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DISSIPATION AND HIGH DISORDER

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Given a field $\{B(x)\}_{x\in\mathbf{Z}^d}$ of independent standard Brownian motions, indexed by $\mathbf{Z}^d$, the generator of a suitable Markov process on $\mathbf{Z}^d, \mathcal{G}$, and sufficiently nice function $\sigma : [0,\infty) \mapsto [0,\infty)$, we consider the influence of the parameter λ on the behavior of the system,

$$\begin{aligned} \mathrm{d}u_t(x) &= (\mathcal{G}u_t)(x)\,\mathrm{d}t + \lambda\sigma\big(u_t(x)\big)\,\mathrm{d}B_t(x) \qquad \big[t > 0,\ x \in \mathbf{Z}^d\big],\\ u_0(x) &= c_0\delta_0(x). \end{aligned}$$

We show that for any $\lambda > 0$ in dimensions one and two the total mass $\sum_{x\in\mathbf{Z}^d} u_t(x)$ converges to zero as $t \to \infty$ while for dimensions greater than two there is a phase transition point $\lambda_c \in (0,\infty)$ such that for $\lambda > \lambda_c$, $\sum_{x\in\mathbf{Z}^d} u_t(x) \to 0$ as $t \to \infty$ while for $\lambda < \lambda_c$, $\sum_{x\in\mathbf{Z}^d} u_t(x) \not\to 0$ as $t \to \infty$.

1. Introduction. Let τ denote a probability density function on $\mathbf{Z}^d$, and consider the linear operator $\mathcal{G}$ defined by

$$(\mathcal{G}h)(x) = \sum_{y\in\mathbf{Z}^d} \big[h(x+y) - h(x)\big]\tau(y), \tag{1.1}$$

for all $x \in \mathbf{Z}^d$ and bounded functions $h : \mathbf{Z}^d \to \mathbf{R}$. We may think of $\mathcal{G}$ as the generator of a rate-one continuous-time random walk $\mathbf{X} := \{X_t\}_{t\geq 0}$ on $\mathbf{Z}^d$, that is, X is a compound Poisson process such that $X_0 = 0$, $\tau(x) = \mathrm{P}\{X_J = x, J < \infty\}$ for all $x \in \mathbf{Z}^d$, and J denotes the first time the process X_t jumps. In order to rule out trivialities, we will assume that X is genuinely d-dimensional. In particular, $J < \infty$ a.s. and $\tau(x) = \mathrm{P}\{X_J = x\}$.

Let $\{B(x)\}_{x\in\mathbf{Z}^d}$ denote a field of independent standard Brownian motions, indexed by $\mathbf{Z}^d$, and consider the system of Itô stochastic ODEs,

$$\mathrm{d}u_t(x) = (\mathcal{G}u_t)(x)\,\mathrm{d}t + \lambda\sigma\big(u_t(x)\big)\,\mathrm{d}B_t(x) \qquad \big[t > 0, x \in \mathbf{Z}^d\big], \tag{1.2}$$

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subject to $u_0(x) := c_0\delta_0(x)$ for all $x \in \mathbf{Z}^d$, where $c_0, \lambda > 0$ are finite and nonrandom numerical quantities. We will think of the number c_0 as fixed, and of λ as a tuning parameter which describes the *level of the noise*.

Here and throughout, we assume that $\sigma : \mathbf{R} \to \mathbf{R}$ is a deterministic Lipschitz-continuous function. It follows from the work of Shiga and Shimizu [21] that the particle system (1.2) has a unique strong solution.

We study the solution to (1.2) under further mild restrictions on the operator $\mathcal{G}$ and the nonlinearity σ. Regarding $\mathcal{G}$, we always assume that τ has mean zero and compact support; the latter is equivalent to the notion that $\mathcal{G}$ is finite range. To summarize, we have

$$\sum_{x\in\mathbf{Z}^d} x_j\tau(x) = 0 \qquad \text{for all } 1 \le j \le d \quad \text{and} \quad \max_{\|x\|>R_0} \tau(x) = 0 \tag{1.3}$$

for some $R_0 \in (1, \infty)$. In order to rule out trivialities, we assume also that $\tau(0) < 1$. Otherwise, (1.2) describes a countable family of independent and/or noninteracting one-dimensional Itô diffusions. We also note that the best-studied example of (1.2) is included here; that is, the case where $\mathcal{G}$ is the discrete Laplacian, $(\mathcal{G}h)(x) = (2d)^{-1}\sum_{y\in\mathbf{Z}^d:|y-x|=1} h(y)$ where $|z| := \sum_{j=1}^d |z_j|$ for all $z \in \mathbf{Z}^d$. Other examples abound.

As regards the nonlinearity, we will always assume that

$$\sigma(0) = 0 \quad \text{and} \quad \mathrm{L}_\sigma := \inf_{z\in\mathbf{R}} \left|\sigma(z)/z\right| > 0. \tag{1.4}$$

The first part of this condition ensures that the solution u to (1.2) is "physical." More precisely, the strict inequality $u_t(x) > 0$ holds for all $t > 0$ and $x \in \mathbf{Z}^d$ almost surely; see Georgiou et al. [13], Lemma 7.1. The second is a "moment intermittency condition" [11, 20].

The *parabolic Anderson model* $\sigma(u) = u$ has been studied a great deal (see Carmona and Molchanov [5]) in part because it arises naturally in other disciplines, and also because it is close to being an exactly-solvable model. In fact, in a few cases, it is exactly solvable; see Borodin and Corwin [1].

Thanks to a comparison argument ([13], Theorem 5.1), Theorem 1.2 of Shiga [20] implies that there exists a number $\lambda_1 > 0$ such that

$$\lim_{t\to\infty} u_t(x) = 0 \qquad \text{a.s. for all } x \in \mathbf{Z}^d, \tag{1.5}$$

if $\lambda > \lambda_1$. One can recast this, somewhat informally, as the assertion that the solution to (1.2) is locally dissipative under *strong disorder*; see Carmona and Hu [2] for the terminology on strong vs. weak disorder.

On the other hand, the theory of Georgiou et al. [13] implies that if $d \ge 3$, then there exists a finite and positive number λ_2 such that

$$\lim_{t\to\infty} \sup_{x\in\mathbf{Z}^d} u_t(x) = 0 \qquad \text{a.s.}, \tag{1.6}$$

whenever $\lambda \in (0, \lambda_2)$. This implies that the solution to (1.2) is uniformly—hence also locally—dissipative under *weak disorder*.

Finally, let us mention that when there is no disorder, that is when $\sigma \equiv 0$, the solution to the Kolmogorov–Fokker–Planck equation (1.2) is simply $u_t(x) = \mathrm{P}\{X_t = -x\}$, which goes to zero uniformly in x as $t \to \infty$ thanks to a suitable form of the local central limit theorem.

