\Psi(t, y') \frac{e^{-2\mu_1 t}}{1 \wedge t^d} e^{-\frac{2c_1}{3} \frac{|x-y|^2 + |x'-y'|^2}{t}}
$$

$$
\int_0^t \left(1 \wedge \frac{(t-s)s}{t}\right)^{-\beta/2} ds, \tag{7.1}
$$

where $\Psi(t, x)$ is as defined in (1.24).

Proof We first derive the following bound: for all $s, r > 0$ and $v, w \in U$,

$$
\Psi(s, v)\Psi(r, w)e^{-\frac{c_1}{6} \frac{|v-w|^2}{r}} \leq C_0 \Psi(s, w). \tag{7.2}
$$

Since U is a $C^{1,\alpha}$-domain for some $\alpha > 0$, (3.9) holds with $a_1 = a_2 = 1$. By (3.9) and the triangle inequality,

$$
\Phi_1(v) \leq c_0(\text{dist}(w, \partial U) + |v - w|) \leq c_0^2 \Phi_1(w) + c_0|v - w|.
$$

It follows that

$$
\Psi(s, v) = 1 \wedge \frac{\Phi_1(v)}{1 \wedge s^{1/2}} \leq 1 \wedge \left(\frac{c_0^2 \Phi_1(w) + c_0|v - w|}{\Phi_1(w)} \cdot \frac{\Phi_1(w)}{1 \wedge s^{1/2}}\right)
$$

$$\leq \left(c_0^2 + \frac{c_0|v - w|}{\Phi_1(w)} \right) \Psi(s, w).$$

Thus,

$$\Psi(s, v)\Psi(r, w) \leq \left(c_0^2 + \frac{c_0|v - w|}{\Phi_1(w)} \right) \left(1 \wedge \frac{\Phi_1(w)}{1 \wedge r^{1/2}} \right) \Psi(s, w)$$

$$\leq \left(c_0^2 + \frac{c_0|v - w|}{1 \wedge r^{1/2}} \right) \Psi(s, w).$$

Moreover, since the function xe^{-x^2} is bounded, we have

$$\Psi(s, v)\Psi(r, w)e^{-\frac{c_1}{6} \frac{|v-w|^2}{r}} \leq \left(c_0^2 + c_0|v - w| + \frac{c_0|v - w|}{r^{1/2}} e^{-\frac{c_1}{6} \frac{|v-w|^2}{r}} \right) \Psi(s, w)$$

$$\leq C_0 \Psi(s, w),$$

where C_0 is a finite constant depending on c_0, c_1 and the diameter of U. This proves (7.2).

In order to prove the lemma, we split the integral over $[0, t]$ into the sum of the integral over $[t/2, t]$ and the integral over $[0, t/2]$. Denote these two integrals by I_1 and I_2, respectively. By (7.2), we have the following two bounds

$$\Psi(t - s, x)\Psi(s, z)e^{-\frac{c_1}{3} \frac{|x-z|^2}{t-s}} \leq C_0 \Psi(s, x),$$

$$\Psi(t - s, x')\Psi(s, z')e^{-\frac{c_1}{3} \frac{|x'-z'|^2}{t-s}} \leq C_0 \Psi(s, x').$$

These bounds and $\Psi(t - s, z)\Psi(t - s, z') \leq 1$ imply that

$$I_1 \leq C_0^2 \int_{t/2}^t ds \, \Psi(s, x)\Psi(s, x')\Psi(s, y)\Psi(s, y')$$

$$\times \iint_{U^2} dz \, dz' \frac{e^{-2\mu_1(t-s)}}{1 \wedge (t - s)^d} e^{-\frac{2c_1}{3} \frac{|x-z|^2+|x'-z'|^2}{t-s}} f(z - z') \frac{e^{-2\mu_1 s}}{1 \wedge s^d} e^{-\frac{2c_1}{3} \frac{|z-y|^2+|z'-y'|^2}{s}}.$$

