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Interpolating the stochastic heat and wave equations with time-independent noise: solvability and exact asymptotics

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Abstract
In this article, we study a class of stochastic partial differential equations with fractional differential operators subject to some time-independent multiplicative Gaussian noise. We derive sharp conditions, under which a unique global $L^p(\Omega)$-solution exists for all $p \geq 2$. In this case, we derive exact moment asymptotics following the same strategy as that in a recent work by Balan et al. (Inst Henri Poincaré Probab Stat. To appear, 2021). In the case when there exists only a local solution, we determine the precise deterministic time, T_2, before which a unique $L^2(\Omega)$-solution exists, but after which the series corresponding to the $L^2(\Omega)$ moment of the solution blows up. By properly choosing the parameters, results in this paper interpolate the known results for both stochastic heat and wave equations.

Keywords Stochastic partial differential equations · *Caputo* derivatives · *Riemann-Liouville* fractional integral · Fractional Laplacian · *Malliavin* calculus · *Skorohod* integral · Exact moment asymptotics · Time-independent Gaussian noise · White noise · Global and local solutions

Mathematics Subject Classification Primary 60H15; Secondary 60H07 · 37H15

Contents

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1 Introduction

In this paper we study the following *stochastic partial differential equation* (SPDE) with fractional differential operators:

$$\begin{cases} \left(\partial_t^b + \frac{\nu}{2}(-\Delta)^{a/2}\right) u(t,x) = I_t^r\left[\sqrt{\theta}\, u(t,x)\, \dot{W}(x)\right] & x \in \mathbb{R}^d, t>0, \\ u(0,\cdot)=1 & b\in(0,1], \\ u(0,\cdot)=1, \quad \partial_t u(0,\cdot)=0 & b\in(1,2), \end{cases} \tag{1.1}$$

where $a \in (0,2]$, $b \in (0,2)$, $r \geq 0$, $\nu > 0$ and $\theta > 0$. Here the noise $W = \left\{W(\phi) : \phi \in \mathcal{D}(\mathbb{R}^d)\right\}$ is a centered and time-independent Gaussian process, defined on a complete probability space $(\Omega, \mathcal{F}, P)$, with mean zero and covariance

$$\mathbb{E}\left[W(\phi)W(\psi)\right] = \int_{\mathbb{R}^d} \mathcal{F}\phi(\xi)\overline{\mathcal{F}\psi(\xi)}\mu(\mathrm{d}\xi) =: \langle \phi, \psi\rangle_{\mathcal{H}},$$

where μ refers to the *spectral measure*, which is assumed to be a nonnegative and nonnegative definite tempered measure on $\mathbb{R}^d$. Let γ be the Fourier transform of μ (see Sect. 3.1), which is also a nonnegative and nonnegative definite measure on $\mathbb{R}^d$ thanks to Bochner's theorem. Throughout the paper, we use $\mathcal{F}\phi(\xi) = \int_{\mathbb{R}^d} \exp(-ix\xi)\phi(x)\mathrm{d}x$ to denote the Fourier transform of a test function ϕ.

In (1.1), $(-\Delta)^{a/2}$ refers to the *fractional Laplacian* of order a, ∂_t^b denotes the *Caputo fractional differential operator*

$$\partial_t^b f(t) := \begin{cases} \dfrac{1}{\Gamma(m-b)} \int_0^t \mathrm{d}\tau \dfrac{f^{(m)}(\tau)}{(t-\tau)^{b+1-m}} & \text{if } m-1<b<m, \\ \dfrac{\mathrm{d}^m}{\mathrm{d}t^m} f(t) & \text{if } b=m, \end{cases}$$

where m is an integer, and I_t^r refers to the *Riemann-Liouville fractional integral* of order $r>0$

$$I_t^r f(t) := \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} f(s)\mathrm{d}s, \quad \text{for } t>0,$$

with the convention that when $r=0$, $I_t^0 = \mathrm{Id}$ reduces to the identity operator. The fundamental solution to (1.1) is expressed explicitly in terms of the *Fox H-function*,

$H^{m,n}_{p,q}(z)$, which is much more complicated than the Green's function for either the heat or wave equation. We denote the fundamental solution as

$$G(t,x) := G_{a,b,r,\nu,d}(t,x), \tag{1.2}$$

where

$$G_{a,b,r,\nu,d}(t,x) = \pi^{-d/2}|x|^{-d}t^{b+r-1}H^{2,1}_{2,3}\left(\frac{|x|^a}{2^{a-1}\nu t^b}\,\middle|\,\begin{matrix}(1,1),\ (b+r,b)\\ (d/2,a/2),\ (1,1),\ (1,a/2)\end{matrix}\right).$$

We direct the reader to Theorem 4.1[1] of [6] for more details. Since we are interested in the constant one initial condition (and zero initial velocity when $b > 1$), Theorem 4.1 (*ibid.*) implies that the corresponding solution to the homogeneous equation (i.e. the solution when there is no driving source) is equal to the constant one. Hence through superposition, (1.1) can be written as the following stochastic integral equation:

$$u(t,x) = 1 + \sqrt{\theta}\int_0^t \left(\int_{\mathbb{R}^d} G(t-s,x-y)u(s,y)W(\delta y)\right)\mathrm{d}s, \tag{1.3}$$

where the stochastic integral is in the *Skorohod* sense; see Definition 3.1 below. In the following, the fundamental solution will exclusively refer to $G(t,x)$, which is indeed a smooth function for $x \neq 0$. Our results rely on the following assumption for the nonnegativity of $G(t,x)$:

Assumption 1.1 (Nonnegativity) Assume that the fundamental solution $G(t,x)$ is nonnegative for all $t > 0$ and $x \in \mathbb{R}^d$.

Remark 1.2 Thanks to Theorem 4.6 of [6] (see also Theorem 3.1 of [5] for the case when $r = 0$), we have the following four groups of sufficient conditions,[2] under either group of which $G(t,\ldots)$ is nonnegative (see Fig. 1 for an illustration) :

1. $d \geq 1$, $b \in (0,1]$, $a \in (0,2]$, $r \geq 0$;
2. $1 \leq d \leq 3$, $1 < b < a \leq 2$, $r > 0$;
3. $1 \leq d \leq 3$, $1 < b = a < 2$, $r > \dfrac{d+3}{2} - b$.

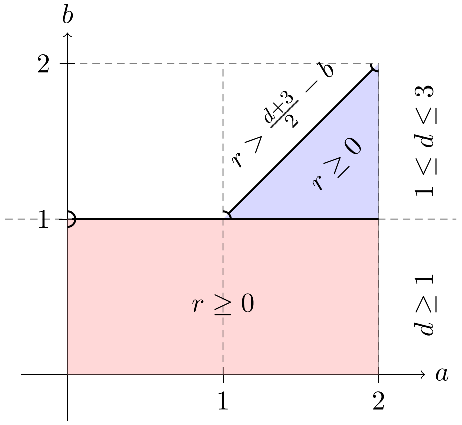
Regarding the noise, we formulate the following assumption in order to cover the Riesz kernel case, the fractional noise and a mixture of them:

Assumption 1.3 (Noise) Let $k \in \{1,\ldots,d\}$ and partition the d-coordinates of $x = (x_1,\ldots,x_d)$ into k distinct groups of size d_i so that $d_1 + \cdots + d_k = d$. Denote $x_{(i)} =$

[1] $G(t,x)$ corresponds to $Y_{a,b,r,\nu,d}(t,x)$ from [6].

[2] Note that when $d \geq 1$, $b = 1$ and $a \in (0,2]$, part (1) of [6, Theorem 4.6] says that the fundamental solution Y, which is the fundamental solution G in this paper, is nonnegative provided $r = 0$ or $r > 1$. Indeed, because in this case Z is always nonnegative, for $r > 0$, Y as a fractional integral of Z (see (4.5), *ibid.*), Y, or our G, should also be nonnegative. We thank Guannan Hu who pointed out to us this observation.

Fig. 1 Illustration of the sufficient conditions (Remark 1.2) for $G(t,\ldots)$ to be nonnegative



$(x_{i_1}, \ldots, x_{i_{d_i}})$ to be the coordinates in the i^{th} partition. Assume that the correlation function of the Gaussian noise is given by

$$\gamma(x) = \prod_{i=1}^{k} \left|x_{(i)}\right|^{-\alpha_i} \qquad \text{with } \alpha_i \in (0, d_i). \tag{1.4}$$

Define $\alpha := \sum_{i=1}^{k} \alpha_i$.

Remark 1.4 *(Spectral density and decomposition)* Recall that the *spectral density* of γ from (1.4), which by definition is $\mathcal{F}\gamma$, takes the following form:

$$\mu(\mathrm{d}\xi) = \varphi(\xi)\mathrm{d}\xi \quad \text{with} \quad \varphi(\xi) = \prod_{i=1}^{k} C_{\alpha_i, d_i} |\xi_{(i)}|^{-(d_i - \alpha_i)}. \tag{1.5}$$

Moreover, in the derivations below, we need to find a nonnegative and nonnegative definite K such that $\gamma = K * K$ where '$*$' denotes the spatial convolution. Indeed, one can choose

$$K(x) = \prod_{i=1}^{k} \beta_{\alpha_i, d_i} \left|x_{(i)}\right|^{-(d_i + \alpha_i)/2}. \tag{1.6}$$

The two constants in both (1.5) and (1.6) are defined as

$$C_{\alpha,d} = \pi^{-d/2} 2^{-\alpha} \frac{\Gamma((d-\alpha)/2)}{\Gamma(\alpha/2)} \quad \text{and} \quad \beta_{\alpha,d} = \pi^{-d/4} \frac{\Gamma((d+\alpha)/4)}{\Gamma((d-\alpha)/4)} \sqrt{\frac{\Gamma((d-\alpha)/2)}{\Gamma(\alpha/2)}}. \tag{1.7}$$

Example 1.5 *(Noises)* We have the following special cases: (1) Setting $k = 1$ in (1.4) and (1.5) recovers the Riesz kernel case. In this case,

$$\gamma(x) = |x|^{-\alpha}, \quad \varphi(x) = C_{\alpha,d}|x|^{-(d-\alpha)} \quad \text{and} \quad K(x) = \beta_{\alpha,d}|x|^{-(d+\alpha)/2}. \tag{1.8}$$

(2) Setting $k = d$ in (1.4) and (1.5) recovers the time-independent fractional noise. The corresponding SHE with such noise was earlier studied by Hu [17]. For this noise, we have that

$$\gamma(x) = \prod_{i=1}^{d} |x_i|^{-\alpha_i}, \quad \varphi(\xi) = \prod_{i=1}^{d} C_{\alpha_i,1}|\xi_i|^{-(1-\alpha_i)} \quad \text{and} \quad K(x) = \prod_{i=1}^{d} \beta_{\alpha_i,1}|x_i|^{-(1+\alpha_i)/2}. \tag{1.9}$$

In a recent work by Balan et al. [1], the same equation as (1.1), but exclusively for the *stochastic wave equation* (SWE), namely, the case when $a = b = \nu = 2$ and $r = 0$, has been studied, where both the well-posedness and the exact moment asymptotics have been obtained. The corresponding *stochastic heat equation* (SHE), namely, the case when $a = 2$, $b = \nu = 1$ and $r = 0$, has been earlier studied by Hu [17], but only for the well-posedness and exclusively for the fractional noise (1.9). The corresponding moment asymptotics have been obtained by X. Chen [8] as a special case by setting $\alpha_0 = 0$. One may check Remark 1.9 of Balan et al. [1] for the explicit expressions in terms of notation of the current paper. In this paper, by working on a more general class of SPDEs, we are able to interpolate the asymptotics for both SWE and SHE; see Sect. 2.2 below for more details. Moreover, we give the sharp conditions under which there exists only a local $L^2(\Omega)$ solution.

The moment asymptotics obtained by X. Chen, such as those in [8, 9], rely crucially on the Feynman-Kac representation of the moments of the solution. However, whenever $b \neq 1$, especially for the case when $b \in (1, 2)$, we are not aware of any such Feynman-Kac formula for the moments. Instead, in the recent work by Balan et al. [1], this difficulty has been overcome by studying the Wiener chaos expansion of the solution. In this paper, we follow the same strategy laid out by Balan *et al* (*ibid.*). The challenge comes from the much more involved parametric form of the fundamental solution.

Now let us state the main results of this paper. The first main result deals with the well-posedness of the SPDE (1.1) (or (1.3)) as stated in the following theorem. For this, we need to introduce the following variational constant (see Sect. 3.2 for more details):

$$\mathcal{M}_{a,d}(f) := \sup_{g \in \mathcal{F}_a} \left\{ \left\langle g^2 * g^2, f \right\rangle_{L^2(\mathbb{R}^d)}^{1/2} - \frac{1}{2}\, \mathcal{E}_a(g, g) \right\}. \tag{1.10}$$

We use the convention that $\mathcal{M}_a(f) := \mathcal{M}_{a,d}(f)$ when the dimension is clear from the context, and $\mathcal{M}_a := \mathcal{M}_a(\gamma)$, where γ is defined in (1.4). It is important to note that by Theorem 3.5, stated and proven below, that $\mathcal{M}_a < \infty$.

Theorem 1.6 (Solvability) *Assume that both Assumptions 1.1 and 1.3 hold.*

(1) (1.1) *has a unique (global) solution $u(t, x)$ in $L^p(\Omega)$ for all $p \geq 2$, $t > 0$, and $x \in \mathbb{R}^d$ provided that*

$$0 < \alpha < \min\left(\frac{a}{b}\left[2(b+r)-1\right], 2a, d\right). \tag{1.11}$$

(2) *Otherwise, if*

$$r \in [0, 1/2) \qquad \textit{and} \qquad 0 < \alpha = \frac{a}{b}[2(b+r)-1] \le d, \tag{1.12}$$

then (1.1) has a local solution in the sense that

(2-i) *For any $p \ge 2$, (1.1) has a unique solution $u(t,x)$ in $L^p(\Omega)$ for all $p \ge 2$ and $x \in \mathbb{R}^d$, but only for $t \in (0, T_p)$ where*

$$T_p := \frac{\nu^{\alpha/a}}{2\theta(p-1)\mathcal{M}_a^{(2a-\alpha)/a}}. \tag{1.13}$$

(2-ii) *For any $t > T_2$, the series (3.9) below diverges, that is, the $L^2(\Omega)$-solution $u(t,x)$ to (1.1) does not exist whenever $t > T_2$.*

The second main result of the paper is about the moment asymptotics. We use $\|\cdot\|_p$ to denote the $L^p(\Omega)$ moments.

Theorem 1.7 *Under Assumptions 1.1 and 1.3, if condition (1.11) holds, then we have that*

$$\begin{aligned}
&\lim_{t_p\to\infty} t_p^{-\beta} \log \|u(t,x)\|_p \\
&= \left(\frac{1}{2}\right)\left(\frac{2a}{2a(b+r)-b\alpha}\right)^{\beta} \\
&\times \left(\theta\nu^{-\alpha/a}\mathcal{M}_a^{\frac{2a-\alpha}{a}}\right)^{\frac{a}{2a(b+r)-b\alpha-a}} \left(2(b+r) - \frac{b\alpha}{a} - 1\right),
\end{aligned} \tag{1.14}$$

where

$$\beta := \frac{2(b+r) - \dfrac{b\alpha}{a}}{2(b+r) - \dfrac{b\alpha}{a} - 1} \qquad \textit{and} \qquad t_p := (p-1)^{1-1/\beta}\, t. \tag{1.15}$$

Proof We prove the matching upper bound (5.1) and the lower bound (6.1) of (1.14) at the end of Sects. 5 and 6 below, respectively, which together prove (1.14). □

As a direct consequence of (1.14), one can send either t or p to infinity as follows:

Corollary 1.8 *Under both Assumptions 1.1 and 1.3, if condition (1.11) holds, then*

(1) *For all $p \geq 2$ fixed, it holds that*

$$\lim_{t\to\infty} t^{-\beta}\log\mathbb{E}\left(|u(t,x)|^p\right) = p(p-1)^{\frac{1}{2(b+r)-\frac{b\alpha}{a}-1}}\left(\frac{1}{2}\right)\left(\frac{2a}{2a(b+r)-b\alpha}\right)^{\beta} \times\left(\theta\nu^{-\alpha/a}\mathcal{M}_a^{\frac{2a-\alpha}{a}}\right)^{\frac{a}{2a(b+r)-b\alpha-a}}\left(2(b+r)-\frac{b\alpha}{a}-1\right); \tag{1.16}$$

(2) *For all $t > 0$ fixed, it holds that*

$$\lim_{p\to\infty} p^{-\beta}\log\mathbb{E}\left(|u(t,x)|^p\right) = t^{\beta}\left(\frac{1}{2}\right)\left(\frac{2a}{2a(b+r)-b\alpha}\right)^{\beta} \times\left(\theta\nu^{-\alpha/a}\mathcal{M}_a^{\frac{2a-\alpha}{a}}\right)^{\frac{a}{2a(b+r)-b\alpha-a}}\left(2(b+r)-\frac{b\alpha}{a}-1\right). \tag{1.17}$$

The paper is organized as follows. In Sect. 2, we first give some concrete examples, where one can find many explicit formulas for either moment asymptotics in the case of global solutions or the expressions for the critical time T_p in the case of local solutions. Then in Sect. 3, we present some preliminaries of the paper, including the *Skorohod* integral, definition of the mild solution, and some asymptotics with corresponding variational constants. We prove part (1) and part (2) of Theorem 1.6 in Sects. 4 and 5, respectively. The upper bound and lower bounds for (1.14) are established in Sects. 5 and 6, respectively. Finally, in the "Appendix"—Sect. 7, we list a few proofs of results that are used in the paper.

2 Examples on solvability and asymptotics

In this section, we will give various examples to illustrate our main results. The cases with $b = 1$ and $r = 0$ are mostly known, which will be pointed out in the example below and will be used as test examples for our results. To the best of our knowledge, all results in this paper for either $b \neq 1, 2$ or $r > 0$ should be new.

2.1 Examples on solvability

In this part, we list some concrete examples regarding the solvability—Theorem 1.6.