Thus, we see that *local* dissipation is a generic property of (1.2), regardless of the strength of the disorder in (1.2). By contrast, the main result of this paper shows that *global* dissipation essentially characterizes the presence of strong disorder. In order to describe our result, consider the *total mass process*

$$m_t(\lambda) := \|u_t\|_{\ell^1(\mathbf{Z}^d)} := \sum_{x \in \mathbf{Z}^d} |u_t(x)| \qquad [t \geq 0].$$

It is well known that $t \mapsto m_t(\lambda)$ is a mean-c_0 continuous $L^2(\mathrm{P})$-martingale. As far as we know, a variation on this observation goes on one hand at least as far back as Spitzer's paper ([22], Proposition 2.3), on discrete (more-or-less linear) interacting particle systems. More closely related variations can be found in the literature on measure-valued diffusions (see Dawson and Perkins [10] for pointers to the literature). The particular case that we need follows from (3.2) below and the fact that $m_t(\lambda) > 0$ for all $t > 0$, a.s. The asserted positivity follows from Lemma 7.1 of Georgiou et al. [13] which implies that

$$u_t(x) > 0 \qquad \text{for all } x \in \mathbf{Z}^d, t > 0, \text{ almost surely.} \tag{1.7}$$

Owing to the martingale convergence theorem, one consequence of positivity is that

$$m_\infty(\lambda) := \lim_{t \to \infty} m_t(\lambda) \tag{1.8}$$

exists a.s. and is finite a.s. for all $\lambda > 0$.

DEFINITION. We say that (1.2) is *globally dissipative* if $m_\infty(\lambda) = 0$ a.s.

Frequently, the probability literature refers to this property as "extinction." We prefer "dissipation" because a correct interpretation of "extinction," in the present setting, might suggest the false claim that $m_t(\lambda) = 0$ a.s. for all t sufficiently large, since as mentioned above, the strict inequality $u_t(x) > 0$ holds for all $t > 0$ and $x \in \mathbf{Z}^d$ almost surely.

The principle result of this paper is the following, which essentially equates global dissipation with the presence of strong disorder.

THEOREM 1.1. *In recurrent dimensions* $[d = 1, 2]$, *the system* (1.2) *is always globally dissipative. In transient dimensions* $[d \geq 3]$, *there is a sharp phase transition*; *namely, there exists a nonrandom number* $\lambda_c \in (0, \infty)$ *such that* (1.2) *is globally dissipative if* $\lambda > \lambda_c$ *and not globally dissipative if* $\lambda \in (0, \lambda_c)$.

REMARK. The case $\lambda = \lambda_c$ is open.

Theorem 1.1 is a qualitative result, but its proof has some quantitative aspects as well. In particular, as part of the proof, we will demonstrate that there exists a finite random variable $V := V(\lambda, \sigma, c_0, d)$ and a nonrandom positive and finite constant $v = v(\lambda, \sigma, c_0, d)$ such that

$$m_t(\lambda) \le V \times \begin{cases} \exp(-vt^{1/3}), & \text{if } d = 1, \\ \exp(-v\sqrt{\log t}), & \text{if } d = 2, \end{cases} \tag{1.9}$$

almost surely for all $t > 1$. We do not know if these bounds are sharp, only that

$$\limsup_{t\to\infty} \frac{1}{t} \log m_t(\lambda) > -\infty, \tag{1.10}$$

with positive probability in all dimensions $d \ge 1$ and for all $\lambda > 0$. However, our methods are in some sense robust: We will prove that some aspects of (1.9) can be carried out in the continuous setting of stochastic partial differential equations as well (see Theorem 4.1).

Let us conclude the Introduction with a few remarks about the literature.

In the case that $d = 1, 2$, the qualititative part of Theorem 1.1 is a part of the folklore of the subject, and follows from well-known ideas about linear interacting particle systems; see, for example, Liggett ([15], Theorem 4.5, page 451) and especially Shiga [20], Remark 4.

Shiga [20], page 793, asserts that "it is plausible that the extinction occurs" when $d \ge 3$. The transient-dimension portion of Theorem 1.1 disproves Shiga's prediction when the noise level is sufficiently low. In the language of interacting particle systems, Theorem 1.1 implies the "survival" of the solution to (1.2) in transient dimensions when λ is small. Our method of proof of system survival is quite different from the more familiar ergodic-theoretic ones and worthy of attention in its own right, for example, compare with Liggett [15], Chapter IX, Section 2.

Throughout, Lip_σ denotes the optimal Lipschitz constant of σ; that is,

$$\mathrm{Lip}_\sigma := \sup_{-\infty<x<y<\infty} \left| \frac{\sigma(x) - \sigma(y)}{x - y} \right|. \tag{1.11}$$

Of course, $\mathrm{Lip}_\sigma < \infty$ by default.

2. Some technical estimates. In this section, we record three elementary technical facts that we will soon need. One (Lemma 2.1) is a variation on very well-known large deviations estimates for Lévy processes. The other two (Lemmas 2.2 and 2.3) contain extremal bounds on subsolutions to a certain infinite family of differential equations.

Let $Y_1, Y_2, \ldots$ be i.i.d. random variables in $\mathbf{Z}^d$ such that $\mathrm{P}\{Y_1 = x\} = \tau(x)$ for all $x \in \mathbf{Z}^d$. In particular, Y_1 has mean zero and moment generating function

$$\varphi(z) := \mathrm{E}\exp(z \cdot Y_1), \tag{2.1}$$

that is finite for all z in an open neighborhood of the origin of $\mathbf{R}^d$.

Let $N := \{N(t)\}_{t\geq 0}$ denote an independent rate-one Poisson process, and consider the compound Poisson process (sometimes also called continuous-time random walk)

$$X_t := \sum_{j=1}^{N(t)} Y_j \qquad [t \geq 0], \tag{2.2}$$

where $\sum_{j=1}^{0} Y_j := 0$. Clearly, $\{X_t\}_{t\geq 0}$ is a Lévy process on $\mathbf{Z}^d$ whose generator $\mathcal{G}$ is defined in (1.1).

LEMMA 2.1. *Under the preceding conditions, for every $q \in (0, \infty)$ there exists $c \in (0, \infty)$ such that*

$$\mathrm{P}\{\|X_t\| > K\} \leq 2d \exp(-cK^2/t), \tag{2.3}$$

uniformly for all $t \geq 1$ and $K \in [0, qt]$.

Lemma 2.1 is basically a version of Hoeffding's inequality [14] in continuous time, and can be obtained from Hoeffding's inequality by first conditioning on $N(t)$. Next, we describe the second, more analytic, portion of this section.

Choose and fix $\alpha, \delta, \gamma > 0$, and define $\mathbf{F}(\alpha, \delta, \gamma)$ to be the collection of all nonnegative continuously-differentiable functions $f : \mathbf{R}_+ \to \mathbf{R}_+$ such that

$$f'(t) \leq -\alpha \sup_{K\in[a,bt]} \left[\frac{f(t) - \exp(-\gamma K^2/t)}{K^\delta}\right] \qquad \text{for all } t \geq 1, \tag{2.4}$$

and some $0 < a < b$. We will reserve the notation $\mathbf{F}(\alpha, \delta, \gamma)$ as this function class throughout the paper.

