NONLINEAR STOCHASTIC TIME-FRACTIONAL DIFFUSION EQUATIONS ON ℝ: MOMENTS, HÖLDER REGULARITY AND INTERMITTENCY

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ABSTRACT. We study the nonlinear stochastic time-fractional diffusion equations in the spatial domain ℝ, driven by multiplicative space-time white noise. The fractional index β varies continuously from 0 to 2. The case $\beta = 1$ (resp. $\beta = 2$) corresponds to the stochastic heat (resp. wave) equation. The cases $\beta \in\ ]0, 1[$ and $\beta \in\ ]1, 2[$ are called *slow diffusion equations* and *fast diffusion equations*, respectively. Existence and uniqueness of random field solutions with measure-valued initial data, such as the Dirac delta measure, are established. Upper bounds on all p-th moments ($p \geq 2$) are obtained, which are expressed using a kernel function $\mathcal{K}(t, x)$. The second moment is sharp. We obtain the Hölder continuity of the solution for the slow diffusion equations when the initial data is a bounded function. We prove the weak intermittency for both slow and fast diffusion equations. In this study, we introduce a special function, the *two-parameter Mainardi functions*, which are generalizations of the one-parameter Mainardi functions.

1. INTRODUCTION

Viscoelasticity is the property of materials that exhibit both viscous and elastic characteristics when undergoing deformation (see e.g. [15, 21, 29]). Viscosity mainly refers to fluids and elasticity to solids. A *linear* theory to bring these two properties together has been achieved using fractional calculus by Mainardi and his coauthors; see [23] for an introduction to this subject. It has wide applications to fields such as chemistry (e.g. [16, 17]), seismology (e.g. [1]), soil mechanics (e.g. [20]), arterial rheology ([12]), biological tissues (e.g. [22]), etc. In this linear model, the system is governed by the partial differential operator

$$\mathcal{L} =\ _xD_\delta^a -\ _tD_*^\beta,$$

where the space derivative $_xD_\delta^a$ is the Riesz-Feller fractional derivative of order a and skewness δ, and the time derivative $_tD_*^\beta$ is a *Caputo derivative* of order $\beta \in\ ]0, 2]$. These three parameters vary in the following ranges:

$$a \in\ ]0, 2]\,, \quad \beta \in\ ]0, 2]\,, \quad |\delta| \leq a \wedge (2 - a)\,,$$

where $a \wedge b := \min(a, b)$. We are interested in this linear model driven by multiplicative space-time white noise:

$$\mathcal{L}u(t, x) = I_t^{\lceil \beta \rceil - \beta} \left[\rho(u(t, x)) \dot{W}(t, x) \right], \quad t \in \mathbb{R}_+^* :=]0, +\infty[, \ x \in \mathbb{R},$$

where $\lceil \beta \rceil$ is the smallest integer not less than β, $\dot{W}$ is the space-time white noise, the function $\rho : \mathbb{R} \mapsto \mathbb{R}$ is Lipschitz continuous, and I_t^α is the *Riemann-Liouville fractional integral* of order α:

$$I_t^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} f(s) \mathrm{d}s, \quad \text{for } t > 0 \text{ and } \alpha > 0,$$

and $I_t^0 = \mathrm{Id}$, where Id denotes the identity operator. When β is an integer, then $I_t^{\lceil \beta \rceil - \beta} = \mathrm{Id}$. The case where $\rho(u) = \lambda u$, $\beta = 1$, and $a = 2$ (hence $\delta = 0$), is called the *parabolic Anderson model*; see [2,3]. The logarithm of the solution gives the *Hopf-Cole solution* to the famous *Kardar-Parisi-Zhang equation* [19].

Due to the time-fractional derivative, the semigroup theory does not work except for the case $\beta = 1$. These studies heavily depend on the properties of the fundamental solutions or the Green functions to $\mathcal{L}u = 0$. In [24], some of these Green functions are obtained through inverse Fourier transform of some special functions, among which the following three cases are more tractable: (1) space-fractional heat equation: $\{0 < a \leq 2, \ \beta = 1\}$; (2) time-fractional heat/wave equation: $\{a = 2, 0 < \beta \leq 2\}$; (3) neutral fractional diffusion equation: $\{0 < a = \beta \leq 2\}$. The first case has been recently studied in [8,9]. In this paper, we will study the second case, i.e., we will study the following nonlinear stochastic time-fractional diffusion equations (formally):

(1.1)
$$\left({}_t D_*^\beta - \frac{\partial^2}{\partial x^2} \right) u(t, x) = I_t^{\lceil \beta \rceil - \beta} \left[\rho\left(u(t, x)\right) \dot{W}(t, x) \right], \quad \beta \in]0, 2], \ t \in \mathbb{R}_+^*, \ x \in \mathbb{R}.$$

The Caputo fractional differential operator ${}_t D_*^\beta$ is defined as

$${}_t D_*^\beta f(t) := \begin{cases} \dfrac{1}{\Gamma(m - \beta)} \displaystyle\int_0^t \mathrm{d}\tau \, \dfrac{f^{(m)}(\tau)}{(t - \tau)^{\beta + 1 - m}} & \text{if } m - 1 < \beta < m, \\[12pt] \dfrac{\mathrm{d}^m}{\mathrm{d}t^m} f(t) & \text{if } \beta = m. \end{cases}$$

We refer to [14,24] for more details of these fractional differential operators. When $\beta = 2$, ${}_t D_*^\beta = \frac{\partial^2}{\partial t^2}$ and (1.1) reduces to the *stochastic wave equation* (SWE)

(1.2)
$$\left(\frac{\partial^2}{\partial t^2} - \kappa^2 \frac{\partial^2}{\partial x^2} \right) u(t, x) = \rho(u(t, x)) \dot{W}(t, x),$$

with the speed of wave propagation $\kappa = 1$. When $\beta = 1$, ${}_t D_*^\beta = \frac{\partial}{\partial t}$ and (1.1) reduces to the *stochastic heat equation* (SHE)

(1.3)
$$\left(\frac{\partial}{\partial t} - \frac{\nu}{2} \frac{\partial^2}{\partial x^2} \right) u(t, x) = \rho(u(t, x)) \dot{W}(t, x),$$

with the diffusion parameter $\nu = 2$. The above two special cases have been studied carefully; see [4–7,10]. The case $\beta \in]0, 1]$ is called the *slow diffusion*, $\beta \in]1, 2]$ the *fast diffusion*, and $\beta = 1$ the *standard diffusion*. In the following we will also call the case $\beta \in]0, 1[$ slow diffusion and the case $\beta \in]1, 2[$ fast diffusion. For the slow

and standard diffusions, we only need to specify the initial data $u(0, x)$. For the fast diffusion, we need to give $\frac{\partial}{\partial t} u(0, x)$ as well. Note that another related equation is the *stochastic fractional heat equation* (SFHE)

$$(1.4) \qquad \left(\frac{\partial}{\partial t} - {}_x D_\delta^a \right) u(t, x) = \rho(u(t, x)) \dot{W}(t, x) ,$$

which has been studied recently in [8, 9]; see also [13, 18].

All investigations on SPDEs of the above kinds require a good study of the corresponding Green functions. By Green functions, we mean the solutions to the equations

$$(1.5) \qquad \begin{cases} \left({}_t D_*^\beta - \dfrac{\partial^2}{\partial x^2} \right) u(t, x) = 0, & t \in \mathbb{R}_+^*, \; x \in \mathbb{R} , \\[2mm] u(0, x) = \delta_0(x) , & x \in \mathbb{R} , \quad \text{if } 0 < \beta \leq 1 , \\[2mm] u(0, x) = 0 , \quad \dfrac{\partial}{\partial t} u(0, x) = \delta_0(x) , & x \in \mathbb{R} , \quad \text{if } 1 < \beta \leq 2 , \end{cases}$$

where δ_0 is the Dirac delta function with a unit mass at zero. We use $G_\beta(t, x)$ to denote these Green functions. The Green functions for slow diffusion equations and their properties can be found in [24]. As far as we know, there is no literature studying the Green functions of the fast diffusion equations. Note that in [24], the Green functions for the fast diffusions are not the ones we need. To obtain the Green functions for the fast diffusion equations, one needs to generalize the one-parameter *Mainardi function* (see [23, 24]) to the two-parameter settings (see (4.4) below), based on which corresponding properties for the Green functions of the fast diffusion equations need to be proved (see Lemma 4.1 below).

If we denote the solution to the homogeneous equation by $J_0(t, x)$ (see (2.3) below), then the rigorous meaning of (1.1), which is the actual equation that we are going to study, is the following stochastic integral equation:

$$(1.6) \qquad \begin{aligned} u(t, x) &= J_0(t, x) + I(t, x), \quad \text{where} \\ I(t, x) &= \iint_{[0,t] \times \mathbb{R}} G_\beta(t - s, x - y) \, \rho(u(s, y)) \, W(\mathrm{d}s, \mathrm{d}y), \end{aligned}$$

where the stochastic integral is the Walsh integral [34]. To motivate the relation between the SPDE (1.1) and the integral equation (1.6), we need the *time-fractional Duhamel's principle* (see [33, Theorem 3.6]). If we replace the right-hand side of (1.1) by a nice forcing term $g(t, x)$. Then by [33, Theorem 3.6], the solution to (1.1) with vanishing initial conditions $\frac{\partial^m}{\partial t^m} u(0, x) = 0$ for $m = 0, \dots, \lceil \beta \rceil - 1$ is

$$u(t, x) = \int_0^t \mathrm{d}s \int_\mathbb{R} \mathrm{d}y \, G_\beta(t - s, x - y) \, {}_t D_+^{\lceil \beta \rceil - \beta} g(s, y),$$

where ${}_t D_+^\alpha$ for $\alpha \geq 0$ is the *Riemann-Liouville fractional derivatives* of order α:

$$ {}_t D_+^\alpha f(t) := \begin{cases} \dfrac{1}{\Gamma(m - \alpha)} \dfrac{\mathrm{d}^m}{\mathrm{d}t^m} \displaystyle\int_0^t \mathrm{d}\tau \, \dfrac{f(\tau)}{(t - \tau)^{\alpha + 1 - m}} & \text{if } m - 1 < \alpha < m , \\[4mm] \dfrac{\mathrm{d}^m}{\mathrm{d}t^m} f(t) & \text{if } \alpha = m . \end{cases}$$

Then if one replaces $g(t, x)$ by $I_t^{\lceil \beta \rceil - \beta} g(t, x)$ and uses the fact that ${}_t D_+^\alpha \circ I_t^\alpha = \mathrm{Id}$ for all $\alpha \geq 0$, then one can see that the solution to

$$\begin{cases} \left({}_t D_*^\beta - \dfrac{\partial^2}{\partial x^2} \right) u(t, x) = I_t^{\lceil \beta \rceil - \beta} \left[g(t, x) \right], & \beta \in \,]0, 2], \ t \in \mathbb{R}_+^*, \ x \in \mathbb{R}, \\[2mm] \dfrac{\partial^m}{\partial t^m} u(0, x) = 0, & \text{for } m = 0, \dots, \lceil \beta \rceil - 1, \end{cases}$$

is

$$u(t, x) = \int_0^t \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y \, G_\beta(t - s, x - y) \, g(s, y).$$

We will establish the existence of random field solutions to (1.6) starting from measure-valued initial conditions. Let μ be a Borel measure and $\mu = \mu_+ - \mu_-$, where, from the Jordan decomposition, $\mu_\pm$ are two nonnegative Borel measures with disjoint support and $|\mu| = \mu_+ + \mu_-$. Define an axillary function

(1.7) $$f_\eta(x) := \exp \left(-\frac{\eta}{2} |x|^{1 + \lfloor \beta \rfloor} \right), \quad \text{for } x \in \mathbb{R},$$

where $\lfloor \beta \rfloor$ is the largest integer not bigger than β. Let $\mathcal{M}(\mathbb{R})$ be the set of signed (regular) Borel measures on $\mathbb{R}$. For $0 < \beta < 2$, define

(1.8) $$\mathcal{M}_T^\beta(\mathbb{R}) := \left\{ \mu \in \mathcal{M}(\mathbb{R}) : \left(|\mu| * f_\eta \right)(x) < +\infty \, , \text{for all } \eta > 0 \text{ and } x \in \mathbb{R} \right\},$$

where $*$ denotes the convolution in the space variable. Then $\mathcal{M}_T^1(\mathbb{R}) = \mathcal{M}_H(\mathbb{R})$, where $\mathcal{M}_H(\mathbb{R})$ is the notation used in [5,6] for the admissible initial data for the SHE (1.3). Note that even though the initial data can be Schwartz distributions for the heat equation without noise, but for the SPDE, initial data cannot go beyond measures; see [4, Theorem 3.2.17] or [7, Theorem 2.22]. We will prove the existence and uniqueness of random field solutions to (1.1) for all initial data in $\mathcal{M}_T^\beta(\mathbb{R})$. As in [6–8], we will obtain similar moment formulas expressed using a special function $\mathcal{K}(t, x)$. For the SHE and the SWE, this kernel function $\mathcal{K}(t, x)$ has an explicit form. But for the space-fractional heat equations [8] and the current time-fractional diffusion equations, we only have some estimates on it. In particular, for the slow diffusion equations, we will obtain both upper and lower bounds on $\mathcal{K}(t, x)$. For the fast diffusion equations, we will only derive some upper bounds.

After establishing the existence of random field solutions, we will study some properties of the solutions. The first property is the sample-path regularity (for the slow diffusion equations). Given a subset $D \subseteq \mathbb{R}_+ \times \mathbb{R}$ and positive constants β_1, β_2, denote by $C_{\beta_1, \beta_2}(D)$ the set of functions $v : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ with the following property: for each compact set $K \subseteq D$, there is a finite constant C such that for all (t, x) and $(s, y) \in K$,

$$|v(t, x) - v(s, y)| \leq C \left(|t - s|^{\beta_1} + |x - y|^{\beta_2} \right).$$

Denote

$$C_{\beta_1-, \beta_2-}(D) := \bigcap_{\alpha_1 \in \,]0, \beta_1[} \bigcap_{\alpha_2 \in \,]0, \beta_2[} C_{\alpha_1, \alpha_2}(D).$$

We will show that for the slow diffusion equations, if the initial data is a bounded function, i.e., $\mu(\mathrm{d}x) = f(x) \mathrm{d}x$ with $f \in L^\infty(\mathbb{R})$, then

(1.9) $$u(\cdot, \circ) \in C_{\frac{2 - \beta}{4}-, \frac{1}{2}-} \left(\mathbb{R}_+ \times \mathbb{R} \right), \quad \text{a.s.}$$

Moreover, if f is bounded and α-Hölder continuous ($\alpha \in \,]0,1[\,$), then

$$(1.10) \quad u(\cdot, \circ) \in C_{\frac{2-\beta}{4}-,\frac{1}{2}-}\left(\mathbb{R}_+^* \times \mathbb{R}\right) \cap C_{\left(\frac{\alpha\beta}{2} \wedge \frac{2-\beta}{4}\right)-,\left(\alpha\beta \wedge \frac{1}{2}\right)-}\left(\mathbb{R}_+ \times \mathbb{R}\right), \quad \text{a.s.}$$

When $\beta = 1$, the above results partially recover the results for the stochastic heat equation in [5]. Note that the regularity results in [5] is more general since the initial data can be measured.

The second property that we are going to study is the intermittency. More precisely, define the *upper and lower (moment) Lyapunov exponents* as

$$(1.11)$$
$$\overline{m}_p(x) := \limsup_{t \to +\infty} \frac{\log \mathbb{E}\left[|u(t,x)|^p\right]}{t} \quad \text{and} \quad \underline{m}_p(x) := \liminf_{t \to +\infty} \frac{\log \mathbb{E}\left[|u(t,x)|^p\right]}{t}.$$

When the initial data are spatially homogeneous (i.e., the initial data are constants), the Lyapunov exponents do not depend on the spatial variable. In this case, a solution is called *fully intermittent* if $m_1 = 0$ and $\underline{m}_2 > 0$ (see [3, Definition III.1.1, on p. 55]). As for the weak intermittency, there are various definitions. For convenience of stating our results, we will call the solution *weakly intermittent of type I* if $\underline{m}_2 > 0$, and *weakly intermittent of type II* if $\overline{m}_2 > 0$. Clearly, the weak intermittency of type I is slightly stronger than the weak intermittency of type II, but weaker than the full intermittency by missing $m_1 = 0$. The weak intermittency of type II is used in [18].

The full intermittency for the SHE and the SFHE, and the weak intermittency of type I for SWE are established in [2], [9] and [7], respectively. Conus, *et al.* prove the weak intermittency of type II for the SWE in [11, Theorem 2.3]. We will establish the weak intermittency of type I for the slow diffusion equations and the weak intermittency of type II for the fast diffusion equations. Moreover, we show that

$$(1.12) \qquad\qquad \overline{m}_p \leq \begin{cases} C\, p^{\frac{4-\beta}{2-\beta}} & \text{if } \beta \in \,]0,1], \\ C\, p^{\frac{8-\beta}{6-\beta}} & \text{if } \beta \in \,]1,2[\,, \end{cases}$$

which reduces to the SHE case (see [2,6,18]) when $\beta = 1$, i.e., $\overline{m}_p \leq C\, p^3$, and to the SWE case (see [7]) when $\beta = 2$, i.e., $\overline{m}_p \leq C\, p^{3/2}$. Note that the above constants C may vary from one inequality to the other.

At the final stage of this work, we notice some recent works by Mijena and Nane [25, 26], who have also studied this equation in a more general setting where the Laplacian is replaced by $-(-\Delta)^{\alpha/2}$ and the space dimension can be any $d < \alpha(2 \wedge \beta^{-1})$. When $\beta \in \,]0,1[$, $\alpha = 2$ and $d = 1$, they obtain the same rate as in (1.12). The main differences of our work from [25, 26] include: (1) our initial data are more general (measures), which entails more calculations; (2) we cover the case $\beta \in \,]1,2[$, to which most efforts in Section 4 are contributed; (3) we derive both upper and lower moment bounds, which can be handy for proving many other results; (4) we prove the weak intermittency of type I for the slow diffusion equations, thanks to our *lower* bound on the second moment.

These studies are far from being conclusive. Many aspects can be improved, such as the Hölder regularity for measure-valued initial data and for fast diffusion equations, full intermittency for both slow and fast diffusion equations, the SPDE

(1.1) without the fractional integral operator, etc. Finally, one interesting question is whether the sample-path comparison principle holds for the slow diffusion equations; see the recent work [9] for the SFHE (1.4) and references therein.

This paper is structured as follows. We first introduce some notation in Section 2. The main results are stated in Section 3. In Section 4, we prove some useful properties of the Green functions. Section 5 gives a general framework on calculating the function $\mathcal{K}(t, x)$, based on which Theorem 3.4 is proved. The proof of the existence and uniqueness results with moment estimates, i.e., Theorem 3.1, is presented in Section 6.