Example 2.1 *(SHE)* By setting $a = 2$, $b = 1$ and $r = 0$ in (1.12), we obtain the following condition for the SHE under which there only exists a local solution:

$$\alpha = 2 \leq d. \tag{2.1}$$

Clearly, the fundamental solutions in this case are nonnegative for all $d \geq 1$. Hence, the picture is slightly more complicated since we need to check all possible dimensions $d \geq 1$. We illustrate possible cases in Fig. 2. In particular, let us explain a few cases:

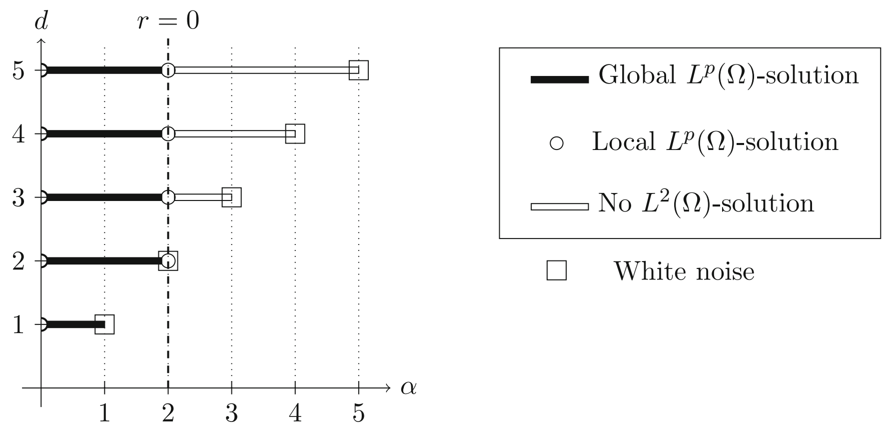


Fig. 2 Solvability for the stochastic heat equation (i.e., $a = 2$, $b = 1$ and $r = 0$) with $p \geq 2$

(a) When $d = 2$, condition (2.1) says that the 2-dimensional SHE driven by white noise has only a local $L^p(\Omega)$ solution. By applying (1.13) to this case, the critical time T_p becomes

$$T_p = \frac{\nu}{2\theta(p-1)\mathcal{M}_{2,2}(\delta_0)}, \qquad p \geq 2. \tag{2.2}$$

Note that in part 2) of Theorem 4.1 of Hu [18], some lower and upper bounds for T_2 were obtained. More precisely, by setting additionally that $\theta = 1$ and $\nu = 1$, Hu (*ibid.*) proved that when $t < 2$, an $L^2(\Omega)$ solution exists but when $t > 2\pi$, the second moment of the solution blows up. It is an interesting exercise to show that

$$2 \leq T_2 = \frac{1}{2\mathcal{M}_{2,2}(\delta_0)} \leq 2\pi, \qquad \text{where } d = 2.$$

This case is covered as a special time-independent case (i.e., $H_0 = 1$) by Chen et al. [11, Theorem 3.4 and Remark 3.13].

(b) Recall that the white noise driven SHE corresponds to when $\alpha = d$. Therefore by examining (1.11) and (1.12), we see that when $d \geq 3$, the SHE driven by white noise no longer has any $L^2(\Omega)$-solution. In addition, local solutions exist only when $\alpha = 2$ and the noise is not white. This is illustrated in Fig. 2 below. In addition, the critical time T_p takes the same expression as (2.2) but one needs to replace δ_0 by γ.

Remark 2.2 Note that we use the Skorohod integral to interpret the multiplication of the solution with the noise in (1.1). Multiplication interpreted in this way is traditionally called the *Wick product* which is consistent with the *Itô* or *Walsh* integral (see, e.g., [14]) when the noise is white in time. One can also interpret this product as the usual product. In order to handle the singularities caused by this multiplication, one needs to carry out certain renormalization processes. In fact, for the standard SHE with white

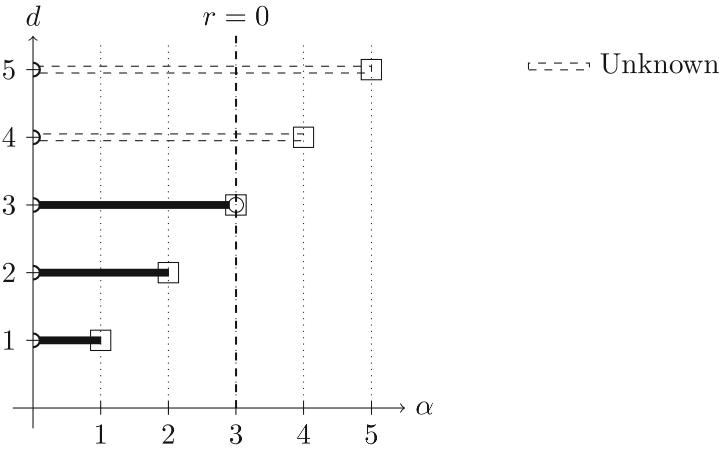


Fig. 3 Solvability for the stochastic wave equation (i.e., $a = b = 2$ and $r = 0$). See Fig. 2 for an additional legend

noise in $\mathbb{R}^d$ (i.e., $a = 2$, $b = 1$, $\alpha = d$ and $r = 0$), Hairer and Labbé constructed pathwise solutions using the regularity structure for both cases $d = 2, 3$ in [15] and [16], respectively. The relation between these two types of solution is left for future work.

Example 2.3 *(SWE)* By setting $a = 2$ and formally setting $b = 2$ in (1.12), we obtain the following condition for the stochastic wave equation under which there only exists a local solution:

$$\alpha = 3 + 2r \leq d \quad \text{and} \quad r \in [0, 1/2]. \tag{2.3}$$

We recall that results in Balan et al. [1] require $d \leq 3$, and likewise, Assumption 1.1 and all known sufficient conditions for the nonnegativity of the fundamental solution (see Remark 1.2) also require $d \leq 3$ in case of $b \in (1, 2)$. With this restriction, conditions (2.3) reduce to

$$\alpha = 3 = d \quad \text{and} \quad r = 0,$$

which says that at dimension $d = 3$, when $\dot{W}$ is a white noise, there exists only a local $L^p(\Omega)$ solution for all $p \geq 2$. See Fig. 3 for an illustration. Moreover, one can check easily that the expression for the critical time T_p in (1.13) in this case reduces to

$$T_p = \frac{\nu^{3/2}}{2\theta(p-1)\sqrt{\mathcal{M}_{2,3}(\delta_0)}}, \quad p \geq 2, \tag{2.4}$$

which is identical to (1.12) (*ibid.*) when setting $\nu = 2$.

Example 2.4 *(Fractional SPDEs with $r = \lceil b \rceil - b$ and $a = 2$)* For the fractional SDPEs with $b \neq 1$, many known works focus on the case when $r = \lceil b \rceil - b$, where $\lceil b \rceil$ is the ceiling function; see, e,g., [4, 21]. To facilitate the discussions here, we will only focus on the case when $a = 2$. In particular, by setting $r = \lceil b \rceil - b$ and $a = 2$, conditions in (1.12) become

$$\begin{cases} \alpha = \dfrac{2}{b} \le d \quad \text{and} \quad b \in [1/2, 1], \\ \alpha = \dfrac{6}{b} \le d \quad \text{and} \quad b \in [3/2, 2). \end{cases} \tag{2.5}$$

When $b = 1$, we have $r = 0$ and the fundamental solution is the standard heat kernel. Hence, Assumption 1.1 is satisfied for all $d \ge 1$. When $b < 1$, sufficient conditions in Remark 1.2 guarantees Assumption 1.1 for all $d \ge 1$. However, when $b > 1$ and $a = 2$, from Remark 1.2 we see that the fundamental solution is nonnegative only for $d \le 3$. The solvability for this case is illustrated in Fig. 4 and the critical time T_p in case of local solution (hence, only for the case when $b \in [1/2, 1]$) is equal to

$$T_p = \frac{\nu^{\alpha/2}}{2\theta(p-1)\mathcal{M}_{2,d}^{2-\alpha/2}}. \tag{2.6}$$

Example 2.5 *(Fractional SPDEs with $r = 0$ and $a = 2$)* In this example, we study the special case of the fractional SPDEs when $r = 0$. The choice of $r = 0$ has been used in, e.g., [5]. We will only consider the case $a = 2$ for simplicity. Now by setting $r = 0$ and $a = 2$ and restricting $b \le 1$, conditions in (1.12) become

$$\alpha = 4 - \frac{2}{b} \le d \quad \text{and} \quad b \in (0, 1]. \tag{2.7}$$

As discussed in Example 2.4, Assumption 1.1 is satisfied for all $d \ge 1$ when $b \le 1$ but only for $d \le 3$ when $b > 1$. The solvability for this case is illustrated in Fig. 5 with T_p given in (2.6). In particular, for the example in the second figure in Fig. 5, namely, when $b = 2/3$ and $\alpha = d = 1$, the white noise driven SHE has a local solution with

$$T_p = \frac{2^{5/2}\sqrt{\nu}}{3(p-1)\theta}, \qquad \text{for all } p \ge 2, \tag{2.8}$$

where we have applied (2.6) and the relation (3.22).

More examples regarding the solvability can be studied in a similar way, which are left to the interested readers.

Example 2.6 *(SHE with fractional Laplacian)* The stochastic heat equation with fractional Laplacian (i.e., the case when $b = 1$, $r = 0$ and $a \in (0, 2]$) has been widely studied in the literature, but possibly with different noises. In this case, the fundamental solutions are transition densities for the alpha-stable jump processes, which are

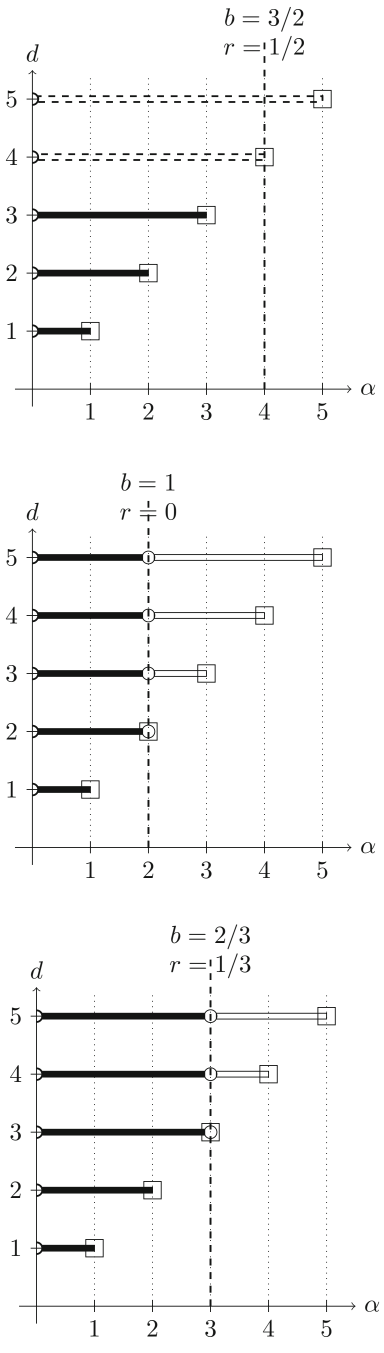


Fig. 4 Solvability for the fractional SPDEs in case of $a = 2$ and $r = \lceil b \rceil - b$. See Figs. 2 and 3 for the legend

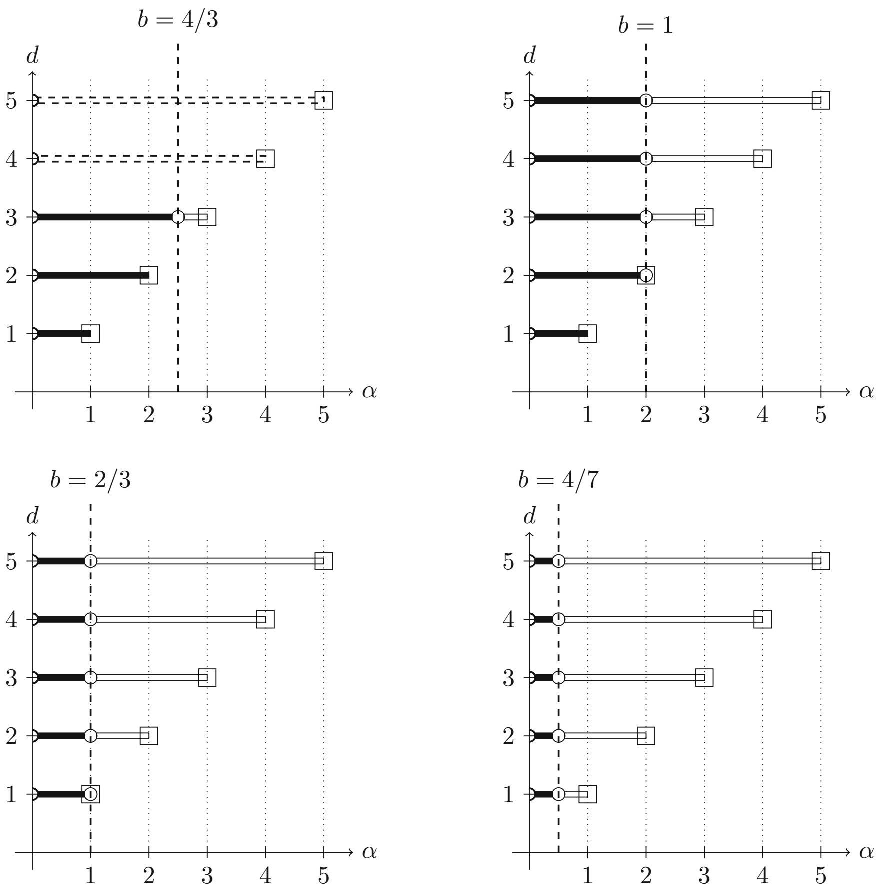


Fig. 5 Solvability for the fractional SPDEs in case of $a = 2$ and $r = 0$. See Figs. 2 and 3 for the legend

necessarily to be nonnegative. This is also consistent with the sufficient conditions for nonnegativity in Remark 1.2. By setting $b = 1$ and $r = 0$ in (1.12), we have the following condition:

$$\alpha = a \leq d$$

The solvability for this case is illustrated in Fig. 6.

2.2 Examples on asymptotics

In this part, we list several examples for the asymptotics when global solutions exist. In particular, we will show that the asymptotics in (1.14) interpolates the corresponding results for both stochastic wave and heat equations.

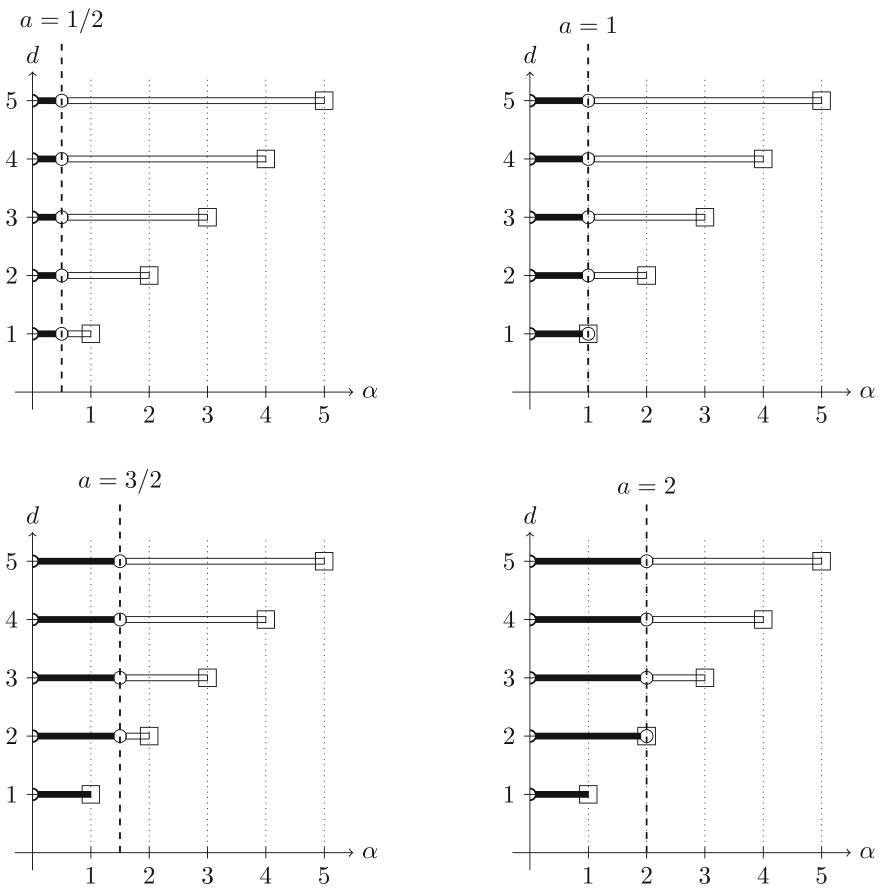


Fig. 6 Solvability for the stochastic heat equation with fractional Laplacian, i.e, the case when $b = 1$ and $r = 0$. See Figs. 2 and 3 for the legend

Example 2.7 *(Asymptotics for SWE)* Even though our results requires b to be strictly less than 2, but by formally setting

$$a = b = \nu = 2 \quad \text{and} \quad r = 0,$$

we have that

$$\beta = \frac{4 - \alpha}{3 - \alpha} \quad \text{and} \quad t_p = (p - 1)^{1/(4-\alpha)} t,$$

and results in (1.14), (1.16), and (1.17) recover the corresponding results for the stochastic wave equation, namely, (1.9), (1.10), and (1.11) of [1], respectively. Due to the importance of white noise and for the future references, here we list two special cases regarding white noise:

(1) The SWE with white noise in $\mathbb{R}$: By further setting $d = \alpha = 1$, we see that

$$\lim_{t\to\infty} \frac{\log \mathbb{E}\left[|u(t,x)|^p\right]}{t^{3/2}} = \frac{p(p-1)^{1/2}\sqrt{\theta}}{3(2\nu)^{1/4}} \quad \text{and} \quad \lim_{p\to\infty} \frac{\log \mathbb{E}\left[|u(t,x)|^p\right]}{p^{3/2}} = \frac{t^{3/2}\sqrt{\theta}}{3(2\nu)^{1/4}}, \tag{2.9}$$

where we have applied (3.22).

(2) The SWE with white noise in $\mathbb{R}^2$: Similarly, by setting $d = \alpha = 2$, we see that

$$\lim_{t\to\infty} \frac{\log \mathbb{E}\left[|u(t,x)|^p\right]}{t^2} = \frac{p(p-1)\theta\mathcal{M}_{2,2}(\delta_0)}{2\nu} \quad \text{and} \quad \lim_{p\to\infty} \frac{\log \mathbb{E}\left[|u(t,x)|^p\right]}{p^2} = \frac{t^2\theta\mathcal{M}_{2,2}(\delta_0)}{2\nu}. \tag{2.10}$$

Example 2.8 *(Asymptotics for SHE)* As for the stochastic heat equation case, by setting

$$a = 2, \quad b = \nu = 1 \quad \text{and} \quad r = 0,$$

we have that

$$\beta = \frac{4-\alpha}{2-\alpha} \quad \text{and} \quad t_p = (p-1)^{2/(4-\alpha)} t,$$

and results in (1.14) and (1.16) recover the corresponding conjectured results for SHE, namely, (1.16) and (1.17) of Balan et al. [1], respectively, which are equivalent to Theorem 1.1 and 1.2 of X. Chen [8] when setting $\alpha_0 = 0$ and using [10, Lemma A.2] to rewrite the constant $\mathcal{E}$ in [8] in terms of $\mathcal{M}_a$. Due to the importance of the white noise case, we list the corresponding asymptotics here. When $\alpha = d = 1$,

$$\lim_{t\to\infty} \frac{\log \mathbb{E}\left(|u(t,x)|^p\right)}{t^3} = \frac{p(p-1)^2\theta^2}{24\nu} \quad \text{and} \quad \lim_{p\to\infty} \frac{\log \mathbb{E}\left(|u(t,x)|^p\right)}{p^3} = \frac{t^3\theta^2}{24\nu}, \tag{2.11}$$

where we have applied (3.22). Note that some upper and lower bounds for the first limit in (2.11) in case of $p = 2$ were earlier obtained by Hu [18, part 1) of Theorem 4.1].