Suppose $f \in \mathbf{F}(\alpha, \delta, \gamma)$ for some finite numbers $\alpha, \delta, \gamma > 0$. Because $f(t) \geq 0$ for all $t > 0$, we can set $K := bt$ in the optimization problem that defines $\mathbf{F}(\alpha, \delta, \gamma)$ in order to conclude that

$$f'(t) \leq \alpha (bt)^{-\delta} \exp(-\gamma b^2 t) \qquad \text{for all } t \geq 1. \tag{2.5}$$

Consequently, f is bounded. The following gives a strong improvement in the case that $\delta < 2$.

LEMMA 2.2. *For every $\delta \in [0, 2)$, $\alpha, \gamma > 0$, and $f \in \mathbf{F}(\alpha, \delta, \gamma)$,*

$$\limsup_{t\to\infty} \frac{\log f(t)}{t^\nu} < 0 \qquad \textit{with } \nu := \frac{2-\delta}{2+\delta}. \tag{2.6}$$

PROOF. Define $\beta := 4/(2+\delta)$ and observe that $\beta \in (1, 2]$ since $\delta \in [0, 2)$. We appeal to (2.4) with $K := t^{\beta/2}$ in order to see that every $f \in \mathbf{F}(\alpha, \delta, \gamma)$ satisfies

$$f'(t) + \alpha t^{-2\delta/(2+\delta)} f(t) \leq \alpha \exp(-\gamma t^\nu), \tag{2.7}$$

uniformly for all t sufficiently large. Define

$$g(t) := \exp(\theta t^\nu) f(t) \qquad [t \geq 0], \tag{2.8}$$

where θ is a fixed parameter that satisfies

$$0 < \theta < \min\left(\gamma, \frac{\alpha}{\nu}\right). \tag{2.9}$$

Then, (2.7) ensures that g satisfies

$$\begin{aligned} g'(t) &= \exp(\theta t^\nu)\big[f'(t) + \theta \nu t^{-2\delta/(2+\delta)} f(t)\big] \\ &< \alpha \exp(-[\gamma - \theta] t^\nu), \end{aligned} \tag{2.10}$$

for all t sufficiently large. This implies that g is bounded, which is another way to state the lemma. □

The preceding proof works also when $\delta = 2$, and shows that in that case every function $f \in \mathbf{F}(\alpha, 2, \gamma)$ is bounded for every $\alpha, \gamma > 0$. But this is vacuous, as we have seen already.

Next, we study the case that $\delta = 2$ more carefully and show among other things that if $f \in \mathbf{F}(\alpha, 2, \gamma)$ for some $\alpha, \gamma > 0$, then $f(t)$ tends to 0 faster than any negative power of $\log t$ as $t \to \infty$.

LEMMA 2.3. *For every* $\alpha, \gamma > 0$ *and* $f \in \mathbf{F}(\alpha, 2, \gamma)$,

$$\limsup_{t \to \infty} \frac{\log f(t)}{(\log t)^{1/2}} < 0. \tag{2.11}$$

PROOF. The argument is similar to the proof of Lemma 2.2, but we need to make a few modifications. Specifically, we now use $K := t^{1/2}(\log t)^{1/4}$, and $g(t) := \exp\{\theta\sqrt{\log t}\} f(t)$ for a sufficiently small constant $\theta > 0$. The remaining details are routine and left to the interested reader. □

3. Proof of Theorem 1.1. The proof is split into separate parts. First, let us define $\{p_t\}_{t \geq 0}$ to be the transition functions of the underlying walk X, that is,

$$p_t(x) := \mathrm{P}\{X_t = x\} \qquad \text{for all } t \geq 0 \text{ and } x \in \mathbf{Z}^d. \tag{3.1}$$

These functions play a role in our analysis, since the solution u to (1.2) can be written in the following integral form:

$$u_t(x) = c_0 p_t(-x) + \sum_{y \in \mathbf{Z}^d} \int_0^t p_{t-s}(y - x) \sigma\big(u_s(y)\big)\, \mathrm{d}B_s(y); \tag{3.2}$$

see Shiga and Shimizu [21]. Now we proceed with the proof, which is split into a number of distinct steps.

3.1. *Proof in recurrent dimensions*. We begin by proving (1.9); Theorem 1.1 follows immediately in recurrent dimensions, that is, when $d \in \{1, 2\}$.

The proof in recurrent dimensions proceeds by estimating fractional moments of $m_t(\lambda)$; see Chapter XII of Liggett [15] for similar ideas in the context of discrete particle systems and Mueller and Tribe [19] in the context of continuous systems.

As was mentioned in the Introduction, it is well known that $\{m_t(\lambda)\}_{t\geq 0}$ is a continuous $L^2(\mathrm{P})$-martingale with $\mathrm{E}[m_t(\lambda)] = c_0$ for all $t \geq 0$. This is obtained by summing (3.2) over $x \in \mathbf{Z}^d$ on both sides in order to see that

$$m_t(\lambda) = c_0 + \lambda \sum_{y\in\mathbf{Z}^d} \int_0^t \sigma\big(u_s(y)\big)\, \mathrm{d}B_s(y) \qquad [t \geq 0]. \tag{3.3}$$

Because $\sigma(0) = 0$ [see (1.4)], it follows that

$$\big|\sigma(z)\big| \leq \mathrm{Lip}_\sigma\, |z| \qquad \text{for all } z \in \mathbf{R}. \tag{3.4}$$

Therefore, the exchange of summation and stochastic integration is a standard consequence of measurability and the fact that

$$\sum_{y\in\mathbf{Z}^d} \int_0^t \mathrm{E}\big(\big|\sigma\big(u_s(y)\big)\big|^2\big)\, \mathrm{d}s \leq \mathrm{Lip}_\sigma^2 \sum_{y\in\mathbf{Z}^d} \int_0^t \mathrm{E}\big(\big|u_s(y)\big|^2\big)\, \mathrm{d}s < \infty, \tag{3.5}$$

for all $t > 0$. See (2.14) of Shiga and Shimizu [21] for a qualitative statement. Indeed, Lemma 8.1 of Georgiou et al. [13] shows a sharper quantitative estimate that, if $\mathrm{Lip}_\sigma^2\, \Upsilon(\beta) < 1$, then

$$\mathrm{E}\|u_t\|^2_{\ell^2(\mathbf{Z}^d)} \leq \frac{\|u_0\|^2_{\ell^2(\mathbf{Z}^d)} \mathrm{e}^{\beta t}}{1 - \mathrm{Lip}_\sigma^2\, \Upsilon(\beta)},$$

where

$$\Upsilon(\beta) := (2\pi)^{-d} \int_{(-\pi,\pi)^d} \frac{\mathrm{d}\xi}{\beta + 2(1 - \mathrm{Re}\,\phi(\xi))}$$

and $\phi(\xi) := \mathrm{E}\exp(i\xi \cdot Y_1)$ [see also (2.1)]. Here, by choosing β large enough, we can make $\mathrm{Lip}_\sigma^2\, \Upsilon(\beta) < 1$. Thus, we can obtain the finiteness of the right-hand side of (3.5).