2. Some preliminaries and notation

Recall that the Green functions $G_\beta(t, x)$ solve (1.5). Note that in [24], the fundamental solution is defined with the initial conditions $u(0, x) = \delta_0(x)$ and $\frac{\partial}{\partial t} u(0, x) = 0$ for all $\beta \in {]}0, 2]$. Let $G_\beta^*(t, x)$, which is also called the Green function, be the solution to (1.5) subject to the initial data

$$u(0, x) = \delta_0(x) \quad \text{and} \quad \frac{\partial}{\partial t} u(0, x) = 0.$$

Here are some special cases. If $\beta = 1$, then $G_\beta(t, x)$ reduces to the heat kernel function, i.e.,

$$(2.1) \qquad G_1(t, x) = \frac{1}{\sqrt{4t}} \exp\left(-\frac{x^2}{4t}\right), \quad \text{for } (t, x) \in \mathbb{R}_+ \times \mathbb{R}.$$

If $\beta = 2$, then $G_\beta(t, x)$ and $G_\beta^*(t, x)$ reduce to the heat kernel functions, i.e.,

$$(2.2) \qquad G_2(t, x) = \frac{1}{2} \mathbb{1}_{\{|x| \leq t\}} \quad \text{and} \quad G_2^*(t, x) = \frac{1}{2} \left(\delta_t(x) + \delta_{-t}(x)\right).$$

For μ and $\nu \in \mathcal{M}_T^\beta(\mathbb{R})$, the solution to the following homogeneous equation

$$\begin{cases} \left({}_t D_*^\beta - \dfrac{\partial^2}{\partial x^2}\right) u(t, x) = 0, & t \in \mathbb{R}_+^*, \ x \in \mathbb{R}, \\ u(0, \cdot) = \mu(\cdot), & \text{if } 0 < \beta \leq 1, \\ u(0, \cdot) = \mu(\cdot), \quad \dfrac{\partial}{\partial t} u(0, \cdot) = \nu(\cdot), & \text{if } 1 < \beta < 2, \end{cases}$$

will always be denoted by $J_0(t, x)$, which is equal to

$$(2.3) \quad J_0(t, x) := \begin{cases} \displaystyle\int_\mathbb{R} \mu(\mathrm{d}y) \, G_\beta(t, x - y), & \text{if } 0 < \beta \leq 1, \\ \displaystyle\int_\mathbb{R} \nu(\mathrm{d}y) \, G_\beta(t, x - y) + \int_\mathbb{R} \mu(\mathrm{d}y) \, G_\beta^*(t, x - y), & \text{if } 1 < \beta < 2. \end{cases}$$

Remark 2.1. For the slow diffusion equations ($0 < \beta \leq 1$), the Green function $G_\beta(t, x)$ is the same as the function $G_{\alpha,\beta}^\theta(x, t)$ in [24, Section 3] with $\alpha = 2$ and $\theta = 0$. For the fast diffusion equations ($1 < \beta < 2$), our function $G_\beta^*(t, x)$ corresponds to the function $G_{\alpha,\beta}^\theta(x, t)$ in [24, Section 3]. In these two cases, the Green functions $G_\beta(t, x)$ and $G_\beta^*(t, x)$, and their properties are mostly known; see [24] and [23, Appendix F]. However, for the fast diffusion equations, the Green function $G_\beta(t, x)$ and its properties need to be proved, which is done in Lemma 4.1 below.

Let $W = \{W_t(A) : A \in \mathcal{B}_b(\mathbb{R}), t \geq 0\}$ be a space-time white noise defined on a complete probability space $(\Omega, \mathcal{F}, P)$, where $\mathcal{B}_b(\mathbb{R})$ is the collection of Borel sets with finite Lebesgue measure. Let

$$\mathcal{F}_t = \sigma(W_s(A) : 0 \leq s \leq t, A \in \mathcal{B}_b(\mathbb{R})) \vee \mathcal{N}, \quad t \geq 0,$$

be the natural filtration augmented by the σ-field $\mathcal{N}$ generated by all P-null sets in $\mathcal{F}$. We use $\|\cdot\|_p$ to denote the $L^p(\Omega)$-norm $(p \geq 1)$. In this setup, W becomes a worthy martingale measure in the sense of Walsh [34], and $\iint_{[0,t] \times \mathbb{R}} X(s, y) W(\mathrm{d}s, \mathrm{d}y)$ is well-defined in this reference for a suitable class of random fields $\{X(s, y), (s, y) \in \mathbb{R}_+ \times \mathbb{R}\}$.

Recall that the rigorous meaning of the SPDE (1.1) is in the integral form (1.6).

Definition 2.2. A process $u = \big(u(t, x), (t, x) \in \mathbb{R}_+^* \times \mathbb{R}\big)$ is called a *random field solution* to (1.1) if

(1) u is adapted, i.e., for all $(t, x) \in \mathbb{R}_+^* \times \mathbb{R}$, $u(t, x)$ is $\mathcal{F}_t$-measurable;

(2) u is jointly measurable with respect to $\mathcal{B}(\mathbb{R}_+^* \times \mathbb{R}) \times \mathcal{F}$;

(3) $\big(G_\beta^2 \star \|\rho(u)\|_2^2\big)(t, x) < +\infty$ for all $(t, x) \in \mathbb{R}_+^* \times \mathbb{R}$, where $\star$ is the convolution in both space and time variables. Moreover, the function $(t, x) \mapsto I(t, x)$ mapping $\mathbb{R}_+^* \times \mathbb{R}$ into $L^2(\Omega)$ is continuous;

(4) u satisfies (1.6) a.s., for all $(t, x) \in \mathbb{R}_+^* \times \mathbb{R}$.

Assume that the function $\rho : \mathbb{R} \mapsto \mathbb{R}$ is globally Lipschitz continuous with Lipschitz constant $\mathrm{Lip}_\rho > 0$. We need some growth conditions on ρ: assume that for some constants $L_\rho > 0$ and $\overline{\varsigma} \geq 0$,

$$(2.4) \qquad |\rho(x)|^2 \leq L_\rho^2 \left(\overline{\varsigma}^2 + x^2\right), \qquad \text{for all } x \in \mathbb{R}.$$

Sometimes we need a lower bound on $\rho(x)$: assume that for some constants $l_\rho > 0$ and $\underline{\varsigma} \geq 0$,

$$(2.5) \qquad |\rho(x)|^2 \geq l_\rho^2 \left(\underline{\varsigma}^2 + x^2\right), \qquad \text{for all } x \in \mathbb{R}.$$

For all $(t, x) \in \mathbb{R}_+^* \times \mathbb{R}$, $n \in \mathbb{N}$ and $\lambda \in \mathbb{R}$, define

$$\mathcal{L}_0(t, x; \lambda) := \lambda^2 G_\beta^2(t, x),$$

$$(2.6) \qquad \mathcal{L}_n(t, x; \lambda) := (\mathcal{L}_0 \star \cdots \star \mathcal{L}_0)(t, x), \quad \text{for } n \geq 1 \ (n \text{ convolutions}),$$

$$(2.7) \qquad \mathcal{K}(t, x; \lambda) := \sum_{n=0}^\infty \mathcal{L}_n(t, x; \lambda).$$

We will use the following conventions to the kernel functions $\mathcal{K}(t, x; \lambda)$:

$$\mathcal{K}(t, x) := \mathcal{K}(t, x; \lambda), \qquad \overline{\mathcal{K}}(t, x) := \mathcal{K}(t, x; L_\rho),$$

$$\underline{\mathcal{K}}(t, x) := \mathcal{K}(t, x; l_\rho), \qquad \widehat{\mathcal{K}}_p(t, x) := \mathcal{K}(t, x; 4\sqrt{p}\, L_\rho), \quad \text{for } p \geq 2.$$

3. MAIN RESULTS

Our first theorem is about the existence, uniqueness and moment estimates of the solutions to (1.1). It possesses a general form as [6, Theorem 2.4], [7, Theorem 2.3], and [8, Theorem 3.1].

Theorem 3.1 (Existence, uniqueness and moments). *Suppose that*
  (i) $0 < \beta < 2$;
  (ii) *the function ρ is Lipschitz continuous and satisfies the growth condition* (2.4);
  (iii) *the initial data are such that $\mu \in \mathcal{M}_T^\beta(\mathbb{R})$ if $\beta \in \,]0,1]$, and μ, $\nu \in \mathcal{M}_T^\beta(\mathbb{R})$ if $\beta \in \,]1,2[$.*

Then the SPDE (1.1) *has a unique (in the sense of versions) random field solution $\{u(t,x) : (t,x) \in \mathbb{R}_+^* \times \mathbb{R}\}$. Moreover, the following statements are true:*
  (1) $(t,x) \mapsto u(t,x)$ *is $L^p(\Omega)$-continuous for all integers $p \geq 2$.*
  (2) *For all even integers $p \geq 2$, all $t > 0$ and $x, y \in \mathbb{R}$,*

$$(3.1) \qquad ||u(t,x)||_p^2 \leq \begin{cases} J_0^2(t,x) + \left(\left[\bar{\varsigma}^2 + J_0^2 \right] \star \overline{\mathcal{K}} \right)(t,x), & \text{if } p = 2, \\ 2J_0^2(t,x) + \left(\left[\bar{\varsigma}^2 + 2J_0^2 \right] \star \widehat{\mathcal{K}}_p \right)(t,x), & \text{if } p > 2. \end{cases}$$

  (3) *If ρ satisfies* (2.5), *then for all $t > 0$ and $x, y \in \mathbb{R}$,*

$$(3.2) \qquad ||u(t,x)||_2^2 \geq J_0^2(t,x) + \left(\left(\underline{\varsigma}^2 + J_0^2 \right) \star \underline{\mathcal{K}} \right)(t,x).$$

The following Theorem 3.2 gives the Hölder continuity of the solution for the slow diffusion equations. We cannot prove the Hölder regularity for the fast diffusion equations due to the less precise results in Proposition 6.4 than those in Proposition 6.3.

Theorem 3.2. *Suppose that $\beta \in \,]0,1]$. If $\mu(\mathrm{d}x) = f(x)\mathrm{d}x$ with $f \in L^\infty(\mathbb{R})$, then*

$$(3.3) \qquad \sup_{(t,x) \in [0,T] \times \mathbb{R}} ||u(t,x)||_p^2 < +\infty, \quad \text{for all } T \geq 0 \text{ and } p \geq 2.$$

Moreover, we have

$$(3.4) \qquad I(\cdot, \circ) \in C_{\frac{2-\beta}{4}-, \frac{1}{2}-}(\mathbb{R}_+ \times \mathbb{R}), \quad a.s.,$$

and (1.9) *holds. If f is bounded and α-Hölder continuous $(\alpha \in \,]0,1[)$, then* (1.10) *holds.*

Proof. The bound (3.3) is a simple consequence of (3.1). The proof of (3.4) is straightforward under (3.3) (see [5, Remark 4.6]). The remaining parts are due to Proposition 6.6. □

In only very few cases, one can derive explicit form for $\mathcal{K}(t,x)$. A first case is when $\beta = 1$; see Example 5.3. A second case is given in Example 5.4. A third case is when $\beta = 2$:

$$\mathcal{K}^{\mathrm{wave}}(t,x;\lambda) = \frac{\lambda^2}{4} I_0 \left(\sqrt{\frac{\lambda^2((\kappa t)^2 - x^2)}{2\kappa}} \right) 1_{\{|x| \leq \kappa t\}},$$

where $I_0(x)$ is the *modified Bessel function of the first kind* of order 0; see [7]. Hence, in order to use the moment bounds in (3.1) and (3.2), we need some good estimates on the kernel function $\mathcal{K}(t,x)$. For this purpose, we define some *reference kernel functions*:

$$(3.5) \qquad \mathcal{G}_\beta(t,x) := \begin{cases} \dfrac{1}{2\,t^{\beta/2}} \exp\left(-\dfrac{|x|}{t^{\beta/2}} \right) & \text{if } 0 < \beta < 1, \\[3mm] \dfrac{1}{\sqrt{4\pi t^\beta}} \exp\left(-\dfrac{x^2}{4t^\beta} \right) & \text{if } 1 \leq \beta < 2. \end{cases}$$

Note that when $1 \le \beta < 2$, $\mathcal{G}_\beta(t,x) = G_1\left(t^\beta, x\right)$. For convenience, when $0 < \beta < 1$, denote

$$(3.6) \qquad \mathcal{G}_{e,\beta}(t,x) := \mathcal{G}_\beta(t,x),$$

where the subscription "e" refers to the exponential function. Clearly, $\mathcal{G}_\beta(t,x)$ is nonnegative and $\int_\mathbb{R} dx\, \mathcal{G}_\beta(t,x) = 1$. For $0 < \beta < 1$, define

$$(3.7) \qquad \mathcal{G}_\beta(t,x) := G_1\left(t^\beta, x\right) = \frac{1}{\sqrt{4\pi t^\beta}} \exp\left(-\frac{x^2}{4\,t^\beta}\right).$$

We need some constants:

$$(3.8) \qquad \widehat{C}_\beta := \frac{2^{\beta/2}}{2^{\beta/2} - 1} \exp\left(-\frac{1}{2^{\beta/2}}\right), \quad \text{for } \beta > 0,$$

and

$$(3.9) \qquad \widetilde{C}_\beta := \begin{cases} \widehat{C}_\beta & \text{if } 0 < \beta < 1, \\ 2^{\frac{\beta-1}{2}} & \text{if } 1 \le \beta < 2. \end{cases}$$

Remark 3.3. The constant $\widehat{C}_\beta$ as a function of $\beta \in \,]0,2]$ is decreasing with $\widehat{C}_2 = 2e^{-1/2} \approx 1.21306$, $\widehat{C}_1 = \left(2 + \sqrt{2}\right) e^{-\frac{1}{\sqrt{2}}} \approx 1.68344$, and $\lim_{\beta \to 0_+} \widehat{C}_\beta = \infty$.

Define

$$(3.10) \qquad \Psi_\beta := \sup_{x \in \mathbb{R}} \frac{G_\beta^2(1,x)}{\mathcal{G}_\beta(1,x)} < +\infty, \quad \text{for } 0 < \beta < 2,$$

and

$$(3.11) \qquad \underline{\Psi}_\beta := \inf_{x \in \mathbb{R}} \frac{G_\beta^2(1,x)}{\mathcal{G}_\beta(1,x)} > 0, \quad \text{for } 0 < \beta < 1.$$

Proposition 5.8 below shows that $\Psi_\beta < +\infty$ and $\underline{\Psi}_\beta > 0$.

Theorem 3.4. *Fix $\lambda > 0$.*
    (1) *For $\beta \in \,]0,2[\,$, there is a finite constant $C := C(\beta, \lambda)$ such that*

$$(3.12) \qquad \mathcal{K}(t,x;\lambda) \le \frac{C}{t^\sigma}\, \mathcal{G}_\beta(t,x)\,(1 + t^\sigma \exp\left(\Upsilon t\right)),$$

where

$$(3.13) \qquad \sigma = \beta/2 + 2(1 - \lceil \beta \rceil) \quad and \quad \Upsilon = \left(\lambda^2\, \Psi_\beta\, \widetilde{C}_\beta\, \Gamma(1 - \sigma)\right)^{\frac{1}{1-\sigma}}.$$

    (2) *For $\beta \in \,]0,1[\,$, there is a constant $\underline{C} := \underline{C}(\beta, \lambda) > 0$ such that*

$$(3.14) \qquad \mathcal{K}(t,x;\lambda) \ge \underline{C}\, \mathcal{G}_\beta(t,x) \exp\left(\underline{\Upsilon}\, t\right),$$

where

$$\underline{\sigma} = \beta/2 - 1 \quad and \quad \underline{\Upsilon} = \left(2^{-1/2}\, \lambda^2\, \underline{\Psi}_\beta\, \Gamma(1 - \underline{\sigma})\right)^{\frac{1}{1-\underline{\sigma}}}.$$

Proof. Apply Proposition 5.8 below with $\lambda G_\beta(t,x)$. Note that introducing the factor λ changes the constants C_0 and $\underline{C}_0$ by a factor λ^2. $\qquad \square$

The last set of results are the weak intermittency and the bounds in (1.12).

Theorem 3.5 (Weak intermittency of type I for slow diffusion equations). *Suppose that $\beta \in {]}0,1{[}$ and $\mu(\mathrm{d}x) = c\,\mathrm{d}x$. If ρ satisfies (2.4) and $|c| + |\overline{\varsigma}| \neq 0$, then*

$$\overline{m}_p \leq \frac{1}{2}\left[2^4\, L_\rho^2\, \widehat{C}_\beta \Psi_\beta\, \Gamma(1-\beta/2)\right]^{\frac{2}{2-\beta}}\, p^{\frac{4-\beta}{2-\beta}}, \quad \text{for all } p \geq 2 \text{ even.}$$

If ρ satisfies (2.5) and $|c| + |\underline{\varsigma}| \neq 0$, then the solution is weakly intermittent of type I:

$$\underline{m}_p \geq \frac{p}{2}\left(2^{-1/2}\, l_\rho^2\, \underline{\Psi}_\beta\, \Gamma(2-\beta/2)\right)^{\frac{2}{4-\beta}}, \quad \text{for all } p \geq 2.$$

Proof. Clearly, in this case, $J_0(t,x) = c$. Hence, by (3.1) and (3.12),

$$\|u(t,x)\|_p^2 \leq c^2 + \frac{C}{t^\sigma}\left(\overline{\varsigma}^2 + 2c^2\right)\left(1 + t^\sigma \exp\left(\left[2^4\, L_\rho^2\, \widehat{C}_\beta\Psi_\beta\, \Gamma(1-\sigma)\right]^{\frac{1}{1-\sigma}} p^{\frac{1}{1-\sigma}}\, t\right)\right),$$

with $\sigma = \beta/2$. Then increase the power by a factor $p/2$. This proves the upper bounds. As for the lower bound, by (3.2) and (3.14),

$$\|u(t,x)\|_p^2 \geq \|u(t,x)\|_2^2 \geq c^2 + C\left(\underline{\varsigma}^2 + c^2\right)\exp\left(\left[2^{-1/2}\, l_\rho^2\, \underline{\Psi}_\beta\, \Gamma(1-\underline{\sigma})\right]^{\frac{1}{1-\underline{\sigma}}} t\right)$$

with $\underline{\sigma} = \frac{\beta}{2} - 1$. This completes the proof.   □

Theorem 3.6 (Weak intermittency of type II for fast diffusion equations). *Suppose that $\beta \in {]}1,2{[}$, $\mu(\mathrm{d}x) = c\mathrm{d}x$ and $\nu(\mathrm{d}x) = c'\mathrm{d}x$. If ρ satisfies (2.4) and $|c| + |c'| + |\overline{\varsigma}| \neq 0$, then*

$$\overline{m}_p \leq \frac{1}{2}\left[2^{9/2}\, L_\rho^2\, \Psi_\beta\, \Gamma(3-\beta/2)\right]^{\frac{2}{6-\beta}}\, p^{\frac{8-\beta}{6-\beta}}, \quad \text{for all } p \geq 2 \text{ even.}$$

Proof. By Lemma 4.1(iii), $J_0(t,x) = c\,t + c'$. The condition $|c| + |c'| \neq 0$ implies $J_0(t,x) \neq 0$ for large t. Hence, by (3.1) and (3.12),

$$\|u(t,x)\|_p^2 \leq (c\,t + c')^2 + \frac{C}{t^\sigma}\left(\overline{\varsigma}^2 + 2(c\,t + c')^2\right)$$

$$\times \left(1 + t^\sigma \exp\left(\left[2^4\widetilde{C}_\beta\, L_\rho^2\, \Psi_\beta\, \Gamma(1-\sigma)\right]^{\frac{1}{1-\sigma}} p^{\frac{1}{1-\sigma}}\, t\right)\right)$$

with $\sigma = \frac{\beta}{2} - 2$. Then increase the power by a factor $p/2$ and use the fact that $\widetilde{C}_\beta \leq \sqrt{2}$.   □

4. Some properties of the Green functions

We need some special functions. The following two-parameter *Mittag-Leffler function*

$$(4.1) \qquad E_{\alpha,\beta}(z) := \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k + \beta)}, \qquad \alpha > 0,\ \beta > 0,$$

is a generalization of exponential function, $E_{1,1}(z) = e^z$; see, e.g., [30, Section 1.2]. Another special case[1] is

$$(4.2) \qquad E_{1/2,1/2}(x) = \frac{1}{\sqrt{\pi}} + x\, e^{x^2}\mathrm{erfc}(-x), \quad \text{for } x \geq 0,$$

---

[1] *Proof of (4.2).* By [27, 41:6:6], $\frac{1}{\sqrt{\pi x^2}} + e^{x^2}\mathrm{erfc}(-x) = \frac{1}{x^2}\sum_{n=1}^\infty \frac{x^n}{\Gamma(n/2)}$. Then apply (5.10).□

where $\mathrm{erf}(x) = \frac{2}{\sqrt{x}} \int_0^x \mathrm{d}y\, e^{-y^2}$ is the *error function* and $\mathrm{erfc}(x) = 1 - \mathrm{erf}(x)$ is the *complementary error function*. We will use the convention that $E_\alpha(z) = E_{\alpha,1}(z)$. A function is called *completely monotonic* if $(-1)^n f^{(n)}(x) \geq 0$ for $n = 0, 1, 2, \dots$; see [35, Definition 4, p. 108]. An important fact [32] that we are going to use is that

(4.3) $x \in \mathbb{R}_+ \mapsto E_{\alpha,\beta}(-x)$ is completely monotonic $\iff$ $0 < \alpha \leq 1 \wedge \beta$.