Example 2.9 *(Asymptotics for SHE with fractional Laplacian)* In this example we restrict ourselves to the case when $b = 1$, $a \in (0, 2]$, $\alpha < d$, and $r = 0$, which is the 1-dimensional SHE with fractional Laplace. With this set up we have

$$\beta = \frac{2a-\alpha}{a-\alpha} \qquad \text{and} \qquad t_p = (p-1)^{\frac{a}{2a-\alpha}} t,$$

and by Corollary 1.8,

$$\lim_{t\to\infty}\frac{\log\mathbb{E}\left(|u(t,x)|^p\right)}{t^{\frac{2a-\alpha}{a-\alpha}}}=p(p-1)^{\frac{a}{a-\alpha}}\left(\frac{1}{2}\right)\left(\frac{2a}{2a-\alpha}\right)^{\frac{2a-\alpha}{a-\alpha}}\left[\theta\nu^{-\alpha/a}\mathcal{M}_{a,d}^{\frac{2a-\alpha}{a}}\right]^{\frac{a}{a-\alpha}}\left(\frac{a-\alpha}{a}\right) \tag{2.12}$$

and

$$\lim_{p\to\infty}\frac{\log\mathbb{E}\left(|u(t,x)|^p\right)}{p^{\frac{2a-\alpha}{a-\alpha}}}=t^{\frac{2a-\alpha}{a-\alpha}}\left(\frac{1}{2}\right)\left(\frac{2a}{2a-\alpha}\right)^{\frac{2a-\alpha}{a-\alpha}}\left[\theta\nu^{-\alpha/a}\mathcal{M}_{a,d}^{\frac{2a-\alpha}{a}}\right]^{\frac{a}{a-\alpha}}\left(\frac{a-\alpha}{a}\right). \tag{2.13}$$

this setup has been studied in [12] for the case of a time-dependent noise where the covariance function is given by

$$\mathbb{E}[\dot{W}(r,x)\dot{W}(s,y)]=|r-s|^{-\alpha_0}\gamma(x-y)$$

and $\gamma(x)$ is defined to be either of the following:

$$\gamma(x):=\begin{cases}|x|^{-\alpha} & \text{where } \alpha\in(0,d) \ \text{ or}\\ \prod_{j=1}^d|x_j|^{\alpha_j} & \text{where } \alpha_j\in(0,1).\end{cases} \tag{2.14}$$

They proved that for $\alpha<\min\{a,d\}$ and let $p\geq 2$,

$$\lim_{t\to\infty}t^{-\frac{2a-\alpha-\alpha\alpha_0}{a-\alpha}}\log\mathbb{E}[|u(t,x)|^p]=p(p-1)^{\frac{a}{a-\alpha}}\mathbf{M}(a,\alpha_0,d,\gamma), \tag{2.15}$$

where the variational constant is given by

$$\mathbf{M}(a,\alpha_0,d,\gamma)=\sup_{g\in\mathcal{A}_{a,d}}\left\{\frac{1}{2}\int_0^1\int_0^1\int_{\mathbb{R}^{2d}}\frac{\gamma(x-y)}{|r-s|^{\alpha_0}}g^2(s,x)g^2(r,y)\mathrm{d}x\mathrm{d}y\mathrm{d}r\mathrm{d}s\right.$$
$$\left.-(2\pi)^{-d}\int_0^1\int_{\mathbb{R}^d}|x|^a|\mathcal{F}g(s,\xi)|^2\mathrm{d}\xi\mathrm{d}s\right\}$$

with[3]

$$\mathcal{A}_{a,d}:=\left\{g(s,x):\int_{\mathbb{R}^d}g^2(s,x)\mathrm{d}x=1,\forall s\in[0,1]\text{ and }(2\pi)^{-d}\right.$$
$$\left.\int_0^1\int_{\mathbb{R}^d}|x|^a|\mathcal{F}g(s,\xi)|^2\mathrm{d}\xi\mathrm{d}s<\infty\right\}.$$

[3] Note that the Fourier transform is defined differently in [12].

By setting $\alpha_0 = 0$ and letting $g(s, x) = g(x) \in \mathcal{F}_a$ be independent in s, which is the time-independent setup, then Equation (2.12) and Lemma 3.9 together recover (2.15). Indeed, by observing (3.29) and (3.33), we see that

$$\mathbf{M}(a, \alpha_0, d, \gamma) = \mathbf{E}_{a,d}\left(\frac{1}{2}\gamma, 2\right)$$
$$= 2^{\frac{-a}{a-\alpha}}\left(\frac{a-\alpha}{a}\right)\left(\frac{2a}{2a-\alpha}\right)^{\frac{2a-\alpha}{a-\alpha}} \mathcal{M}_{a,d}(\gamma, 1)^{\frac{2a-\alpha}{a-\alpha}} \tag{2.16}$$

and by rewriting (2.15) with (2.16) yields (2.12). Finally, we note that condition (2.14) can be relaxed to allow white noise in one dimensional case, namely, $\alpha = d = 1$. In this case, one can simply replace α and d in both (2.12) and (2.20) by 1 and in addition replace $\mathcal{M}_{a,d}$ by $\mathcal{M}_{a,1}(\delta_0)$.

Example 2.10 *(Asymptotics for SPDEs with $r = \lceil b \rceil - b$ and white noise)* In this example, we consider the case when $a = 2, d = \alpha = 1$ (white noise), and $r = \lceil b \rceil - b$. As seen in Example 2.4, there exists a global solution. In this case,

$$\beta = \frac{4\lceil b \rceil - b}{4\lceil b \rceil - b - 2} \qquad \text{and} \qquad t_p = (p-1)^{\frac{2}{4\lceil b \rceil - b}}\, t,$$

and by (3.22) and Corollary 1.8,

$$\lim_{t\to\infty} \frac{\log \mathbb{E}\left(|u(t,x)|^p\right)}{t^{\frac{4\lceil b \rceil - b}{4\lceil b \rceil - b - 2}}} = p(p-1)^{\frac{2}{4\lceil b \rceil - b - 2}} \times \left(\frac{9\theta^2}{8\nu}\right)^{\frac{1}{4\lceil b \rceil - b - 2}} (4\lceil b \rceil - b - 2)\,(4\lceil b \rceil - b)^{-\frac{4\lceil b \rceil - b}{4\lceil b \rceil - b - 2}}, \tag{2.17}$$

and

$$\lim_{p\to\infty} \frac{\log \mathbb{E}\left(|u(t,x)|^p\right)}{p^{\frac{4\lceil b \rceil - b}{4\lceil b \rceil - b - 2}}} = t^{\frac{4\lceil b \rceil - b}{4\lceil b \rceil - b - 2}} \left(\frac{9\theta^2}{8\nu}\right)^{\frac{1}{4\lceil b \rceil - b - 2}} (4\lceil b \rceil - b - 2)\,(4\lceil b \rceil - b)^{-\frac{4\lceil b \rceil - b}{4\lceil b \rceil - b - 2}}. \tag{2.18}$$

Example 2.11 *(Asymptotics for SPDEs with $r = 0$ and white noise)* From Example 2.5, we see that when $a = 2$, $r = 0$, $d = \alpha = 1$ (white noise), the global solution exists when $b \in (2/3, 2)$. In this case, we have that

$$\beta = \frac{3b}{3b-2} \qquad \text{and} \qquad t_p = (p-1)^{2/(3b)}\, t,$$

and by (3.22) and Corollary 1.8,

$$\lim_{t\to\infty} \frac{\log \mathbb{E}\left(|u(t,x)|^p\right)}{t^{3b/(3b-2)}} = p(p-1)^{\frac{2}{3b}}\left(b - \frac{2}{3}\right) b^{-\frac{3b}{3b-2}} \left(\frac{\theta^2}{8\nu}\right)^{\frac{1}{3b-2}} \tag{2.19}$$

and

$$\lim_{p\to\infty}\frac{\log\mathbb{E}\left(|u(t,x)|^p\right)}{p^{3b/(3b-2)}}=t^{\frac{3b}{3b-2}}\left(b-\frac{2}{3}\right)b^{-\frac{3b}{3b-2}}\left(\frac{\theta^2}{8\nu}\right)^{\frac{1}{3b-2}}. \tag{2.20}$$

3 Some preliminaries

3.1 Skorohod integral and mild solution

We start with a nonnegative and nonnegative definite tempered measure Γ with density γ in the sense that $\Gamma(\mathrm{d}x)=\gamma(x)\mathrm{d}x$ and

$$\int_{\mathbb{R}^d}\Gamma(\mathrm{d}x)(\phi*\widetilde{\phi})(x)\geq 0 \quad \text{for all } \phi\in\mathscr{S}(\mathbb{R}^d)$$

where $\widetilde{\phi}(x):=\phi(-x)$. According to the Bochner theorem, there exists a nonnegative and nonnegative definite measure μ, often referred as the *spectral measure* on $\mathbb{R}^d$ whose Fourier transform (in the weak sense) is Γ, namely, that for any $\phi\in\mathcal{D}(\mathbb{R}^d)$ (the space of test functions),

$$\int_{\mathbb{R}^d}\Gamma(\mathrm{d}x)\phi(x)=\int_{\mathbb{R}^d}\mu(\mathrm{d}\xi)\mathcal{F}\phi(\xi).$$

Since μ is nonnegative definite, the following functional

$$C(\phi,\psi)=\int_{\mathbb{R}}\mathcal{F}\phi(\xi)\overline{\mathcal{F}\psi(\xi)}\mu(\mathrm{d}\xi), \qquad \text{for all } \phi,\psi\in\mathcal{D}(\mathbb{R}^d) \tag{3.1}$$

is nonnegative-definite and thus one can associate it with a zero-mean Gaussian processes, $W:=\left\{W(\phi):\phi\in\mathcal{D}(\mathbb{R}^d)\right\}$, with the covariance functional of W given by (3.1). In other words,

$$\mathbb{E}\left(W(\phi)W(\psi)\right)=\int_{\mathbb{R}}\mathcal{F}\phi(\xi)\overline{\mathcal{F}\psi(\xi)}\mu(\mathrm{d}\xi)=:\langle\phi,\psi\rangle_{\mathcal{H}}.$$

Let $\mathcal{H}$ be the completion of $\mathcal{D}(\mathbb{R}^d)$ with respect to $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ and thus we see $\phi\mapsto W(\phi)$ is an isometry from $\mathcal{D}(\mathbb{R}^d)$ to $L^2(\Omega)$, that is, $\mathbb{E}\left(W(\phi)^2\right)=\|\phi\|^2_{\mathcal{H}}$ for $\phi\in\mathcal{D}(\mathbb{R}^d)$. One can extend this isometry from $\mathcal{D}(\mathbb{R}^d)$ to $\mathcal{H}$. We refer the interested readers to [14] and references therein.

We denote δ the *Skorohod integral* with respect to W and denote its domain by $\mathrm{Dom}(\delta)$. u is called *Skorohod integrable* if $u\in\mathrm{Dom}(\delta)$, in which case we write $\delta(u)=\int_{\mathbb{R}^d}u(x)W(\delta x)$ and by isometry, $\mathbb{E}\left(\delta(u)^2\right)=\mathbb{E}(\|u\|^2_{\mathcal{H}})$. For a complete treatment of the Skorohod integral, see Nualart et al. [22].

Definition 3.1 *(Mild, local and global solutions)*

(1) For $T \in (0, \infty]$, a random field $u = \{u(t, x) : t \in (0, T),\ x \in \mathbb{R}^d\}$ is called *a mild solution* to the equation (1.1) if for all $x \in \mathbb{R}^d$ and s, t fixed with $0 < s \leq t < T$, $y \to G(t-s, x-y)u(s, y)$ is Skorohod integrable and the following stochastic integral equation holds almost surely

$$u(t, x) = 1 + \sqrt{\theta} \int_0^t \left(\int_{\mathbb{R}^d} G(t-s, x-y)u(s, y)W(\delta y) \right) \mathrm{d}s. \tag{3.2}$$

(2) Let $u(t, x)$ be a mild solution to (1.1) (or (3.2)) and fix $p \geq 1$. We call $u(t, x)$ a *global* $L^p(\Omega)$*-solution*, or simply an $L^p(\Omega)$*-solution* if

$$\|u(t, x)\|_p < \infty \qquad \text{for all } t > 0 \text{and } x \in \mathbb{R}^d. \tag{3.3}$$

(3) If there exist $0 < T_1 \leq T_2 < \infty$ such that $\|u(t, x)\|_p$ is finite for all $t \in (0, T_1)$ and $x \in \mathbb{R}^d$, but $\|u(t, x)\|_p$ diverges to infinity whenever $t > T_2$, the mild solution $u(t, x)$ in this case is called a *local* $L^p(\Omega)$*-solution*.

Note that through construction of the Skorohod integral δ, a mild solution is necessarily to be an $L^2(\Omega)$-solution. For more details, one may check, e.g., Nualart [22, Chapter 3].

Through the standard *Picard iteration* scheme, the solution can be expressed by the following *Wiener chaos expansion* [4]:

$$u(t, x) = 1 + \sum_{k=1}^{\infty} \theta^{k/2} I_k(f_k(\cdot, x, t)), \tag{3.4}$$

where $I_k : \mathcal{H}^{\otimes k}\left((\mathbb{R}^d)^k\right) \to \mathcal{H}_k$ is the k^{th} order Skorohod integral and $\mathcal{H}_k$ is the k^{th} Wiener chaos space and the kernels $f_k(\cdots, x, t)$, obtained through the iteration, are equal to

$$\begin{aligned}
&f_n(x_1, \ldots, x_k; x, t) \\
&\quad = \int_0^t \int_0^{t_n} \cdots \int_0^{t_2} G(t-t_n, x-x_n) \cdots G(t_2 - t_1, x_2 - x_1)\mathrm{d}t_1 \cdots \mathrm{d}t_n \\
&\quad = \int_0^t \int_0^{t_n} \cdots \int_0^{t_2} G(t_1, x-x_n) \cdots G(t_n - t_{n-1}, x_2 - x_1)\mathrm{d}t_1 \cdots \mathrm{d}t_n.
\end{aligned}$$

For ease of notation, throughout this article, we may write the above integrals as

[4] Wiener chaos expansion has been widely to solve the linear stochastic partial differential equations. We direct interested readers to [3, Section 5] for a presentation of this procedure.

$$
\begin{aligned}
f_n(x_1, \ldots, x_n; x, t) &= \int_{[0,t]^n_<} G(t - t_n, x - x_n) \cdots G(t_2 - t_1, x_2 - x_1) \mathrm{d}\vec{t} \\
&= \int_{[0,t]^n_<} G(t_1, x - x_n) \cdots G(t_n - t_{n-1}, x_2 - x_1) \mathrm{d}\vec{t},
\end{aligned}
$$

where $[0, t]^n_< := \{(t_1, \ldots, t_n) \in [0, t]^n : t_1 < \cdots < t_n\}$. As usual, we use $\widetilde{f}_n(\cdot, x, t)$ to denote the symmetrization of $f_n(\cdot, x, t)$:

$$
\begin{aligned}
\widetilde{f}_n(\cdot; x, t) &= \frac{1}{n!} \sum_{\rho \in \Sigma_n} f_n\left(x_{\rho(1)}, \ldots, x_{\rho(n)}\right) \\
&= \frac{1}{n!} \sum_{\rho \in \Sigma_n} \int_{[0,t]^n_<} G\left(t - t_n, x - x_{\rho(n)}\right) \cdots G\left(t_2 - t_1, x_{\rho(2)} - x_{\rho(1)}\right) \mathrm{d}\vec{t},
\end{aligned}
$$

where Σ_n is the set of all permutations of $\{1, \ldots, n\}$. By setting $t_{n+1} = t$, the Fourier transform of the kernels, f_n, is given by

$$
\mathcal{F} f_n(\cdot; x, t)(\xi_1, \ldots, \xi_n) = e^{-i\left(\sum_{j=1}^n \xi_j\right)\cdot x} \int_{[0,t]^n_<} \prod_{k=1}^{n} \overline{\mathcal{F}G(t_{k+1} - t_k, \ldots)\left(\sum_{j=1}^{k} \xi_j\right)} \mathrm{d}\vec{t}. \tag{3.5}
$$

Recall the notation above in (1.2) that $G(t, x) = G_{a,b,r,d}(t, x)$. The following scaling properties for both $\mathcal{F}G(t, \ldots)(\xi)$ and $\left\| \widetilde{f}_n(\cdot, x, t) \right\|_{\mathcal{H}^{\otimes n}}$ play an important role in the paper.

Lemma 3.2 *For any $c, t > 0$, $n \geq 1$, $\xi, \xi_1, \ldots, \xi_n \in \mathbb{R}^d$, the following scaling properties hold:*

$$
\begin{aligned}
&\mathcal{F}G(t, \cdot)(c\xi) = c^{-\frac{a}{b}(b+r-1)} \mathcal{F}G\left(c^{\frac{a}{b}} t, \ldots\right)(\xi) \quad \text{and} \quad \mathcal{F}G(ct, \cdot)(\xi) \\
&\quad = c^{b+r-1} \mathcal{F}G(t, \cdot)(c^{b/a} \xi), &(3.6)\\
&\mathcal{F}\widetilde{f}_n(\cdot, 0, ct)(\xi_1, \ldots, \xi_n) = c^{n(b+r)} \mathcal{F}\widetilde{f}_n(\cdot, 0, t)(c^{b/a}\xi_1, \ldots, c^{b/a}\xi_n), &(3.7)\\
&\left\| \widetilde{f}_n(\cdot, 0, t) \right\|^2_{\mathcal{H}^{\otimes n}} = t^{[2(b+r) - b\alpha/a]n} \left\| \widetilde{f}_n(\cdot, 0, 1) \right\|^2_{\mathcal{H}^{\otimes n}}. &(3.8)
\end{aligned}
$$

Proof The scaling properties in (3.6) are direct consequences of the explicit expression of $\mathcal{F}G(t, \cdot)(\xi)$ as in [6, (4.8)]. Property (3.7) is an easy exercise of change of variables on (3.5). Property (3.8) is a direct consequence of (3.6), (3.7), and the scaling property of the spectral measure μ. We leave the details for the interested readers. □

Finally, let us recall the following standard result about the existence and uniqueness of the solution to (1.1) (or (3.2)) when it can be written as the Wiener chaos expansion (3.4).