Because of (3.5) and the Itô isometry, $m(\lambda) := \{m_t(\lambda)\}_{t\geq 0}$ is also an $L^2(\mathrm{P})$-martingale, and the quadratic variation process of $m(\lambda)$ is described by

$$\langle m(\lambda)\rangle_t = \lambda^2 \sum_{y\in\mathbf{Z}^d} \int_0^t \big|\sigma\big(u_s(y)\big)\big|^2\, \mathrm{d}s = \lambda^2 \int_0^t \|\sigma \circ u_s\|^2_{\ell^2(\mathbf{Z}^d)}\, \mathrm{d}s. \tag{3.6}$$

Therefore, (1.4) and (3.4) together yield the a.s. inequalities

$$\lambda^2 \mathrm{L}_\sigma^2 \int_0^t \|u_s\|^2_{\ell^2(\mathbf{Z}^d)}\, \mathrm{d}s \leq \langle m(\lambda)\rangle_t \leq \lambda^2\, \mathrm{Lip}_\sigma^2 \int_0^t \|u_s\|^2_{\ell^2(\mathbf{Z}^d)}\, \mathrm{d}s, \tag{3.7}$$

valid for all $t > 0$.

Since $m_t(\lambda) \ge u_t(0)$, equation (7.2) of Georgiou et al. [13] guarantees that for every $T > 0$ there exists $C_T := C_T(\lambda) \in (0, \infty)$ such that

$$\mathrm{P}\left\{\inf_{t\in[0,T]} m_t(\lambda) < \varepsilon\right\} \le C_T \varepsilon^{\log\log(1/\varepsilon)/C_T} \qquad \text{for all } \varepsilon \in (0,1). \tag{3.8}$$

This shows in particular that $\sup_{t\in[0,T]} |m_t(\lambda)| \in L^p(\mathrm{P})$ for all $p \in (-\infty, 0), T \in (0,\infty)$. Consequently, we may apply Itô's formula to see that for all $\eta \in (0,1)$,

$$\begin{aligned}\left[m_t(\lambda)\right]^\eta &= c_0^\eta + \eta \int_0^t \left[m_s(\lambda)\right]^{\eta-1} \mathrm{d}m_s(\lambda) \\ &\quad - \frac{\lambda^2\eta(1-\eta)}{2}\int_0^t \left[m_s(\lambda)\right]^\eta \frac{\|\sigma\circ u_s\|^2_{\ell^2(\mathbf{Z}^d)}}{\|u_s\|^2_{\ell^1(\mathbf{Z}^d)}}\,\mathrm{d}s,\end{aligned} \tag{3.9}$$

almost surely, where the stochastic integrals are bona fide continuous $L^2(\mathrm{P})$-martingales. In particular,

$$\mathrm{E}\left(\left[m_t(\lambda)\right]^\eta\right) = c_0^\eta - \frac{\lambda^2\eta(1-\eta)}{2}\int_0^t \mathrm{E}\left(\left[m_s(\lambda)\right]^\eta \frac{\|\sigma\circ u_s\|^2_{\ell^2(\mathbf{Z}^d)}}{\|u_s\|^2_{\ell^1(\mathbf{Z}^d)}}\right)\mathrm{d}s, \tag{3.10}$$

for every $t > 0$ and $\eta \in (0,1)$. The preceding is true also for $\eta \ge 1$, but we care only about values of η in $(0,1)$.

Because of (1.4), $\|\sigma\circ u_s\|_{\ell^2(\mathbf{Z}^d)} \ge \mathrm{L}_\sigma \|u_s\|_{\ell^2(\mathbf{Z}^d)}$. Therefore, the nonrandom function $t \mapsto \mathrm{E}([m_t(\lambda)]^\eta)$ is continuously differentiable and solves

$$f'(t) \le -\frac{\lambda^2\eta(1-\eta)\mathrm{L}_\sigma^2}{2}\mathrm{E}\left(\left[m_t(\lambda)\right]^\eta R_t^2\right) \qquad \text{for all } t > 0, \tag{3.11}$$

where

$$R_t := \frac{\|u_t\|_{\ell^2(\mathbf{Z}^d)}}{\|u_t\|_{\ell^1(\mathbf{Z}^d)}}. \tag{3.12}$$

For every real number $K \ge 1$, let

$$\mathcal{B}(K) := \left\{x \in \mathbf{Z}^d : \|x\| \le K\right\}. \tag{3.13}$$

There exists a positive and finite constant $c := c(d)$ such that the cardinality of $\mathcal{B}(K)$ is at least $c^{-1}K^d$, uniformly for all $K \ge 1$. Therefore, by the Cauchy–Schwarz inequality,

$$\begin{aligned}\|u_t\|^2_{\ell^2(\mathbf{Z}^d)} &\ge \sum_{x\in\mathcal{B}(K)} \left[u_t(x)\right]^2 \\ &\ge cK^{-d}\left(\sum_{x\in\mathcal{B}(K)} u_t(x)\right)^2 \\ &= cK^{-d}\left(\|u_t\|_{\ell^1(\mathbf{Z}^d)} - \sum_{x\notin\mathcal{B}(K)} u_t(x)\right)^2,\end{aligned} \tag{3.14}$$

for every $t, K > 0$. Consequently,

$$R_t^2 \geq cK^{-d}\left(1 - \frac{\sum_{x\notin\mathcal{B}(K)} u_t(x)}{\sum_{x\in\mathbf{Z}^d} u_t(x)}\right)^2 \geq cK^{-d}\left(1 - 2\frac{\sum_{x\notin\mathcal{B}(K)} u_t(x)}{\sum_{x\in\mathbf{Z}^d} u_t(x)}\right), \tag{3.15}$$

and hence (3.11) implies that

$$\begin{aligned} f'(t) &\leq -\frac{c\lambda^2\eta(1-\eta)\mathrm{L}_\sigma^2}{2K^d}\left(f(t) - 2\mathrm{E}\left[\frac{\sum_{x\notin\mathcal{B}(K)} u_t(x)}{(\sum_{x\in\mathbf{Z}^d} u_t(x))^{1-\eta}}\right]\right) \\ &\leq -\frac{c\lambda^2\eta(1-\eta)\mathrm{L}_\sigma^2}{2K^d}\left(f(t) - 2\mathrm{E}\left[\left(\sum_{x\notin\mathcal{B}(K)} u_t(x)\right)^\eta\right]\right); \end{aligned} \tag{3.16}$$

the last line holds merely because

$$\left\{\sum_{x\notin\mathcal{B}(K)} u_t(x)\right\}^{1-\eta} \leq \left\{\sum_{x\in\mathbf{Z}^d} u_t(x)\right\}^{1-\eta}. \tag{3.17}$$