Let $W_{\lambda,\mu}(z)$ be the *two-parameter Wright function* of order λ defined as follows:

$$W_{\lambda,\mu}(z) := \sum_{n=0}^\infty \frac{z^n}{n!\,\Gamma(\lambda n + \mu)}\,, \qquad \text{for } \lambda > -1,\ \mu \in \mathbb{C} \text{ and } z \in \mathbb{C}\,;$$

see, e.g., [23, Appendix F] and references therein. We define the *two-parameter Mainardi functions* of order $\lambda \in [0, 1[$ by

(4.4)

$$M_{\lambda,\mu}(z) := W_{-\lambda,\mu-\lambda}(-z) = \sum_{n=0}^\infty \frac{(-z)^n}{n!\,\Gamma(\mu - (n+1)\lambda)}\,, \qquad \text{for } \mu \in \mathbb{C} \text{ and } z \in \mathbb{C}\,,$$

and we will use the convention that $M_\lambda(z) = M_{\lambda,1}(z)$. In particular, $M_{1/2}(z) = \frac{1}{\sqrt{\pi}} \exp(-z^2/4)$. The one-parameter Mainardi functions $M_\lambda(z)$ are used by Mainardi et al. in [23, 24]. This two-parameter extension is necessary for the Green function $G_\beta(t, x)$ of the fast diffusions.

Lemma 4.1 (Properties of the Green functions $G_\beta(t,x)$ and $G_\beta^*(t,x)$). *For $\beta \in\ ]0, 2[$, the following properties hold:*

(i) *The Green function $G_\beta(t, x)$ has the following explicit form:*

(4.5)

$$G_\beta(t, x) = \frac{t^{\lceil \beta \rceil - 1 - \beta/2}}{2}\, M_{\beta/2, \lceil \beta \rceil}\left(\frac{|x|}{t^{\beta/2}}\right) = \begin{cases} \dfrac{t^{-\beta/2}}{2} M_{\beta/2}\left(\dfrac{|x|}{t^{\beta/2}}\right)\,, & \text{if } 0 < \beta \leq 1, \\[3mm] \dfrac{t^{1-\beta/2}}{2} M_{\beta/2, 2}\left(\dfrac{|x|}{t^{\beta/2}}\right)\,, & \text{if } 1 < \beta < 2. \end{cases}$$

The function $G_\beta^*(t, x)$ has the same form as (4.5) except that all $\lceil \beta \rceil$'s in (4.5) should be replaced by 1, i.e.,

$$G_\beta^*(t, x) = \frac{t^{-\beta/2}}{2} M_{\beta/2}\left(\frac{|x|}{t^{\beta/2}}\right), \quad \text{for } 1 < \beta < 2.$$

(ii) *$G_\beta(t, x)$ has the following scaling property:*

(4.6) $$G_\beta(t, x) = t^{\lceil \beta \rceil - 1 - \beta/2} G_\beta\left(1, \frac{x}{t^{\beta/2}}\right).$$

The scaling property of $G_\beta^*(t, x)$ is the same as (4.6) except that the $\lceil \beta \rceil$ in (4.6) should be replaced by 1.

(iii) *For any $t > 0$ fixed, both functions $x \mapsto G_\beta(t, x)$ and $x \mapsto G_\beta^*(t, x)$ are symmetric and nonnegative, i.e., $G_\beta(t, x) = G_\beta(t, -x) \geq 0$ and $G_\beta^*(t, x) = G_\beta^*(t, -x) \geq 0$, for all $x \in \mathbb{R}$. Moreover,*

(4.7) $$\int_\mathbb{R} \mathrm{d}x\, G_\beta(t, x) = t^{\lceil \beta \rceil - 1} \quad and \quad \int_\mathbb{R} \mathrm{d}x\, G_\beta^*(t, x) = 1\,.$$

In particular, the functions $x \mapsto G_\beta(t,x)$ *with* $\beta \in {]0,1]}$ *and* $x \mapsto G_\beta^*(t,x)$ *are probability densities.*

(iv) $G_\beta(1,x)$ *has the following asymptotic property:*

$$(4.8) \qquad G_\beta(1,x) = \frac{1}{2} M_{\beta/2,\lceil\beta\rceil}\left(|x|\right) \approx A\,|x|^a\,e^{-b|x|^c}, \quad as\ |x| \to +\infty\,,$$

where

$$(4.9) \qquad A = \pi^{-1/2} 2^{\frac{2\lceil\beta\rceil-3}{2-\beta}} (2-\beta)^{\frac{\beta-2\lceil\beta\rceil}{2}} \beta^{\frac{\beta(1+\beta-2\lceil\beta\rceil)}{2(2-\beta)}} \geq \frac{1}{\sqrt{8\pi}}\,,$$

$$(4.10) \qquad a = \frac{1+\beta-2\lceil\beta\rceil}{2-\beta} \leq 0\,,$$

$$(4.11) \qquad c = \frac{2}{2-\beta} > 1\,,$$

$$(4.12) \qquad b = (2-\beta)\,2^{-2/(2-\beta)} \beta^{\beta/(2-\beta)} \in {]0,1[}\,.$$

$G_\beta^*(1,x)$ *has the same asymptotic property except that all* $\lceil\beta\rceil$'s *in* (4.8), (4.9) *and* (4.10) *should be replaced by 1, the range of* a *is* $\ ]-1/2,+\infty[$ *and the range of* A *is* $[1/\sqrt{8e\pi},1/2[$. *See Figure 1 for the plots of these parameters as functions of* β.

(v) $G_\beta(t,x)$ *satisfies the following moment formula:*

$$(4.13) \quad \int_{\mathbb{R}} \mathrm{d}x\,|x|^a G_\beta(t,x) = \frac{\Gamma(a+1)}{\Gamma\left(\frac{a\beta}{2}+\lceil\beta\rceil\right)}\,t^{a\beta/2+\lceil\beta\rceil-1}\,, \quad for\ a > -1\ and\ t \geq 0\,.$$

The moment formula for $G_\beta^*(t,x)$ *is the same as* (4.13) *except that all* $\lceil\beta\rceil$'s *should be replace d by 1.*

(vi) *The Fourier transform of the Green function* $G_\beta(t,x)$ *is*

$$(4.14) \qquad \int_{\mathbb{R}} \mathrm{d}x\,e^{-i\xi x} G_\beta(t,x) = t^{\lceil\beta\rceil-1}\,E_{\beta,\lceil\beta\rceil}\left(-t^\beta\xi^2\right)\,, \quad for\ t > 0\ and\ \xi \in \mathbb{R}\,.$$

The Fourier transform of $G_\beta^*(t,x)$ *is the same as* (4.14) *except that all* $\lceil\beta\rceil$'s *in* (4.14) *should be replaced by 1.*

(vii) *the Laplace transform of the function* $\mathbb{R}_+ \ni x \mapsto G_\beta(1,x)$ *is*

$$(4.15) \qquad \int_0^\infty \mathrm{d}x\,e^{-zx} G_\beta(1,x) = \frac{1}{2} E_{\beta/2,\lceil\beta\rceil}(-z)\,, \quad for\ all\ z \in \mathbb{C}\,.$$

The Laplace transform of $\mathbb{R}_+ \ni x \mapsto G_\beta^*(1,x)$ *is the same as* (4.15) *except that the* $\lceil\beta\rceil$ *in* (4.15) *should be replaced by 1.*

(viii) *The function* $x \mapsto G_\beta(t,x)$ *attains its maximum value at* $x=0$:

$$(4.16) \qquad \sup_{x\in\mathbb{R}} G_\beta(t,x) = G_\beta(t,0) = \frac{t^{\lceil\beta\rceil-1-\beta/2}}{2}\Gamma\left(\lceil\beta\rceil-\frac{\beta}{2}\right)^{-1}\,.$$

The function $x \mapsto G_\beta^*(t,x)$ *attains two symmetric maximums that move apart from the origin with time.*

(ix) *The function* $x \mapsto G_\beta(t,x)$ *is continuous at* $x=0$ *but in general not differentiable there. Its* n-*th derivatives are equal to*

$$(4.17) \quad \frac{\partial^n}{\partial x^n} G_\beta(t,x) = \begin{cases} \dfrac{(-1)^n t^{\lceil\beta\rceil-1-(n+1)\beta/2}}{2} M_{\beta/2,\lceil\beta\rceil-n\beta/2}\left(\dfrac{x}{t^{\beta/2}}\right) & if\ x > 0\,, \\[2mm] \dfrac{t^{\lceil\beta\rceil-1-(n+1)\beta/2}}{2} M_{\beta/2,\lceil\beta\rceil-n\beta/2}\left(-\dfrac{x}{t^{\beta/2}}\right) & if\ x < 0\,. \end{cases}$$

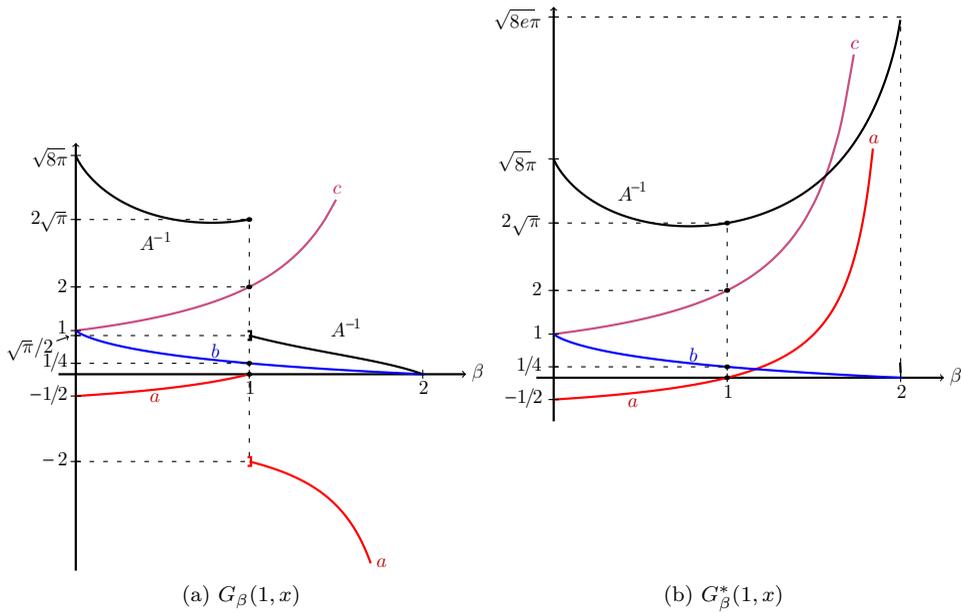(a) $G_\beta(1,x)$ (b) $G^*_\beta(1,x)$

Figure 1. The parameters of the asymptotics of the functions $G_\beta(1,x)$ and $G^*_\beta(1,x)$.

Proof. Denote

$$G^\star_\beta(t,x) := \begin{cases} G_\beta(t,x) & \text{if } \beta \in \,]0,1], \\ G^*_\beta(t,x) & \text{if } \beta \in \,]1,2[. \end{cases}$$

All these properties for $G^\star_\beta(t,x)$ can be found in [24] and [23, Appendix F]. The expression (4.5) for $G^\star_\beta(t,x)$ can be found in [24, (4.23)]. The scaling property (4.6) for $G^\star_\beta(t,x)$ can be found in [24, (3.7)]. The asymptotic property of $G^\star_\beta(t,x)$ can be found in [24, (4.29), (4.30)]. The moment formula (4.13) for $G^\star_\beta(t,x)$ can be found in [24, (4.31)], where one can extend integer n to all $a > -1$. The Fourier transform of $G^\star_\beta(t,x)$ can be found in [24, (4.21)]. The Laplace transform (4.15) of $G^\star_\beta(t,x)$ is due to the Laplace transform of the Wright function of the second kind (see e.g., [23, (F. 25), on p. 248]): $W_{-\lambda,\mu}(-x) \xrightarrow{\mathcal{L}} E_{\lambda,\lambda+\mu}(-z)$ for $0 < \lambda < 1$, which implies

$$(4.18) \qquad M_{\lambda,\mu}(x) \xrightarrow{\mathcal{L}} E_{\lambda,\mu}(-z)\,, \quad \text{for } 0 < \lambda < 1\,.$$

The statements in both (iii) and (viii) for $G^\star_\beta(t,x)$ can be found in [24, p. 22].

It remains to prove properties of the Green functions $G_\beta(t,x)$ with $\beta \in \,]1,2[$. Since the arguments for $G_\beta(t,x)$ with $\beta \in \,]0,1]$ are similar to those for $G_\beta(t,x)$ with $\beta \in \,]1,2[$, in the following, we will prove both cases altogether. We will mostly follow the arguments by Mainardi et al. in [24]. Let $\widehat{f}$ and $\widetilde{g}$ denote the Fourier transform in the space variable and the Laplace transform in the time variable,

respectively. Apply the Fourier transform on the initial data of (1.5):

$$\begin{cases} \widehat{G_\beta}(0_+,\xi) = 1 \,, & \text{if } 0 < \beta \le 1 \,, \\ \widehat{G_\beta}(0_+,\xi) = 0 \,, \quad \dfrac{\partial}{\partial t}\widehat{G_\beta}(0_+,\xi) = 1 \,, & \text{if } 1 < \beta < 2 \,. \end{cases}$$

Apply both the Fourier and the Laplace transforms on both sides of the main equation in (1.5):

$$\widetilde{\widehat{G_\beta}}(s,\xi)s^\beta - s^{\beta-\lceil\beta\rceil} + \xi^2 \widetilde{\widehat{G_\beta}}(s,\xi) = 0 \,,$$

where we have used the equivalent definition of the Caputo fractional differential operator of order β through the Laplace transform (see [24, (2.12)]):

$$\mathcal{L}\left[{}_t D_*^\beta f(t)\right](s) = s^\beta \widetilde{f} - \sum_{k=0}^{m-1} s^{\beta-1-k}\, f^{(k)}(0_+) \,, \quad \text{if } m-1 < \beta \le m \,.$$

Hence,

$$\widetilde{\widehat{G_\beta}}(s,\xi) = \frac{s^{\beta-\lceil\beta\rceil}}{s^\beta + \xi^2} \,, \quad \text{for } 0 < \beta < 2 \,.$$

By the scaling rules for the Fourier and Laplace transforms, we have that

$$G_\beta(bt, ax) \xrightarrow{\mathcal{F}} \frac{1}{a}\widehat{G_\beta(bt,\cdot)}(\xi/a) \xrightarrow{\mathcal{L}} \frac{1}{ab}\widetilde{\widehat{G_\beta}}(s/b, \xi/a) = \frac{1}{ab}\frac{\left(\frac{s}{b}\right)^{\beta-\lceil\beta\rceil}}{\left(\frac{s}{b}\right)^\beta + \left(\frac{\xi}{a}\right)^2}$$

$$= \frac{1}{ab^{1-\lceil\beta\rceil}}\frac{s^{\beta-\lceil\beta\rceil}}{s^\beta + \left(\frac{b^{\beta/2}\xi}{a}\right)^2} \xrightarrow{\mathcal{L}^{-1}} \frac{1}{ab^{1-\lceil\beta\rceil}}\widehat{G_\beta(t,\cdot)}\left(\frac{b^{\beta/2}}{a}\xi\right)$$

$$\xrightarrow{\mathcal{F}^{-1}} b^{-\beta/2+\lceil\beta\rceil-1}G_\beta\left(t, \frac{a}{b^{\beta/2}}x\right) \,,$$

which proves the scaling property (4.6). Now use the following Laplace transform (see [30, (1.80, p. 21)])

$$\int_0^\infty \mathrm{d}t\, e^{-st}t^{\alpha k+\beta-1}E_{\alpha,\beta}^{(k)}\left(\pm\lambda t^\alpha\right) = \frac{k!\, s^{\alpha-\beta}}{\left(s^\alpha \mp \lambda\right)^{k+1}} \,, \qquad \Re(s) > |\lambda|^{1/\alpha} \,,$$

where $E_{\alpha,\beta}^{(k)}(y) = \frac{\mathrm{d}^k}{\mathrm{d}y^k}E_{\alpha,\beta}(y)$. We see that $\widehat{G_\beta}(t,\xi) = t^{\lceil\beta\rceil-1} E_{\beta,\lceil\beta\rceil}\left(-\xi^2 t^\beta\right)$, which proves (4.14). Then an application of the inverse Fourier transform using Lemma 4.5 gives the Green function (4.5). As a consequence, the function $x \mapsto G_\beta(t,x)$ is symmetric and

$$\int_{\mathbb{R}} \mathrm{d}x\, G_\beta(t,x) = \widehat{G_\beta}(t,0) = t^{\lceil\beta\rceil-1} \,,$$

which proves (4.7). By the scaling property and the symmetry of $x \mapsto G_\beta(t,x)$,

$$\int_{\mathbb{R}} \mathrm{d}x\, |x|^n G_\beta(t,x) = 2\int_0^\infty \mathrm{d}x\, x^n G_\beta(t,x) = 2\, t^{\frac{n\beta}{2}+\lceil\beta\rceil-1}\int_0^\infty \mathrm{d}y\, y^n\, G_\beta(1,y).$$

Then the moment formula (4.13) is proved by applying Lemma 4.4.

The asymptotic property of $G_\beta(1,x)$ is a direct consequence of the asymptotics of the Wright function (see [23, (F.3), p. 238]): For $0 < \lambda < 1$, and $\mu \in \mathbb{R}$,

$$(4.19) \quad M_{\lambda,\lambda+\mu}(x) = W_{-\lambda,\mu}(-x) \approx A_0\, Y_\lambda(x)^{1/2-\mu}\exp\left(-Y_\lambda(x)\right), \quad \text{as } x \to +\infty \,,$$

where

$$Y_\lambda(x) = (1-\lambda)\,\lambda^{\frac{\lambda}{1-\lambda}}x^{\frac{1}{1-\lambda}} \quad \text{and} \quad A_0 = \left(2\pi\,(1-\lambda)\right)^{-1/2} \,.$$

Note that the constant A_0 can be found in [36, 37].