Theorem 3.3 *Fix any $T \in (0, \infty]$. Suppose that $f_n(\cdot, x, t) \in \mathcal{H}^{\otimes n}$ for any $t \in (0, T)$, $x \in \mathbb{R}^d$ and $n \geq 1$. Then (1.1) (or (3.2)) has a unique $L^2(\Omega)$-solution on $(0, T) \times \mathbb{R}^d$*

if and only if the series (3.4) converges in $L^2(\Omega)$ for any $(t, x) \in (0, T) \times \mathbb{R}^d$, which is equivalent to the convergence of the series (3.9). In this case, the solution is given by (3.4) with the second moment given by

$$\mathbb{E}\left(u(t,x)^2\right) = \sum_{n\geq 0} \theta^n n! \left\| \widetilde{f}_n(\cdot, x, t) \right\|^2_{\mathcal{H}^{\otimes n}} \quad \textit{for all } (t,x) \in (0,T) \times \mathbb{R}^d. \tag{3.9}$$

3.2 Some asymptotics and variational constants

Recall that the correlation function γ satisfies Assumption 1.3 and that the corresponding spectral measure is μ; see Remark 1.4. Define

$$\rho_{\nu,a}(\gamma) = \sup_{\|f\|_{L^2(\mathbb{R}^d)}=1} \int_{\mathbb{R}^d} \left[\int_{\mathbb{R}^d} \frac{f(x+y)f(y)}{\sqrt{1+\frac{\nu}{2}|x+y|^a}\sqrt{1+\frac{\nu}{2}|y|^a}} \mathrm{d}y \right]^2 \mu(\mathrm{d}x) \tag{3.10}$$

and

$$\begin{aligned} \mathcal{M}_{a,d}(\gamma,\theta) &:= \sup_{g\in\mathcal{F}_a} \left\{ \left(\iint_{\mathbb{R}^{2d}} g^2(x) g^2(y) \gamma(x+y) \mathrm{d}x \mathrm{d}y \right)^{1/2} - \frac{\theta}{2}\, \mathcal{E}_a(g,g) \right\} \\ &= \sup_{g\in\mathcal{F}_a} \left\{ \left\langle g^2 * g^2, \gamma \right\rangle^{1/2}_{L^2(\mathbb{R}^d)} - \frac{\theta}{2}\, \mathcal{E}_a(g,g) \right\}, \end{aligned} \tag{3.11}$$

where

$$\mathcal{E}_a(g,g) := (2\pi)^{-d} \int_{\mathbb{R}^d} |\xi|^a |\mathcal{F}g(\xi)|^2 \mathrm{d}\xi \quad \text{and} \tag{3.12}$$

$$\mathcal{F}_a := \left\{ f \in L^2(\mathbb{R}^d) :\ \|f\|_{L^2(\mathbb{R}^d)} = 1,\ \mathcal{E}_a(f,f) < \infty \right\}. \tag{3.13}$$

We often omit the dimension d in $\mathcal{M}_{a,d}$ when it is clear from context. We use the convention that $\mathcal{M}_a(f) := \mathcal{M}_a(f, 1)$ to be consistent with notation (1.10). By a similar argument as the proof of [1, Lemma 2.3], one can show that

$$\mathcal{M}_a(\Theta\gamma, \theta) = \Theta^{\frac{a}{2a-\alpha}} \theta^{-\frac{\alpha}{2a-\alpha}} \mathcal{M}_a(\gamma, 1), \qquad \text{for all } \theta \text{ and } \Theta > 0. \tag{3.14}$$

For the Riesz kernel case (see Example 1.5), Bass, Chen and Rosen [2] established that when $a \in (0, 2]$, $\nu = 2$ and $\alpha < \min\{2a, d\}$,

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{n} \log \left[\frac{1}{(n!)^2} \int_{(\mathbb{R}^d)^n} \left(\sum_{\sigma\in\Sigma_n} \prod_{k=1}^n \frac{1}{1 + |\sum_{j=k}^n \xi_{\sigma(j)}|^a} \right)^2 \mu(\mathrm{d}\vec{\xi}\,) \right] \\ &= \log\left(\rho_{2,a}\left(|\cdot|^{-\alpha}\right)\right), \end{aligned} \tag{3.15}$$

and [5]

$$\rho_{2,a}\left(|\cdot|^{-\alpha}\right) = \mathcal{M}_a^{2-(\alpha/a)}\left(|\cdot|^{-\alpha}, 2\right), \tag{3.16}$$

where

$$\mu(\mathrm{d}\vec{\xi}\ ) = \prod_{j=1}^{n} \mu(\mathrm{d}\xi_j) = \prod_{j=1}^{n} \varphi(\xi_j)\mathrm{d}\xi_j. \tag{3.17}$$

We first apply some scaling arguments to accommodate the parameter ν in both (3.15) and (3.16), the proof of which can be found in "Appendix":

Lemma 3.4 (The Riesz kernel case) *If $\gamma(x) = |x|^{-\alpha}$ for some $\alpha \in (0, d)$, then for any $\nu > 0$ and $a \in (0, 2]$,*

$$\rho_{\nu,a}\left(|\cdot|^{-\alpha}\right) = \left(\frac{\nu}{2}\right)^{-\alpha/a} \mathcal{M}_a^{2-(\alpha/a)}\left(|\cdot|^{-\alpha}, 2\right) = \nu^{-\alpha/a}\mathcal{M}_a^{2-(\alpha/a)}(|\cdot|^{-\alpha}), \tag{3.18}$$

and

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n} \log\left[\frac{1}{(n!)^2}\int_{(\mathbb{R}^d)^n}\left(\sum_{\sigma\in\Sigma_n}\prod_{k=1}^{n}\frac{1}{1+\frac{\nu}{2}|\sum_{j=k}^{n}\xi_{\sigma(j)}|^a}\right)^2 \mu(\mathrm{d}\vec{\xi}\ )\right]\\ = \log\left(\rho_{\nu,a}\left(|\cdot|^{-\alpha}\right)\right).\end{aligned} \tag{3.19}$$

More generally we have the following theorem:

Theorem 3.5 *Suppose that the correlation function γ satisfies Assumption 1.3 and is such that $\alpha < \min\{2a, d\}$. Then both (3.18) and (3.19) hold with $|\cdot|^{-\alpha}$ and μ replaced by γ and μ as in (1.5), respectively. More precisely, it holds that*

$$\rho_{\nu,a}(\gamma) = \nu^{-\alpha/a}\mathcal{M}_a^{2-(\alpha/a)}(\gamma) < \infty, \tag{3.20}$$

and

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n} \log\left[\frac{1}{(n!)^2}\int_{(\mathbb{R}^d)^n}\left(\sum_{\sigma\in\Sigma_n}\prod_{k=1}^{n}\frac{1}{1+\frac{\nu}{2}|\sum_{j=k}^{n}\xi_{\sigma(j)}|^a}\right)^2 \mu(\mathrm{d}\vec{\xi}\ )\right]\\ = \log\left(\rho_{\nu,a}\left(\gamma\right)\right).\end{aligned} \tag{3.21}$$

Remark 3.6 It is often very difficult to obtain the exact value for the variational constant $\mathcal{M}_{a,d}(\gamma)$. To the best of our knowledge, only in case of $a = 2$ and $\alpha = d = 1$ (white noise), one can compute explicitly that

$$\mathcal{M}_{2,1}(\delta_0) = (3/4)(1/6)^{1/3}, \tag{3.22}$$

[5] In Theorem 1.5 or eq. (1.20) of Bass et al. [2], the factor $(2\pi)^{-d}$ should not be present.

Table 1 Notation correspondence

	Laplace		Noise			Moment	Variational Const.
Bass et al [2]	β	2	σ	$\lvert\cdot\rvert^{-\sigma}$	$\varphi_{d-\sigma}$	p	Λ_σ
Current paper	a	ν	α	$\gamma(\cdot)$	φ	2	$\mathcal{M}_a\left(\lvert\cdot\rvert^{-\alpha},2\right)$

which is a consequence of Chen and Li [13, Lemma 7.2] with $p = 2$. When $d \geq 2$, the value of $\mathcal{M}_{2,d}(\delta_0)$ can be expressed using the best constant for the classical *Gagliardo-Nirenberg* inequality; see Remark 3.13 of Chen et al. [11] for more details.

Sketch of the proof of Theorem 3.5 The proof of this theorem follows essentially the identical proof as Bass et al. [2], which is exclusively for the Riesz kernel. One simplification is that we only need to handle the case $p = 2$ thanks to the hypercontractivity property. For our slight extension to the noise given in Assumption 1.3, there is no need to repeat their paper. Instead we will only point out the differences and necessary changes. For your convenience, the correspondence of parameters between Bass et al. [2] and the current paper is listed in the following Table 1.

Theorem 3.5 is proven by showing the following claims: for γ given in Assumption 1.3,

(i) $\rho_{\nu,a}(\gamma) < \infty$;

(ii) $\liminf_{n\to\infty} \frac{1}{n}\log\left[\frac{1}{(n!)^2}\int_{(\mathbb{R}^d)^n}\left(\sum_{\sigma\in\Sigma_n}\prod_{k=1}^n \frac{1}{1+\frac{\nu}{2}\left|\sum_{j=k}^n \xi_{\sigma(j)}\right|^a}\right)^2 \mu(\mathrm{d}\vec{\xi}\,)\right]$
$\geq \log\left(\rho_{\nu,a}(\gamma)\right)$;

(iii) $\limsup_{n\to\infty} \frac{1}{n}\log\left[\frac{1}{(n!)^2}\int_{(\mathbb{R}^d)^n}\left(\sum_{\sigma\in\Sigma_n}\prod_{k=1}^n \frac{1}{1+\frac{\nu}{2}\left|\sum_{j=k}^n \xi_{\sigma(j)}\right|^a}\right)^2 \mu(\mathrm{d}\vec{\xi}\,)\right]$
$\leq \log\left(\rho_{\nu,a}(\gamma)\right)$;

(iv) $\mathcal{M}_a(\gamma) < \infty$;

(v) $\rho_{\nu,a}(\gamma) = \nu^{-\alpha/a}\mathcal{M}_a^{2-(\alpha/a)}(\gamma)$.

Part (i) which corresponds to Lemma 1.6 (*ibid.*) is established by Lemma 3.7 below.

Following exactly the same arguments as those in Sect. 3 (*ibid.*) with $\varphi_{d-\sigma}$ (*ibid.*) replaced by our φ as in (1.5), one can prove part (ii) for $\nu = 2$. Then an application of the scaling property as the proof of Lemma 3.4 shows the general case $\nu > 0$.

The proof of the upper bound, namely part (iii), is more challenging. This part corresponds to Sects. 5 and 6 (*ibid.*). By examining these two sections carefully, we need to make some changes in Sect. 5 (*ibid.*), where as the arguments in Sect. 6 (*ibid.*) follow unchanged. For Sect. 5 (*ibid.*), we need to use the following decomposition of φ as opposed to (5.4) (*ibid.*):

$$\varphi(\xi) = \prod_{i=1}^k C_{\alpha_i,d_i}|\xi_{(i)}|^{-(d_i-\alpha_i)} = \prod_{i=1}^k C_{\alpha_i,d_i}\left(\mathcal{P}_i * \mathcal{P}_i\right)\left(\xi_{(i)}\right),$$

with

$$\mathcal{P}_i\left(\xi_{(i)}\right)=\beta_{d_i-\alpha_i,d_i}\left|\xi_{(i)}\right|^{-(d_i-\alpha_i/2)};$$

see (1.7) for the constants. Or equivalently,

$$\varphi(\xi)=(\mathcal{P}*\mathcal{P})(\xi)\quad\text{with}\quad\mathcal{P}(\xi):=\prod_{i=1}^{k}\sqrt{C_{\alpha_i,d_i}}\,\beta_{d_i-\alpha_i,d_i}\left|\xi_{(i)}\right|^{-(d_i-\alpha_i/2)}.$$

Now (5.5) (*ibid.*) should be written as

$$\mathcal{P}_{\beta,\epsilon}(\xi)=\hat{h}(\epsilon\xi)\prod_{i=1}^{k}\frac{\sqrt{C_{\alpha_i,d_i}}\,\beta_{d_i-\alpha_i,d_i}}{\beta+\left|\xi_{(i)}\right|^{-(d_i-\alpha_i/2)}},\quad\text{for all }\beta,\epsilon\geq 0,$$

where $h(\cdot)$ is defined in (5.2) (*ibid.*). So $\mathcal{P}(\xi)=\mathcal{P}_{0,0}(\xi)$ and

$$\left(\mathcal{P}_{\beta,\epsilon}*\mathcal{P}_{\beta,\epsilon}\right)(\xi)\leq\left(\mathcal{P}_{\beta,0}*\mathcal{P}_{\beta,0}\right)(\xi)\leq\left(\mathcal{P}_{0,0}*\mathcal{P}_{0,0}\right)(\xi)=\varphi(\xi);$$

see (5.6) (*ibid.*). With these changes, one can update accordingly the proof of Lemma 5.1 (*ibid.*) without any difficulty. Then the rest of Sect. 5 (*ibid.*) follows unchanged. In this way, we establish part (iii) for $\nu=2$. Finally, a scaling argument as in part (ii) proves part (iii) for all $\nu>0$.

Parts (iv) and (v) correspond to Sect. 7 (*ibid.*). In particular, part (iv) corresponds to Lemma 7.1 (*ibid.*). Note that we only need to study the case $p=2$. By (3.24) with $\varphi(x-y)$ replaced by $\gamma(x-y)$, we see that

$$\begin{aligned}\int_{(\mathbb{R}^d)^2}g^2(x)g^2(-y)\gamma(x-y)\mathrm{d}x\mathrm{d}y&\leq C\left\|\tilde{g}^2\right\|_{L^{2d/(2d-\alpha)}(\mathbb{R}^d)}\left\|g^2\right\|_{L^{2d/(2d-\alpha)}(\mathbb{R}^d)}\\&=C\left\|g\right\|^4_{L^{4d/(2d-\alpha)}(\mathbb{R}^d)},\end{aligned}\tag{3.23}$$

where $\tilde{g}^2(x)=g(-x)$. Note that we must have $\alpha<2a$ to ensure that the right hand side of the above is finite. This is seen by applying $h=\gamma$ in Lemma 3.7. Thus equation (7.1) (*ibid.*) can be applied in our setting . The rest of the proof of Lemma 7.1 (*ibid.*) remains unchanged.

It remains to update the proof of Theorem 1.5 in Sect. 7 (*ibid.*). For this, one needs only to update the four appearances of $1/(|\cdot|^{\sigma})$ in (7.15), (7.22) and (7.23) (*ibid.*) to $\gamma(\cdot)$. Note that the factor $(2\pi)^{-d(p+1)}$ in the first equation of (7.22) (*ibid.*) should be $(2\pi)^{-dp}$. With this, we complete the sketch proof of Theorem 3.5. □

Note that the proof of [2, Lemma 1.6] relies on inequality (1.27) on p. 630 (*ibid.*), which was a consequence of *Sobolev's inequality*. For the more general noises studied in this paper, we can no longer apply this inequality. Instead, we prove the following lemma using the *weak Young's inequality* (see, e.g., [20, p.107]) as a generalization of Lemma 1.6 (*ibid.*). Even though we only need the case $p=2$, the following lemma is proven for all $p\geq 2$.

Lemma 3.7 *For any f, g, h with $h \geq 0$, for φ given as in (1.5) (see also Assumption 1.3), and for all $p \geq 2$, it holds that*

$$\left(\int_{\mathbb{R}^d}\left[\int_{\mathbb{R}^d}\frac{|f(x+y)g(y)|}{\sqrt{h(x+y)}\sqrt{h(y)}}\mathrm{d}y\right]^p\varphi(x)\mathrm{d}x\right)^{1/p}$$
$$\leq C\,\|f\|_{L^2(\mathbb{R}^d)}\,\|g\|_{L^2(\mathbb{R}^d)}\,\left\|h^{-1}\right\|_{L^{pd/\alpha}(\mathbb{R}^d)}.$$

Proof By observing the proof of Lemma 1.6 of [2], we only need to prove that

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}F(y)G(x)\varphi(x-y)\,\mathrm{d}y\mathrm{d}x \leq C\,\|F\|_{L^{2d/(d+\alpha)}(\mathbb{R}^d)}\,\|G\|_{L^{2d/(d+\alpha)}(\mathbb{R}^d)}\,, \quad (3.24)$$

where

$$F(x) = \frac{|f(x)|}{(h(x))^{p/2}} \qquad \text{and} \qquad G(x) = \frac{|g(x)|}{(h(x))^{p/2}}.$$

Note that when $\varphi(x) = C|x|^{-(d-\alpha)}$, (3.24) is nothing but (1.27) (*ibid.*). Here we need to handle more general φ as given in (1.5). To prove (3.24), we need to apply the *weak version of Young's inequality* (see, e.g., [20, eq. (7) on p. 107]), which says that for all $p, q, r > 1$ with $1/p + 1/q + 1/r = 2$, it holds that

$$\left|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}a(x)b(x-y)c(y)\mathrm{d}x\mathrm{d}y\right| \leq K_{p,q,r,d}\,\|a\|_{L^p(\mathbb{R}^d)}\,\|b\|_{q,w}\,\|c\|_{L^r(\mathbb{R}^d)}\,, \quad (3.25)$$

where

$$\|b\|_{q,w} := \sup_A |A|^{-1/q'}\int_A |b(x)|\mathrm{d}x, \qquad \text{with } 1/q + 1/(q') = 1,$$

and A is an arbitrary Borel set of finite measure $|A| < \infty$. Now we apply (3.25) with

$$a = F,\quad c = G,\quad b = \varphi,\quad p = r = 2d/(d+\alpha),\quad \text{and}\quad q = \frac{d}{d-\sum_{j=1}^d \alpha_j} = d/(d-\alpha).$$

By (3.25) above, it suffices to prove that $\|\varphi\|_{q,w}$ is finite with $q = d/(d-\alpha)$ and $1/q' = 1 - 1/q = \alpha/d$.

Recall that according to Assumption 1.3, the d coordinates are partitioned into k groups. Define $A_R := A_1 \times \cdots \times A_k$ where $A_i = B_{R,d_i}(0)$ is the ball in $\mathbb{R}^{d_i}$ centered at the origin with radius R. With this we have that

$$\int_{A_i}|x_{(i)}|^{-(d_i-\alpha_i)}\mathrm{d}x_{(i)} = |\mathbb{S}^{d_i-1}|\,\frac{R^{\alpha_i}}{\alpha_i}, \quad (3.26)$$

where we have used polar coordinated to calculate the integral and $\left|\mathbb{S}^{d_i-1}\right| = 2\pi^{d_i/2}/\Gamma(d_i/2)$ is the surface area of the unit sphere in $\mathbb{R}^{d_i}$ (clearly, when $d_i = 1$,

$|S^0| = 2$). Moreover, by the formula for the volume of balls in $\mathbb{R}^{d_i}$, we see that

$$|A_i| = \frac{\pi^{d_i/2}}{\Gamma\left(1+\frac{d_i}{2}\right)} R^{d_i} = |\mathbb{S}^{d_i-1}| \frac{R^{d_i}}{d_i}. \tag{3.27}$$

Recall that $1/q' = \alpha/d$. Then a combination of (3.26), (3.27) and (1.5) shows that

$$\begin{aligned} |A_R|^{-1/q'} \int_{A_R} \varphi(x) \mathrm{d}x &= \prod_{i=1}^{k} C_{\alpha_i,d_i} |A_i|^{-1/q'} \int_{A_i} |x_{(i)}|^{-(d_i-\alpha_i)} \mathrm{d}x_{(i)} \\ &= \prod_{i=1}^{k} C_{\alpha_i,d_i} \alpha_i^{-1} |S^{d_i-1}|^{1-\alpha/d} R^{\alpha_i - \frac{d_i}{d}\alpha} d_i^{\alpha/d} \\ &= \prod_{i=1}^{k} C_{\alpha_i,d_i} \alpha_i^{-1} |S^{d_i-1}|^{1-\alpha/d} d_i^{\alpha/d} =: K, \end{aligned}$$

where the constants C_{α_i,d_i} are defined in (1.7) and the final constant K does not depend on R. Finally, by symmetry of φ, we have that

$$\|\varphi\|_{q,w} = \sup_{R>0} |A_R|^{-1/q'} \int_{A_R} \varphi(x) \mathrm{d}x = K < \infty. \tag{3.28}$$

Hence, $\varphi \in L_{q,w}(\mathbb{R}^d)$ with $q = d/(d-\alpha)$. This completes the proof of Lemma 3.7. □

Remark 3.8 Note that when there is only one partition (i.e., $k = 1$), or equivalently when γ itself is the Riesz kernel, by [20, (6) on p. 107], we see that

$$\left\| |\cdot|^{-(d-\alpha)} \right\|_{\frac{d}{d-\alpha},w} = \alpha^{-1} |S^{d-1}|^{1-\alpha/d} d^{\alpha/d},$$

which is consistent with the norm we find in (3.28) up to a constant $C_{\alpha,d}$.