Then, using the bound $\Psi(s, x) \leq \Psi(t/2, x) \leq \sqrt{2}\Psi(t, x)$ for $t/2 \leq s \leq t$ and Lemma 5.1, we obtain

$$I_1 \leq C\Psi(t, x)\Psi(t, x')\Psi(t, y)\Psi(t, y) \frac{e^{-2\mu_1 t}}{1 \wedge t^d} e^{-\frac{2c_1}{3} \frac{|x-y|^2+|x'-y'|^2}{t}}$$

$$\int_{t/2}^t \left(1 \wedge \frac{(t - s)s}{t} \right)^{-\beta/2} ds.$$

For I_2, we use the bounds

$$\Psi(t - s, z)\Psi(s, y)e^{-\frac{c_1}{6} \frac{|z-y|^2}{s}} \leq C_0 \Psi(t - s, y),$$

$$\Psi(t - s, z')\Psi(s, y')e^{-\frac{c_1}{6}\frac{|z'-y'|^2}{s}} \le C_0\Psi(t - s, y'),$$

and $\Psi(s, z)\Psi(s, z') \le 1$ to get that

$$I_2 \le C_0^2 \int_0^{t/2} ds\ \Psi(t - s, x)\Psi(t - s, x')\Psi(t - s, y)\Psi(t - s, y')$$

$$\times \iint_{U^2} dz\ dz'\ \frac{e^{-2\mu_1(t-s)}}{1 \wedge (t-s)^d}e^{-c_1\frac{|x-z|^2+|x'-z'|^2}{t-s}}f(z-z')\frac{e^{-2\mu_1 s}}{1 \wedge s^d}e^{-c_1\frac{|z-y|^2+|z'-y'|^2}{2s}}.$$

For $0 \le s \le t/2$, we apply the identity (5.1) with t and s replaced by $t' = t + s$ and $s' = 2s$ respectively, and with $v = z - x$ and $w = y - x$. This gives

$$e^{-c_1\frac{|x-z|^2+|x'-z'|^2}{t-s}}e^{-c_1\frac{|z-y|^2+|z'-y'|^2}{2s}}$$

$$= e^{-c_1\frac{|x-y|^2+|x'-y'|^2}{t+s}}e^{-c_1\frac{|(z-x)-\frac{t'-s'}{t'}(y-x)|^2+|(z'-x')-\frac{t'-s'}{t'}(y'-x')|^2}{\tau'}}$$

$$\le e^{-\frac{2c_1}{3}\frac{|x-y|^2+|x'-y'|^2}{t}}e^{-c_1\frac{|(z-x)-\frac{t'-s'}{t'}(y-x)|^2+|(z'-x')-\frac{t'-s'}{t'}(y'-x')|^2}{\tau'}},$$

where $\tau' = (t' - s')s'/t'$. It is easy to see that for $0 \le s \le t/2$,

$$(1 \wedge \tau')^{d-\beta/2} \le 2^d \left(1 \wedge \frac{(t-s)s}{t}\right)^{d-\beta/2}.$$

Hence, we can follow the proof of Lemma 5.1(i) and use the bound $\Psi(t - s, x) \le \sqrt{2}\Psi(t, x)$ for $0 \le s \le t/2$ to deduce that

$$I_2 \le C'\Psi(t, x)\Psi(t, x')\Psi(t, y)\Psi(t, y)\frac{e^{-2\mu_1 t}}{1 \wedge t^d}e^{-\frac{2c_1}{3}\frac{|x-y|^2+|x'-y'|^2}{t}}$$

$$\int_0^{t/2} \left(1 \wedge \frac{(t-s)s}{t}\right)^{-\beta/2} ds.$$

The proof of Lemma 7.1 is complete. $\qquad\square$

We can now strengthen the bounds (6.10) and (6.11) for $\mathcal{K}_D^\lambda(t, x, x', y, y')$ in the case of bounded $C^{1,\alpha}$-domains. Note that $\Psi(t, x) = 0$ for all $x \in \partial U$, hence (7.3) and (7.4) below provide more precise estimates than (6.10) and (6.11) for x, x', y, y' that are close to ∂U.

Proposition 7.2 *If U is a bounded $C^{1,\alpha}$-domain for some $\alpha > 0$. Then:*

(i) *There exist positive finite constants C, c, c' such that for all $t > 0$ and x, x', y, $y' \in U$,*

$$
\begin{aligned}
\mathcal{K}_D^\lambda(t, x, x', y, y') &\leq \frac{C\lambda^2}{1 \wedge t^d} \Psi(t, x) \Psi(t, x') \Psi(t, y) \Psi(t, y') \\
&\times e^{-\frac{2c_1}{3}\left(\frac{|x-y|^2}{t} + \frac{|x'-y'|^2}{t}\right)} e^{2t\left(c\lambda^2 + c'\lambda^{\frac{4}{2-\beta}} - \mu_1\right)}.
\end{aligned}
\tag{7.3}
$$