The Laplace transform in (4.15) is proved by (4.18). Bernstein's theorem on monotone functions (see Theorem 4.6) and (4.3) prove the positivity of $G_\beta(1, x)$ for $x \geq 0$. Then by symmetry of $G_\beta(1, x)$, $G_\beta(1, x) \geq 0$ for all $x \in \mathbb{R}$. By (4.15) and the property of the Laplace transform, we see that

$$\mathcal{L}\left[\frac{\mathrm{d}}{\mathrm{d}x}G_\beta(1, \cdot)\right](z) = \frac{z}{2}E_{\beta/2, \lceil \beta \rceil}(-z) - G_\beta(1, 0) = -E_{\beta/2, \lceil \beta \rceil - \beta/2}(-z),$$

where we have also used the fact that $G_\beta(1, 0) = [2\Gamma(\lceil \beta \rceil - \beta/2)]^{-1}$ and the recurrence relation of the Mittag-Leffler function $E_{\alpha, \beta}(z) = \Gamma(\beta)^{-1} + zE_{\alpha, \alpha+\beta}(z)$. Notice the function $E_{\beta/2, \lceil \beta \rceil - \beta/2}(-x)$ is completely monotone for $x \in \mathbb{R}_+$ because $\lceil \beta \rceil - \beta/2 \geq \beta/2$ and $\beta/2 \in]0, 1]$. Hence, by the same reason for the positivity of $G_\beta(1, x)$, we can conclude the nonpositivity of $\frac{\mathrm{d}}{\mathrm{d}x}G_\beta(1, x)$, which proves that the global maximum of $G_\beta(t, x)$ is achieved at $x = 0$.

As for (4.17), by differentiating term-by-term (see also [23, (F. 8), p. 239]), we see that $\frac{\mathrm{d}}{\mathrm{d}z}W_{\lambda, \mu}(z) = W_{\lambda, \lambda+\mu}(z)$, from which one can easily derive that

$$(4.20) \qquad \frac{\mathrm{d}^n}{\mathrm{d}z^n}M_{\lambda, \mu}(z) = (-1)^n M_{\lambda, \mu - n\lambda}(z).$$

Hence, (4.17) follows. This completes the proof of Lemma 4.1. □

Remark 4.2. Note that in general, the function $x \mapsto G_\beta(t, x)$ is not differentiable at $x = 0$. But we have

$$\frac{\partial^n}{\partial x^n}G_\beta(t, 0-) = (-1)^n \frac{\partial^n}{\partial x^n}G_\beta(t, 0+)$$
$$= \frac{(-1)^n t^{\lceil \beta \rceil - 1 - (n+1)\beta/2}}{2}\Gamma\left(\lceil \beta \rceil - \frac{\beta(n+1)}{2}\right)^{-1},$$

because $M_{\beta/2, \lceil \beta \rceil - n\beta/2}(0-) = M_{\beta/2, \lceil \beta \rceil - n\beta/2}(0+) = \Gamma\left(\lceil \beta \rceil - \frac{\beta(n+1)}{2}\right)^{-1}$. When $\beta = 1$ and $n \geq 1$ is an odd integer, then $M_{1/2, 1-n/2}(0) = \Gamma\left(\frac{1-n}{2}\right)^{-1} = 0$, which explains why the heat kernel function (2.1) is smooth at $x = 0$.

Remark 4.3 (Wave equation case $\beta = 2$). By definition of $M_{\lambda, \mu}(z)$ in (4.4), the parameter λ should be strictly less than 1. Hence, the Green functions $G_\beta(t, x)$ and $G_\beta^*(t, x)$ in (4.5) do not cover the case where $\beta = 2$. However, the wave equation case $\beta = 2$ is a limiting case as $\beta \uparrow 2$, which can be seen from Figure 3. Another way to see this is through the Fourier transform (4.14). By letting $\beta = 2$ in (4.14), one has that

$$\int_\mathbb{R} \mathrm{d}x\, e^{-i\xi x}G_2(t, x) = tE_{2,2}(-t^2\xi^2) = \frac{\sin(t\xi)}{\xi},$$
$$\int_\mathbb{R} \mathrm{d}x\, e^{-i\xi x}G_2^*(t, x) = E_{2,1}(-t^2\xi^2) = \cos(t\xi),$$

which equal the Fourier transforms of the wave kernel functions: $\frac{1}{2}1_{\{|x| \leq t\}}$ and $\frac{1}{2}(\delta_t(x) + \delta_{-t}(x))$, respectively. Hence, in the limiting case, we have (2.2).

We draw some of these Green functions $G_\beta(1, x)$ in Figure 2. The range of x is from -5 to 5. From these graphs, one can see that when β tends to 2, the Green
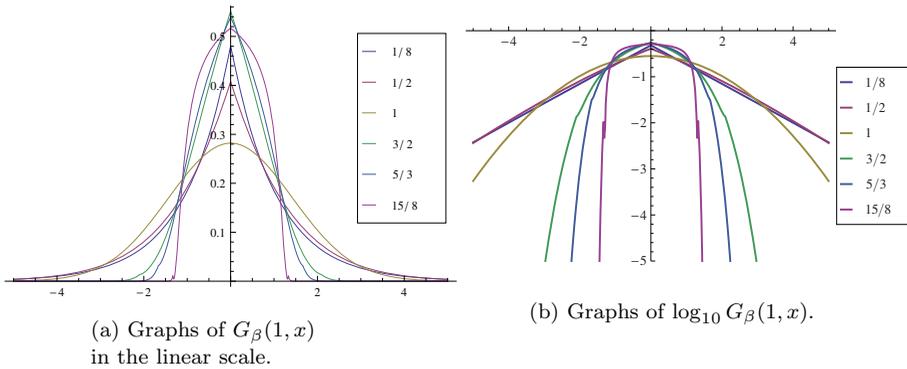
(a) Graphs of $G_\beta(1, x)$ in the linear scale.

(b) Graphs of $\log_{10} G_\beta(1, x)$.

Figure 2. Some graphs with $\beta = 1/8$, $1/2$, 1, $3/2$, $5/3$ and $15/8$.

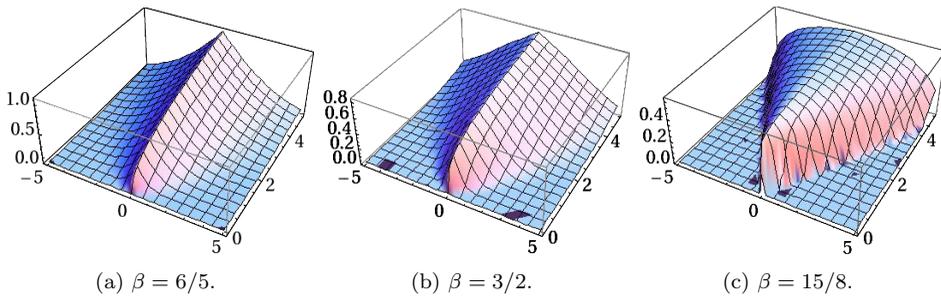(a) $\beta = 6/5$.   (b) $\beta = 3/2$.   (c) $\beta = 15/8$.

Figure 3. Graphs of the Green functions $G_\beta(t, x)$ for $1 < \beta < 2$.

function tends to the wave kernel function $\frac{1}{2}1_{\{|x| \leq 1\}}$. Note that these graphs are plotted by concatenating the truncated summations for $n \leq 23$ in the asymptotic representation (4.8), and hence there are some truncation errors, which can be seen, in these graphs.

In Figure 3, we draw some Green functions in space-time coordinates for the fast diffusion equations ($\beta \in \,]1, 2[\,$). The ranges for x and t are $[-5, 5]$ and $]0, 5]$, respectively. When β tends to 2, these graphs become closer to the wave kernel function $G_2(t, x) = \frac{1}{2}1_{\{|x| \leq t\}}$.

At the end of this section, we list some technical results used in the proof of Lemma 4.1.

Lemma 4.4. *The following integral holds:*

$$\int_0^\infty \mathrm{d}x \, x^a M_{\lambda, \mu}(x) = \frac{\Gamma(a+1)}{\Gamma(\lambda a + \mu)}, \qquad \text{for } a > -1,\ \lambda \in [0, 1[\,,\ \text{and } \mu \in \mathbb{C}\,.$$

Proof. By the integral representation of the Wright function,

$$W_{\lambda, \mu}(z) = \frac{1}{2\pi i} \int_{H_a} \frac{\mathrm{d}t}{t^\mu} \exp\left(t + zt^{-\lambda}\right), \qquad \lambda > -1, \quad \mu \in \mathbb{C}\,,$$

where H_a denotes the Hankel contour (see [23, (F.2), on p. 238] for more details). Notice that $M_{\lambda,\mu}(x) = W_{-\lambda,\mu-\lambda}(-x)$. Then

$$
\begin{aligned}
\int_0^\infty \mathrm{d}x \, x^a M_{\lambda,\mu}(x) &= \int_0^\infty \mathrm{d}x \, x^a \left[\frac{1}{2\pi i} \int_{H_a} \frac{\mathrm{d}t}{t^{\mu-\lambda}} e^{t-xt^\lambda} \right] \\
&= \frac{1}{2\pi i} \int_{H_a} \mathrm{d}t \left[\int_0^\infty \mathrm{d}x \, e^{-xt^\lambda} x^a \right] \frac{e^t}{t^{\mu-\lambda}} \\
&= \frac{1}{2\pi i} \int_{H_a} \mathrm{d}t \, \frac{\Gamma(a+1)}{t^{\lambda a+\lambda}} \frac{e^t}{t^{\mu-\lambda}} \\
&= \frac{\Gamma(a+1)}{2\pi i} \int_{H_a} \mathrm{d}t \, \frac{e^t}{t^{\lambda a+\mu}} = \frac{\Gamma(a+1)}{\Gamma(\lambda a + \mu)} \,,
\end{aligned}
$$

where we have used the definition of the Gamma function in the third equality (which requires that $a > -1$) and in the last step we have used the Hankel integral representation of the Gamma function $\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_{H_a} \mathrm{d}t \, e^t t^{-z}$; see, e.g., [28, 5.9.1, p. 139]. $\qquad \square$

Lemma 4.5. *The Fourier transform of the function $M_{\lambda,\mu}(|x|)$ is*

$$
\begin{aligned}
\mathcal{F} \left[\frac{1}{2} M_{\lambda,\mu}(|\cdot|) \right](\xi) &= \int_{\mathbb{R}} \mathrm{d}x \, e^{-ix\xi} M_{\lambda,\mu}(|x|) \\
&= E_{2\lambda,\mu}\left(-\xi^2\right), \quad \textit{for all } \lambda \in [0,1[\ \textit{and } \mu \in \mathbb{C} \,.
\end{aligned}
$$

Proof. By developing in series the cosine function and the moment formula in Lemma 4.4,

$$
\begin{aligned}
\mathcal{F} \left[\frac{1}{2} M_{\lambda,\mu}(|\cdot|) \right](\xi) &= \int_0^\infty \mathrm{d}x \, \cos(\xi x) M_{\lambda,\mu}(x) \\
&= \sum_{n=0}^\infty (-1)^n \frac{\xi^{2n}}{(2n)!} \int_0^\infty \mathrm{d}x \, x^{2n} M_{\lambda,\mu}(x) \\
&= \sum_{n=0}^\infty \frac{\left(-\xi^2\right)^n}{\Gamma(2n\lambda+\mu)} = E_{2\lambda,\mu}\left(-\xi^2\right).
\end{aligned}
$$

$\qquad \square$

Theorem 4.6 (Bernstein's theorem [35, Theorem 12a]). *A necessary and sufficient condition that $f(x)$ should be completely monotonic in $0 \leq x < +\infty$ is that $f(x) = \int_0^\infty e^{-xt} \mathrm{d}\alpha(t)$, where $\alpha(t)$ is bounded and nondecreasing and the integral converges for $0 \leq x < +\infty$.*

5. Calculations of $\mathcal{K}(t,x)$ and proof of Theorem 3.4

Let $G : \mathbb{R}_+ \times \mathbb{R}^d \mapsto \mathbb{R}$ with $d \in \mathbb{N}$, $d \geq 1$ be a Borel measurable function.

Assumption 5.1. *The function $G : \mathbb{R}_+ \times \mathbb{R}^d \mapsto \mathbb{R}$ has the following properties:*

(1) *There is a nonnegative function $\mathcal{G}(t,x)$, called reference kernel function, and constants $C_0 > 0$, $\sigma < 1$ such that*

$$
(5.1) \qquad G(t,x)^2 \leq \frac{C_0}{t^\sigma} \, \mathcal{G}(t,x), \quad \textit{for all } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d.
$$

(2) *The reference kernel function $\mathcal{G}(t,x)$ satisfies the following sub-semigroup property: for some constant $C_1 > 0$,*

$$(5.2) \quad \int_{\mathbb{R}^d} \mathrm{d}y \, \mathcal{G}\left(t, x-y\right) \mathcal{G}\left(s, y\right) \leq C_1 \, \mathcal{G}\left(t+s, x\right), \quad \text{for all } t, s > 0 \text{ and } x \in \mathbb{R}^d.$$

Define

$$\mathcal{L}_0\left(t, x\right) := G(t, x)^2, \quad \text{for all } (t, x) \in \mathbb{R}_+^* \times \mathbb{R}^d.$$

Recall that "$\star$" denotes the convolution in both space and time variables (space-time convolution). For all $n \in \mathbb{N}^* := \mathbb{N} \setminus \{0\}$ and all $(t, x) \in \mathbb{R}_+^* \times \mathbb{R}^d$, define

$$\mathcal{L}_n\left(t, x\right) := \underbrace{(\mathcal{L}_0 \star \cdots \star \mathcal{L}_0)}_{n+1 \text{ times of } \mathcal{L}_0}(t, x),$$

$$(5.3) \qquad\qquad \mathcal{K}\left(t, x\right) := \sum_{n=0}^{\infty} \mathcal{L}_n\left(t, x\right).$$

Denote

$$B_n\left(t; \sigma, C_0, C_1\right) := C_0^n C_1^{n-1} \frac{\Gamma\left(1-\sigma\right)^n}{\Gamma\left(n(1-\sigma)\right)} \, t^{n(1-\sigma)-1}, \qquad \text{for all } n \geq 0.$$

For simplicity, we write $B_n\left(t; \sigma, C_0, C_1\right)$ simply by $B_n(t)$.

Proposition 5.2. *Under Assumption 5.1, the following properties are true:*
(i) $\mathcal{L}_n(t, x)$ is nonnegative and satisfies the following inequality:

$$(5.4) \qquad \mathcal{L}_n(t, x) \leq B_{n+1}(t)\mathcal{G}(t, x), \quad \text{for all } n \geq 0 \text{ and } (t, x) \in \mathbb{R}_+^* \times \mathbb{R}^d.$$

Moreover, (5.4) becomes an equality if both (5.1) and (5.2) are equalities.
(ii) For all $t > 0$ and $\lambda > 0$, the following series $\sum_{n=1}^{\infty} \mathcal{L}_n(t, x)$ converges uniformly over $x \in \mathbb{R}^d$ and hence $\mathcal{K}(t, x)$ in (5.3) is well-defined.
(iii) $B_n(t)$ are nonnegative and for all $m \in \mathbb{N}^*$, $\sum_{n=0}^{\infty} B_n(t)^{1/m} < +\infty$.
(iv) For all $t \geq 0$ and $x \in \mathbb{R}^d$,

$$(5.5) \qquad\qquad \mathcal{K}\left(t, x\right) \leq \mathcal{G}(t, x) \frac{\gamma}{t^\sigma} E_{1-\sigma, 1-\sigma}\left(\gamma t^{1-\sigma}\right)$$

$$(5.6) \qquad\qquad \leq \frac{C}{t^\sigma} \, \mathcal{G}(t, x) \left(1 + t^\sigma \exp\left(\gamma^{\frac{1}{1-\sigma}} t\right)\right),$$

where $\gamma = C_0 C_1 \Gamma(1-\sigma)$ and the constant $C = C\left(\sigma, \gamma\right)$ can be chosen as

$$(5.7) \qquad\qquad C\left(\sigma, \gamma\right) := \gamma \sup_{t \geq 0} \frac{E_{1-\sigma, 1-\sigma}\left(\gamma \, t^{1-\sigma}\right)}{1 + t^\sigma \exp\left(\gamma^{\frac{1}{1-\sigma}} t\right)} < +\infty.$$

Moreover, (5.5) becomes equality if both (5.1) and (5.2) are equalities.
(v) If there exist a kernel function $\underline{\mathcal{G}}(t, x)$ and some constants $\underline{C}_0 > 0$, $\underline{C}_1 > 0$, and $\underline{\sigma} < 1$ such that for all $t, s > 0$ and $x \in \mathbb{R}^d$,

$$G(t, x)^2 \geq \underline{C}_0 \, t^{-\underline{\sigma}} \, \underline{\mathcal{G}}(t, x),$$

and $\underline{\mathcal{G}}(t, x)$ satisfies the sup-semigroup property

$$\int_{\mathbb{R}^d} \mathrm{d}y \, \underline{\mathcal{G}}\left(t, x-y\right) \underline{\mathcal{G}}\left(s, y\right) \geq \underline{C}_1 \, \underline{\mathcal{G}}\left(t+s, x\right),$$

then for all $t \geq 0$ and $x \in \mathbb{R}^d$,

$$(5.8) \qquad \mathcal{K}(t,x) \geq \mathcal{G}(t,x) \, \frac{\gamma}{t^\sigma} E_{1-\sigma,1-\sigma}\left(\gamma t^{1-\sigma}\right)$$

$$(5.9) \qquad \geq C \, \mathcal{G}(t,x) \, \exp\left(\gamma^{\frac{1}{1-\sigma}} t\right) ,$$

where $\gamma = C_0 C_1 \Gamma(1-\sigma)$ and

$$C = C\left(\sigma,\gamma\right) := \gamma \inf_{t \geq 0} \frac{E_{1-\sigma,1-\sigma}\left(\gamma t^{1-\sigma}\right)}{t^\sigma \exp\left(\gamma^{\frac{1}{1-\sigma}} t\right)} > 0 .$$

Proof. (i) The nonnegativity is clear. The case $n = 0$ is trivially true. Suppose that the relation (5.4) holds up to $n - 1$. Then by the Beta integral,

$$
\begin{aligned}
\mathcal{L}_n(t,x) &= \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}y \, \mathcal{L}_{n-1}(t-s, x-y) \, G^2(s,y) \\
&\leq C_0 \int_0^t \mathrm{d}s \, B_n(t-s) s^{-\sigma} \int_{\mathbb{R}^d} \mathrm{d}y \, \mathcal{G}(t-s, x-y) \, \mathcal{G}(s,y) \\
&\leq C_0^{n+1} C_1^n \, \mathcal{G}(t,x) \frac{\Gamma(1-\sigma)^n}{\Gamma(n(1-\sigma))} \int_0^t \mathrm{d}s \, (t-s)^{n(1-\sigma)-1} s^{-\sigma} \\
&= C_0^{n+1} C_1^n \frac{\Gamma(1-\sigma)^{n+1}}{\Gamma((n+1)(1-\sigma))} \, t^{(n+1)(1-\sigma)-1} \, \mathcal{G}(t,x) \\
&= B_{n+1}(t) \mathcal{G}(t,x) .
\end{aligned}
$$

(ii) It is a special case of (iii).

(iii) The nonnegativity is clear. By (5.4),

$$\mathcal{L}_n(t,x) \leq B_{n+1}(t) t^{-\sigma} \sup_{x \in \mathbb{R}^d} \mathcal{G}(t,x) < +\infty .$$

Thus, if the series $\sum_n B_n(t)^{1/m}$ converges, then it does so uniformly over $x \in \mathbb{R}^d$. Denote $\beta := 1 - \sigma$. Use the ratio test

$$\left(\frac{B_n(t)}{B_{n-1}(t)}\right)^{1/m} = \left(C_0 C_1 \Gamma(\beta) t^\beta\right)^{1/m} \left(\frac{\Gamma((n-1)(1-\sigma))}{\Gamma(n(1-\sigma))}\right)^{1/m} .$$

By the asymptotic expansion of the Gamma function ([28, 5.11.2, p. 140]),

$$\frac{\Gamma((n-1)(1-\sigma))}{\Gamma(n(1-\sigma))} \approx \left(\frac{e}{\beta}\right)^\beta \left(1 - \frac{1}{n}\right)^{(n-1)\beta} \frac{1}{n^\beta} \approx \frac{1}{(\beta n)^\beta}$$

for large n. Now clearly, $\beta > 0$ since $\sigma < 1$. Hence, for all $t > 0$ and for large n,

$$\left(\frac{B_n(t)}{B_{n-1}(t)}\right)^{1/m} \approx \left(C_0 C_1 \Gamma(\beta) t^\beta\right)^{1/m} \frac{1}{(\beta n)^{\beta/m}},$$

which tends to zero as $n \to +\infty$.