In order to compare our results with known results (see, e.g., Example 2.9), let us introduce another commonly used variational constant

$$\begin{aligned} \mathbf{E}_{a,d}(\gamma,\theta) &:= \sup_{g\in\mathcal{F}_a} \left\{ \iint_{\mathbb{R}^{2d}} g^2(x) g^2(y) \gamma(x+y) \mathrm{d}x\mathrm{d}y - \frac{\theta}{2}\, \mathcal{E}_a(g,g) \right\} \\ &= \sup_{g\in\mathcal{F}_a} \left\{ \left\langle g^2 * g^2, \gamma \right\rangle_{L^2(\mathbb{R}^d)} - \frac{\theta}{2}\, \mathcal{E}_a(g,g) \right\}. \end{aligned} \tag{3.29}$$

By using the same techniques used to derive (3.14), one can show that for any $\Theta > 0$ and $\theta > 0$ that

$$\mathbf{E}_{a,d}(\Theta\gamma,\theta) = \Theta^{\frac{a}{a-\alpha}} \theta^{-\frac{\alpha}{a-\alpha}} \mathbf{E}_{a,d}(\gamma,1). \tag{3.30}$$

The relation between $\mathbf{E}_{a,d}(\gamma, \theta)$ and $\mathcal{M}_{a,d}(\gamma, \theta)$ can be established in a similar way as [10, Lemma A.2], which is stated in the following lemma:

Lemma 3.9 *Under Assumption 1.3 and assuming $\alpha < \min\{a, d\}$, the following three expressions hold:*

$$\mathbf{E}_{a,d}(\Theta\gamma, \theta) = \Theta^{\frac{a}{a-\alpha}} \theta^{-\frac{\alpha}{a-\alpha}} \left(\frac{2\alpha}{a}\right)^{\alpha/(a-\alpha)} \frac{a-\alpha}{a} \sigma(a, d, \alpha)^{a/(a-\alpha)}, \tag{3.31}$$

$$\mathcal{M}_{a,d}(\Theta\gamma, \theta) = \Theta^{\frac{a}{2a-\alpha}} \theta^{-\frac{\alpha}{2a-\alpha}} \left(\frac{\alpha}{a}\right)^{\alpha/(2a-\alpha)} \frac{2a-\alpha}{2a} \sigma(a, d, \alpha)^{a/(2a-\alpha)}, \tag{3.32}$$

$$\mathbf{E}_{a,d}(\Theta\gamma, \theta) = \left(\frac{a-\alpha}{a}\right) 2^{\alpha/(a-\alpha)} \left(\frac{2a-\alpha}{2a}\right)^{-(2a-\alpha)/(a-\alpha)} \mathcal{M}_{a,d}(\Theta\gamma, \theta)^{\frac{2a-\alpha}{a-\alpha}} \tag{3.33}$$

where $\sigma(a, d, \alpha)$ is defined in the following Lemma.

We need to prove two lemmas in order to prove Lemma 3.9.

Lemma 3.10 *Under Assumption 1.3, for any $f \in L^2(\mathbb{R}^d)$ with $\mathcal{E}_a(f, f) < \infty$, it holds that*

$$\int_{\mathbb{R}^d} f^2(x)\gamma(x)\mathrm{d}x \le C \|f\|_2^{2-(2\alpha/a)} \mathcal{E}_a(f, f)^{\alpha/a}, \tag{3.34}$$

where the constant C only depends on a, d and α with $\alpha < \min\{a, d\}$. Denote the best constant in (3.34) by $\sigma(a, d, \alpha)$.

Proof The proof of this result follows the scheme laid out in the proof of [7, Lemma A.3]. By the same techniques presented above in Lemma 3.5, one can show the following quantity is finite:

$$\begin{aligned}\Lambda &:= \sup_{h\in\mathcal{F}_a} \left\{ \int_{\mathbb{R}^d} h^2(x)\gamma(x)\mathrm{d}x - \frac{1}{2}(2\pi)^{-d} \int_{\mathbb{R}^d} |x|^a |\mathcal{F}h(x)|^2 \mathrm{d}x \right\} \\ &= \sup_{h\in\mathcal{F}_a} \left\{ \int_{\mathbb{R}^d} h^2(x)\gamma(x)\mathrm{d}x - \frac{1}{2}\mathcal{E}_a(h, h) \right\} < \infty.\end{aligned}$$

Fix an arbitrary $f \in \mathcal{F}_a$. Clearly, $\|f\|_2 = 1$ and $\mathcal{E}_a(f, f) < \infty$. Let C_f be the constant such that

$$\int_{\mathbb{R}^d} f^2(x)\gamma(x)\mathrm{d}x = C_f \mathcal{E}_a(f, f)^{\alpha/a}.$$

Now for $g(x) := t^{d/2} f(tx)$, it is easy to see that $\|g\|_2 = 1$ and

$$\mathcal{E}_a(g, g) = t^a \mathcal{E}_a(f, f) \quad \text{and} \quad \int_{\mathbb{R}^d} g^2(x)\gamma(x)\mathrm{d}x = t^\alpha \int_{\mathbb{R}^d} f^2(x)\gamma(x)\mathrm{d}x.$$

From this we can deduce that

$$\int_{\mathbb{R}^d} g^2(x)\gamma(x)\mathrm{d}x = C_f \mathcal{E}_a(g,g)^{\alpha/a}.$$

Next we note that

$$\begin{aligned}\Lambda &\geq \int_{\mathbb{R}^d} g^2(x)\gamma(x)\mathrm{d}x - \frac{1}{2}\mathcal{E}_a(g,g)\\ &= t^\alpha \int_{\mathbb{R}^d} f^2(x)\gamma(x)\mathrm{d}x - \frac{1}{2}t^a \mathcal{E}_a(f,f)\\ &= C_f t^\alpha \mathcal{E}_a(f,f)^{\alpha/a} - \frac{1}{2}t^a\mathcal{E}_a(f,f)\\ &= C_f\left(t\mathcal{E}_a(f,f)^{1/a}\right)^\alpha - \frac{1}{2}\left(t\mathcal{E}_a(f,f)^{1/a}\right)^a.\end{aligned}$$

Since $t > 0$, then $t\mathcal{E}_a(f,f)^{1/a}$ runs through all of $\mathbb{R}_+$ and thus we have that

$$\Lambda \geq \sup_{x>0}\left\{C_f x^\alpha - \frac{1}{2}x^a\right\} = \frac{a-\alpha}{a}C_f^{a/(a-\alpha)}\left(\frac{2\alpha}{a}\right)^{\alpha/(a-\alpha)}.$$

Note that this reduces to the equation present in the proof of Lemma A.3 [7] when $a = 2$. By taking the sup over all $f \in \mathcal{F}_a$ we see that

$$\infty > \Lambda \geq \frac{a-\alpha}{a}\sigma(a,d,\alpha)^{a/(a-\alpha)}\left(\frac{2\alpha}{a}\right)^{\alpha/(a-\alpha)}.$$

where

$$\sup_{f\in\mathcal{F}_a} C_f = \sigma(a,d,\alpha)$$

and finally we conclude that for any $f \in \mathcal{F}_a$

$$\int_{\mathbb{R}^d} f^2(x)\gamma(x)\mathrm{d}x \leq \sigma(a,d,\alpha)\mathcal{E}_a(f,f)^{\alpha/a} < \infty. \tag{3.35}$$

For arbitrary $f \in L^2(\mathbb{R}^2)$ with $\mathcal{E}_a(f,f) < \infty$ we apply (3.35) to $f/\|f\|_2$ and see that

$$\int_{\mathbb{R}^d} f^2(x)\gamma(x)\mathrm{d}x \leq \sigma(a,d,\alpha)\,\|f\|_2^{2-(2\alpha/a)}\,\mathcal{E}_a(f,f)^{\alpha/a}$$

which again reduces to the equation A.4 [7] when $a = 2$. □

Lemma 3.11 *For any $f \in \mathcal{F}_a$ and for $\alpha < \min\{a, d\}$ we have*

$$\int_{\mathbb{R}^{2d}} \gamma(x+y) f^2(x) f^2(y) \mathrm{d}x\mathrm{d}y \le \sigma(a,d,\alpha)\mathcal{E}_a(f,f)^{\alpha/a} \tag{3.36}$$

and $\sigma(a,d,\alpha)$ is the sharpest such constant.

Proof Suppose that $f \in L^2(\mathbb{R}^d)$ and suppose that $\mathcal{E}_a(f,f) < \infty$ and let $y \in \mathbb{R}^d$ be arbitrary. Recall the translation property of the Fourier transform

$$|\mathcal{F}f(\cdot)(\xi)| = |\mathcal{F}f(\cdot - y)(\xi)|.$$

Then by applying a change of variables and recalling (3.34) we see that

$$\begin{aligned}\int_{\mathbb{R}^d} f^2(x)\gamma(x+y)\mathrm{d}x &= \int_{\mathbb{R}^d} f^2(x-y)\gamma(x)\mathrm{d}x \\ &\le \sigma(a,d,\alpha)\, \|f(\cdot - y)\|_2^{2-(2\alpha/a)}\, \mathcal{E}_a\big(f(\cdot - y), f(\cdot - y)\big)^{\alpha/a} \\ &= \sigma(a,d,\alpha)\, \|f\|_2^{2-(2\alpha/a)}\, \mathcal{E}_a\big(f,f\big)^{\alpha/a},\end{aligned}$$

and in return,

$$\sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} f^2(x)\gamma(x+y) \le \sigma(a,d,\alpha)\, \|f\|_2^{2-(2\alpha/a)}\, \mathcal{E}_a(f,f)^{\alpha/a}.$$

Next, notice that

$$\begin{aligned}\int_{\mathbb{R}^{2d}} f^2(x) f^2(y)\gamma(x+y)\mathrm{d}x\mathrm{d}y &= \int_{\mathbb{R}^d} \mathrm{d}x f^2(x) \int_{\mathbb{R}^d} \mathrm{d}y f^2(y)\gamma(x+y) \\ &\le \sigma(a,d,\alpha)\, \|f\|_2^{4-(2\alpha/a)}\, \mathcal{E}_a(f,f)^{\alpha/a}\end{aligned}$$

and when $f \in \mathcal{F}_a$, and thus $\|f\|_2 = 1$, we see that this reduces to

$$\int_{\mathbb{R}^{2d}} \gamma(x+y) f^2(x) f^2(y)\mathrm{d}x\mathrm{d}y \le \sigma(a,d,\alpha)\mathcal{E}_a(f,f)^{\alpha/a}.$$

We note that the sharpness of $\sigma(a,d,\alpha)$ follows immediately from Lemma 3.10. In addition, this reduces to equation (A.1) [10] for the time independent case when $a = 2$. □

Proof of Lemma 3.9 We only prove the case of $\Theta = \theta = 1$, the general case can be proven by applying the scaling properties (3.14) and (3.30).

We have that

$$\begin{aligned}E_{a,d}(\gamma,1) &\le \sup_{g\in\mathcal{F}_a}\left\{\sigma(a,d,\alpha)\mathcal{E}_a(g,g)^{\alpha/a}-\frac{1}{2}\left(\mathcal{E}_a(g,g)^{1/a}\right)^a\right\}\\ &\le \sup_{x>0}\left\{\sigma(a,d,\alpha)x^\alpha-\frac{1}{2}x^a\right\}\\ &=\left(\frac{2\alpha}{a}\right)^{\alpha/(a-\alpha)}\frac{a-\alpha}{a}\sigma(a,d,\alpha)^{a/(a-\alpha)} \qquad (3.37)\end{aligned}$$

and similarly

$$\begin{aligned}\mathcal{M}_{a,d}(\gamma,1) &\le \sup_{x>0}\left\{\sigma(a,d,\alpha)^{1/2}x^{\alpha/2}-\frac{1}{2}x^a\right\}\\ &=\left(\frac{\alpha}{a}\right)^{\alpha/(2a-\alpha)}\frac{2a-\alpha}{2a}\sigma(a,d,\alpha)^{a/(2a-\alpha)}. \qquad (3.38)\end{aligned}$$

Recalling Lemma 3.11 above, one can choose $0<\epsilon<\sigma(a,d,\alpha)$ and $f\in\mathcal{F}_a$ such that

$$\int_{\mathbb{R}^{2d}}\gamma(x+y)f^2(x)f^2(y)\mathrm{d}x\mathrm{d}y \ge (\sigma(a,d,\alpha)-\epsilon)\mathcal{E}_a(f,f)^{\alpha/a}.$$

Now define

$$g(x)=t^{d/2}f(tx).$$

Notice that

$$\begin{aligned}E_{a,d}(\gamma,1) &\ge \int_{\mathbb{R}^{2d}}g^2(x)g^2(y)\gamma(x-y)\mathrm{d}x\mathrm{d}y-\frac{1}{2}\mathcal{E}_a(g,g)\\ &=t^\alpha\int_{\mathbb{R}^{2d}}\gamma(x-y)f^2(x)f^2(y)-\frac{1}{2}t^a\mathcal{E}_a(f,f)\\ &\ge(\sigma(a,d,\alpha)-\epsilon)t^\alpha\mathcal{E}_a(f,f)^{\alpha/a}-\frac{1}{2}t^a\mathcal{E}_a(f,f)\end{aligned}$$

and this is true for all $t>0$ so we can say that

$$\begin{aligned}E_{a,d}(\gamma,1) &\ge \sup_{x>0}\left\{(\sigma(a,d,\alpha)-\epsilon)x^\alpha-\frac{1}{2}x^a\right\}\\ &=\left(\frac{2\alpha}{a}\right)^{\alpha/(a-\alpha)}\frac{a-\alpha}{a}(\sigma(a,d,\alpha)-\epsilon)^{a/(a-\alpha)}\end{aligned}$$

and be letting $\epsilon\to 0$ gives us

$$E_{a,d}(\gamma,1)\ge\left(\frac{2\alpha}{a}\right)^{\alpha/(a-\alpha)}\frac{a-\alpha}{a}\sigma(a,d,\alpha)^{a/(a-\alpha)} \qquad (3.39)$$

and if we combine this with (3.37) then we see that

$$E_{a,d}(\gamma, 1) = \left(\frac{2\alpha}{a}\right)^{\alpha/(a-\alpha)} \frac{a-\alpha}{a} \sigma(a, d, \alpha)^{a/(a-\alpha)}. \tag{3.40}$$

Similarly we show that

$$\mathcal{M}_{a,d}(\gamma, 1) = \left(\frac{\alpha}{a}\right)^{\alpha/(2a-\alpha)} \frac{2a-\alpha}{2a} \sigma(a, d, \alpha)^{a/(2a-\alpha)}. \tag{3.41}$$

Lastly, by combining (3.40) and (3.41), we see that

$$E_{a,d}(\gamma, 1) = \left(\frac{a-\alpha}{a}\right) (2)^{\alpha/(a-\alpha)} \left(\frac{2a-\alpha}{2a}\right)^{-(2a-\alpha)/(a-\alpha)} \mathcal{M}_{a,d}(\gamma, 1)^{(2a-\alpha)/(a-\alpha)}. \tag{3.42}$$

Equations (3.41), (3.40) and (3.42) recover equations (A.2), (A.3) and (A.4) of [10] respectively when $a = 2$. □

4 Existence and uniqueness of the solution

In this section, we will prove part (1) of Theorem 1.6. The proof will need the following lemma:

Lemma 4.1 (Lemma 3.5 of *[1]*) *If* $H : [0, \infty) \to [0, \infty)$ *is a non-decreasing function, then*

$$2 \int_0^\infty e^{-2t} H^2(t) \mathrm{d}t \le \left(\int_0^\infty e^{-t} H(t) \mathrm{d}t\right)^2. \tag{4.1}$$

The proof of Theorem 1.6 follows the same strategy as [1, Section 3] with minor changes such as

$$\frac{1}{1+|\xi|^2} \quad \text{replaced by} \quad \frac{1}{1+\frac{\nu}{2}|\xi|^a}. \tag{4.2}$$

Nevertheless, for completeness, here we streamline and reorganize this proof as follows.

Proof of Theorem 1.6 We first introduce some notation. Let $L(x)$ be the Laplace transform of $G(\cdot, x)$ evaluated at one and calculate its Fourier transform $\mathcal{F}L(\xi)$ as follows:

$$L(x) = \int_0^\infty e^{-t} G(t, x) \mathrm{d}t \quad \text{and} \quad \mathcal{F}L(\xi) = \int_0^\infty e^{-t} \mathcal{F}G(t, \cdot)(\xi) \mathrm{d}t = \frac{1}{1+\frac{\nu}{2}|\xi|^a}; \tag{4.3}$$

see the proof of Theorem 4.1 of [6] for the last equality. Similarly, let $L_n(\vec{y})$ to be the Laplace transform of $\widetilde{f}_n(\vec{y}, 0, \cdot)$ evaluated at one, namely,

$$\begin{aligned} L_n(\vec{y}) &:= n! \int_0^\infty e^{-t} \widetilde{f}_n(\vec{y}; 0, t) \mathrm{d}t \\ &= \sum_{\sigma \in \Sigma_n} \int_0^\infty e^{-t} \int_{[0,t]^n_<} \prod_{k=1}^n G(s_k - s_{k-1}, y_{\sigma(k)} - y_{\sigma(k-1)}) \mathrm{d}\vec{s} \mathrm{d}t \end{aligned}$$

with the convention that $s_0 = 0$ and $y_{\sigma(0)} = 0$. By the relation of convolution and the Laplace transform (or through a change of variables), we see that

$$L_n(\vec{y}) = \sum_{\sigma \in \Sigma_n} L\left(y_{\sigma(1)}\right) L\left(y_{\sigma(2)} - y_{\sigma(1)}\right) \cdots L\left(y_{\sigma(n)} - y_{\sigma(n-1)}\right). \tag{4.4}$$

Hence, from (4.3),

$$\mathcal{F} L_n(\vec{\xi}) = \sum_{\sigma \in \Sigma_n} \prod_{k=1}^n \frac{1}{1 + \frac{\nu}{2} \left| \sum_{j=k}^n \xi_{\sigma(j)} \right|^a}. \tag{4.5}$$

Moreover, define

$$\begin{aligned} H_n(t, \vec{x}) &= n! \int_{(\mathbb{R}^d)^n} \prod_{k=1}^n K(x_k - y_k) \widetilde{f}_n(\vec{y}; 0, t) \mathrm{d}\vec{y} \\ &= \sum_{\sigma \in \Sigma_n} \int_{[0,t]^n_<} \int_{(\mathbb{R}^d)^n} \prod_{k=1}^n K(x_k - y_k) \prod_{k=1}^n G(s_k - s_{k-1}, y_{\sigma(k)} - y_{\sigma(k-1)}) \mathrm{d}\vec{y} \mathrm{d}\vec{s}, \end{aligned} \tag{4.6}$$

where recall that K is defined in (1.6). Under the nonnegativity assumption—Assumption 1.1, we see that for any $\vec{x} \in \mathbb{R}^{nd}$ fixed, the function $t \to H_n(t, \vec{x})$ is a non-decreasing function for $t \geq 0$. For this function, we are about to apply Lemma 4.1.