By Jensen's inequality and Lemma 2.1, we can find $c \in (0, \infty)$ such that

$$\begin{aligned} \mathrm{E}\left[\left(\sum_{x\notin\mathcal{B}(K)} u_t(x)\right)^\eta\right] &\leq \left(\mathrm{E}\left[\sum_{x\notin\mathcal{B}(K)} u_t(x)\right]\right)^\eta \\ &= \left(\sum_{x\notin\mathcal{B}(K)} c_0 p_t(x)\right)^\eta \\ &\leq (2c_0 d)^\eta \exp(-c\eta K^2/t), \end{aligned} \tag{3.18}$$

uniformly for all $K \in [1, t]$. Therefore,

$$f'(t) \leq -\frac{c\lambda^2\eta(1-\eta)\mathrm{L}_\sigma^2}{2} \sup_{K\in[1,t]}\left(\frac{f(t) - 2(2c_0 d)^\eta \exp(-c\eta K^2/t)}{K^d}\right)$$

and so with $C = (2(2c_0 d)^\eta)^{-1}$, we have

$$Cf'(t) \leq -\frac{c\lambda^2\eta(1-\eta)\mathrm{L}_\sigma^2}{2} \sup_{K\in[1,t]}\left(\frac{Cf(t) - \exp(-c\eta K^2/t)}{K^d}\right), \tag{3.19}$$

uniformly for all $t \geq 1$. In other words, Cf is an element of $\mathbf{F}(\alpha, d, c\eta)$, where $\alpha := c\lambda^2\eta(1-\eta)\mathrm{L}_\sigma^2/2$. Because of this fact, we may employ Lemmas 2.2 and 2.3 in order to deduce the existence of constants $V := V(\eta, \lambda) \in (1, \infty)$, $v := v(\eta, \lambda) \in (0, \infty)$ such that for all $t \geq 1$,

$$\mathrm{E}([m_t(\lambda)]^\eta) \leq V \times \begin{cases} \exp(-vt^{1/3}), & \text{if } d = 1, \\ \exp(-v\sqrt{\log t}), & \text{if } d = 2. \end{cases} \tag{3.20}$$

If $U_1, \ldots, U_n$ is a nonnegative supermartingale, then Doob's inequality tells us that $\lambda \mathrm{P}\{\max_{1\le j\le n} U_j > \lambda\} \le \mathrm{E}(U_1)$ for all $\lambda > 0$. Since $\{[m_s(\lambda)]^\eta\}_{s\ge t}$ is a continuous nonnegative supermartingale for every fixed $t > 0$, Doob's inequality and a standard approximation argument together yield

$$\mathrm{P}\Big\{\sup_{s\ge t} m_s(\lambda) > a\Big\} \le a^{-\eta}\mathrm{E}\big([m_t(\lambda)]^\eta\big), \tag{3.21}$$

for all $t, a > 0$ and $\eta \in (0,1)$. When $d = 1$, this and (3.20) together imply that

$$\begin{aligned} P_n &:= \mathrm{P}\Big\{\sup_{s\ge n-1} m_s(\lambda) > \exp\big(-vn^{1/3}\big)\Big\} \\ &\le V\exp\big(v\big(\eta n^{1/3} - (n-1)^{1/3}\big)\big), \end{aligned} \tag{3.22}$$

for all integers $n \ge 1$. Since $\sum_{n=1}^\infty P_n < \infty$, the Borel–Cantelli lemma implies the existence of an integer-valued random variable n_0 such that

$$\sup_{s\ge n-1} m_s(\lambda) \le \exp\big(-vn^{1/3}\big) \qquad \text{for all } n > n_0 \text{ a.s.} \tag{3.23}$$

If $t > n_0$ is an arbitrary number, random or otherwise, then we can find a unique integer $n \ge n_0$ such that $n - 1 \le t \le n$. Then clearly

$$m_t(\lambda) \le \sup_{s\ge n-1} m_s(\lambda) \le \exp\big(-vn^{1/3}\big) \le \exp\big(-vt^{1/3}\big) \qquad \text{a.s.} \tag{3.24}$$

This inequality yields the first bound in (1.9), whence Theorem 1.1 when $d = 1$.

The proof of part 2 of (1.9) is essentially the same as the proof of part 1, but when $d = 2$ we use the second estimate in (3.20) instead of the first one there. This proves Theorem 1.1 for $d = 2$.

3.2. *Proof in transient dimensions*: *Existence of a unique phase transition*. In the second step of the proof, we show the existence of a unique phase transition. In principle, the proof is valid regardless of the value of the ambient dimension. However, it will turn out that the phase transition is nontrivial only when $d \ge 3$.

Let us write the solution to (1.2) as $u_t(x;\lambda)$, in order to emphasize the dependence of the solution on the size λ of the underlying noise. Recall that $\lambda > 0$ is a free parameter. Therefore, the preceding constructs $\lambda \mapsto u_\bullet(\bullet;\lambda)$ as a coupling of stochastic processes, as well.

According to a comparison theorem of Cox et al. [7], for all integers $N \ge 1$, and all real $t > 0$,

$$\mathrm{E}\exp\Big(-\sum_{x\in\mathcal{B}(N)} u_t(x;\bar\lambda)\Big) \ge \mathrm{E}\exp\Big(-\sum_{x\in\mathcal{B}(N)} u_t(x;\lambda)\Big), \tag{3.25}$$

as long as $\bar\lambda \ge \lambda > 0$. We let $N \uparrow \infty$, and appeal to the monotone convergence theorem, in order to see that $\mathrm{E}\exp(-m_t(\bar\lambda)) \ge \mathrm{E}\exp(-m_t(\lambda))$ for all $t > 0$, as long as $\bar{\lambda} \ge \lambda > 0$. Now let $t \to \infty$ in order to deduce from the dominated convergence theorem that

$$\mathrm{E}\exp(-m_\infty(\bar{\lambda})) \ge \mathrm{E}\exp(-m_\infty(\lambda)), \tag{3.26}$$

as long as $\bar{\lambda} \ge \lambda > 0$. In other words, $\lambda \mapsto \mathrm{E}\exp(-m_\infty(\lambda))$ is nondecreasing. Thus,

$$\lambda_c := \sup\{\lambda > 0 : \mathrm{E}e^{-m_\infty(\lambda)} < 1\} = \inf\{\lambda > 0 : \mathrm{E}e^{-m_\infty(\lambda)} = 1\}, \tag{3.27}$$

where $\inf\varnothing := +\infty$ and $\sup\varnothing := 0$. By the nonnegativity of $m_\infty(\lambda)$, we can also write

$$\lambda_c = \sup\{\lambda > 0 : m_\infty(\lambda) > 0 \text{ with positive probability}\} \tag{3.28}$$

$$= \inf\{\lambda > 0 : m_\infty(\lambda) = 0 \text{ a.s.}\}. \tag{3.29}$$

This proves the existence of a unique $\lambda_c \in [0, \infty]$ with the properties mentioned in Theorem 1.1. The already-verified portion of the proof implies that $\lambda_c = 0$ when $d = 1, 2$. The next two parts of the proof will show the nontriviality of λ_c in transient dimensions; namely, that $0 < \lambda_c < \infty$ when $d \ge 3$. This endeavor will complete the proof.