(iv) The bound (5.5) is because

$$(5.10) \qquad \sum_{k=1}^\infty \frac{z^k}{\Gamma(\alpha k)} = z E_{\alpha,\alpha}(z) ,$$

and the bounds in (i):

$$\mathcal{K}(t,x) \leq \mathcal{G}(t,x) \sum_{n=1}^{\infty} B_n(t) = \frac{1}{C_1 t} \mathcal{G}(t,x) \sum_{n=1}^{\infty} \frac{\left(C_0 C_1 \Gamma(1-\sigma) t^{1-\sigma}\right)^n}{\Gamma(n(1-\sigma))}$$

$$= C_0 \, \Gamma(1-\sigma) \, t^{-\sigma} \, \mathcal{G}(t,x) \, E_{1-\sigma,1-\sigma} \left(C_0 C_1 \Gamma(1-\sigma) t^{1-\sigma}\right).$$

As for (5.6), we only need to show that the constant C defined in (5.7) is finite. Let

$$f(t) = \frac{E_{1-\sigma,1-\sigma}\left(\gamma \, t^{1-\sigma}\right)}{1 + t^\sigma \exp\left(\gamma^{\frac{1}{1-\sigma}} t\right)} \, .$$

By Lemma 5.9 with real nonnegative value $z = \gamma t^{1-\sigma}$ and $p = 1$:

$$\gamma E_{1-\sigma,1-\sigma}\left(\gamma \, t^{1-\sigma}\right) = \frac{1}{1-\sigma} \gamma^{\frac{1}{1-\sigma}} \, t^\sigma \exp\left(\gamma^{\frac{1}{1-\sigma}} t\right) + O\left(\frac{1}{|t|^{1-\sigma}}\right), \qquad t \to +\infty,$$

we see that $\lim_{t\to+\infty} f(t) = \frac{1}{1-\sigma} \gamma^{\frac{1}{1-\sigma}}$. Then because the Mittag-Leffler function is an entire function on complex plain [14, Theorem 4.1, p. 68], we can conclude that $\sup_{t\geq 0} f(t) < +\infty$.

(v) The proof is similar to (i) and (iv). We only need to show that C is strictly positive. Because the function $g(t) = E_{1-\sigma,1-\sigma}^{-1}\left(\gamma \, t^{1-\sigma}\right) \, t^\sigma \exp\left(\gamma^{\frac{1}{1-\sigma}} t\right)$ is continuous over $t \in [0,+\infty]$ with $g(0) = 0$ and $\lim_{t\to+\infty} g(t) = (1-\sigma) \, \gamma^{\frac{\sigma}{\sigma-1}} < +\infty$, this function is bounded from above for $t \in [0,+\infty]$ and hence $\inf_{t\geq 0} g^{-1}(t) > 0$. This completes the proof of Proposition 5.2. $\qquad\square$

Example 5.3. For the heat kernel $p_\nu(t,x) = (2\pi\nu t)^{-1/2} \exp\left(-\frac{x^2}{2\nu t}\right)$ with $\nu > 0$, Assumption 5.1 holds with both inequalities (5.1) and (5.2) replaced by equalities, and

$$C_0 = \frac{1}{\sqrt{4\pi\nu}}, \quad \sigma = \frac{1}{2}, \quad \mathcal{G}(t,x) = p_{\nu/2}(t,x), \quad C_1 = 1.$$

Then, $\gamma = (4\nu)^{-1/2}$. Therefore, by (4.2) and $\text{erfc}(-x) = 2\Phi(\sqrt{2}x)$ where $\Phi(x)$ is the distribution function of the standard normal distribution, Proposition 5.2 implies that

$$\mathcal{K}(t,x) = \frac{\mathcal{G}(t,x)}{\sqrt{4\nu t}} \left[\frac{1}{\sqrt{\pi}} + \frac{\sqrt{t}}{\sqrt{4\nu}} \text{erfc}\left(-\frac{\sqrt{t}}{\sqrt{4\nu}}\right) e^{\frac{t}{4\nu}}\right]$$

$$= p_{\frac{\nu}{2}}(t,x) \left[\frac{1}{\sqrt{4\pi\nu t}} + \frac{1}{2\nu} \Phi\left(\frac{\sqrt{t}}{\sqrt{2\nu}}\right) e^{\frac{t}{4\nu}}\right],$$

which recovers the results in [6].

Example 5.4. Let us consider the following SPDE:

$$\begin{cases} \left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)^2 u(t,x) = \rho(u(t,x))\dot{W}(t,x), & (t,x) \in \mathbb{R}_+^* \times \mathbb{R}, \\ u(0,\cdot) = \mu(\cdot). \end{cases}$$

The Green function is $G(t,x) = \frac{\sqrt{t}}{\sqrt{4\pi}} \exp\left(-\frac{x^2}{4t}\right)$; see [31, Section 9.2.5-2]. Assumption 5.1 holds with both inequalities (5.1) and (5.2) replaced by equalities, and

$$C_0 = \frac{1}{\sqrt{8\pi}}, \quad \sigma = -\frac{3}{2}, \quad \mathcal{G}(t,x) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right), \quad C_1 = 1.$$

Then, $\gamma = \frac{1}{\sqrt{8\pi}}\Gamma(5/2) = \frac{3\sqrt{2}}{16}$. Therefore, Proposition 5.2 implies that

$$\mathcal{K}(t,x) = \frac{3\sqrt{2}}{16}\,t^{3/2}\,\mathcal{G}(t,x)\,E_{5/2,5/2}\left(\frac{3\sqrt{2}}{26}\,t^{5/2}\right).$$

In particular, if $\rho(u) = u$, then $\mathbb{E}\left(u^2(t,x)\right) = J_0^2(t,x) + (J_0^2 \star \mathcal{K})(t,x)$, where $J_0(t,x) = (\mu * G(t,\cdot))(x)$. Note that the initial data can be more general than the SHE (1.3): It can be any distribution μ such that it is the (distributional) derivative of some measures in $\mathcal{M}_H(\mathbb{R})$, i.e., for some $\mu_0 \in \mathcal{M}_H(\mathbb{R})$, $\mu = \mu_0'$. More details of this SPDE, which will not be pursued here, are left to interested readers.

Here are three natural choices of the reference kernel functions $\mathcal{G}(t,x)$:

(1) the Gaussian kernel function

$$\mathcal{G}_g(t,x) := (4\pi t)^{-d/2}\exp\left(-\frac{|x|^2}{4t}\right), \quad \text{for all } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d,\, d \geq 1,$$

where $|x|^2 = x_1^2 + \cdots + x_d^2$;

(2) the Poisson kernel function:

$$\mathcal{G}_p(t,x) := c_n \frac{t}{(t^2 + |x|^2)^{(d+1)/2}}, \quad \text{for all } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d,\, d \geq 1,$$

where $c_n = \pi^{-(n+1)/2}\,\Gamma((n+1)/2)$;

(3) the exponential kernel function $\mathcal{G}_{e,\beta}(t,x)$ defined in (3.6).

Clearly, we have the following scaling properties for these reference kernel functions:

$$\mathcal{G}_g(t,x) = t^{-d/2}\,\mathcal{G}_g(1,t^{-1/2}\,x),$$
$$\mathcal{G}_p(t,x) = t^{-d}\,\mathcal{G}_p(1,t^{-1}\,x),$$
$$\mathcal{G}_{e,\beta}(t,x) = t^{-\beta/2}\mathcal{G}_{e,\beta}(1,t^{-\beta/2}x).$$

Both $\mathcal{G}_g(t,x)$ and $\mathcal{G}_p(t,x)$ satisfy part (2) of Assumption 5.1 with $C_1 = 1$ and "$\leq$" replaced by "$=$". By Lemma 5.10 below, $\mathcal{G}_{e,\beta}(t,x)$ satisfies part (2) of Assumption 5.1 with $C_1 = \widehat{C}_\beta$, where $\widehat{C}_\beta$ is defined in (3.8).

Proposition 5.5 (Gaussian reference kernel). *Suppose the function* $G : \mathbb{R}_+ \times \mathbb{R}^d \mapsto \mathbb{R}$ *satisfies the following two properties:*

(i) *The scaling property: for some constants* $\gamma_1 \in \mathbb{R}$ *and* $\gamma_2 \geq 1/2$,

$$G(t,x) = t^{\gamma_1}G\left(1,\frac{x}{t^{\gamma_2}}\right), \quad \text{for all } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d.$$

(ii) *The function* $x \mapsto G(1,x)$ *is bounded such that* $\sup_{x\in\mathbb{R}^d}\frac{G(1,x)^2}{\mathcal{G}_g(1,x)} < +\infty$.

Then $G(t,x)$ *satisfies Assumption 5.1 with* $\mathcal{G}(t,x) = \mathcal{G}_g(t^{2\gamma_2},x)$ *and*

$$(5.11) \qquad C_0 = \sup_{x\in\mathbb{R}^d}\frac{G(1,x)^2}{\mathcal{G}_g(1,x)}, \quad C_1 = 2^{d(\gamma_2-1/2)}, \quad \text{and} \quad \sigma = -(2\gamma_1 + d\gamma_2).$$

Proof. Notice that by the scaling properties of $\mathcal{G}_g(t,x)$ and $G(t,x)$, we have that

$$\sup_{(t,x)\in\mathbb{R}_+\times\mathbb{R}^d}\frac{G(t,x)^2}{t^{-\sigma}\mathcal{G}(t,x)} = \sup_{(t,x)\in\mathbb{R}_+\times\mathbb{R}^d}\frac{G(t,x)^2}{t^{-\sigma}\mathcal{G}_g(t^{2\gamma_2},x)} = \sup_{y\in\mathbb{R}^d}\frac{G(1,y)^2}{\mathcal{G}_g(1,y)},$$

which is finite by (ii). Hence, part (1) of Assumption 5.1 is satisfied with the constants C_0 and σ defined in (5.11). As for part (2) of Assumption 5.1, by the semigroup property of $\mathcal{G}_g(t,x)$, we have

$$\int_{\mathbb{R}^d} \mathrm{d}y\, \mathcal{G}(t, x-y)\mathcal{G}(s,y) = \int_{\mathbb{R}^d} \mathrm{d}y\, \mathcal{G}_g\left(t^{2\gamma_2}, y\right) \mathcal{G}_g\left(s^{2\gamma_2}, x-y\right) = \mathcal{G}_g\left(t^{2\gamma_2} + s^{2\gamma_2}, x\right).$$

Notice that the function $[0,1] \ni r \mapsto f(r) = (1-r)^{2\gamma_2} + r^{2\gamma_2}$ is convex because $2\gamma_2 \geq 1$. By solving $f'(r) = 0$, we find that

$$\min_{r \in [0,1]} f(r) = f(1/2) = 2^{1-2\gamma_2} \quad \text{and} \quad \max_{r \in [0,1]} f(r) = f(1) = f(0) = 1.$$

Hence,

$$(5.12) \qquad 2^{1-2\gamma_2}(t+s)^{2\gamma_2} \leq t^{2\gamma_2} + s^{2\gamma_2} = (t+s)^{2\gamma_2} f\left(\frac{s}{t+s}\right) \leq (t+s)^{2\gamma_2}.$$

Finally,

$$\mathcal{G}_g\left(t^{2\gamma_2} + s^{2\gamma_2}, x\right) = \left[4\pi\left(t^{2\gamma_2} + s^{2\gamma_2}\right)\right]^{-d/2} \exp\left(-\frac{|x|^2}{4\left(t^{2\gamma_2} + s^{2\gamma_2}\right)}\right)$$

$$\leq 2^{d(\gamma_2 - 1/2)} \left[4\pi\left((t+s)^{2\gamma_2}\right)\right]^{-d/2} \exp\left(-\frac{|x|^2}{4(t+s)^{2\gamma_2}}\right)$$

$$= 2^{d(\gamma_2 - 1/2)}\, \mathcal{G}\left((t+s)^{2\gamma_2}, x\right),$$

which completes the proof of Proposition 5.5. $\qquad \square$

We will not use the Poisson reference kernel in this paper. We prove the following result for the future reference.

Proposition 5.6 (Poisson reference kernel). *Suppose the function* $G : \mathbb{R}_+ \times \mathbb{R}^d \mapsto \mathbb{R}$ *satisfies the following two properties:*

(i) *The scaling property: for some constants* $\gamma_1 \in \mathbb{R}$ *and* $0 < \gamma_2 \leq 1$,

$$G(t,x) = t^{\gamma_1} G\left(1, \frac{x}{t^{\gamma_2}}\right), \quad \text{for all } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d.$$

(ii) *The function* $x \mapsto G(1,x)$ *is bounded such that* $\sup_{x \in \mathbb{R}^d} \frac{G(1,x)^2}{\mathcal{G}_p(1,x)} < +\infty$.

Then $G(t,x)$ *satisfies Assumption 5.1 with* $\mathcal{G}(t,x) = \mathcal{G}_p(t^{\gamma_2}, x)$ *and*

$$(5.13) \qquad C_0 = \sup_{x \in \mathbb{R}^d} \frac{G(1,x)^2}{\mathcal{G}_p(1,x)}, \quad C_1 = 2^{1-\gamma_2}, \quad \text{and} \quad \sigma = -(2\gamma_1 + d\gamma_2).$$

Proof. Notice that by the scaling properties of $\mathcal{G}_p(t,x)$ and $G(t,x)$, we have that

$$\sup_{(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d} \frac{G(t,x)^2}{t^{-\sigma}\mathcal{G}(t,x)} = \sup_{(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d} \frac{G(t,x)^2}{t^{-\sigma}\mathcal{G}_p(t^{\gamma_2}, x)} = \sup_{y \in \mathbb{R}^d} \frac{G(1,y)^2}{\mathcal{G}_p(1,y)},$$

which is finite due to (ii). Hence, part (1) of Assumption 5.1 is satisfied with the constants C_0 and σ defined in (5.13). As for part (2) of Assumption 5.1, by the semigroup property of $\mathcal{G}_p(t,x)$, we have

$$\int_{\mathbb{R}^d} \mathrm{d}y\, \mathcal{G}(t, x-y)\mathcal{G}(s,y) = \int_{\mathbb{R}^d} \mathrm{d}y\, \mathcal{G}_p\left(t^{2\gamma_2}, y\right) \mathcal{G}_p\left(s^{2\gamma_2}, x-y\right) = \mathcal{G}_p\left(t^{2\gamma_2} + s^{2\gamma_2}, x\right).$$

Then because $0 < \gamma_2 \leq 1$,

$$(5.14) \qquad\qquad (t+s)^{\gamma_2} \leq t^{\gamma_2} + s^{\gamma_2} \leq 2^{1-\gamma_2}(t+s)^{\gamma_2}.$$

Therefore,

$$\mathcal{G}_p\left(t^{\gamma_2} + s^{\gamma_2}, x\right) = \frac{c_n\left(t^{\gamma_2} + s^{\gamma_2}\right)}{\left((t^{\gamma_2} + s^{\gamma_2})^2 + |x|^2\right)^{(d+1)/2}}$$

$$\leq \frac{c_n\, 2^{1-\gamma_2}\,(t+s)^{\gamma_2}}{\left((t+s)^{2\gamma_2} + |x|^2\right)^{(d+1)/2}} = 2^{1-\gamma_2}\,\mathcal{G}_p((t+s)^{\gamma_2}, x),$$

which completes the proof of Proposition 5.6. $\qquad\qquad\qquad\qquad\square$

Proposition 5.7 (Exponential reference kernel). *Let $\beta \in\,]0,2]$. Suppose the function $G : \mathbb{R}_+ \times \mathbb{R} \mapsto \mathbb{R}$ satisfies the following two properties:*

(i) *The scaling property:*

$$G(t,x) = t^{-\beta/2} G\left(1, \frac{x}{t^{\beta/2}}\right), \quad \text{for all } (t,x) \in \mathbb{R}_+ \times \mathbb{R}.$$

(ii) *The function $x \mapsto G(1,x)$ is bounded such that $\sup_{x \in \mathbb{R}} \frac{G(1,x)^2}{\mathcal{G}_{e,\beta}(1,x)} < +\infty.$*

Then $G(t,x)$ satisfies Assumption 5.1 with $\mathcal{G}(t,x) = \mathcal{G}_{e,\beta}(t,x)$ and

$$(5.15) \qquad\qquad C_0 = \sup_{x \in \mathbb{R}} \frac{G(1,x)^2}{\mathcal{G}_{e,\beta}(1,x)}, \quad C_1 = \widehat{C}_\beta, \quad \text{and} \quad \sigma = \frac{\beta}{2},$$

where $\widehat{C}_\beta$ is defined in (3.8).

Proof. Notice that by the scaling properties of $\mathcal{G}_{e,\beta}(t,x)$ and $G(t,x)$, we have that

$$\sup_{(t,x) \in \mathbb{R}_+ \times \mathbb{R}} \frac{G(t,x)^2}{t^{-\sigma} \mathcal{G}(t,x)} = \sup_{y \in \mathbb{R}} \frac{G(1,y)^2}{\mathcal{G}_{e,\beta}(1,y)},$$

which is finite due to (ii). Hence, part (1) of Assumption 5.1 is satisfied with the constants C_0 and σ defined in (5.15). Part (2) of Assumption 5.1 is due to Lemma 5.10 below with $C_1 = \widehat{C}_\beta$. This completes the proof of Proposition 5.7. $\qquad\square$

Now we apply Proposition 5.2 to the Green functions $G_\beta(t,x)$ with $0 < \beta < 2$. More precisely, we will apply Proposition 5.5 (resp. 5.7) with $\mathcal{G}_\beta(t,x)$ defined in (3.5) in the case of fast (resp. slow) diffusions for the upper bounds of $\mathcal{K}(t,x)$, and Proposition 5.5 with $\mathcal{G}_\beta(t,x)$ defined in (3.7) in the case of slow diffusion for the lower bound of $\mathcal{K}(t,x)$. Recall the constants Ψ_β and $\underline{\Psi}_\beta$ defined in (3.10) and (3.11), respectively, and the constant $\widetilde{C}_\beta$ defined in (3.9).