Step 1. We first compute the corresponding part to the right-hand side of (4.1). By Fubini's theorem,

$$\begin{aligned} \int_0^\infty e^{-t} H_n(t, \vec{x}) \mathrm{d}t &= \sum_{\sigma \in \Sigma_n} \int_0^\infty e^{-t} \int_{[0,t]^n_<} \int_{(\mathbb{R}^d)^n} \prod_{k=1}^n K(x_k - y_k) \\ &\quad \times \prod_{k=1}^n G(s_k - s_{k-1}, y_{\sigma(k)} - y_{\sigma(k-1)}) \mathrm{d}\vec{y} \mathrm{d}\vec{s} \mathrm{d}t \\ &= \int_{\mathbb{R}^{dn}} \prod_{k=1}^n K(x_k - y_k) L_n(\vec{y}) \mathrm{d}\vec{y}. \end{aligned}$$

Then an application of the Plancherel's theorem and the fact that $K * K = \gamma$ shows that

$$\int_{(\mathbb{R}^d)^n} \left[\int_0^\infty e^{-t} H_n(t, \vec{x}) \mathrm{d}t\right]^2 \mathrm{d}\vec{x} = \int_{(\mathbb{R}^d)^n} |\mathcal{F} L_n(\vec{\xi})|^2 \mu(\mathrm{d}\vec{\xi}\ ). \tag{4.7}$$

One may check the proof of Lemma 3.3 of [1] for more details.

Step 2. Now we compute the corresponding part to the left-hand side of (4.1). First, using the fact that $K * K = \gamma$, we see that

$$\| \widetilde{f}_n(\cdot, 0; t) \|^2_{\mathcal{H}^{\otimes n}} = \frac{1}{(n!)^2} \int_{(\mathbb{R}^d)^n} H_n^2(t, \vec{x}) \mathrm{d}\vec{x}; \tag{4.8}$$

one may check the proof of Lemma 3.4 of [1] for more details. By the scaling property for $\mathcal{F}\widetilde{f}_n$ in (3.7), one can show that

$$\int_0^\infty e^{-t} \| \widetilde{f}_n(\cdot, 0, t) \|^2_{\mathcal{H}^{\otimes n}} \mathrm{d}t = \frac{2^{n(2(b+r)-b\alpha/a)}}{(n!)^2} \int_0^\infty 2e^{-2t} \int_{\mathbb{R}^{nd}} H_n(t, \vec{x})^2 \mathrm{d}\vec{x} \mathrm{d}t; \tag{4.9}$$

see "Appendix" for the proof.

Step 3. Now we can apply Fubini's theorem and Lemma 4.1 to the function $t \to H_n(t, \vec{x})$ to see that

$$\int_0^\infty 2e^{-2t} \int_{\mathbb{R}^{nd}} H_n(t, \vec{x})^2 \mathrm{d}\vec{x} \mathrm{d}t \le \int_{\mathbb{R}^{nd}} \left[\int_0^\infty e^{-t} H_n(t, \vec{x}) \mathrm{d}t\right]^2 \mathrm{d}\vec{x}. \tag{4.10}$$

Therefore, combining (4.7), (4.8), and (4.10) shows that

$$\begin{aligned} \int_0^\infty e^{-t} \| \widetilde{f}_n(\cdot, 0, t) \|^2_{\mathcal{H}^{\otimes n}} \mathrm{d}t &\le \frac{2^{n(2(b+r)-b\alpha/a)}}{(n!)^2} \int_{(\mathbb{R}^d)^n} |\mathcal{F} L_n(\vec{\xi})|^2 \mu(\mathrm{d}\vec{\xi}\ ) \\ &=: \frac{2^{n(2(b+r)-b\alpha/a)}}{(n!)^2} T_n(\nu, a), \end{aligned} \tag{4.11}$$

where

$$T_n(\nu, a) = \int_{(\mathbb{R}^d)^n} \left[\sum_{\sigma \in \Sigma_n} \prod_{k=1}^n \frac{1}{1 + \frac{\nu}{2} \left| \sum_{j=k}^n \xi_{\sigma(j)} \right|^a} \right]^2 \mu(\mathrm{d}\vec{\xi}\ ). \tag{4.12}$$

By the same arguments as those of Lemma 3.6 of [1] with the replacement (4.2), we see that

$$T_n(\nu,a) \le (n!)^2 C_\mu^n(\nu,a) \qquad \text{with} \quad C_\mu(\nu,a) := \int_{(\mathbb{R}^d)} \left(\frac{1}{1+\frac{\nu}{2}|\xi|^a}\right)^2 \mu(\mathrm{d}\xi). \tag{4.13}$$

Notice that

$$\int_{(\mathbb{R}^d)} \left(\frac{1}{1+|\xi|^a}\right)^2 \mu(\mathrm{d}\xi) = C\int_0^\infty \frac{\rho^{\alpha-1}}{(1+\rho^a)^2}\mathrm{d}\rho < \infty \quad \Longleftrightarrow \quad 2a-\alpha+1 > 1. \tag{4.14}$$

Therefore, conditions in (1.11) imply that $C_\mu(\nu,a) < \infty$.
Combining (4.11) and (4.13) gives that

$$\int_0^\infty e^{-t} \left\| \widetilde{f}_n(\cdot,0;t)\right\|^2_{\mathcal{H}^{\otimes n}} \mathrm{d}t \le 2^{n\left[2(b+r)-\frac{b\alpha}{a}\right]} C_\mu^n(\nu,a) < +\infty. \tag{4.15}$$

Step 4. From the scaling property (3.8) we see that

$$\begin{aligned}\int_0^\infty e^{-t} \left\| \widetilde{f}_n\,(\cdot,0,t)\right\|^2_{\mathcal{H}^{\otimes n}} \mathrm{d}t &= \left\| \widetilde{f}_n(\cdot,0,1)\right\|^2_{\mathcal{H}^{\otimes n}} \int_0^\infty e^{-t} t^{\left[2(b+r)-\frac{b\alpha}{a}\right]n} \mathrm{d}t \\ &= \left\| \widetilde{f}_n(\cdot,0,1)\right\|^2_{\mathcal{H}^{\otimes n}} \Gamma\left(\left[2(b+r)-\frac{b\alpha}{a}\right]n+1\right),\end{aligned} \tag{4.16}$$

which entails another part of the conditions in (1.11):

$$2(b+r) - \frac{b\alpha}{a} > 0. \tag{4.17}$$

From (4.15) and (4.16), we deduce that

$$\begin{aligned}\left\| \widetilde{f}_n(\cdot,0,1)\right\|^2_{\mathcal{H}^{\otimes n}} &= \frac{1}{\Gamma\left(\left[2(b+r)-\frac{b\alpha}{a}\right]n+1\right)} \int_0^\infty e^{-t} \left\| \widetilde{f}_n(\cdot,0,t)\right\|^2_{\mathcal{H}^{\otimes n}} \mathrm{d}t \\ &\le \frac{\left(2^{\left[2(b+r)-\frac{b\alpha}{a}\right]} C_\mu(\nu,a)\right)^n}{\Gamma\left(\left[2(b+r)-\frac{b\alpha}{a}\right]n+1\right)} \le C^n \frac{\left(2^{\left[2(b+r)-\frac{b\alpha}{a}\right]} C_\mu(\nu,a)\right)^n}{(n!)^{2(b+r)-b\alpha/a}},\end{aligned}$$

where the constant C depends only on the value of $2(b+r) - b\alpha/a$ and the last inequality is due to Stirling's formula (see (5.2) below).
Because of the constant one initial condition, $\|u(t,x)\|_2 = \|u(t,0)\|_2$ for all $x \in \mathbb{R}^d$ and $t > 0$. Therefore, by (3.9), (3.8), and the above inequality,

$$
\begin{aligned}
\|u(t,x)\|_2^2 &= \sum_{n\geq 0} \theta^n n!\, t^{[2(b+r)-b\alpha/a]n} \left\| \widetilde{f}_n(\cdot,0,1)\right\|_{\mathcal{H}^{\otimes n}}^2 \\
&\leq \sum_{n\geq 0} \theta^n C^n t^{[2(b+r)-b\alpha/a]n} \frac{\left(2^{\left[2(b+r)-\frac{b\alpha}{a}\right]} C_\mu(\nu,a)\right)^n}{(n!)^{2(b+r)-1-b\alpha/a}},
\end{aligned} \tag{4.18}
$$

which is finite provided that (see (1.11))

$$
2(b+r)-1-b\alpha/a > 0. \tag{4.19}
$$

Finally, an application of the Minkowski inequality and the hypercontractivity (see [1, Theorem B.1] or [19] for the case of the SHE) shows that for all $p \geq 2$,

$$
\begin{aligned}
\|u(t,x)\|_p &\leq \sum_{n\geq 0} \theta^{n/2}(p-1)^{n/2}\sqrt{n!}\left\| \widetilde{f}_n(\cdot,0,t)\right\|_{\mathcal{H}^{\otimes n}} \\
&\leq \sum_{n\geq 0} \theta^{n/2} C^{n/2}(p-1)^{n/2} t^{[2(b+r)-b\alpha/a]n/2} \frac{\left(2^{\left[2(b+r)-\frac{b\alpha}{a}\right]} C_\mu(\nu,a)\right)^{n/2}}{(n!)^{\frac{1}{2}[2(b+r)-1-b\alpha/a]}} < \infty.
\end{aligned} \tag{4.20}
$$

Therefore, under condition (1.11), (1.1) has an unique $L^p(\Omega)$-solution $u(t,x)$ for all $p \geq 2$, $t > 0$ and $x \in \mathbb{R}^d$. This proves part (1) of Theorem 1.6.

Step 5. The proof of part (2) of Theorem 1.6 will be postponed to part (ii) of Lemma 5.1 below. □

5 Upper bound of the asymptotics

In this section, we will give the proof of part 2 of Theorem 1.6 and establish the upper bound of (1.14) (under Assumptions 1.1 and 1.3, and condition (1.11)), namely,

$$
\begin{aligned}
&\limsup_{t_p\to\infty} t_p^{-\beta} \log \|u(t,x)\|_p \\
&\leq \left(\frac{1}{2}\right)\left(\frac{2a}{2a(b+r)-b\alpha}\right)^{\beta}\left(\theta\nu^{-\alpha/a}\mathcal{M}_a^{\frac{2a-\alpha}{a}}\right)^{\frac{a}{2a(b+r)-b\alpha-a}}\left(2(b+r)-\frac{b\alpha}{a}-1\right).
\end{aligned} \tag{5.1}
$$

As for the upper bound, we will first establish the corresponding result for $p=2$ in Lemma 5.2 and then apply the hypercontractivity property given by Theorem B.1 in [1] to obtain the general case for $p \geq 2$. To prove the next lemma, we will need the following equality

$$
\lim_{n\to\infty}\frac{1}{n}\log\left(\frac{\Gamma(an+1)}{(n!)^a}\right) = a\log(a), \quad \text{for all } a > 0, \tag{5.2}
$$

which is a direct consequence of Stirling's formula.

Lemma 5.1 *Assume Assumptions 1.1 and 1.3 hold and in addition that* $\alpha < \min\{2a, d\}$. *Let* ρ *be the constant defined in (3.10). Then the following identities hold true:*

$$\begin{aligned}
&\lim_{n\to\infty}\frac{1}{n}\log\left(\|\widetilde{f}_n(\cdot,0,1)\|^2_{\mathcal{H}^{\otimes n}}\Gamma\left(\left[2(b+r)-\frac{b\alpha}{a}\right]n+1\right)\right)\\
&\quad=\log\left(2^{2(b+r)-\frac{b\alpha}{a}}\rho\right)\quad \textit{and} \qquad (5.3)\\
&\lim_{n\to\infty}\frac{1}{n}\log\left((n!)^{2(b+r)-\frac{b\alpha}{a}}\|\widetilde{f}_n(\cdot,0,1)\|^2_{\mathcal{H}^{\otimes n}}\right)\\
&\quad=\log\left(\left(\frac{2}{2(b+r)-\frac{b\alpha}{a}}\right)^{2(b+r)-\frac{b\alpha}{a}}\right)+\log\rho. \qquad (5.4)
\end{aligned}$$

Proof The proof follows the same arguments as those of [1, Lemma 4.3]. Nevertheless, we sketch the proof here for completeness. Recall the definition of $T_n(\nu,a)$ defined in (4.12). From (3.21), we see that

$$\lim_{n\to\infty}\frac{1}{n}\log\left[\frac{T_n(\nu,a)}{(n!)^2}\right]=\log\left(\rho_{\nu,a}\right). \qquad (5.5)$$

As a consequence of (4.11) and (4.16) in the proof of Theorem 1.6, we see that

$$\|\widetilde{f}_n(\cdot,0,1)\|^2_{\mathcal{H}^{\otimes n}}\Gamma\left(\left[2(b+r)-\frac{b\alpha}{a}\right]n+1\right)\le\frac{2^{n\left[2(b+r)-\frac{b\alpha}{a}\right]}}{(n!)^2}T_n(\nu,a).$$

Combining this and (3.21) we see that

$$\begin{aligned}
\limsup_{n\to\infty}\frac{1}{n}\log&\left[\|\widetilde{f}_n(\cdot,0,1)\|^2_{\mathcal{H}^{\otimes n}}\Gamma\left(\left[2(b+r)-\frac{b\alpha}{a}\right]n+1\right)\right]\\
&\le\log\left(2^{\left[2(b+r)-\frac{b\alpha}{a}\right]}\right)+\lim_{n\to\infty}\frac{1}{n}\log\left(\frac{T_n(\nu,a)}{(n!)^2}\right)\\
&=\log\left(2^{\left[2(b+r)-\frac{b\alpha}{a}\right]}\right)+\log(\rho_{\nu,a}),
\end{aligned}$$

which proves the upper bound for (5.3).

Now we prove the lower bound for (5.3). Let τ and $\widetilde{\tau}$ be independent exponential random variables with mean one. In the following, we will compute $\mathbb{E}[J_n(\tau,\widetilde{\tau})]$ in two ways, where

$$J_n(t,t'):=\int_{\mathbb{R}^{nd}}H_n(t,x)H_n(t',x)\mathrm{d}x,\qquad t,t'>0;$$

see (4.6) for the definition of the function H_n. Notice that using the above notation, (4.8) can be rewritten as

$$\left\| \widetilde{f}_n(\cdot, 0; t)\right\|^2_{\mathcal{H}^{\otimes n}} = \frac{1}{(n!)^2} \int_{(\mathbb{R}^d)^n} H_n(t, \vec{x})^2 \mathrm{d}\vec{x} = \frac{1}{(n!)^2} J_n(t, t).$$

On the one hand, the Cauchy-Schwartz inequality implies that

$$J_n(t, t') \le J_n(t, t)^{1/2} J_n(t', t')^{1/2} = t^{[2(b+r)-b\alpha/a]n/2} (t')^{[2(b+r)-b\alpha/a]n/2} J_n(1, 1).$$

where we have used the scaling property of $J_n(t, t)$ inherited from that of $\left\| \widetilde{f}_n(\cdot, 0; t)\right\|_{\mathcal{H}^{\otimes n}}$ as in (3.8). Hence,

$$\begin{aligned}\mathbb{E}[J_n(\tau, \widetilde{\tau})] &\le \mathbb{E}\left[\tau^{[2(b+r)-b\alpha/a]n/2}\right] \mathbb{E}\left[\widetilde{\tau}^{[2(b+r)-b\alpha/a]n/2}\right] J_n(1, 1)\\ &= \Gamma\left(\frac{[2(b+r)-b\alpha/a]n}{2} + 1\right)^2 (n!)^2 \left\| \widetilde{f}_n(\cdot, 0; 1)\right\|^2_{\mathcal{H}^{\otimes n}}.\end{aligned}$$

On the other hand, by (4.7) and (4.12), we see that

$$\begin{aligned}\mathbb{E}[J_n(\tau, \widetilde{\tau})] &= \int_0^\infty \int_0^\infty e^{-t} e^{-\widetilde{t}} J_n(t, \widetilde{t}) \mathrm{d}t \mathrm{d}\widetilde{t}\\ &= \int_{\mathbb{R}^{dn}} \left[\int_0^\infty e^{-t} H_n(t, x) \mathrm{d}t\right]^2 \mathrm{d}x = T_n(\nu, a).\end{aligned}$$

Therefore,

$$\frac{T_n(\nu, a)}{(n!)^2} \le \Gamma\left(\frac{[2(b+r)-b\alpha/a]n}{2} + 1\right)^2 \left\| \widetilde{f}_n(\cdot, 0; 1)\right\|^2_{\mathcal{H}^{\otimes n}}. \tag{5.6}$$

Now, an application of Stirling's formula as in (5.2) to see that, as $n \to \infty$,

$$\Gamma\left(\frac{[2(b+r)-b\alpha/a]n}{2} + 1\right)^2 \sim \Gamma\left([2(b+r)-b\alpha/a]n + 1\right) 2^{-[2(b+r)-b\alpha/a]n} C_n, \tag{5.7}$$

where $C_n = 2^{-1}[2(b+r) - b\alpha/a]^{1/2}(2\pi n)^{1/2}$. Then an application of (5.5), (5.6) and (5.7) proves the lower bound of (5.3). Lastly, (5.4) follows from (5.3) and the limit (5.2). This proves Lemma 5.1. □

Now we are ready to prove part (2) of Theorem 1.6.

Proof of part (2) of Theorem 1.6 The critical case happens when the exponent of $n!$ in (4.20) vanishes, namely,

$$\alpha = \frac{a}{b}[2(b+r) - 1].$$

Among the three inequalities in (1.11), we also need to make sure that the minimum is achieved by $\frac{a}{b}[2(b+r)-1]$, for which, we need to additionally require $\frac{a}{b}[2(b+r)-1] \le d$ and

$$\frac{a}{b}[2(b+r)-1] < 2a \quad \Longleftrightarrow \quad r \in [0, 1/2).$$

The reason for having a strict inequality above is our need to apply (5.4) and Theorem 3.5 later on in this proof. Putting these conditions together gives the conditions stated in (1.12).