3.3. *Proof in transient dimensions*: *Supercritical phase*. In this section, we consider only dimensions $d \ge 3$, and demonstrate that $m_\infty(\lambda) = 0$ a.s. if λ is sufficiently large. This immediately proves that

$$\lambda_c < \infty. \tag{3.30}$$

We follow carefully Shiga's proof of his Theorem 1.2 ([20], pages 800–806), keeping track of the various sums and estimating them by elementary means in order to find that for all λ sufficiently large (κ small, in the notation of Shiga) there exists a constant $c \in (0, \infty)$ such that

$$\sup_{x \in \mathbf{Z}^d} \mathrm{P}\{u_t(x) > e^{-t/c}\} \le c e^{-t/c} \qquad \text{for all } t \ge 1. \tag{3.31}$$

Among other things, this readily implies the following weak formulation of a "local extinction result":

$$\limsup_{t \to \infty} \frac{\log u_t(x)}{t} < 0 \qquad \text{a.s. for all } x \in \mathbf{Z}^d. \tag{3.32}$$

This is another way to say that the "almost-sure Lyapunov exponent of the solution is negative." When $\sigma(x) = x$ and $\mathcal{G}$ is the discrete Laplacian, that is, when τ is the uniform distribution on the graph neighbors of the origin in $\mathbf{Z}^d$—(3.32) is known to hold with a limit in place of a lim inf; see Carmona and Molchanov [5]. The most complete results, in this case, can be found in Carmona et al. [3] and Cranston et al. [8]. More generally still, Shiga [20] considered the same class of nonlinear functions σ as we do, and established (3.32) with a proper limit in place of a liminf.

We now suppose that λ is large enough to ensure the validity of (3.31), and derive (3.30) as follows. Recall $\mathcal{B}(K)$ from (3.13) and let $|\mathcal{B}(K)|$ denote its cardinality. Setting $A(t) = \{\max_{x\in\mathcal{B}(t^2)} u_t(x) > \frac{\eta}{|\mathcal{B}(t^2)|}\}$, we have by Chebyshev's inequality

$$\begin{aligned} \mathrm{P}\{m_t(\lambda) > 2\eta\} &= \mathrm{P}\{\{m_t(\lambda) > 2\eta\} \cap A(t)\} + \mathrm{P}\{\{m_t(\lambda) > 2\eta\} \cap A(t)^c\} \\ &\le \mathrm{P}\{A(t)\} + \mathrm{P}\Bigg\{\sum_{x\notin\mathcal{B}(t^2)} u_t(x) > \eta\Bigg\} \\ &\le \mathrm{P}\{A(t)\} + \eta^{-1} \sum_{x\notin\mathcal{B}(t^2)} E[u_t(x)], \end{aligned} \tag{3.33}$$

for all $\eta > 0$ and $t > 1$. Since $|\mathcal{B}(t^2)| \sim \text{const} \cdot t^{2d}$ as $t \to \infty$, Shiga's estimate (3.31) ensures that

$$\mathrm{P}\Bigg\{\max_{x\in\mathcal{B}(t^2)} u_t(x) > \frac{\eta}{|\mathcal{B}(t^2)|}\Bigg\} = O(t^{2d})\mathrm{e}^{-t/c} \qquad \text{as } t \to \infty, \tag{3.34}$$

whereas (3.2) and Lemma 2.1 together ensure that there exist finite and positive constants c_1 and c_2 such that

$$\sum_{x\notin\mathcal{B}(t^2)} \mathrm{E}[u_t(x)] = c_0\mathrm{P}\{\|X_t\| > t^2\} \le c_1\mathrm{e}^{-c_2 t}, \tag{3.35}$$

for all $t > 1$ sufficiently large. Thus, $m_t(\lambda) \to 0$ in probability as $t \to \infty$, and hence $m_\infty(\lambda) = 0$ a.s. for all λ sufficiently large; (3.30) follows.

3.4. *Proof in transient dimensions*: *Subcritical phase*. We continue to assume that $d \ge 3$, and now prove that $\lambda_c > 0$.

Let $\{X'_t\}_{t\ge 0}$ be an independent copy of the continuous-time random walk X whose generator, we recall, is $\mathcal{G}$, and define

$$\Upsilon(0) := \int_0^\infty \mathrm{P}\{X_t = X'_t\}\,\mathrm{d}t. \tag{3.36}$$

This is the total expected local time of the symmetrized walk $X - X'$ at the origin of $\mathbf{Z}^d$. It is well known that $\Upsilon(0)$ is finite because $X - X'$ is a d-dimensional nontrivial random walk, and hence transient; see Chung and Fuchs [6]. In fact, if r is the probability of return to the origin for $X - X'$ then $\Upsilon(0)$ has an exponential distribution with parameter $2(1 - r)$.

Choose and fix any $\lambda > 0$ that satisfies

$$\lambda < \big[\mathrm{Lip}_\sigma \sqrt{\Upsilon(0)}\big]^{-1}. \tag{3.37}$$

According to Proposition 8.3 of Georgiou et al. [13],

$$\sup_{t\ge 0} \mathrm{E}(|m_t(\lambda)|^2) \le 2c_0^2\left(\frac{1+\varepsilon}{1-\varepsilon}\right), \tag{3.38}$$

where $0 < \varepsilon := \lambda^2 \operatorname{Lip}_\sigma^2 \Upsilon(0) < 1$. The Paley–Zygmund inequality is the following form of the Cauchy–Schwarz inequality:

$$\mathrm{P}\{W \geq c_0/2\} \geq \frac{c_0^2}{4\mathrm{E}(W^2)}, \tag{3.39}$$

valid for every nonnegative mean-c_0 random variable $W \in L^2(\mathrm{P})$. We choose $W := m_t(\lambda)$ to see that

$$\delta := \inf_{t \geq 0} \mathrm{P}\{m_t(\lambda) \geq c_0/2\} > 0, \tag{3.40}$$

as long as λ satisfies (3.37). Thus, $\mathrm{P}\{m_\infty(\lambda) \geq c_0/2\} \geq \delta > 0$ for all such values of λ, and hence $\lambda_c \geq [\operatorname{Lip}_\sigma \sqrt{\Upsilon(0)}]^{-1} > 0$, as desired.