Proposition 5.8. (1) *Proposition 5.2(i)–(iv) hold for $G_\beta(t,x)$ with $\beta \in\,]0,2[$ and*

$$d = 1\,, \quad \sigma = \beta/2 + 2(1 - \lceil \beta \rceil), \quad \mathcal{G}(t,x) = \mathcal{G}_\beta(t,x)\,, \quad C_0 = \Psi_\beta\,, \quad C_1 = \widetilde{C}_\beta\,.$$

(2) *Proposition 5.2(v) holds for $G_\beta(t,x)$ with $\beta \in\,]0,1[$ and*

$$d = 1, \quad \underline{\sigma} = \frac{\beta}{2} - 1, \quad \underline{\mathcal{G}}(t,x) = \underline{\mathcal{G}}_\beta(t,x), \quad \underline{C}_0 = \underline{\Psi}_\beta, \quad \underline{C}_1 = 2^{\frac{\beta-2}{4}}.$$

Proof. (1) We begin with the case where $\beta \in\,]1,2[\,$. By (4.6), $G_\beta(t,x)$ satisfies the scaling property with $\gamma_2 = \beta/2 \geq 1/2$ and $\gamma_1 = 1 - \beta/2$. Notice that

$$\Psi_\beta = \sup_{y \geq 0} \frac{\sqrt{\pi}}{2}\, \exp\left(y^2/4\right)\, M_{\beta/2}^2(y)\,.$$

Since the parameter c in (4.8) is strictly bigger than 2 (see also Figure 1), we see that

$$\lim_{y \to +\infty} \frac{\sqrt{\pi}}{2} \, \exp\left(y^2/4\right) M_{\beta/2}^2(y) = 0 \, .$$

Since the function $y \mapsto \exp\left(y^2/2\right) M_{\beta/2}^2(y)$ is an entire function, we see that the above supremum does exist. Therefore, one can apply Proposition 5.5 with $d = 1$, $C_0 = \Psi_\beta$, $C_1 = 2^{(\beta-1)/2}$, $\sigma = \beta/2 - 2 < -1$, and the above γ_1 and γ_2.

The proof for the slow diffusion equations can be proved similarly using Proposition 5.7 with $\gamma_2 = -\gamma_1 = \beta/2 \leq 1/2$, $C_0 = \Psi_\beta = \sup_{y \geq 0} \, 1/2 \, e^y M_{\beta/2}^2(y)$, $C_1 = \widehat{C}_\beta$, and $\sigma = \beta/2 < 1$.

(2) We claim that if $\beta \in \,]0,1[\,$, then for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$ and $s \geq 0$, we have that

$$(5.16) \qquad\qquad G_\beta^2(t,x) \geq \underline{\Psi}_\beta \, t^{1-\beta/2} \, \mathcal{G}_\beta(t,x),$$

$$(5.17) \qquad\qquad \int_\mathbb{R} \mathrm{d}y \, \mathcal{G}_\beta(t, x-y) \mathcal{G}_\beta(s,y) \geq 2^{\frac{\beta-2}{4}} \, \mathcal{G}_\beta(t+s,x).$$

By the scaling property (4.6), $\sup_{(t,x) \in \mathbb{R}_+ \times \mathbb{R}} \frac{t^{1-\beta/2} \, \mathcal{G}_\beta(t,x)}{G_\beta^2(t,x)} = \sup_{y \in \mathbb{R}} \frac{\mathcal{G}_\beta(1,y)}{G_\beta^2(1,y)}$, which is finite by the same reasoning as above, where the parameter c in (4.11) is strictly less than 2 in this case. Thus, $\underline{C}_0 = \underline{\Phi}_\beta > 0$ and (5.16) follows with $\underline{\sigma} = \beta/2 - 1 < -1/2$. The inequality (5.17) is proved by the semigroup property of the heat kernel function and (5.14). So $\underline{C}_1 = 2^{(\beta-2)/4}$. Then apply Proposition 5.5. This completes the proof of Proposition 5.8. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

At the end of this section, we list two technical results that are used in this section.

Lemma 5.9 (Theorem 1.3, p. 32 in [30]). *If $0 < \alpha < 2$, β is an arbitrary complex number and μ is an arbitrary real number such that $\pi\alpha/2 < \mu < \pi \wedge (\pi\alpha)$, then for an arbitrary integer $p \geq 1$ the following expression holds:*

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha} z^{(1-\beta)/\alpha} \exp\left(z^{1/\alpha}\right)$$
$$- \sum_{k=1}^{p} \frac{z^{-k}}{\Gamma(\beta - \alpha k)} + O\left(|z|^{-1-p}\right), \quad |z| \to \infty, \quad |\arg(z)| \leq \mu \, .$$

Lemma 5.10. *Suppose $\beta \in \,]0,2]$. The exponential reference kernel function $\mathcal{G}_{e,\beta}(t,x)$ defined in (3.6) satisfies the sub-semigroup property, i.e., for all $t \geq 0$, $s \geq 0$ and $x \in \mathbb{R}$,*

$$\left(\mathcal{G}_{e,\beta}(t,\cdot) * \mathcal{G}_{e,\beta}(s,\cdot)\right)(x) \leq \widehat{C}_\beta \, \mathcal{G}_{e,\beta}(t+s,x) \, ,$$

where the constant $\widehat{C}_\beta$ is defined in (3.8).

Proof. Fix $a > b > 0$ and let $\theta = \beta/2$. Because

$$\int_\mathbb{R} \frac{1}{4a^\theta b^\theta} \exp\left(-\frac{|x-y|}{a^\theta} - \frac{|y|}{b^\theta}\right) \mathrm{d}y = \frac{1}{2(a^\theta + b^\theta)} \frac{a^\theta \, \exp\left(-\frac{|x|}{a^\theta}\right) - b^\theta \, \exp\left(-\frac{|x|}{b^\theta}\right)}{a^\theta - b^\theta} \, ,$$

we only need to prove that

(5.18)
$$\frac{a^\theta \, \exp\left(-\frac{|x|}{a^\theta}\right) - b^\theta \, \exp\left(-\frac{|x|}{b^\theta}\right)}{a^\theta - b^\theta} \le \widehat{C}_{2\theta} \exp\left(-\frac{|x|}{(a+b)^\theta}\right), \quad \text{for all } (t, x) \in \mathbb{R}_+ \times \mathbb{R}.$$

By setting $r = b/a$ and $\eta = |x|/a^\theta$, (5.18) is equivalent to

$$\frac{e^{-\eta} - r^\theta e^{-\eta/r^\theta}}{1 - r^\theta} \le \widehat{C}_{2\theta} \, \exp\left(-\frac{\eta}{(1+r)^\theta}\right), \quad \text{for all } r \in \,]0, 1[\text{ and } \eta \ge 0 \,.$$

Denote

$$f(r, \eta) = \frac{e^{-\eta} - r^\theta e^{-\eta/r^\theta}}{1 - r^\theta} \exp\left(\frac{\eta}{(1+r)^\theta}\right).$$

Some simple calculations show that

$$\lim_{r \to 0_+} f(r, \eta) = 1 \quad \text{and} \quad \lim_{r \to 1_-} f(r, \eta) = (1+\eta)e^{-(1-2^{-\theta})\eta},$$

and also

$$\lim_{\eta \to 0_+} f(r, \eta) = 1 \quad \text{and} \quad \lim_{\eta \to +\infty} f(r, \eta) = 0.$$

Fix $r \in \,]0, 1[\,$. By solving

$$\frac{\partial f(r, \eta)}{\partial \eta} = \frac{e^{\eta\left(-r^{-\beta} + (1+r)^{-\beta} - 1\right)} \left(e^\eta\left((1+r)^\beta - r^\beta\right) - \left((1+r)^\beta - 1\right)e^{\eta/r^\beta}\right)}{(1+r)^\beta(1 - r^\beta)} = 0,$$

which has one finite solution

$$\eta = -\frac{r^\theta}{1 - r^\theta} \log \frac{(1+r)^\theta - 1}{(1+r)^\theta - r^\theta},$$

we find the local maximum of the function $\eta \mapsto f(r, \eta)$. This local maximum is indeed the global maximum. Hence

$$f(r, \eta) \le h(r) := \frac{(1+r)^\theta}{(1+r)^\theta - 1} \exp\left[\frac{1 - \left(\frac{r}{1+r}\right)^\theta}{1 - r^\theta} \log\left(1 - \frac{1 - r^\theta}{(1+r)^\theta - r^\theta}\right)\right].$$

Since $h'(r) \ge 0$ for all $r \in \,]0, 1[$ and $\theta \in \,]0, 1]$, we have that

$$h(r) \le \lim_{r \to 1} h(r) = \frac{2^\theta}{2^\theta - 1} \exp\left(-\frac{1}{2^\theta}\right) = \widehat{C}_{2\theta}.$$

Therefore, $f(r, \beta) \le \widehat{C}_{2\theta}$. This proves (5.18).

Apply (5.18) with $a = (t \vee s)^{\beta/2}$, $b = (t \wedge s)^{\beta/2}$ and $\theta = \beta/2 \in \,]0, 1]$, and use $(t + s)^{\beta/2} \le t^{\beta/2} + s^{\beta/2}$ to obtain

$$\int_{\mathbb{R}} dy \, \mathcal{G}_{e,\beta}(t, x - y) \, \mathcal{G}_{e,\beta}(s, y) \le \widehat{C}_\beta \, \mathcal{G}_{e,\beta}(t + s, x),$$

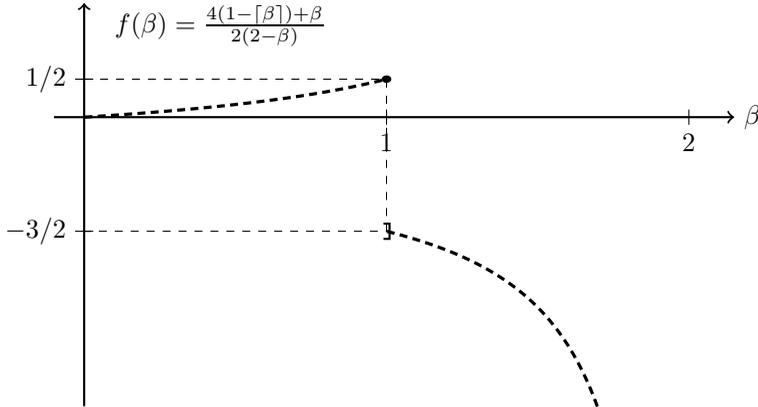which completes the proof of Lemma 5.10. □

Figure 4. Plot of the function $f(\beta)$.

6. Proof of Theorem 3.1

The proof of Theorem 3.1 will be presented at the end of this section. Before proving Theorem 3.1, we need several results. The first one is related to the tails of the Green functions. The corresponding results for the SHE, the SFHE, and the SWE can be found in [6, Proposition 5.3], [8, Proposition 4.7], and [7, Lemma 3.2], respectively. First, we need some notation: for $\tau > 0$, $\alpha > 0$ and $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}$, denote

$$B_{t,x,\tau,\alpha} := \left\{ (t',x') \in \mathbb{R}_+^* \times \mathbb{R} : 0 \leq t' \leq t + \tau, \ |x - x'| \leq \alpha \right\}.$$

Proposition 6.1. *Suppose that $\beta \in {]0,2[}$. Then for all $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}$, there exists a constant $A > 0$ such that for all $(t',x') \in B_{t,x,1/2,1}$ and all $s \in [0,t'[$ and $y \in \mathbb{R}$ with $|y| \geq A$, we have that $G_\beta(t' - s, x' - y) \leq G_\beta(t + 1 - s, x - y)$.*

Proof. Fix $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}$. By the scaling and asymptotic properties of the Green function $G_\beta(\cdot,\cdot)$, we know that

$$
\frac{G_\beta(t + 1 - s, x - y)}{G_\beta(t' - s, x' - y)} = \left(\frac{t' - s}{t + 1 - s} \right)^{\beta/2 + 1 - \lceil \beta \rceil} \frac{G_\beta\left(1, \frac{x-y}{(t+1-s)^{\beta/2}}\right)}{G_\beta\left(1, \frac{x'-y}{(t'-s)^{\beta/2}}\right)}
$$

$$
\approx \left(\frac{t' - s}{t + 1 - s} \right)^{\frac{\beta}{2} + 1 - \lceil \beta \rceil + \frac{a\beta}{2}} \frac{|x - y|^a}{|x' - y|^a}
$$

$$
\times \exp\left(\frac{b|x' - y|^c}{(t' - s)^{\beta c/2}} - \frac{b|x - y|^c}{(t + 1 - s)^{\beta c/2}} \right),
$$

as $|y| \to +\infty$ where $a = \frac{1 + \beta - 2\lceil \beta \rceil}{2 - \beta}$, $b \in {]0,1[}$ and $c > 1$ (see (4.10), (4.12) and (4.11)). Denote

$$f(\beta) = \frac{\beta(1 + a)}{2} + 1 - \lceil \beta \rceil = \frac{4(1 - \lceil \beta \rceil) + \beta}{2(2 - \beta)},$$

which is plotted in Figure 4. Simple calculations show that $f(\beta) > 0$ if $0 < \beta \leq 1$,

and $f(\beta) < 0$ if $1 < \beta < 2$. Notice that

$$(6.1) \quad \frac{t+1-s}{t'-s} = 1 + \frac{t+1-t'}{t'-s} \geq 1 + \frac{t+1-t'}{t'} \geq \frac{t+1}{t+1/2} = 1 + \frac{1}{2t+1} > 1 \ .$$

Hence, if $1 < \beta < 2$, then

$$\left(\frac{t'-s}{t+1-s}\right)^{f(\beta)} = \left(\frac{t+1-s}{t'-s}\right)^{|f(\beta)|} \geq 1 \ .$$

If $0 < \beta \leq 1$, we have that

$$\left(\frac{t'-s}{t+1-s}\right)^{f(\beta)} \geq \left(\frac{t'-s}{t+1}\right)^{|f(\beta)|} = (t+1)^{-|f(\beta)|} \exp\left(|f(\beta)| \, \log(t'-s)\right).$$

Assume $n > 1$. When $|y| \geq |x| + n$, we have $|x-y| > n$ and then

$$\frac{n}{n+1} \leq \frac{|x-y|}{|x-y|+1} \leq \frac{|x-y|}{|x'-y|} \leq \frac{|x-y|}{|x-y|-1} \leq \frac{n}{n-1} \ .$$

Since $\frac{n}{n+1} > \frac{n-1}{n}$ for all $n > 1$, we have that

$$(6.2) \quad \frac{|x-y|^a}{|x'-y|^a} \geq \left(1 - \frac{1}{n}\right)^{|a|} \ ,$$

which holds for all $a \in \mathbb{R}$.

The above bounds (6.1) and (6.2) imply that

$$\left(1 - \frac{1}{n}\right)\frac{|x'-y|^c}{(t'-s)^{\beta c/2}} \geq \left(1 + \frac{1}{2t+1}\right)^{\beta c/2}\left(1 - \frac{1}{n}\right)^{c+1}\frac{|x-y|^c}{(t+1-s)^{\beta c/2}} \ .$$

By choosing n large enough, in particular,

$$n > \left(1 - \left[1 + \frac{1}{2t+1}\right]^{-\frac{\beta c}{2(c+1)}}\right)^{-1} \ ,$$

we have that

$$\eta := \left(1 + \frac{1}{2t+1}\right)^{\beta c/2}\left(1 - \frac{1}{n}\right)^{c+1} > 1 \ .$$

Thus,

$$\exp\left(\frac{b|x'-y|^c}{(t'-s)^{\beta c/2}} - \frac{b|x-y|^c}{(t+1-s)^{\beta c/2}}\right)$$

$$\geq \exp\left(b(\eta-1)\frac{|x-y|^c}{(t+1-s)^{\beta c/2}} + \frac{b|x'-y|^c}{n(t'-s)^{\beta c/2}}\right)$$

$$\geq \exp\left(b(\eta-1)\frac{|x-y|^c}{(t+1)^{\beta c/2}} + \frac{b\,(n-1)^c}{n(t'-s)^{\beta c/2}}\right).$$

Finally, if $1 < \beta < 2$, then

$$\frac{G_\beta(t+1-s, x-y)}{G_\beta(t'-s, x'-y)}$$

$$\geq \left(1 - \frac{1}{n}\right)^{|a|}\exp\left(b(\eta-1)\frac{|x-y|^c}{(t+1)^{\beta c/2}} + \frac{b\,(n-1)^c}{n(t+1)^{\beta c/2}}\right) \to +\infty \ ,$$

as $|y| \to +\infty$. Hence, we can choose a large constant A, such that for all $|y| \geq A$, the inequality

$$\frac{G_\beta(t+1-s, x-y)}{G_\beta(t'-s, x'-y)} > 1$$

holds for all $(t', x') \in B_{t,x,1/2,1}$ and $s \in [0, t']$. If $0 < \beta \leq 1$, then

$$\frac{G_\beta(t+1-s, x-y)}{G_\beta(t'-s, x'-y)} \geq (t+1)^{-f(\beta)} \left(1 - \frac{1}{n}\right)^{|a|}$$
$$\times \exp\left(b(\eta-1)\frac{|x-y|^c}{(t+1)^{\beta c/2}} + \frac{b(n-1)^c}{n(t'-s)^{\beta c/2}} + f(\beta)\log(t'-s)\right).$$

The function

$$g(t) = \frac{C_1}{t} + C_2 \log t, \quad \text{for } t > 0 \text{ and } C_1, C_2 > 0,$$

has its global minimum at $t_0 := C_1/C_2$:

$$\min_{t \in \mathbb{R}_+^*} g(t) = g(t_0) = C_2 (1 + \log(C_1/C_2)),$$

because $g'(t) = C_2 t^{-2}(t - t_0)$, which is negative when $t < t_0$ and positive when $t > t_0$. Hence,

$$\frac{b(n-1)^c}{n(t'-s)^{\beta c/2}} + f(\beta)\log(t'-s) = \frac{b(n-1)^c}{n(t'-s)^{\beta c/2}} + \frac{2f(\beta)}{\beta c}\log\left[(t'-s)^{\beta c/2}\right]$$
$$\geq \frac{2f(\beta)}{\beta c}\left(1 + \log\left(\frac{2f(\beta)n}{b\beta c(n-1)^c}\right)\right).$$

Therefore,

$$\frac{G_\beta(t+1-s, x-y)}{G_\beta(t'-s, x'-y)} \geq (t+1)^{-f(\beta)}\left(1 - \frac{1}{n}\right)^{|a|}$$
$$\times \exp\left(b(\eta-1)\frac{|x-y|^c}{(t+1)^{\beta c/2}} + \frac{2f(\beta)}{\beta c}\left(1 + \log\left(\frac{2f(\beta)n}{b\beta c(n-1)^c}\right)\right)\right) \to +\infty,$$

as $|y| \to +\infty$. We can choose a large constant A, such that for all $|y| \geq A$, all $(t', x') \in B_{t,x,1/2,1}$ and $s \in [0, t']$,

$$\frac{G_\beta(t+1-s, x-y)}{G_\beta(t'-s, x'-y)} > 1.$$

This completes the proof of Proposition 6.1. $\qquad\qquad\qquad\qquad\qquad\square$

The second set of results, Propositions 6.3 and 6.4, give some continuity properties of the Green functions. We need a bound of the two-parameter Mittag-Leffler functions, which will be used in the proof of Proposition 6.3.