We start by proving part *(2-i)*. Let u_λ for $\lambda > 0$ be the solution of the SPDE (1.1) with θ replaced with λ and $u = u_\theta$. By the hypercontractivity property (see [1, Lemma B.1]), we have that

$$\|u(t,x)\|_p \le \left\|u_{(p-1)\theta}(t,x)\right\|_2, \quad \text{for all } p \ge 2, t > 0 \text{and } x \in \mathbb{R}^d. \tag{5.8}$$

Now by recalling Theorem 3.3 and by applying $\alpha = \frac{a}{b}[2(b+r)-1]$ and the scaling property (3.8), we see that

$$\begin{aligned}\left\|u_{(p-1)\theta}(t,x)\right\|_2^2 &= \sum_{n\ge 0} [\theta(p-1)]^n\, n! \left\| \tilde{f}_n(\cdot, 0; t)\right\|_{\mathcal{H}^{\otimes n}}^2 \\ &= \sum_{n\ge 0} [t\theta(p-1)]^n\, n! \left\| \tilde{f}_n(\cdot, 0; 1)\right\|_{\mathcal{H}^{\otimes n}}^2 \\ &=: \sum_{n\ge 0} [t\theta(p-1)]^n\, R_n,\end{aligned}$$

with $R_n = n! \left\| \tilde{f}_n(\cdot, 0; 1)\right\|_{\mathcal{H}^{\otimes n}}^2$. By the Cauchy-Hadamard theorem, this series converges for $\theta t(p-1) < \limsup_{n\to\infty} |R_n|^{-1/n}$. However, by (5.4) and Theorem 3.5, we see that

$$\lim_{n\to\infty} \frac{1}{n} \log(|R_n|) = \log(2\rho) = \log\left(2\nu^{-\alpha/a} \mathcal{M}_a^{(2a-\alpha)/a}\right).$$

Therefore, $\limsup_{n\to\infty} |R_n|^{-1/n} = (2\nu^{-\alpha/a}\mathcal{M}_a^{(2a-\alpha)/a})^{-1}$ and $\|u(t,x)\|_p$ converges for

$$t < \frac{1}{2\theta(p-1)\nu^{-\alpha/a}\mathcal{M}_a^{(2a-\alpha)/a}} =: T_p\,; \qquad \text{see (1.13).}$$

To show part *(2-ii)*, we use the Cauchy-Hadamard theorem and the same techniques above to see that the radius of convergence of the series

$$\|u(t,x)\|_2^2 = \sum_{n\ge 0} \theta^n n! \left\| \tilde{f}_n(\cdot, 0; t)\right\|_{\mathcal{H}^{\otimes n}}^2,$$

is precisely T_2. This completes the proof of part (2) of Theorem 1.6. □

Lemma 5.2 *Assume Assumptions 1.1 and 1.3 hold. Let ρ be the constant defined in (3.10). Under condition (1.11), we have that*

$$\begin{aligned}
&\lim_{t\to\infty} \frac{1}{t^\beta} \log \mathbb{E}(|u(t,x)|^2) \\
&= \left(\frac{2a}{2a(b+r)-b\alpha}\right)^\beta (\theta\rho)^{\frac{a}{2a(b+r)-b\alpha-a}} \left(2(b+r)-\frac{b\alpha}{a}-1\right).
\end{aligned}$$

Proof By part (1) of Theorem 1.6, there is an $L^2(\Omega)$ solution $u(t,x)$. By the scaling property (3.8),

$$\begin{aligned}
\mathbb{E}(|u(t,x)|^2) &= \sum_{n\ge 0} \theta^n (n!) \left\| \widetilde{f}_n(\cdot,0,t)\right\|^2_{\mathcal{H}^{\otimes n}} \\
&= \sum_{n\ge 0} \theta^n (n!) t^{(2(b+r)-b\alpha/a)n} \left\| \widetilde{f}_n(\cdot,0,1)\right\|^2_{\mathcal{H}^{\otimes n}} \\
&= \sum_{n\ge 0} z_n R_n t^{(2(b+r)-b\alpha/a)n}
\end{aligned}$$

where

$$R_n = (n!)^{2(b+r)-b\alpha/a} \left\| \widetilde{f}_n(\cdot,0,1)\right\|^2_{\mathcal{H}^{\otimes n}} \quad \text{and} \quad z_n = \frac{\theta^n}{(n!)^{2(b+r)-(b\alpha/a)-1}}.$$

Notice that (5.4) above says that

$$\frac{1}{n}\log(R_n) \to \log\left(\left(\frac{2}{2(b+r)-\frac{b\alpha}{a}}\right)^{2(b+r)-\frac{b\alpha}{a}} \rho\right) \quad \text{as } n\to\infty.$$

Now define R to be

$$R = \left(\frac{2}{2(b+r)-\frac{b\alpha}{a}}\right)^{2(b+r)-\frac{b\alpha}{a}} \rho.$$

We want to find a β and A so that

$$\lim_{t\to\infty} \frac{1}{t^\beta} \log \sum_{n\ge 0} z_n R^n \left(t^{(2(b+r)-b\alpha/a)}\right)^n = A.$$

Indeed, by the following limit (see [1, Lemma A.3]),

$$\lim_{t\to\infty} t^{-1/\gamma} \log \sum_{n\ge 0} (n!)^{-\gamma} t^n = \gamma, \qquad \text{for all } \gamma > 0,$$

we see that

$$\begin{aligned}&\lim_{t\to\infty}\left[\frac{1}{(\theta R)t^{2(b+r)-b\alpha/a}}\right]^{\frac{1}{2(b+r)-(b\alpha/a)-1}}\log\sum_{n\geq 0}\frac{\left[(\theta R)t^{2(b+r)-b\alpha/a}\right]^n}{(n!)^{2(b+r)-(b\alpha/a)-1}}\\&=2(b+r)-(b\alpha/a)-1,\end{aligned}$$

which, by an easy algebraic manipulation, is equivalent to

$$\begin{aligned}&\lim_{t\to\infty}\left[\frac{1}{t^{2(b+r)-b\alpha/a}}\right]^{\frac{1}{2(b+r)-(b\alpha/a)-1}}\log\sum_{n\geq 0}\frac{\left[(\theta R)t^{2(b+r)-b\alpha/a}\right]^n}{(n!)^{2(b+r)-(b\alpha/a)-1}}\\&=[2(b+r)-(b\alpha/a)-1](\theta R)^{\frac{1}{2(b+r)-(b\alpha/a)-1}}.\end{aligned}$$

Hence,

$$\beta=\frac{2a(b+r)-b\alpha}{2a(b+r)-(b\alpha)-a}\quad\text{and}\quad A=[2(b+r)-(b\alpha/a)-1](\theta R)^{\frac{a}{2a(b+r)-b\alpha-a}}.$$

Finally, an application of [1, Lemma A.2] shows that

$$\begin{aligned}&\lim_{t\to\infty}\frac{1}{t^{\beta}}\log\mathbb{E}(|u(t,x)|^2)\\&=\theta^{\frac{a}{2a(b+r)-b\alpha-a}}\left(\frac{2a}{2a(b+r)-b\alpha}\right)^{\beta}\rho^{\frac{a}{2a(b+r)-b\alpha-a}}\left(2(b+r)-\frac{b\alpha}{a}-1\right),\end{aligned}$$

which proves Lemma 5.2. □

Now we are ready to prove (5.1).

Proof of (5.1) By the hypercontractivity (5.8) and the scaling property (3.8), we see that for all $p\geq 2$,

$$\begin{aligned}\|u(t,0)\|_p^2\leq\left\|u_{(p-1)\theta}(t,0)\right\|_2^2&=\sum_{n\geq 0}(n!)^n\theta^n(p-1)^n\left\|\widetilde{f}_n(\cdot,0,t)\right\|_{\mathcal{H}^{\otimes n}}^2\\&=\sum_{n\geq 0}(n!)^n\theta^n\left\|\widetilde{f}_n\left(\cdot,0,t(p-1)^{\frac{1}{2(b+r)-b\alpha/a}}\right)\right\|_{\mathcal{H}^{\otimes n}}^2\\&=\left\|u\left(t(p-1)^{\frac{1}{2(b+r)-b\alpha/a}},0\right)\right\|_2^2.\end{aligned}$$

Hence,

$$\|u(t,0)\|_p\leq\left\|u\left(t(p-1)^{\frac{1}{2(b+r)-b\alpha/a}},0\right)\right\|_2=:\left\|u(t_p,0)\right\|_2,\tag{5.9}$$

where t_p is defined in (1.15). Finally, an application of Lemma 5.2 proves (5.1). □

6 Lower bound of the asymptotics

In this section, we will prove the lower bound of (1.14), namely,

$$\begin{aligned}&\liminf_{t_p\to\infty} t_p^{-\beta}\log\|u(t,x)\|_p\\&\quad\geq\left(\frac{1}{2}\right)\left(\frac{2a}{2a(b+r)-b\alpha}\right)^{\beta}\\&\quad\times\left(\theta\nu^{-\alpha/a}\mathcal{M}_a^{\frac{2a-\alpha}{a}}\right)^{\frac{a}{2a(b+r)-b\alpha-a}}\left(2(b+r)-\frac{b\alpha}{a}-1\right).\end{aligned}\tag{6.1}$$

Through out this section, we assume that Assumptions 1.1 and 1.3, and condition (1.11) hold.

We start by defining the function $W_n(t,\phi)$ on $(0,\infty)\times L^2_{\mathbb{C}}(\mu)$ by

$$\begin{aligned}W_n(t,\phi):=&\int_{[0,t]^n_<}\int_{(\mathbb{R}^d)^n}\prod_{k=1}^n\phi(\xi_k)\prod_{k=1}^n\mathcal{F}G(s_k-s_{k-1},\cdot)(\xi_k+\cdots+\xi_n)\mu(\mathrm{d}\xi_1)\\&\cdots\mu(\mathrm{d}\xi_n)\mathrm{d}s.\end{aligned}\tag{6.2}$$

with $s_0=0$. We now give conditions under which W_n is well defined.

Lemma 6.1 *If the measure μ satisfies the relation in (4.13), then $W_n(t,\phi)$ is well defined and for any $d\geq 1$, $t>0$ and $\phi\in L^2_{\mathbb{C}}(\mu)$. Moreover,*

$$\int_0^\infty e^{-t}W_n(t,\phi)\mathrm{d}t=\int_{(\mathbb{R}^d)^n}\prod_{k=1}^n\phi(\xi_k)\prod_{k=1}^n\frac{1}{1+\frac{\nu}{2}|\xi_k+\cdots+\xi_n|^a}\mu(\mathrm{d}\xi)\cdots\mu(\mathrm{d}\xi_n).\tag{6.3}$$

Proof The proof follows the exact arguments as those in the proof of Lemma 6.2 of [1] except that one needs to use the following Laplace transform:

$$\int_0^\infty e^{-t}\mathcal{F}G(t,\cdot)(\xi)\ \mathrm{d}t=\frac{1}{1+\frac{\nu}{2}|\xi|^a};$$

see the second equation in (4.3). □

The next proposition is a restatement of Proposition 6.3 of [1]. The proof follows the same proof as that of Proposition 6.3 of [1]. We will not repeat it here.

Proposition 6.2 *For $f\in\mathcal{H}$, $t>0$, and $p,q>1$ with $p^{-1}+q^{-1}=1$, it holds that*

$$\|u(t,0)\|_p\geq\exp\left\{-\frac{1}{2}(q-1)\|f\|^2_{\mathcal{H}}\right\}\left|\sum_{n\geq 0}\theta^{n/2}W_n(t,\mathcal{F}f)\right|\tag{6.4}$$

and as a consequence, the series $\left|\sum_{n\geq 0}\theta^{n/2}W_n(t,\mathcal{F}f)\right|$ *converges provided that* $\|u(t,0)\|_p<\infty$*, which is the case under Theorem 1.6.*

Now we are going to apply a scaling argument to (6.4) in order to put t and p together, from which we can determine the constants t_p and β defined in (1.15).

Proposition 6.3 *For* $p,q>1$, $p^{-1}+q^{-1}=1$ *and for any* $f\in\mathcal{H}$,

$$\|u(t,0)\|_p \geq \exp\left\{-\frac{1}{2}t_p^{\beta}\,\|f\|_{\mathcal{H}}^2\right\}\left|\sum_{n\geq 0}\theta^{n/2}W_n\left(t_p^{\beta},\mathcal{F}f\right)\right|, \tag{6.5}$$

where the constants β *and* t_p *are defined in (1.15).*

Proof From Proposition 6.2, we see that for any $f\in\mathcal{H}$, the inequality (6.4) holds. For some constants $V, W>0$, which will be determined in this proof, let $f_\tau(x):=\tau^V f(\tau^W x)$ be a scaled version of f. It is clear that $f_\tau\in\mathcal{H}$. By some elementary scaling arguments (see the proof of the Lemma 6.4 of [1]), one can show that

$$\|f_\tau\|_{\mathcal{H}}^2=\tau^{2(V-dW)+W\alpha}\,\|f\|_{\mathcal{H}}^2 \quad\text{and} \tag{6.6}$$

$$W_n\left(t,\mathcal{F}f_\tau\right)=\tau^{n\left[V-W\left((d-\alpha)+\frac{a}{b}(b+r)\right)\right]}W_n\left(t\tau^{\frac{a}{b}W},\mathcal{F}f\right). \tag{6.7}$$

Hence, an application of Proposition 6.2 to f_τ shows that

$$\begin{aligned}\|u(t,0)\|_p &\geq \exp\left\{-\frac{1}{2}(q-1)\,\|f_\tau\|_{\mathcal{H}}^2\right\}\left|\sum_{n\geq 0}\theta^{n/2}W_n(t,\mathcal{F}f_\tau)\right|\\ &=\exp\left\{-\frac{1}{2}(q-1)\tau^{2(V-dW)+W\alpha}\,\|f\|_{\mathcal{H}}^2\right\}\\ &\quad\left|\sum_{n\geq 0}\theta^{n/2}\tau^{n\left[V-W\left((d-\alpha)+\frac{a}{b}(b+r)\right)\right]}W_n\left(t\tau^{\frac{a}{b}W},\mathcal{F}f\right)\right|.\end{aligned}$$

Comparing the above lower bound with that in (6.5), we obtain the following two relations with three unknowns W, V and τ:

$$V-W\left((d-\alpha)+\frac{a}{b}(b+r)\right)=0, \tag{6.8}$$

$$(p-1)^{-1}\tau^{2(V-dW)+W\alpha}=t\tau^{\frac{a}{b}W}. \tag{6.9}$$

Since (6.9) should hold for all $t>0$ and $p\geq 2$, one can choose $\tau=(p-1)t$ to reduce the relation (6.9) to

$$\tau^{2(V-dW)+W\alpha}=\tau^{\frac{a}{b}W+1},$$

which then gives the following equation

$$2(V - dW) + W\alpha = 1 + \frac{a}{b}W. \tag{6.10}$$

Now solve the linear equations (6.8) and (6.10) for W and V to see that

$$\begin{cases} W = \dfrac{b}{a}(\beta - 1), \\ V = \left(\dfrac{a}{b}(b+r) - \alpha + d\right)\dfrac{b}{a}(\beta - 1), \end{cases} \qquad \text{with } \beta := \frac{2(b+r) - \dfrac{b\alpha}{a}}{2(b+r) - \dfrac{b\alpha}{a} - 1}.$$

Therefore, the scaling f_τ and t_p^β should be

$$f_\tau(x) = \tau^{\left(\frac{a}{b}(b+r)-\alpha+d\right)\frac{b}{a}(\beta-1)} f\left(\tau^{\frac{b}{a}(\beta-1)}x\right) \quad \text{and} \quad t_p^\beta := t\tau^{\frac{a}{b}W} = (p-1)^{\beta-1}t^\beta,$$

respectively. This completes the proof of Proposition 6.3. □

To remove the absolute value sign in Proposition 6.3, we want to identify all the $f \in \mathcal{H}$ for which $W_n(t, \mathcal{F}f)$ is nonnegative. In fact, if we consider the space

$$\mathcal{H}_+ = \{f \in \mathcal{H} : f \text{ is nonnegative and nonnegative definite}\},$$

then by Plancherel's theorem, for $f \in \mathcal{H}_+$,

$$\begin{aligned} W_n(t, \mathcal{F}f) &= \int_{[0,t]^n_<} \int_{\mathbb{R}^{nd}} \prod_{k=1}^n (f * \gamma)(x_k) \prod_{k=1} G(s_k - s_{k-1}, x_k - x_{k-1}) \mathrm{d}\vec{x}\mathrm{d}\vec{s} \\ &=: U_n(t, f) \geq 0 \end{aligned}$$

with the convention that $s_0 = 0$ and $x_0 = 0$, where we have used the fact that the fundamental solution $G(t, x)$ is nonnegative (under Assumption 1.1). Now define

$$W_n(t) := \sup_{f \in \mathcal{H},\, \|f\|_{\mathcal{H}}=1} W_n(t, \mathcal{F}(f)) \quad \text{and} \quad U_n(t) := \sup_{f \in \mathcal{H}_+, \|f\|_{\mathcal{H}}=1} U_n(t, f).$$

It is clear that $W_n(t) \geq U_n(t) \geq 0$.

Proposition 6.4 *If τ is an exponential random variable with mean one, then*

$$\liminf_{n \to \infty} \frac{1}{n} \log \mathbb{E}(U_n(\tau)) \geq \log(\rho^{1/2}),$$

where ρ is the constant defined in (3.10).

Proof We start by letting τ be an exponential random variable with mean one. With this,

$$\mathbb{E}\left(U_n(\tau)\right)=\int_0^\infty e^{-t}U_n(t)\mathrm{d}t\geq\sup_{f\in\mathcal{H}_+,\,\|f\|_{\mathcal{H}}=1}\int_0^\infty e^{-t}U_n(t,f)\mathrm{d}t.$$

For any $f\in\mathcal{H}_+$ with $\|f\|_{\mathcal{H}}=1$, by Bochner's theorem, $\mathcal{F}f$ is nonnegative and nonnegative definite, which further implies that $\mathcal{F}f$ is even. In addition, Lemma 6.1 gives us that

$$\begin{aligned}\int_0^\infty e^{-t}U_n(t,f)\mathrm{d}t&=\int_0^\infty e^{-t}W_n(t,\mathcal{F}f)\mathrm{d}t\\&=\int_{\mathbb{R}^{dn}}\prod_{k=1}^n\mathcal{F}f(\xi_k)\prod_{k=1}^n\frac{1}{1+\frac{\nu}{2}|\xi_k+\cdots+\xi_n|^a}\mu(\mathrm{d}\xi_1)\cdots\mu(\mathrm{d}\xi_n).\end{aligned}$$

Notice that the right hand side of the above equation takes the form as [2, Equation (3.3)]. By the same arguments that follow in the proof of Theorem 2.3 *ibid* with the replacement (4.2), we have that

$$\liminf_{n\to\infty}\frac{1}{n}\log\int_{\mathbb{R}^{dn}}\prod_{k=1}^n(\mathcal{F}f)(\xi_k)\prod_{k=1}^n\frac{1}{1+\frac{\nu}{2}|\xi_k+\cdots+\xi_n|^a}\mu(\mathrm{d}\xi_1)\cdots\mu(\mathrm{d}\xi_n)\geq\log\rho(\mathcal{F}f),$$

where $\rho(\cdot)$ (check (3.10) for a comparison) is defined as,

$$\rho(g):=\sup_{\|h\|_{L^2(\mathbb{R}^d)}=1}\int_{\mathbb{R}^d}g(\xi)\left[\int_{\mathbb{R}^d}\frac{h(\xi+\eta)h(\eta)}{\sqrt{1+\frac{\nu}{2}|\xi+\eta|^a}\sqrt{1+\frac{\nu}{2}|\eta|^a}}\mathrm{d}\eta\right]\mu(\mathrm{d}\xi),\quad(6.11)$$

for all nonnegative and nonnegative-definite functions $g\in L^2\left(\mu(\mathrm{d}\xi)\right)$. Before we proceed, we first make a few comments:

(1) $g\in L^2\left(\mu\left(\mathrm{d}\xi\right)\right)$ if and only if $\mathcal{F}^{-1}g\in\mathcal{H}$. Moreover, $\|g\|_{L^2(\mu)}=\left\|\mathcal{F}^{-1}g\right\|_{\mathcal{H}}$.
(2) For $g\in L^2\left(\mu\left(\mathrm{d}\xi\right)\right)$, g is nonnegative definite if and only if $\mathcal{F}^{-1}g\in\mathcal{H}_+$.
(3) For $g\in L^2\left(\mu\left(\mathrm{d}\xi\right)\right)$, g is nonnegative if and only if $\mathcal{F}^{-1}g$ is nonnegative definite.
(4) Finding the best nonnegative and nonnegative-definite function g with $\|g\|_{L^2(\mu)}=1$ to maximize $\rho(g)$ is equivalent to finding the best nonnegative and nonnegative-definite $f\in\mathcal{H}_+$ with $\|f\|_{\mathcal{H}}=1$ to maximize $\rho\left(\mathcal{F}f\right)$.
(5) $\rho(g)$ is well defined (i.e., finite) because for any $h\in L^2(\mathbb{R}^d)$ with $\|h\|_{L^2(\mathbb{R}^d)}=1$,

$$\begin{aligned}&\left|\int_{\mathbb{R}^d}g(\xi)\left[\int_{\mathbb{R}^d}\frac{h(\xi+\eta)h(\eta)}{\sqrt{1+\frac{\nu}{2}|\xi+\eta|^a}\sqrt{1+\frac{\nu}{2}|\eta|^a}}\mathrm{d}\eta\right]\mu(\mathrm{d}\xi)\right|\\&\leq\left(\int_{\mathbb{R}^d}g(\xi)^2\mu(\mathrm{d}\xi)\right)^{1/2}\left(\int_{\mathbb{R}^d}\left[\int_{\mathbb{R}^d}\frac{h(\xi+\eta)h(\eta)}{\sqrt{1+\frac{\nu}{2}|\xi+\eta|^a}\sqrt{1+\frac{\nu}{2}|\eta|^a}}\mathrm{d}\eta\right]^2\mu(\mathrm{d}\xi)\right)^{1/2}\\&\leq\|\mathcal{F}g\|_{\mathcal{H}}\sqrt{\rho_{\nu,a}}<\infty,\end{aligned}$$

where the upper bound does not depends on h, and $\rho_{\nu,a}$ is the constant defined in (3.10) which is finite due to Theorem 3.5 (see (3.20)).