(ii) *There exist positive finite constants $\bar{C}$, $\bar{c}$, $\tilde{c}$ such that for all $t > 0$, for all x, x', y, $y' \in U$,*

$$
\begin{aligned}
\mathcal{K}_D^\lambda(t, x, x', y, y') &\geq \frac{\bar{C}\lambda^2}{1 \wedge t^d} \Psi(t, x) \Psi(t, x') \Psi(t, y) \Psi(t, y') \\
&\times e^{-16c_2 \frac{|x-x'|^2}{t}} e^{-12c_2\left(\frac{|x-y|^2}{t} + \frac{|x'-y'|^2}{t}\right)} e^{2t\left(\bar{c}\lambda^2 + \tilde{c}\lambda^{\frac{4}{2-\beta}} - \mu_1\right)}.
\end{aligned}
\tag{7.4}
$$

Proof

(i). Similarly to the proof of Proposition 6.5(i), the upper bound (7.3) can be proved by applying Lemmas 7.1, 5.3 and 4.6.

(ii). To prove the lower bound (7.4), we claim that for each $n \geq 0$,

$$
\begin{aligned}
&\widetilde{G}_D^{\triangleright(n+1)}(t, x, x', y, y') \\
&\geq C^{n+1} \Psi(t, x) \Psi(t, x') \Psi(t, y) \Psi(t, y') \\
&\times \frac{e^{-2\mu_1 t}}{1 \wedge t^d} e^{-16c_2 \frac{|x-x'|^2}{t}} e^{-12c_2 \frac{|x-y|^2 + |x'-y'|^2}{t}} \sum_{k=0}^{n} \frac{t^{n-k\beta/2}}{(n-k)!(k!)^{1-\beta/2}}.
\end{aligned}
\tag{7.5}
$$

Indeed, for $n = 0$, (7.5) follows from Proposition 3.6(i). For $n \geq 1$, we can prove (7.5) by modifying the proof in Proposition 6.5(ii) and we outline the major changes as follows. Instead of $V(x)$ defined in (3.13), we now consider

$$
\widetilde{V}(x) = \begin{cases} \overline{\mathscr{C}(x + (\varepsilon_0/4)\xi_{y_i}, \xi_{y_i}, \varepsilon_0/4)} & \text{in case (1) of (3.13)}, \\ \overline{B(x, \varepsilon_0/4)} & \text{in case (2) of (3.13)}, \end{cases}
$$

so that $\widetilde{V}(x) \subset V(x) \subset U$ and there exists $\delta_0 > 0$ such that for all $x \in U$, for all $z \in \widetilde{V}(x)$, $\text{dist}(z, \partial U) \geq \delta_0$. Hence, (3.9) implies that

$$
\inf_{x \in U} \inf_{z \in \widetilde{V}(x)} \Phi_1(z) \geq c_0^{-1} \delta_0 > 0.
\tag{7.6}
$$

For any $x \in U$, define $\tilde{x}$ by

$$
\tilde{x} = \begin{cases} x + (\varepsilon_0/4)\xi_{y_i} & \text{in case (1) of (3.13)}, \\ x & \text{in case (2) of (3.13)}. \end{cases}
$$

Then, as in Lemma 6.4, we can show that there exists $C_0 > 0$ such that for all $t > 0$, for all $x \in U$, for all $0 \le r \le s \le \varepsilon_0/4$, for all $z \in \widetilde{V}(x) \cap B(\tilde{x}, s)$,

$$\text{Vol}(\widetilde{V}(x) \cap B(\tilde{x}, s) \cap B(z, r)) \ge C_0 (1 \wedge r)^d. \qquad (7.7)$$

For $n = 1$, similarly to the proof in Proposition 6.5(ii), we have

$$\widetilde{G}_D^{\triangleright 2}(t, x, x', y, y')$$
$$\ge \int_{t/4}^{t/2} ds \iint_{[\widetilde{V}(x) \cap B(\tilde{x}, \sqrt{t})]^2} dz\, dz'\, \widetilde{G}_D(t - s, x, x', z, z') f(z - z') \widetilde{G}_D(s, z, z', y, y')$$
$$\ge C \Psi(t, x) \Psi(t, x') \Psi(t, y) \Psi(t, y') \frac{e^{-2\mu_1 t}}{(1 \wedge t^d)^2} e^{-16c_2 \frac{|x-x'|^2}{t}} e^{-12c_2 \frac{|x-y|^2 + |x'-y'|^2}{t}}$$
$$\times \int_{t/4}^{t/2} ds \iint_{[\widetilde{V}(x) \cap B(\tilde{x}, \sqrt{t})]^2} dz\, dz'\, (1 \wedge \Phi_1(z))^2 (1 \wedge \Phi_1(z'))^2 |z - z'|^{-\beta}.$$

Then, by using (7.6) and (7.7), we obtain (7.5) for $n = 1$.