Lemma 6.2. *If $0 < \alpha < 1$ and $\beta \geq \alpha$, then there exists a constant $C_{\alpha,\beta} > 0$ such that*

$$(6.3) \qquad\qquad 0 < E_{\alpha,\beta}(-x^\alpha) \leq \frac{C_{\alpha,\beta}}{1 + x^\alpha}, \qquad \text{for all } x \geq 0.$$

Proof. Nonnegativity is due to (4.3). The upper bound is due to [30, Theorem 1.6, on p. 35] with $z = -x^\alpha$. Clearly $\arg(z) = \pi$ satisfies the required condition. $\qquad\square$

Proposition 6.3. *Suppose* $0 < \beta < 1$*. Let* $C_{\beta,2}$ *be the universal constant in Lemma 6.2. Then the following two properties hold:*

(i) *For all* $t > 0$ *and* $x, y \in \mathbb{R}$,

$$(6.4) \quad \iint_{\mathbb{R}_+ \times \mathbb{R}} \mathrm{d}r \mathrm{d}z \, (G_\beta(t - r, x - z) - G_\beta(t - r, y - z))^2 = \frac{4C_{\beta,2}}{\pi} t^{1-\beta}|x - y|.$$

(ii) *For all* $s, t \in \mathbb{R}_+^*$ *with* $s \le t$*, and* $x \in \mathbb{R}$,

$$(6.5) \quad \int_0^s \mathrm{d}r \int_{\mathbb{R}} \mathrm{d}z \, (G_\beta(t - r, x - z) - G_\beta(s - r, x - z))^2 = 2C_{\beta,2}(t - s)^{1-\beta/2} \,,$$

and

$$(6.6) \quad \int_s^t \mathrm{d}r \int_{\mathbb{R}} \mathrm{d}z \, G_\beta^2(t - r, x - z) = \frac{C_{\beta,2}}{2}(t - s)^{1-\beta/2} \,.$$

Proof. (i) Fix $t > 0$. By Plancherel's theorem and (4.14), the left-hand side (l.h.s.) of (6.4) is equal to

$$\frac{1}{2\pi} \int_0^t \mathrm{d}r \int_{\mathbb{R}} \mathrm{d}\xi \, \left| e^{-i\xi x} E_{\beta,1}\left(-(t - r)^\beta \xi^2\right) - e^{-i\xi y} E_{\beta,1}\left(-(t - r)^\beta \xi^2\right) \right|^2$$

$$= \frac{1}{2\pi} \int_0^t \mathrm{d}r \int_{\mathbb{R}} \mathrm{d}\xi \, E_{\beta,1}^2\left(-(t - r)^\beta \xi^2\right) \, \left| e^{-i\xi x} - e^{-i\xi y} \right|^2$$

$$= \frac{1}{\pi} \int_{\mathbb{R}} \mathrm{d}\xi \, (1 - \cos(\xi(x - y))) \int_0^t \mathrm{d}r \, E_{\beta,1}^2\left(-(t - r)^\beta \xi^2\right).$$

By (4.3),

$$\int_0^t \mathrm{d}r \, E_{\beta,1}^2\left(-(t - r)^\beta \xi^2\right) \le E_{\beta,1}(0) \int_0^t \mathrm{d}r \, E_{\beta,1}\left(-(t - r)^\beta \xi^2\right) = t \, E_{\beta,2}\left(-t^\beta \xi^2\right),$$

where the last equality can be obtained by integration term-by-term (see also [30, (1.99), p. 24]). Then use the bound (6.3) and the fact that $1 - \cos(x) \le 2 \wedge (x^2/2)$ for all $x \in \mathbb{R}$ to see that the l.h.s. of (6.4) is bounded by

$$\frac{C_{\beta,2}t^{1-\beta}}{\pi} \int_{\mathbb{R}} \mathrm{d}\xi \, \frac{2 \wedge [(x - y)\xi/\sqrt{2}]^2}{\xi^2} = \frac{\sqrt{2} \, C_{\beta,2}t^{1-\beta}}{\pi}|x - y| \int_0^\infty \mathrm{d}u \, \frac{2 \wedge u^2}{u^2}$$

$$= \frac{4C_{\beta,2}t^{1-\beta}}{\pi}|x - y|.$$

(ii) Denote the l.h.s. of (6.5) by I. Apply Plancherel's theorem and use (4.3) to obtain

$$I = \frac{1}{2\pi} \int_0^s \mathrm{d}r \int_{\mathbb{R}} \mathrm{d}\xi \, \left| e^{-i\xi x} E_{\beta,1}\left(-(t - r)^\beta \xi^2\right) - e^{-i\xi x} E_{\beta,1}\left(-(s - r)^\beta \xi^2\right) \right|^2$$

$$= \frac{1}{2\pi} \int_0^s \mathrm{d}r \int_{\mathbb{R}} \mathrm{d}\xi \, \left| E_{\beta,1}\left(-(t - r)^\beta \xi^2\right) - E_{\beta,1}\left(-(s - r)^\beta \xi^2\right) \right|^2$$

$$\le \frac{1}{2\pi} \int_0^s \mathrm{d}r \int_{\mathbb{R}} \mathrm{d}\xi \, 2E_{\beta,1}(0) \left[E_{\beta,1}\left(-(t - r)^\beta \xi^2\right) - E_{\beta,1}\left(-(s - r)^\beta \xi^2\right) \right].$$

Integration term-by-term gives that

$$\int_0^s \mathrm{d}r \, E_{\beta,1}\left(-(t - r)^\beta \xi^2\right) = \int_{t-s}^t \mathrm{d}r \, E_{\beta,1}\left(-r^\beta \xi^2\right)$$

$$= t \, E_{\beta,2}\left(-t^\beta \xi^2\right) - (t - s) E_{\beta,2}\left(-(t - s)^\beta \xi^2\right) \,.$$

Hence, by (4.3) again,

$$I \leq \frac{1}{\pi} \int_{\mathbb{R}} \mathrm{d}\xi \left(t E_{\beta,2} \left(-t^\beta \xi^2 \right) - s E_{\beta,2} \left(-s^\beta \xi^2 \right) - (t-s) E_{\beta,2} \left(-(t-s)^\beta \xi^2 \right) \right)$$

$$\leq \frac{1}{\pi}(t-s) \int_{\mathbb{R}} \mathrm{d}\xi \left(E_{\beta,2} \left(-t^\beta \xi^2 \right) + E_{\beta,2} \left(-(t-s)^\beta \xi^2 \right) \right).$$

Then by the bound in (6.3) and the integral $\int_{\mathbb{R}} \mathrm{d}\xi \frac{1}{1+c^2\xi^2} = \pi/|c|$ for $c \neq 0$, we find that

$$I \leq C_{\beta,2}(t-s) \left(\frac{1}{t^{\beta/2}} + \frac{1}{(t-s)^{\beta/2}} \right) \leq 2 C_{\beta,2}(t-s)^{1-\beta/2}.$$

As for (6.6), by similar reasoning, we have

$$\int_s^t \mathrm{d}r \int_{\mathbb{R}} \mathrm{d}z \, G_\beta^2 \left(t-r, x-z \right) \leq \frac{1}{2\pi} \int_s^t \mathrm{d}r \int_{\mathbb{R}} \mathrm{d}\xi \, E_{\beta,1}^2 \left(-(t-r)^\beta \xi^2 \right)$$

$$\leq \frac{1}{2\pi} \int_s^t \mathrm{d}r \int_{\mathbb{R}} \mathrm{d}\xi \, E_{\beta,1} \left(-(t-r)^\beta \xi^2 \right) \leq \frac{t-s}{2\pi} \int_{\mathbb{R}} \mathrm{d}\xi \, E_{\beta,2} \left(-(t-s)^\beta \xi^2 \right)$$

$$\leq \frac{t-s}{2\pi} \int_{\mathbb{R}} \mathrm{d}\xi \, \frac{C_{\beta,2}}{1+(t-s)^\beta \xi^2} = \frac{C_{\beta,2}}{2}(t-s)^{1-\beta/2} \,,$$

which completes the proof of Proposition 6.3. $\square$

For the fast diffusion equations, we are only able to prove the following less precise results in Proposition 6.4 due to the lack of complete monotonicity for $E_{\alpha,\beta}(-x)$ with $\alpha > 1$; see (4.3) for the necessary and sufficient conditions for $E_{\alpha,\beta}(-x)$ to be completely monotonic.

Proposition 6.4. *For all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$ and $1 < \beta < 2$, we have*

$$\lim_{(t',x') \to (t,x)} \iint_{\mathbb{R}_+ \times \mathbb{R}} \mathrm{d}s \mathrm{d}y \, \left(G_\beta \left(t'-s, x'-y \right) - G_\beta \left(t-s, x-y \right) \right)^2 = 0.$$

Proof. We only need to consider the case where $t > 0$. Fix $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}$. Denote $\Lambda := \sup_{x \in \mathbb{R}} G_\beta(1,x)$. We are going to apply the Lebesgue dominated convergence theorem. Clearly, by the continuity of the Green functions, for all $(s,y) \in \mathbb{R}_+^* \times \mathbb{R}$,

$$G_\beta \left(t'-s, x'-y \right) - G_\beta \left(t-s, x-y \right) \to 0 \,, \quad \text{as } (t',x') \to (t,x).$$

We need to find an integrable bound. Choose $A > 0$ according to Proposition 6.1 and suppose that $(t',x') \in B_{t,x,1/2,1}$. If $|y| > A$, since $1 - \beta/2 > 0$, by Proposition 6.1,

$$\left| G_\beta \left(t'-s, x'-y \right) - G_\beta \left(t-s, x-y \right) \right|^2 \leq 4 G_\beta^2 \left(t+1-s, x-y \right)$$

$$\leq 4\Lambda(t+1-s)^{1-\beta/2} G_\beta \left(t+1-s, x-y \right)$$

$$\leq 4\Lambda(t+1)^{1-\beta/2} G_\beta \left(t+1-s, x-y \right).$$

If $|y| \leq A$, we have that

$$\left| G_\beta \left(t'-s, x'-y \right) - G_\beta \left(t-s, x-y \right) \right|^2 \leq 2 G_\beta^2 \left(t'-s, x'-y \right) + 2 G_\beta^2 \left(t-s, x-y \right)$$

$$\leq 2\Lambda^2 \left[(t'-s)^{2-\beta} + (t-s)^{2-\beta} \right]$$

$$\leq 4\Lambda^2 (t+1)^{2-\beta} \,.$$

Hence,

$$|G_\beta\left(t'-s, x'-y\right) - G_\beta\left(t-s, x-y\right)|^2$$
$$\leq 4\Lambda(t+1)^{1-\beta/2}G_\beta\left(t+1-s, x-y\right)\mathbf{1}_{\{|y|>A\}} + 4\Lambda^2(t+1)^{2-\beta}\mathbf{1}_{\{|y|<A,\,0\leq s\leq t+1\}}\ .$$

Denote this upper bound by $f(s, y)$. Clearly, this upper bound is integrable:

$$\int_{\mathbb{R}_+}\mathrm{d}s\int_{\mathbb{R}}\mathrm{d}y\, f(s,y) \leq 4\Lambda(t+1)^{1-\beta/2}\iint_{\mathbb{R}_+\times\mathbb{R}}\mathrm{d}s\mathrm{d}y\, G_\beta\left(t+1-s, x-y\right)$$
$$+ 4\Lambda^2(t+1)^{2-\beta}\iint_{[0,t+1]\times[-A,A]}\mathrm{d}s\mathrm{d}y$$
$$= 2\Lambda(t+1)^{3-\beta/2} + 8A\Lambda^2(t+1)^{3-\beta} < +\infty\ .$$

Therefore, this proposition is proved by the Lebesgue dominated convergence theorem. $\qquad\square$

The third result, Proposition 6.6, is about solutions to the homogeneous equation. We need to prove a lemma first. Recall the function $f_\eta(x)$ defined in (1.7).

Lemma 6.5. *Suppose $\beta \in\,]0, 2[\,$. Let $b \in\,]0, 1[\,$ be the constant defined in (4.12). Then for all $\eta > 0$, the three functions*

$$f_\eta\left(\frac{x}{t^{\beta/2}}\right), \quad G_\beta(t,x)f_b^{-1}\left(\frac{x}{t^{\beta/2}}\right), \quad \text{and} \quad G_\beta^*(t,x)f_b^{-1}\left(\frac{x}{t^{\beta/2}}\right)$$

are Lipschitz continuous over $\mathbb{R}_+^* \times \mathbb{R}$; that is, for all $x, y \in \mathbb{R}$ and $t, s \geq \epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that

$$(6.7) \qquad \left|f_\eta\left(\frac{x}{t^{\beta/2}}\right) - f_\eta\left(\frac{y}{s^{\beta/2}}\right)\right| \leq C_\epsilon\left(|x-y| + |t-s|\right)\ ,$$

$$(6.8) \qquad \left|G_\beta(t,x)f_b^{-1}\left(\frac{x}{t^{\beta/2}}\right) - G_\beta(s,y)f_b^{-1}\left(\frac{y}{s^{\beta/2}}\right)\right| \leq C_\epsilon\left(|x-y| + |t-s|\right)\ ,$$

$$(6.9) \qquad \left|G_\beta^*(t,x)f_b^{-1}\left(\frac{x}{t^{\beta/2}}\right) - G_\beta^*(s,y)f_b^{-1}\left(\frac{y}{s^{\beta/2}}\right)\right| \leq C_\epsilon\left(|x-y| + |t-s|\right).$$

Proof. (i) We first prove (6.7). Denote

$$g(t,x) = f_\eta\left(\frac{x}{t^{\beta/2}}\right) = \exp\left(-\frac{\eta}{2t^{\beta c/2}}|x|^c\right)\ ,$$

where $c = \frac{2}{2-\beta}$. Fix $x \neq 0$. Clearly,

$$\left|\frac{\partial}{\partial x}g(t,x)\right| = \frac{\eta\,c|x|^{c-1}}{2\,t^{\beta c/2}}g(t,x) \leq \frac{c\,\eta}{2\,t^{\beta/2}}\sup_{y\in\mathbb{R}}|y|^{c-1}f_\eta(y)$$

and

$$\left|\frac{\partial}{\partial t}g(t,x)\right| = \frac{\eta\beta c|x|^c}{4\,t^{\beta c/2-1}}g(t,x) \leq \frac{\eta\,\beta\,c}{4\,t}\sup_{y\in\mathbb{R}}|y|^c f_\eta(y)\ .$$

Note that the two suprema are finite because $c > 1$. Hence, by the mean value theorem,

$$|g(t,x) - g(s,y)| \leq |g(t,x) - g(t,y)| + |g(t,y) - g(s,y)|$$
$$\leq \frac{C_1}{\epsilon^{\beta/2}}|x-y| + \frac{C_2}{\epsilon}|t-s|$$

for $xy \geq 0$ (i.e., x and y have the same sign) and $t, s \geq \epsilon > 0$, where

$$C_1 = \frac{\eta}{2-\beta}\sup_{y\in\mathbb{R}}|y|^{\frac{\beta}{2-\beta}}f_\eta(y) \quad \text{and} \quad C_2 = \frac{\eta\beta}{2(2-\beta)}\sup_{y\in\mathbb{R}}|y|^{\frac{2}{2-\beta}}f_\eta(y)\ .$$

When x and y have different signs, we use the fact that

$$|g(t,x) - g(t,y)| \leq |g(t,x) - g(t,0)| + |g(t,0) - g(t,y)|$$

$$\text{(6.10)} \qquad \leq \frac{C_1}{\epsilon^{\beta/2}}\left(|x| + |y|\right) = \frac{C_1}{\epsilon^{\beta/2}}\,|x - y|\,.$$

(ii) Now let us prove (6.8). Denote

$$h(t,x) = G_\beta(t,x) f_b^{-1}\left(\frac{x}{t^{\beta/2}}\right).$$

We assume that both x and y are nonnegative. The case where $xy < 0$ can be covered by a similar argument as in (6.10). By (4.17),

$$\frac{\partial}{\partial x} h(t,x) = -\frac{t^{\lceil\beta\rceil - 1 - \beta}}{2} M_{\beta/2, \lceil\beta\rceil - \beta/2}\left(\frac{x}{t^{\beta/2}}\right) \exp\left(\frac{b}{2}\left|\frac{x}{t^{\beta/2}}\right|^c\right)$$

$$+ \frac{bct^{\lceil\beta\rceil - 1 - \beta}}{4}\left|\frac{x}{t^{\beta/2}}\right|^{c-1} \exp\left(\frac{b}{2}\left|\frac{x}{t^{\beta/2}}\right|^c\right) M_{\beta/2, \lceil\beta\rceil}\left(\frac{x}{t^{\beta/2}}\right).$$

Since the exponent of the asymptotic of $M_{\lambda,\theta}(x)$ in (4.19) depends only on the first parameter λ, we have that

$$C_{\alpha,\theta} := \sup_{y \in \mathbb{R}} |y|^\alpha \left|M_{\beta/2,\theta}(y)\right| \exp\left(\frac{b}{2}|y|^c\right) < +\infty\,, \quad \text{for all } \theta \in \mathbb{R} \text{ and } \alpha > 0\,.$$

Note that $\lceil\beta\rceil - 1 - \beta \leq 0$. Therefore,

$$\left|\frac{\partial}{\partial x} h(t,x)\right| \leq C_\epsilon'\,, \quad \text{for all } x \in \mathbb{R} \text{ and } t \geq \epsilon > 0,$$

where

$$C_\epsilon' = \frac{\epsilon^{\lceil\beta\rceil - 1 - \beta}}{2} C_{0,\lceil\beta\rceil - \beta/2} + \frac{b\,c\,\epsilon^{\lceil\beta\rceil - 1 - \beta}}{4} C_{c-1,\lceil\beta\rceil}\,.$$

By (4.20), we have

$$\frac{\partial}{\partial t} G_\beta(t,x) = \frac{\lceil\beta\rceil - 1 - \beta/2}{2} t^{\lceil\beta\rceil - 2 - \beta/2} M_{\beta/2, \lceil\beta\rceil}\left(\frac{x}{t^{\beta/2}}\right)$$

$$+ \frac{\beta}{2}\left(\frac{x}{t^{\beta/2}}\right) t^{\lceil\beta\rceil - 2 - \beta/2} M_{\beta/2, \lceil\beta\rceil - \beta/2}\left(\frac{x}{t^{\beta/2}}\right).$$

Notice that

$$\frac{\partial}{\partial t} \exp\left(\frac{b}{2}\left|\frac{x}{t^{\beta/2}}\right|^c\right) = -\frac{\beta\,c\,b}{4}\left(\frac{x}{t^{\beta/2}}\right)^c t^{-1} \exp\left(\frac{b}{2}\left|\frac{x}{t^{\beta/2}}\right|^c\right).$$

Hence,

$$\frac{\partial}{\partial t} h(t,x) = \frac{\lceil\beta\rceil - 1 - \beta/2}{2} t^{\lceil\beta\rceil - 2 - \beta/2} M_{\beta/2, \lceil\beta\rceil}\left(\frac{x}{t^{\beta/2}}\right) \exp\left(\frac{b}{2}\left|\frac{x}{t^{\beta/2}}\right|^c\right)$$

$$+ \frac{\beta}{2}\frac{x}{t^{\beta/2}} t^{\lceil\beta\rceil - 2 - \beta/2} M_{\beta/2, \lceil\beta\rceil - \beta/2}\left(\frac{x}{t^{\beta/2}}\right) \exp\left(\frac{b}{2}\left|\frac{x}{t^{\beta/2}}\right|^c\right)$$

$$- \frac{\beta\,c\,b}{8}\left(\frac{x}{t^{\beta/2}}\right)^c t^{\lceil\beta\rceil - 2 - \beta/2} M_{\beta/2, \lceil\beta\rceil}\left(\frac{x}{t^{\beta/2}}\right) \exp\left(\frac{b}{2}\left|\frac{x}{t^{\beta/2}}\right|^c\right).$$

Note that $\lceil\beta\rceil - 2 - \beta/2 \leq 0$. Therefore,

$$\left|\frac{\partial}{\partial x} h(t,x)\right| \leq C_\epsilon''\,, \quad \text{for all } x \in \mathbb{R} \text{ and } t \geq \epsilon > 0,$$

where

$$C_\epsilon'' = \epsilon^{\lceil\beta\rceil - 2 - \beta/2} \left(\frac{|\lceil\beta\rceil - 1 - \beta/2|}{2} C_{0,\lceil\beta\rceil} + \frac{\beta}{2} C_{1,\lceil\beta\rceil - \beta/2} + \frac{\beta \, c \, b}{8} C_{c,\lceil\beta\rceil} \right).$$

Finally, apply the mean value theorem to conclude this case. The argument is the same as (i).