3.5. *Proof of* (1.10). We conclude this section by establishing the quantitative lower bound (1.10) that is valid in all dimensions. Throughout this discussion, $\lambda > 0$ is held fixed.

Notice that

$$\|\sigma \circ u_s\|_{\ell^2(\mathbf{Z}^d)} \leq \operatorname{Lip}_\sigma^2 \|u_s\|_{\ell^2(\mathbf{Z}^d)} \leq \operatorname{Lip}_\sigma^2 \|u_s\|_{\ell^1(\mathbf{Z}^d)}, \tag{3.41}$$

almost surely for all $s > 0$. Therefore, we may apply (3.10) to see that the function

$$f(t) := \mathrm{E}([m_t(\lambda)]^\eta) \qquad [t > 0] \tag{3.42}$$

solves the differential inequality

$$f'(t) \geq -\tfrac{1}{2}\lambda^2 \eta(1-\eta) \operatorname{Lip}_\sigma^2 f(t), \tag{3.43}$$

for all $t > 0$ and $\eta \in (0, 1)$ subject to $f(0) = c_0^\eta$. Therefore,

$$\mathrm{E}([m_t(\lambda)]^\eta) \geq c_0^\eta \exp\left(-\frac{\lambda^2 \eta(1-\eta) \operatorname{Lip}_\sigma^2}{2} t\right), \tag{3.44}$$

for all $t > 0$ and $\eta \in (0, 1)$. We apply the preceding with $t > 1$ and $\eta := \eta_t := 1/t$ in order to see that

$$\begin{aligned} c_0^\eta \exp\left(-\frac{\lambda^2 \operatorname{Lip}_\sigma^2}{2}\right) &\leq c_0^\eta \exp\left(-\frac{\lambda^2 \eta_t(1-\eta_t) \operatorname{Lip}_\sigma^2}{2} t\right) \\ &\leq \mathrm{E}([m_t(\lambda)]^{\eta_t}) \\ &\leq \mathrm{E}([m_t(\lambda)]^{\eta_t}; m_t(\lambda) \geq \mathrm{e}^{-ct}) + \mathrm{e}^{-c}, \end{aligned} \tag{3.45}$$

for all $c > 0$. We apply the preceding with an arbitrary choice of

$$c > \tfrac{1}{2}\lambda^2 \operatorname{Lip}_\sigma^2. \tag{3.46}$$

Since $\eta_t \in (0, 1)$, Hölder's inequality yields

$$\begin{aligned} \mathrm{E}([m_t(\lambda)]^{\eta_t}; m_t(\lambda) \geq \mathrm{e}^{-ct}) &\leq [\mathrm{E}(m_t(\lambda))]^{1/t} [\mathrm{P}\{m_t(\lambda) \geq \mathrm{e}^{-ct}\}]^{(t-1)/t} \\ &= c_0^{1/t} [\mathrm{P}\{m_t(\lambda) \geq \mathrm{e}^{-ct}\}]^{(t-1)/t}. \end{aligned} \tag{3.47}$$

In this way, we find that, as long as c satisfies (3.46),

$$\begin{aligned} \mathrm{P}\{m_t(\lambda) \geq \mathrm{e}^{-ct}\} &\geq c_0^{1/(1-t)}\left[\exp\left(-\frac{\lambda^2 \operatorname{Lip}_\sigma^2}{2}\right) - \mathrm{e}^{-c}\right]^{t/(t-1)} \\ &\to \exp\left(-\frac{\lambda^2 \operatorname{Lip}_\sigma^2}{2}\right) - \mathrm{e}^{-c} > 0, \end{aligned} \tag{3.48}$$

as $t \to \infty$. This implies (1.10). $\square$

4. The stochastic heat equation on the real line. We conclude this paper by showing how one can adjust our methods in order to study continuous stochastic partial differential equations (SPDEs). Indeed, let $\xi := \{\xi_t(x)\}_{t>0, x\in\mathbf{R}}$ denote a space–time white noise; that is, a centered generalized Gaussian noise with covariance measure,

$$\operatorname{Cov}[\xi_t(x), \xi_s(y)] = \delta_0(t-s)\delta_0(x-y) \qquad (s, t > 0, x, y \in \mathbf{R}). \tag{4.1}$$

We consider the SPDE

$$\dot{\psi}_t(x) = \tfrac{1}{2}\psi_t''(x) + \sigma(\psi_t(x))\xi_t(x), \tag{4.2}$$

valid for all $t > 0$ and $x \in \mathbf{R}$, subject to a nonrandom initial profile $\psi_0 \in L^\infty(\mathbf{R})$, with $\psi_0 \geq 0$. The nonlinearity σ is, as before, a deterministic Lipschitz-continuous function that satisfies (1.4).

It is well known ([23], Chapter 3), that the SPDE (4.2) has a unique continuous (weak) solution ψ that satisfies

$$\sup_{t\in[0,T]} \sup_{x\in\mathbf{R}} \mathrm{E}(|\psi_t(x)|^k) < \infty, \tag{4.3}$$

for all $T > 0$ and $k \geq 1$. That solution ψ is also known to have the following integral formulation ([23], Chapter 3):

$$\psi_t(x) = (G_t * \psi_0)(x) + \int_{(0,t)\times\mathbf{R}} G_{t-s}(y-x)\sigma(\psi_s(y))\xi(\mathrm{d}s\,\mathrm{d}y), \tag{4.4}$$

where the stochastic integral is a Walsh integral ([23], Chapter 2) and G denotes the heat kernel

$$G_t(x) := \frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{x^2}{2t}\right) \qquad (t > 0, x \in \mathbf{R}). \tag{4.5}$$

Then we have the following.

THEOREM 4.1. *Suppose in addition that*: (i) $\limsup_{|x|\to\infty} x^{-2}\log\psi_0(x) < 0$ *and* (ii) $\|\psi_0\|_{L^1(\mathbf{R})} > 0$. *Then*, $\psi_t \in L^1(\mathbf{R})$ *a.s. for all* $t > 0$, *and*

$$\limsup_{t\to\infty} \frac{1}{t^{1/3}} \log\|\psi_t\|_{L^1(\mathbf{R})} < 0 \qquad \text{a.s.} \tag{4.6}$$

PROOF. By Mueller's comparison theorem [16, 17], $\psi_t(x) \geq 0$ for all $t \geq 0$ and $x \in \mathbf{R}$ off a single null set. Therefore,

$$\mathcal{M}_t := \|\psi_t\|_{L^1(\mathbf{R})} = \int_{-\infty}^{\infty} \psi_t(x)\,\mathrm{d}x. \tag{4.7}$$

An a priori estimate, similar to those in Dalang and Mueller [9], can be used to show that since $\sigma(0) = 0$ and $\psi_0 \in L^1(\mathbf{R})$, $\psi_t \in L^1(\mathbf{R})$ a.s. for all $t > 0$. Moreover, we can integrate both sides of (3.2) [dx] in order to see that $t \mapsto \mathcal{M}_t$ a.s. solves the following for all $t > 0$:

$$\mathcal{M}_t = \mathcal{M}_0 + \int_{(0,t)\times\mathbf{R}} \sigma\big(\psi_s(y)\big)\xi(\mathrm{d}s\,\mathrm{d}y). \tag{4.8}$$

The exchange of the Lebesgue integral and the stochastic integral is justified by an appeal to a stochastic Fubini theorem ([23], Theorem 2.6, page 296).