For $n \ge 2$, we first restrict the integrals of $s_1, s_2, s_3, \ldots, s_n$ to the smaller intervals $\left[(1 - \frac{1}{4n})\frac{t}{2}, \frac{t}{2}\right]$, $\left[(1 - \frac{1}{2n})\frac{t}{2}, s_1 - \frac{t}{8n}\right]$, $\left[(1 - \frac{2}{2n})\frac{t}{2}, s_2 - (s_1 - s_2)\right]$, $\ldots$, $\left[(1 - \frac{n-1}{2n})\frac{t}{2}, s_{n-1} - (s_{n-1} - s_{n-1})\right]$, respectively, so that $\frac{t-s_1}{5n} \le \frac{t}{8n} \le s_1 - s_2 \le s_2 - s_3 \le \cdots \le s_{n-1} - s_n$. We can then modify the proof of Proposition 6.5(ii) by considering

$$
\begin{aligned}
z_1, z_1' \in A_1 \quad &:= \widetilde{V}(x) \cap B(\tilde{x}, \sqrt{(t - s_1)/(5n)} \wedge (\varepsilon_0/4)), \\
z_2 \in A_2 \quad &:= \widetilde{V}(x) \cap B(\tilde{x}, \sqrt{s_1 - s_2} \wedge (\varepsilon_0/4)) \cap B(z_1, \sqrt{s_1 - s_2} \wedge (\varepsilon_0/4)), \\
z_2' \in A_2' \quad &:= \widetilde{V}(x) \cap B(\tilde{x}, \sqrt{s_1 - s_2} \wedge (\varepsilon_0/4)) \cap B(z_1', \sqrt{s_1 - s_2} \wedge (\varepsilon_0/4)), \\
&\qquad\vdots \\
z_n \in A_n \quad &:= \widetilde{V}(x) \cap B(\tilde{x}, \sqrt{s_{n-1} - s_n} \wedge (\varepsilon_0/4)) \cap B(z_{n-1}, \sqrt{s_{n-1} - s_n} \wedge (\varepsilon_0/4)), \\
z_n' \in A_n' \quad &:= \widetilde{V}(x) \cap B(\tilde{x}, \sqrt{s_{n-1} - s_n} \wedge (\varepsilon_0/4)) \cap B(z_{n-1}', \sqrt{s_{n-1} - s_n} \wedge (\varepsilon_0/4)).
\end{aligned}
$$

Then, along the lines of the proof of Proposition 6.5(ii), we can deduce (7.5) using (7.6) and (7.7) above.

Finally, recall that

$$\mathcal{K}_D^\lambda(t, x, x', y, y') = \lambda^2 \sum_{n=0}^{\infty} \lambda^{2n} \widetilde{G}_D^{\triangleright(n+1)}(t, x, x', y, y').$$

By using (7.5), interchanging the order of summation, and using (4.11), we can obtain the lower bound (7.4) as in the proof of Proposition 6.5(ii). This completes the proof of Proposition 7.2. $\qquad\square$

7.2 Proof of Theorem 1.7

Proof of Theorem 1.7 In Remark 1.8, we have seen that ν satisfies condition (1.25) if and only if $J^*_{2c_1/3}(t, x) < \infty$ for all $t > 0$ and $x \in U$. The proof of the existence, p-th moment bounds and uniqueness of the solution is similar to the proof of Theorem 1.4 in Sect. 5, with the use of Lemma 7.1 instead of Lemma 5.1 (i). The correlation bounds follow immediately from Propositions 6.2 and 7.2. □

7.3 Proof of corollary 1.9

Proof of Corollary 1.9 Note that the Dirichlet kernel kernel for $\mathscr{L}$ on $U = \prod_{i=1}^{m} U_i$ is given by

$$G_D^U(t, x, y) = \prod_{i=1}^{m} G_D^{U_i}(t, x_i, y_i)$$

for $t > 0$ and $x = (x_1, \ldots, x_m)$, $y = (y_1, \ldots, y_m) \in U$, where $G_D^{U_i}(t, x_i, y_i)$ is the Dirichlet heat kernel for $\mathscr{L}_i$ on the C^{1,α_i}-domain U_i. This and Proposition 3.6 applied to each $G_D^{U_i}(t, x_i, y_i)$ imply the following heat kernel estimate