(iii) (6.9) can be proved in the same way. We will not repeat here. This completes the proof of Lemma 6.5. $\qquad\square$

Proposition 6.6. *Suppose that $0 < \beta < 2$ and $\mu \in \mathcal{M}_T^\beta (\mathbb{R})$. Denote*

$$J_1(t,x) := (G_\beta(t,\cdot) * \mu)(x) \quad and \quad J_2(t,x) := (G_\beta^*(t,\cdot) * \mu)(x).$$

(1) Both functions $J_i(t,x)$, $i = 1,2$, are locally Lipschitz continuous on $\mathbb{R}_+^* \times \mathbb{R}$, that is, for all compact sets $K \subseteq \mathbb{R}_+^* \times \mathbb{R}$, there exits a constant $C_K > 0$ such that

$$|J_i(t,x) - J_i(s,y)| = C_K \left(|t - s| + |x - y| \right), \quad for \ all \ (t,x) \ and \ (s,y) \in K.$$

Hence, the solution $J_0(t,x)$ in (2.3) is locally Lipschitz continuous on $\mathbb{R}_+^* \times \mathbb{R}$.

(2) If $0 < \beta \le 1$ and if $\mu(\mathrm{d}x) = f(x)\mathrm{d}x$ where f is α-Hölder continuous with $\alpha \in \,]0,1]$, then $J_1(\cdot, \circ) \in C_{\alpha\beta/2, \, \alpha\beta} (\mathbb{R}_+ \times \mathbb{R})$.

A similar proof for part (2) for SHE can be found in [6, Lemma 3.8].

Proof. (1) We first show the Lipschitz continuity of the function $(t,x) \mapsto J_1(t,x)$ for $0 < \beta < 2$. Let $\epsilon = \inf \{s : (s,y) \in K\}$, $T = \sup \{s : (s,y) \in K\}$ and $k = \sup \{|y| : (s,y) \in K\}$. Since K is a compact set of $\mathbb{R}_+^* \times \mathbb{R}$, we know that $\epsilon > 0$, $T < +\infty$ and $k < +\infty$. Suppose $\epsilon \le t, s \le T$ and $x,y \in [-k,k]$. Notice that

$$|J_1(t,x) - J_1(s,y)| \le \int_{\mathbb{R}} |\mu|(\mathrm{d}z) \, |G_\beta(t, x - z) - G_\beta(s, y - z)|$$

and

$$\left| G_\beta(t, x - z) - G_\beta(s, y - z) \right|$$
$$\le \left| G_\beta(t, x - z) f_b^{-1} \left(\frac{x - z}{t^{\beta/2}} \right) - G_\beta(s, y - z) f_b^{-1} \left(\frac{y - z}{s^{\beta/2}} \right) \right| f_b \left(\frac{x - z}{t^{\beta/2}} \right)$$
$$+ \left| f_b \left(\frac{x - z}{t^{\beta/2}} \right) - f_b \left(\frac{y - z}{s^{\beta/2}} \right) \right| G_\beta(s, y - z) f_b^{-1} \left(\frac{y - z}{s^{\beta/2}} \right),$$

where $b > 0$ is defined in (4.12). By Lemma 6.5, there is a constant $C_\epsilon > 0$ such that

$$|G_\beta(t, x - z) - G_\beta(s, y - z)| \le C_\epsilon \left(|t - s| + |x - y| \right).$$

By the asymptotics of $G_\beta(s,y)$ with s fixed, we know that for some constant $C > 0$,

$$(6.11) \quad G_\beta(s, y - z) f_b^{-1} \left(\frac{y - z}{s^{\beta/2}} \right) \le C \, s^{\lceil\beta\rceil - 1 - \beta/2} f_{b/2} \left(\frac{y - z}{s^{\beta/2}} \right), \quad for \ all \ z \in \mathbb{R}.$$

Notice that $f_b \left(\frac{x}{t^{\beta/2}} \right) \le f_{b \, \epsilon^{\beta c/2}}(x)$ if $t \ge \epsilon$, where $c = \frac{2}{2 - \beta}$. Since

$$-\frac{1}{2} \le \lceil\beta\rceil - 1 - \beta/2 \le \frac{1}{2},$$

we have

$$s^{\lceil\beta\rceil - 1 - \beta/2} \le \sqrt{T \vee \epsilon^{-1}}, \quad for \ \epsilon \le s \le T,$$

where $a \vee b := \max(a, b)$. Therefore,

$$
|J_1(t,x) - J_1(s,y)| \leq C_\epsilon \left[\left(|\mu| * f_{b\ \epsilon^{\beta c/2}} \right)(x) + C\sqrt{T \vee \epsilon^{-1}} \left(|\mu| * f_{b\ \epsilon^{\beta c/2}/2} \right)(y) \right]
$$
$$
\times \left(|t-s| + |x-y| \right),
$$

for all $x, y \in \mathbb{R}$ and $t, s \geq \epsilon$. The function $x \mapsto (|\mu| * f_\eta)(x)$ is well-defined because $\mu \in \mathcal{M}_T^\beta(\mathbb{R})$. Moreover, it is continuous, which can be easily proved by the dominated convergence theorem thanks to the continuity and boundedness of $f_\eta(x)$.

As for the function $J_2(t,x)$, we simply change the power of s in (6.11) by $-\beta/2$ and so

$$
s^{-\beta/2} \leq \epsilon^{-\beta/2} \leq \epsilon^{-1} , \quad \text{for } s \geq \epsilon \text{ and } 1 < \beta < 2 .
$$

Hence, we need to replace the term $\sqrt{T \vee \epsilon^{-1}}$ by ϵ^{-1}. Clearly, $\sqrt{T \vee \epsilon^{-1}} \vee \epsilon^{-1} = \sqrt{T} \vee \epsilon^{-1}$. Finally, we can choose the following constant for both $J_1(t,x)$ and $J_2(t,x)$:

$$
C_K = 2C_\epsilon \left(\sup_{x \in [-k,k]} \left(|\mu| * f_{b\ \epsilon^{\beta c/2}} \right)(x) + C \left(\sqrt{T} \vee \epsilon^{-1} \right) \sup_{x \in [-k,k]} \left(|\mu| * f_{b\ \epsilon^{\beta c/2}/2} \right)(x) \right)
$$
$$
< +\infty .
$$

(2) Fix (t,x) and $(t', x') \in \mathbb{R}_+ \times \mathbb{R}$ with $t' > t$. Then we have that

$$
|J_1(t,x) - J_1(t',x')| \leq |J_1(t,x) - J_1(t',x)| + |J_1(t',x) - J_1(t',x')|
$$
$$
:= I_1(t,t';x) + I_1(t';x,x').
$$

By a change of variables and the Hölder continuity of f, for some constant $C > 0$,

$$
I_1(t,t';x) = \left| \int_{\mathbb{R}} \mathrm{d}y \ \left(G_\beta(t, x-y) - G_\beta(t', x-y) \right) f(y) \right|
$$
$$
= \left| \int_{\mathbb{R}} \mathrm{d}z \ G_\beta(1, z) \left(f(x - t^{\beta/2} z) - f(x - (t')^{\beta/2} z) \right) \right|
$$
$$
\leq C \left| t^{\beta/2} - (t')^{\beta/2} \right|^\alpha \int_{\mathbb{R}} \mathrm{d}z \ G_\beta(1, z) |z|^\alpha,
$$

where the integral is finite by (4.13). By subadditivity of the function $x \in \mathbb{R}_+ \mapsto x^{\beta/2}$, $(t')^{\beta/2} - t^{\beta/2} \leq |t' - t|^{\beta/2}$. The arguments for $I_2(t'; x, x')$ are similar. We will not repeat them here. This completes the proof of Proposition 6.6. $\square$

The last result, Lemma 6.7, is about the initial data. Similar results for the SHE, the SFHE, and the SWE can be proved in [6, Lemma 3.9], [8, Lemma 4.9] and [7, Lemma 3.4], respectively. Recall that $J_0(t,x)$ is the solution to the homogeneous equation; see (2.3).

Lemma 6.7. *Suppose* $0 < \beta < 2$. *For all* μ *and* $\nu \in \mathcal{M}_T^\beta(\mathbb{R})$, *all compact sets* $K \subseteq \mathbb{R}_+^* \times \mathbb{R}$,

$$
\sup_{(t,x) \in K} \left([1 + J_0^2] \star \mathcal{K} \right)(t,x) < \infty.
$$

Proof. We need only consider the part $J_0^2 \star \mathcal{K}$ because the part $1 \star \mathcal{K}$ can be obtained by the special case where $\mu(\mathrm{d}x) = \mathrm{d}x$. Assume that $\mu \geq 0$. For general μ, we simply replace μ below by $|\mu|$. The case $\beta = 1$ is covered by [6, Lemma 3.9]. Note that by

(5.6) and Proposition 5.8, for the two constants c_1 and $c_2 > 0$, one has that for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$,

$$(6.12) \qquad \mathcal{K}(t, x) \leq \mathcal{G}_\beta(t, x) h(t), \quad \text{with} \quad h(t) := c_1 \left(t^{-\sigma} + e^{c_2 \, t} \right),$$

where $\mathcal{G}_\beta(t, x)$ is defined in (3.5) and $\sigma < 1$ is defined in (3.13). In the following, denote $\bar{z} = (z_1 + z_2)/2$ and $\Delta z = z_1 - z_2$.

Slow diffusions. Fix $\beta \in \,]0, 1[$. By the same argument as Proposition 5.8, for some nonnegative constant $C_\beta < \infty$, $G_\beta(t, x) \leq C_\beta \, \mathcal{G}_{e,\beta}(t, x)$ for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$. Thus,

$$J_0^2(s, y) \leq C_\beta^2 \iint_{\mathbb{R}^2} \mu(\mathrm{d}z_1)\mu(\mathrm{d}z_2) \, \mathcal{G}_{e,\beta}(s, y - z_1)\mathcal{G}_{e,\beta}(s, y - z_2).$$

Because $K(t, x) \leq C_\beta \, \mathcal{G}_{e,\beta}(t, x) \, h(t)$, we see that

$$\left(J_0^2 \star \mathcal{K} \right)(t, x) \leq C_\beta^2 \int_0^t \mathrm{d}s \, h(t - s) \int_{\mathbb{R}} \mathrm{d}y \iint_{\mathbb{R}^2} \mu(\mathrm{d}z_1)\mu(\mathrm{d}z_2)$$
$$\times \, \mathcal{G}_{e,\beta}(s, y - z_1) \, \mathcal{G}_{e,\beta}(s, y - z_2) \, \mathcal{G}_{e,\beta}(t - s, x - y).$$

By the inequality

$$(6.13) \qquad \left| \frac{a+b}{2} \right| + \left| \frac{a-b}{2} \right| \leq |a| + |b|,$$

we see that $\mathcal{G}_{e,\beta}(s, y - z_1) \, \mathcal{G}_{e,\beta}(s, y - z_2) \leq \mathcal{G}_{e,\beta}(s, y - \bar{z}) \, \mathcal{G}_{e,\beta}(s, 1/2 \, \Delta z)$. Then we integrate over $\mathrm{d}y$ using Lemma 5.10,

$$\left(J_0^2 \star \mathcal{K} \right)(t, x) \leq C_\beta^2 \, \widehat{C}_\beta \int_0^t \mathrm{d}s \, h(t - s)$$
$$\times \iint_{\mathbb{R}^2} \mu(\mathrm{d}z_1)\mu(\mathrm{d}z_2) \, \mathcal{G}_{e,\beta}(s, 1/2 \, \Delta z) \, \mathcal{G}_{e,\beta}\left(2^{1/\beta}t, x - \bar{z} \right).$$

By (6.13) again,

$$|z_1 - z_2| + |(x - z_1) + (x - z_2)| \geq \left| \frac{1}{2} \left[(x - z_1) + (x - z_2) - (z_1 - z_2) \right] \right|$$
$$+ \left| \frac{1}{2} \left[(x - z_1) + (x - z_2) + (z_1 - z_2) \right] \right|$$
$$= |x - z_1| + |x - z_2|,$$

and thus

$$\mathcal{G}_{e,\beta}(s, 1/2 \, \Delta z) \, \mathcal{G}_{e,\beta}\left(2^{1/\beta}t, x - \bar{z} \right)$$
$$= \frac{1}{4\sqrt{2} \, (st)^{\beta/2}} \exp\left(-\frac{|z_1 - z_2|}{2 \, s^{\beta/2}} - \frac{|(x - z_1) + (x - z_2)|}{\sqrt{8} \, t^{\beta/2}} \right)$$
$$\leq \frac{1}{4\sqrt{2} \, (st)^{\beta/2}} \exp\left(-\frac{|z_1 - z_2|}{\sqrt{8} \, t^{\beta/2}} - \frac{|(x - z_1) + (x - z_2)|}{\sqrt{8} \, t^{\beta/2}} \right)$$
$$\leq \frac{1}{4\sqrt{2} \, (st)^{\beta/2}} \exp\left(-\frac{|x - z_1|}{\sqrt{8} \, t^{\beta/2}} - \frac{|x - z_2|}{\sqrt{8} \, t^{\beta/2}} \right)$$
$$= 4\sqrt{2} \, \mathcal{G}_{e,\beta}\left(2^{6/\beta}t, x - z_1 \right) \, \mathcal{G}_{e,\beta}\left(2^{6/\beta}t, x - z_2 \right).$$

Denote $I(t,x) = \left(\mu * \mathcal{G}_{e,\beta}\left(2^{6/\beta}t, \cdot\right)\right)(x)$. Clearly, $\mu \in \mathcal{M}_T^\beta(\mathbb{R})$ implies that $I(t,x) < +\infty$ for all $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}$. Therefore,

$$\left(J_0^2 \star \mathcal{K}\right)(t,x) \le 4\sqrt{2}\, C_\beta^2\, \widehat{C}_\beta\, I(t,x)^2 \int_0^t \mathrm{d}s\, h(t-s).$$

Clearly, the ds-integral is integrable because $\sigma < 1$.

Fast diffusions. Fix $\beta \in\,]1,2[$. By (2.3), we only need to consider two cases: Case I — $\mu = 0$ and $\nu \ne 0$, and Case II — $\mu \ne 0$ and $\nu = 0$.

We first consider Case I. By the same arguments as in Proposition 5.8, for some nonnegative constant $C_\beta < +\infty$, $G_\beta(t,x) \le C_\beta t\, G_1(t^\beta, x)$. Thus,

$$J_0^2(s,y) \le C_\beta^2 s^2 \iint_{\mathbb{R}^2} \nu(\mathrm{d}z_1)\nu(\mathrm{d}z_2)\, G_1(s^\beta, y-z_1)G_1(s^\beta, y-z_2).$$

By (6.12), we see that

$$\left(J_0^2 \star \mathcal{K}\right)(t,x) \le C_\beta^2 \int_0^t \mathrm{d}s\, h(t-s)s^2 \int_\mathbb{R} \mathrm{d}y \iint_{\mathbb{R}^2} \nu(\mathrm{d}z_1)\nu(\mathrm{d}z_2)$$
$$\times\, G_1(s^\beta, y-z_1)\, G_1(s^\beta, y-z_2)\, G_1((t-s)^\beta, x-y).$$

By [6, Lemma 5.4],

$$G_1(s^\beta, y-z_1)\, G_1(s^\beta, y-z_2) = G_1\left(\frac{s^\beta}{2}, y-\bar{z}\right) G_1\left(2s^\beta, \Delta z\right)$$
$$\le \sqrt{2}\, G_1\left(s^\beta, y-\bar{z}\right) G_1\left(2s^\beta, \Delta z\right).$$

Integrate over dy using the semigroup property of the heat kernel function to obtain

$$\left(J_0^2 \star \mathcal{K}\right)(t,x) \le C_\beta^2 \sqrt{2} \int_0^t \mathrm{d}s\, h(t-s)s^2$$
$$\times \iint_{\mathbb{R}^2} \nu(\mathrm{d}z_1)\nu(\mathrm{d}z_2)G_1(2s^\beta, \Delta z)G_1((t-s)^\beta + s^\beta, x-\bar{z}).$$

By (5.12) and [6, Lemma 5.5],

$$G_1((t-s)^\beta + s^\beta, x-\bar{z})\, G_1(2s^\beta, \Delta z) \le 2^{\frac{\beta-1}{2}} G_1(t^\beta, x-\bar{z})\, G_1(2s^\beta, \Delta z)$$
$$\le 2^{1+\beta/2} \frac{t^{\beta/2}}{s^{\beta/2}} G_1(4t^\beta, x-z_1)G_1(4t^\beta, x-z_2).$$

Denote $I(t,x) = \int_\mathbb{R} \nu(\mathrm{d}z)\, G_1(4t^\beta, x-z)$. Clearly, $\nu \in \mathcal{M}_T^\beta(\mathbb{R})$ implies that $I(t,x) < +\infty$ for all $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}$. Therefore,

$$\left(J_0^2 \star \mathcal{K}\right)(t,x) \le C_\beta^2\, 2^{\frac{\beta+3}{2}} t^{\beta/2} I(t,x)^2 \int_0^t \mathrm{d}s\, s^{2-\beta/2} h(t-s).$$

Clearly, the ds-integral is integrable because $\sigma < 1$ and $2 - \beta/2 > -1$.

As for Case II, by the same argument as Proposition 5.8, for some nonnegative constant $C_\beta < \infty$, $G_\beta^*(t,x) \le C_\beta\, g(t,x)$ for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$. Therefore, this case can be proved by the same arguments as the slow diffusion case with $G_\beta(t,x)$ replaced by $G_\beta^*(t,x)$.

Finally, we remark that in both cases, by the continuity of the function $\mathbb{R}_+^* \times \mathbb{R} \ni (t,x) \mapsto I(t,x)$ (see Proposition 6.6), for all compact sets $K \subseteq \mathbb{R}_+^* \times \mathbb{R}$, $\sup_{(t,x)\in K} I(t,x)^2 < +\infty$. This completes the whole proof of Lemma 6.7. $\square$

Proof of Theorem 3.1. The proof follows the same six steps as those in the proof of [6, Theorem 2.4] with some minor changes:

Both proofs rely on estimates on the kernel function $\mathcal{K}(t, x)$. Instead of an explicit formula for the SHE (see [6, Proposition 2.2]), Theorem 3.4 ensures the finiteness of $\mathcal{K}(t, x)$ and provides a bound on it.

In the Picard iteration scheme, i.e., Steps 1–4 in the proof of [6, Theorem 2.4], we need to check the $L^p(\Omega)$-continuity of the stochastic integral, which then guarantees that at the next step, the integrand is again in $\mathcal{P}_2$, via [6, Proposition 3.4]. Here, the statement of [6, Proposition 3.4] is still true by replacing in its proof [6, Proposition 3.5] by either Proposition 6.3 for the slow diffusion equations or Proposition 6.4 for the fast diffusion equations, and replacing [6, Proposition 5.3] by Proposition 6.1.

In the first step of the Picard iteration scheme, the following property, which determines the set of the admissible initial data, needs to be verified: for all compact sets $K \subseteq \mathbb{R}_+ \times \mathbb{R}$,

$$\sup_{(t,x)\in K} \left(\left[1 + J_0^2\right] \star G_\beta^2 \right)(t, x) < +\infty.$$

For the SHE, this property is proved in [6, Lemma 3.9]. Here, Lemma 6.7 gives the desired result with minimal requirements on the initial data. This property, together with the calculation of the upper bound on $\mathcal{K}(t, x)$ in Theorem 3.4, guarantees that all the $L^p(\Omega)$-moments of $u(t, x)$ are finite. This property is also used to establish uniform convergence of the Picard iteration scheme, hence $L^p(\Omega)$–continuity of $(t, x) \mapsto I(t, x)$.

The proof of (3.2) is identical to that of the corresponding property in [6, Theorem 2.4]. This completes the proof of Theorem 3.1. □

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