The identity (4.8) is the continuous analogue of (3.3), and shows that, parallel to the discrete setting, the total mass $\mathcal{M}$ is a nonnegative, continuous $L^2(\Omega)$-martingale with mean $\mathcal{M}_0$ and quadratic variation,

$$\langle\mathcal{M}\rangle_t = \int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \big|\sigma\big(\psi_s(y)\big)\big|^2\,\mathrm{d}y. \tag{4.9}$$

In particular, (1.4) and the Lipschitz continuity of σ together yield the following: For all $t > 0$,

$$\mathrm{L}_\sigma^2 \int_0^t \|\psi_s\|_{L^2(\mathbf{R})}^2\,\mathrm{d}s \leq \langle\mathcal{M}\rangle_t \leq \mathrm{Lip}_\sigma^2 \int_0^t \|\psi_s\|_{L^2(\mathbf{R})}^2\,\mathrm{d}s \qquad \text{a.s.} \tag{4.10}$$

By Itô's formula, if $\eta \in (0, 1)$ is nonrandom, then almost surely for all $t > 0$,

$$\mathcal{M}_t^\eta = \mathcal{M}_0^\eta + \eta\int_0^t \mathcal{M}_s^{\eta-1}\,\mathrm{d}\mathcal{M}_s + \frac{\eta(\eta-1)}{2}\int_0^t \mathcal{M}_s^{\eta-2}\,\mathrm{d}\langle\mathcal{M}\rangle_s. \tag{4.11}$$

The appeal to Itô's formula, and the fact that the preceding stochastic integral is a bona fide martingale, both follow immediately from the fact that $\mathrm{E}(\sup_{s\in[0,t]} \psi_s^{-\mu}) < \infty$ for all $t > 0$ and $\mu > 0$; see Mueller and Nualart [18].

We integrate both sides of the preceding display [dP]—in a similar vein as was done for (3.10) and (3.11)—in order to obtain the following:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E}\big(\mathcal{M}_t^\eta\big) \leq -\frac{\eta(1-\eta)\mathrm{L}_\sigma^2}{2}\mathrm{E}\big(\mathcal{M}_t^\eta \cdot \mathcal{R}_t\big), \tag{4.12}$$

where

$$\mathcal{R}_s := \frac{\|\psi_s\|_{L^2(\mathbf{R})}^2}{\|\psi_s\|_{L^1(\mathbf{R})}^2} \qquad (s > 0). \tag{4.13}$$

Since ψ_s has finite (negative and positive) moments of all orders, $\mathcal{R}_s$ does too. Now we choose and fix an arbitrary nonrandom constant $K > 0$, and argue as in (3.15) to see that

$$\mathcal{R}_s \geq \frac{1}{2K}\left(1 - \frac{2}{\|\psi_s\|_{L^1(\mathbf{R})}} \int_{|x|>K} \psi_s(x)\,\mathrm{d}x\right), \tag{4.14}$$

for all $s \geq 0$. In particular,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E}(\mathcal{M}_t^\eta) &\leq -\frac{\eta(1-\eta)\mathrm{L}_\sigma^2}{4K}\mathrm{E}(\mathcal{M}_t^\eta) \\
&\quad + \frac{\eta(1-\eta)\mathrm{L}_\sigma^2}{2K}\mathrm{E}\left(\|\psi_t\|_{L^2(\mathbf{R})}^{\eta-1} \cdot \int_{|x|>K} \psi_t(x)\,\mathrm{d}x\right) \\
&\leq -\frac{\eta(1-\eta)\mathrm{L}_\sigma^2}{4K}\mathrm{E}(\mathcal{M}_t^\eta) + \frac{\eta(1-\eta)\mathrm{L}_\sigma^2}{2K}\mathrm{E}\left(\left[\int_{|x|>K} \psi_t(x)\,\mathrm{d}x\right]^\eta\right),
\end{aligned} \tag{4.15}$$

since $[\int_{|x|>K} \psi_t(x)\,\mathrm{d}x / \|\psi_t\|_{L^1(\mathbf{R})}]^{1-\eta} \leq 1$. In order to estimate the last quantity in the preceding display, we appeal to Jensen's inequality:

$$\begin{aligned}
\mathrm{E}\left(\left[\int_{|x|>K} \psi_t(x)\,\mathrm{d}x\right]^\eta\right) &\leq \left[\mathrm{E}\int_{|x|>K} \psi_t(x)\,\mathrm{d}x\right]^\eta \\
&= \left[\int_{|x|>K} (G_t * \psi_0)(x)\,\mathrm{d}x\right]^\eta;
\end{aligned} \tag{4.16}$$

valid since $\mathrm{E}[\psi_t(x)] = (G_t * \psi_0)(x)$ by (3.2). Now a few lines of elementary calculations show that because ψ_0 decays as a Gaussian function, we can find finite and positive constants c_1 and c_2—independently of t—such that

$$\mathrm{E}\left(\left[\int_{|x|>K} \psi_t(x)\,\mathrm{d}x\right]^\eta\right) \leq c_1 \mathrm{e}^{-c_2 K^2/t}. \tag{4.17}$$

Because of (4.15), this proves that

$$f(t) := \mathrm{E}(\|\psi_t\|_{L^1(\mathbf{R})}^\eta) \tag{4.18}$$

satisfies the pointwise inequality

$$f'(t) \leq -\frac{\eta(1-\eta)\mathrm{L}_\sigma^2}{4K} f(t) + \frac{c_1\eta(1-\eta)\mathrm{L}_\sigma^2}{2K}\exp\left(-\frac{K^2}{c_2 t}\right). \tag{4.19}$$

Consequently, there exist finite and positive constants C, α, and γ such that $f \in \mathbf{F}(\alpha, 1, \gamma)$, whence $\log f(t) \leq -Ct^{1/3}$ for all $t \gg 1$, thanks to Lemma 2.2, and hence that

$$\mathrm{E}(\|\psi_t\|_{L^1(\mathbf{R})}^\eta) \leq C_1 \exp\left(-\frac{t^{1/3}}{C_1}\right) \qquad \text{for all } t \geq C_1. \tag{4.20}$$

Since $t \mapsto \|\psi_t\|_{L^1(\mathbf{R})}^\eta$ is a nonnegative supermartingale [see (4.8)], we apply Doob's inequality and a Borel–Cantelli argument to complete the proof. $\square$

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