$$G_D^U(t, x, y) \leq C_1 \Psi^*(t, x) \Psi^*(t, y) \frac{e^{-\mu_1 t}}{1 \wedge t^{d/2}} e^{-c_1 \frac{|x-y|^2}{t}}$$

with suitable constants C_1 and c_1, where Ψ^* is as defined in (1.32) and μ_1 is the sum of the leading eigenvalues of the Dirichlet operators $\mathscr{L}_i$. By the proof of (7.2) in Lemma 7.1, for each i, we have

$$\left(1 \wedge \frac{\Phi_1^{U_i}(v_i)}{1 \wedge s^{1/2}}\right)\left(1 \wedge \frac{\Phi_1^{U_i}(w_i)}{1 \wedge r^{1/2}}\right) e^{-\frac{c_1}{6} \frac{|v_i - w_i|^2}{r}} \leq C_{0,i} \left(1 \wedge \frac{\Phi_1^{U_i}(w_i)}{1 \wedge s^{1/2}}\right)$$

for all $s, r > 0$ and $v_i, w_i \in U_i$, where $C_{0,i}$ is a constant. Taking products over $i \in \{1, \ldots, m\}$ gives

$$\Psi^*(s, v) \Psi^*(r, w) e^{-\frac{c_1}{6} \frac{|v-w|^2}{r}} \leq C_0 \Psi^*(s, w)$$

for all $s, r > 0$ and $v, w \in U$, where $C_0 = \prod_{i=1}^{m} C_{0,i}$. Then, the same proof of Lemma 7.1 shows that the estimate (7.1) holds with every Ψ replaced by Ψ^*. This implies that the estimate (7.3) also holds with every Ψ replaced by Ψ^*. Using these estimates, the statements (i) and (ii) in Theorem 1.7 with Ψ replaced by Ψ^* can be proved for the product domain the same way they are proved in Theorem 1.7. Condition (1.31) ensures that $J^*_c(t, x) < \infty$ for all $t > 0$ and $x \in U$. □

7.4 Some auxiliary results related to examples in Sect. 2.1

Lemma 7.3 *For $v > -1$, let $J_v(\cdot)$ be the Bessel function of first kind and of order v and let z_0 be any positive zero of $J_v(\cdot)$. Then,*

$$\sup_{r \in (0,1)} \frac{r^{-v} J_v(rz_0)}{1-r} < \infty.$$

Proof Set $f(r) = \frac{r^{-v} J_v(rz_0)}{1-r}$, which is a continuous function on $(0, 1)$. By Eq. 10.7.3 of [34], we see that $\lim_{r \to 0} f(r) = \Gamma(v)^{-1} \left(\frac{z_0}{2}\right)^v < \infty$. Because all zeros of $J_v(\cdot)$ are simple (see Section 10.21 *ibid*), we see that $\lim_{r \to 1} f(r) = -z_0 J_v'(z_0) < \infty$. Therefore, $f(r)$ is a continuous function on $[0, 1]$, from which the desired result follows. □

Remark 7.4 In the setting of Example 2.2, the leading eigenfunction is given by

$$\Phi_1(x) = \frac{1}{C_d} |x|^{(2-d)/2} J_{(d-2)/2} (z_0|x|),$$

where z_0 is the first positive zero of the Bessel function $J_{(d-2)/2}(x)$, and C_d is some normalization constant. One may refer to §34.2 in Chapter III of [42] for the case $d = 2$ and Section H in Chapter 2 of [21] for the general case $d \geq 3$. In particular, the leading eigenfunction $\Phi_1(x)$ corresponds to $F_k^{lm}(x)$ in Theorem 2.66 (*ibid.*) with $k = 0, l = 1$ and $m = 1$. Here are a few comments:

(1) The multiplicity for the leading eigenvalue z_0 is one since $d_k = 1$; see Corollary 2.55 (*ibid.*).
(2) $Y_0^1(x)$ in Theorem 2.66 (*ibid.*) is a constant function.
(3) In Fig. 2(2.2), we have chosen the following normalization constant:

$$C_d = \lim_{r \to 0} r^{(2-d)/2} J_{(n-2)/2}(z_0 r) = 2^{-d/2} d \, \Gamma (1 + d/2)^{-1} z_0^{(d-2)/2}, \quad (7.8)$$

where the limit is due to [34, Eq. 10.7.3 on p. 223], so that $\max_{|x| \leq 1} \Psi(1, x) = 1$.

The following Lemma is used in Example 2.3.

Lemma 7.5 R_1 and R_2 are two simple zeros for $Z(r)$ defined in Example 2.3.

Proof It is straightforward to check that R_1 and R_2 are two zeros of $Z(r)$. To show that they are simple, one needs to prove that $Z'(r) \neq 0$ for $r = R_1$ and R_2. By Eq. 10.6.3 of [34], we see that

$$z_0^{-1} Z'(r) = J_0(R_1 z_0) Y_0'(rz_0) - J_0'(rz_0) Y_0(R_1 z_0)$$
$$= -J_0(R_1 z_0) Y_1(rz_0) + J_1(rz_0) Y_0(R_1 z_0).$$

Table 3 Three cases in the proof of Lemma 7.5

(a) Case 1.	J_0	J_1	Y_0	Y_1	(b) Case 2.	J_0	J_1	Y_0	Y_1	(c) Case 3.	J_0	J_1	Y_0	Y_1
$R_1 z_0$	$\neq 0$	—	$= 0$	$\neq 0$	$R_1 z_0$	$= 0$	$\neq 0$	$\neq 0$	—	$R_1 z_0$	$\neq 0$	—	$\neq 0$	—
$R_2 z_0$	$\neq 0$	—	$= 0$	$\neq 0$	$R_2 z_0$	$= 0$	$\neq 0$	$\neq 0$	—	$R_2 z_0$	$\neq 0$	—	$\neq 0$	—

By setting $r = R_1$ and applying Eq. 10.5.2 (*ibid.*), we have that

$$z_0^{-1} Z'(R_1) = -J_0(R_1 z_0) Y_1(R_1 z_0) + J_1(R_1 z_0) Y_0(R_1 z_0) = \frac{2}{\pi R_1 z_0} \neq 0,$$

which proves that R_1 is a simple zero of $Z(r)$.

Let $F(z)$ denote the function in (2.7). By setting $r = R_2$, we have

$$z_0^{-1} Z'(R_2) = -J_0(R_1 z_0) Y_1(R_2 z_0) + J_1(R_2 z_0) Y_0(R_1 z_0). \tag{7.9}$$

Because $J_0^2(z) + Y_0^2(z) > 0$ for all $z > 0$ (see, e.g., Eq. 10.9.30 *ibid*), we see that if $Y_0(R_1 z_0) = 0$, then $J_0(R_1 z_0) \neq 0$. Because all zeros of $Y_\nu(\cdot)$ are simple, we see that $Y_1(R_1 z_0) = -Y_0'(R_1 z_0) \neq 0$. But since z_0 is a zero of $F(z)$, i.e.,

$$0 = J_0(R_1 z_0) Y_0(R_2 z_0) - J_0(R_2 z_0) Y_0(R_1 z_0) = J_0(R_1 z_0) Y_0(R_2 z_0).$$

Hence $Y_0(R_2 z_0) = 0$, which further implies that both $J_0(R_2 z_0) \neq 0$ and $Y_1(R_2 z_0) = -Y_0'(R_2 z_0) \neq 0$. This proves Case 1 in Table 3a. Applying the same arguments with $Y_0(R_1 z_0) = 0$ replaced by $Y_0(R_2 z_0) = 0$, $J_0(R_1 z_0) = 0$, and $J_0(R_2 z_0) = 0$, we see that only three cases can happen, which are listed in the following Table 3:

Cases 1 and 2: From the expression of $Z'(R_2)$ in (7.9), we see that in these two cases, $Z'(R_2) \neq 0$.

Case 3: Since both $Y_0(R_1 z_0)$ and $Y_0(R_2 z_0)$ are nonzero, and since $F(z_0) = 0$, we see that

$$J_0(R_1 z_0) = \frac{Y_0(R_1 z_0) J_0(R_2 z_0)}{Y_0(R_2 z_0)}.$$

This and Eq. 10.5.2 (*ibid.*) imply that

$$z_0^{-1} Z'(R_2) = \frac{Y_0(R_1 z_0)}{Y_0(R_2 z_0)} \left(-J_0(R_2 z_0) Y_1(R_2 z_0) + J_1(R_2 z_0) Y_0(R_2 z_0) \right)$$
$$= \frac{Y_0(R_1 z_0)}{Y_0(R_2 z_0)} \frac{2}{\pi R_2 z_0} \neq 0.$$

Therefore, combining the above three cases, we have proved that R_2 is a simple zero of $Z(r)$. □

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