Note that since both μ and g are nonnegative in (6.11), the supremum in (6.11) has to be achieved by some nonnegative function h. Hence, we may assume h is also nonnegative for the remainder of this proof. With this being said, we see that for any $f \in \mathcal{H}_+$ with $\|f\|_{\mathcal{H}} = 1$,

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{E}(U_n(\tau)) \geq \log \rho(\mathcal{F} f).$$

We now need to calculate

$$\sup_{f\in\mathcal{H}_+,\ \|f\|_{\mathcal{H}}=1} \rho\left(\mathcal{F} f\right).$$

Consider a nonnegative function $h \in L^2(\mathbb{R}^d)$. The function $h(\cdot)/\sqrt{1+\frac{\nu}{2}|\cdot|^a} \in L^2(\mathbb{R}^d)$ so that $g_h(x) := (2\pi)^{d/2}\mathcal{F}^{-1}\left(\frac{h(\cdot)}{\sqrt{1+\frac{\nu}{2}|\cdot|^a}}\right)(x)$ is well defined. Under these conditions, $g_h \in W^{1,a}(\mathbb{R}^d)$ with

$$\begin{aligned}\|g_h\|_{W^{1,a}(\mathbb{R}^d)} &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}(1+|\xi|^a)|\mathcal{F} g_h(\xi)|^2 \mathrm{d}\xi = \int_{\mathbb{R}^d}\frac{1+|\xi|^a}{1+\frac{\nu}{2}|\xi|^a}|h(\xi)|^2\mathrm{d}\xi \\ &\leq C_\nu \|h\|^2_{L^2(\mathbb{R}^d)} < \infty.\end{aligned}$$

Notice that

$$\begin{aligned}\int_{\mathbb{R}^d}\frac{h(\xi+\eta)h(\eta)}{\sqrt{1+\frac{\nu}{2}|\xi+\eta|^a}\sqrt{1+\frac{\nu}{2}|\eta|^a}}\mathrm{d}\eta &= (2\pi)^{-d}\int_{\mathbb{R}^d}\mathcal{F} g_h(\xi-\eta)\mathcal{F} g_h(-\eta)\mathrm{d}\eta \\ &= (2\pi)^{-d}(\mathcal{F} g_h * \widetilde{\mathcal{F} g_h})(\xi),\end{aligned}$$

where we have used the notation that $\widetilde{h}(x) = h(-x)$. Since $h(\cdot)$ is real valued, we see that $\widetilde{\mathcal{F} g_h} = \overline{\mathcal{F} g_h} = \mathcal{F}\bar{g_h}$. Using this and the fact that $\mathcal{F}(fg) = (2\pi)^{-d}\mathcal{F}(f) * \mathcal{F}(g)$, we see that $(2\pi)^{-d}(\mathcal{F} g_h * \widetilde{\mathcal{F} g_h})(\xi) = \mathcal{F}\left[|g_h|^2\right](\xi)$. Hence, from (3.20), we see that $\mathcal{F} g_h * \widetilde{\mathcal{F} g_h} \in L^2\left(\mu(d\xi)\right)$ or equivalently $|g_h|^2 \in \mathcal{H}$. Then,

$$\begin{aligned}\rho(\mathcal{F} f) &= \sup_{\|h\|_{L^2(\mathbb{R}^d)}=1}(2\pi)^{-d}\int_{\mathbb{R}^d}\mathcal{F} f(\xi)(\mathcal{F} g_h * \widetilde{\mathcal{F} g_h})(\xi)\mu(\mathrm{d}\xi) \\ &= \sup_{\|h\|_{L^2(\mathbb{R}^d)}=1}\int_{\mathbb{R}^d}\mathcal{F} f(\xi)\mathcal{F}[|g_h|^2](\xi)\mu(\mathrm{d}\xi) \\ &= \sup_{\|h\|_{L^2(\mathbb{R}^d)}=1}\langle f, |g_h|^2\rangle_{\mathcal{H}}.\end{aligned}$$

With this we see that

$$\sup_{f\in\mathcal{H},\|f\|_{\mathcal{H}=1}}\rho(\mathcal{F}f)=\sup_{f\in\mathcal{H},\|f\|_{\mathcal{H}=1}}\sup_{\|h\|_{L^2}=1}\langle f,|g_h|^2\rangle_{\mathcal{H}}=\sup_{\|h\|_{L^2}=1}\langle|g_h|^2,|g_h|^2\rangle_{\mathcal{H}}^{1/2} \tag{6.12}$$

where the optimal f is chosen to be $|g_h|^2/\left\||g_h|^2\right\|_{\mathcal{H}}$. Now we claim that

$$|g_h|^2\in\mathcal{H}_+, \tag{6.13}$$

which implies that the supremum in (6.12) can be restricted to $\mathcal{H}_+$ and

$$\sup_{f\in\mathcal{H},\ \|f\|_{\mathcal{H}}=1}\rho(\mathcal{F}f)=\sup_{f\in\mathcal{H}_+,\ \|f\|_{\mathcal{H}}=1}\rho(\mathcal{F}f)=\sup_{\|h\|_{L^2}=1}\left\langle|g_h|^2,|g_h|^2\right\rangle_{\mathcal{H}}^{1/2}.$$

Then notice that

$$\mathcal{F}(|g_h|^2)(\xi)=(2\pi)^{-d}(\mathcal{F}g_h*\widetilde{\mathcal{F}g_h})(\xi)=\int_{\mathbb{R}^d}\frac{h(\xi+\eta)h(\eta)}{\sqrt{1+\frac{\nu}{2}|\xi+\eta|^a}\sqrt{1+\frac{\nu}{2}|\eta|^a}}\mathrm{d}\eta. \tag{6.14}$$

Hence, from (3.10) and (6.14), we see that

$$\sup_{\|h\|_{L^2(\mathbb{R}^d)}=1}\left\langle|g_h|^2,|g_h|^2\right\rangle_{\mathcal{H}}=\sup_{\|h\|_{L^2(\mathbb{R}^d)}=1}\int_{\mathbb{R}^d}\left|\mathcal{F}(|g_h|^2)(\xi)\right|^2\mu(\mathrm{d}\xi)=\rho_{\nu,a}$$

which then leads us to the lower bound:

$$\liminf_{n\to\infty}\frac{1}{n}\log\mathbb{E}(U_n(\tau))\geq\log\left(\rho_{\nu,a}^{1/2}\right).$$

Therefore, it remains to proving (6.13). First notice that from the above we see that

$$\int_{\mathbb{R}^d}\left|\mathcal{F}\left(|g_h|^2\right)(\xi)\right|^2\mu(\mathrm{d}\xi)\leq\rho_{\nu,a}<\infty\quad\Longrightarrow\quad|g_h(\cdot)|^2\in\mathcal{H}.$$

Moreover, since h is nonnegative, from (6.14), we see that $\mathcal{F}\left(|g_h|^2\right)(\cdot)$ is also nonnegative. The Bochner-Schwarz theorem then implies that $|g_h(\cdot)|^2$ is nonnegative definite. It is clear that $|g_h(\cdot)|^2$ is nonnegative. This shows that $|g_h(\cdot)|^2\in\mathcal{H}_+$. This completes the whole proof of Proposition 6.4. □

Lemma 6.5 *It holds that*

$$\liminf_{n\to\infty}\frac{1}{n}\log\left[(n!)^{\frac{b}{a}(a-\frac{\alpha}{2}+\frac{a}{b}r)}U_n(1)\right]\geq\log\left(\frac{\rho_{\nu,a}^{1/2}}{\left(\frac{b}{a}[a-\frac{\alpha}{2}+\frac{a}{b}r]\right)^{\frac{b}{a}(a-\frac{\alpha}{2}+\frac{a}{b}r)}}\right). \tag{6.15}$$

Proof Let τ be an exponential random variable with mean one. Notice that by some elementary scaling arguments, we have that for all $t > 0$,

$$W_n(t) = t^{n\frac{b}{a}(a-\frac{\alpha}{2}+\frac{a}{b}r)} W_n(1) \quad \text{and} \quad U_n(t) = t^{n\frac{b}{a}(a-\frac{\alpha}{2}+\frac{a}{b}r)} U_n(1), \tag{6.16}$$

which then imply that

$$\mathbb{E}\left[U_n(\tau)\right] = \mathbb{E}\left(\tau^{n\frac{b}{a}(a-\frac{\alpha}{2}+\frac{a}{b}r)}\right) U_n(1) = \Gamma\left(n\frac{b}{a}\left(a-\frac{\alpha}{2}+\frac{a}{b}r\right)+1\right) U_n(1).$$

Then by Proposition 6.4,

$$\liminf_{n\to\infty} \frac{1}{n} \log\left[\Gamma\left(n\frac{b}{a}\left(a-\frac{\alpha}{2}+\frac{a}{b}r\right)+1\right) U_n(1)\right] \geq \log\left(\rho_{\nu,a}^{1/2}\right). \tag{6.17}$$

Therefore, (6.15) is proven by noticing that, thanks to (5.2),

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{1}{n} \log\left(\frac{\Gamma\left(n\frac{b}{a}\left(a-\frac{\alpha}{2}+\frac{a}{b}r\right)+1\right)}{(n!)^{\frac{b}{a}(a-\frac{\alpha}{2}+\frac{a}{b}r)}}\right) \\
&\quad = \frac{b}{a}\left(a-\frac{\alpha}{2}+\frac{a}{b}r\right) \log\left(\frac{b}{a}\left(a-\frac{\alpha}{2}+\frac{a}{b}r\right)\right),
\end{aligned}$$

where the condition $\frac{b}{a}\left(a-\frac{\alpha}{2}+\frac{a}{b}r\right) > 0$ is guaranteed by (4.19) (or (1.11)).

We need one last lemma before the proof of the lower bound:

Lemma 6.6 *For any $k, \theta > 0$, there exists a constant $c_1 = c_1(\alpha, \mathcal{M}, k, \theta) > 0$ such that, by setting $n_t = [c_1 t]$, it holds that*

$$\liminf_{t\to\infty} \frac{1}{t} \log(k^{n_t}\theta^{n_t/2} U_{n_t}(t)) \geq \left(k\sqrt{\theta}\right)^{\frac{2a}{2ab-\alpha b+2ar}} \left(\rho_{\nu,a}^{1/2}\right)^{\frac{2a}{2ab-\alpha b+2ar}}. \tag{6.18}$$

Proof Fix an arbitrary $\epsilon > 0$. Lemma 6.5 guarantees the existence of an $N_\epsilon > 0$ so for all $n > N_\epsilon$

$$(n!)^{\frac{b}{a}(a-\frac{\alpha}{2}+\frac{a}{b}r)} U_n(1) \geq \exp\left(n(\log(R)-\epsilon)\right) = R^n e^{-n\epsilon} \tag{6.19}$$

where

$$R = \rho_{\nu,a}^{1/2} \left(\frac{b}{a}\left[a-\frac{\alpha}{2}+\frac{a}{b}r\right]\right)^{-\frac{b}{a}(a-\frac{\alpha}{2}+\frac{a}{b}r)}.$$

Now fix a $c > 0$ and let $n_t := [ct]$. Notice that $n_t \geq N_\epsilon$ for any $t > t_\epsilon := (N_\epsilon + 1)/c$. For $t > t_\epsilon$, from (6.19), we have

$$k^{n_t}\theta^{n_t/2}U_{n_t}(t) = k^{n_t}\theta^{n_t/2}t^{\frac{b}{a}(a-\frac{\alpha}{2}+\frac{a}{b}r)n_t}U_{n_t}(1) \geq \frac{k^{n_t}\theta^{n_t/2}t^{\frac{b}{a}(a-\frac{\alpha}{2}+\frac{a}{b}r)n_t}}{(n_t!)^{\frac{b}{a}(a-\frac{\alpha}{2}+\frac{a}{b}r)}}R^{n_t}e^{-n_t\epsilon}. \tag{6.20}$$

Notice that $[ct]/t \to c$ as $t \to \infty$ which means that $n_t/t \to c$ as $t \to \infty$. With this we can say

$$\liminf_{t\to\infty}\frac{1}{t}\log\left(k^{n_t}\theta^{n_t/2}U_{n_t}(t)\right) = c\liminf_{t\to\infty}\frac{1}{n_t}\log\left(k^{n_t}\theta^{n_t/2}U_{n_t}(t)\right) =: I(n_t).$$

Now, by (6.20), we have that

$$\begin{aligned}
I(n_t) &\geq c\liminf_{t\to\infty}\frac{1}{n_t}\log\left((kR\sqrt{\theta})^{n_t}\frac{t^{\frac{b}{a}(a-\frac{\alpha}{2}+\frac{a}{b}r)n_t}}{(n_t!)^{\frac{b}{a}(a-\frac{\alpha}{2}+\frac{a}{b}r)}}\right) - c\epsilon \\
&= c\log(k\sqrt{\theta}R) + c\liminf_{t\to\infty}\frac{1}{n_t}\log\left[\left(\frac{t}{n_t}\right)^{\frac{b}{a}(a-\frac{\alpha}{2}+\frac{a}{b}r)n_t}\frac{n_t^{\frac{b}{a}(a-\frac{\alpha}{2}+\frac{a}{b}r)n_t}}{(n_t!)^{\frac{b}{a}(a-\frac{\alpha}{2}+\frac{a}{b}r)}}\right] - c\epsilon \\
&= c\log(k\sqrt{\theta}R) - c\frac{b}{a}\left(a-\frac{\alpha}{2}+\frac{a}{b}r\right)\log(c) + c\frac{b}{a}\left(a-\frac{\alpha}{2}+\frac{a}{b}r\right) \\
&\quad \liminf_{t\to\infty}\frac{1}{n_t}\log\left(\frac{n_t^{n_t}}{(n_t)!}\right) - c\epsilon \\
&= c\log(k\sqrt{\theta}R) - c\frac{b}{a}\left(a-\frac{\alpha}{2}+\frac{a}{b}r\right)\log(c) + c\frac{b}{a}\left(a-\frac{\alpha}{2}+\frac{a}{b}r\right) - c\epsilon
\end{aligned}$$

and letting ϵ tend to 0 we see that

$$I(n_t) \geq c\left[\log(k\sqrt{\theta}R) - \frac{b}{a}\left(a-\frac{\alpha}{2}+\frac{a}{b}r\right)\log(c) + \frac{b}{a}\left(a-\frac{\alpha}{2}+\frac{a}{b}r\right)\right] =: h(c).$$

In order to maximize $h(c)$, notice that

$$h'(c) = 0 \iff c^* = (k\sqrt{\theta}R)^{\frac{a}{b(a-\frac{\alpha}{2}+\frac{a}{b}r)}}.$$

After plugging c^* and replacing R we arrive at the following inequality

$$I(n_t) \geq (k\sqrt{\theta})^{\frac{a}{b(a-\frac{\alpha}{2}+\frac{a}{b}r)}}(\rho_{\nu,a}^{1/2})^{\frac{a}{b(a-\frac{\alpha}{2}+\frac{a}{b}r)}},$$

which proves (6.18) after some simplification. □

We are now ready to prove (6.1).

Proof of (6.1) By proposition 6.3, for and $p, q > 0$ with $p^{-1} + q^{-1} = 1$ we have that

$$\|u(t,0)\|_p \geq \exp\left\{-\frac{1}{2} t_p^{\beta} \, \|f\|_{\mathcal{H}}^2\right\} \left| \sum_{n \geq 0} \theta^{n/2} W_n(t_p^{\beta}, \mathcal{F} f) \right|.$$

We now take the supremum over all $f \in \mathcal{H}_+$ with $\|f\|_{\mathcal{H}} = k > 0$. Recall that $W_n(t, \phi) = k^n W_n(t, \phi/k)$ for $\phi \in L^2(\mu)$ and the non-negativity of $W_n(t, \cdot)$ on $\mathcal{H}_+$. Let $c > 0$.

$$\begin{aligned}
\|u(t,0)\|_p &\geq \exp\left\{-\frac{1}{2} t_p^{\beta} k^2\right\} \sup_{f \in \mathcal{H}_+ \ \|f\|_{\mathcal{H}} = k} \sum_{n \geq 0} \theta^{n/2} W_n(t_p^{\beta}, \mathcal{F} f) \\
&= \exp\left\{-\frac{1}{2} t_p^{\beta} k^2\right\} \sup_{f \in \mathcal{H}_+ \ \|f\|_{\mathcal{H}} = 1} \sum_{n \geq 0} k^n \theta^{n/2} W_n(t_p^{\beta}, \mathcal{F} f) \\
&\geq \exp\left\{-\frac{1}{2} t_p^{\beta} k^2\right\} \sup_{f \in \mathcal{H}_+ \ \|f\|_{\mathcal{H}} = 1} k^{n_t} \theta^{n_t/2} U_{n_t}(t_p^{\beta}, f)
\end{aligned}$$

where $n_t = [c t_p^{\beta}]$. Now by choosing c as in Lemma 6.6 we get that

$$\begin{aligned}
&\liminf_{t_p \to \infty} t_p^{-\beta} \log \|u(t,x)\|_p \geq \liminf_{t_p \to \infty} \left(\frac{-\frac{1}{2} t_p^{\beta} k^2}{t_p^{\beta}} + \frac{1}{t_p^{\beta}} \log \right. \\
&\quad \left. \left[\sup_{f \in \mathcal{H}_+ \ \|f\|_{\mathcal{H}} = 1} k^{n_t} \theta^{n_t/2} U_{n_t}(t_p^{\beta}, f) \right] \right) \\
&\quad = -\frac{1}{2} k^2 + k^{\frac{2a}{2ab - \alpha b + 2ar}} (\rho\theta)^{\frac{a}{2ab - \alpha b + 2ar}} =: h(k).
\end{aligned}$$

By maximizing h for $k > 0$, we see that h is maximized at the point

$$k^* = \left(\frac{2ab - \alpha b + 2ar}{2aB} \right)^{\frac{2ab - \alpha b + 2ar}{2a - 2[2ab - \alpha b + 2ar]}} \qquad \text{with } B = (\theta\rho)^{\frac{a}{2ab - \alpha b + 2ar}}.$$

Inserting k^* into h gives us that

$$h(k^*) = B^{\beta} \left(\frac{2a}{2ab - \alpha b + 2ar} \right)^{\beta} \left(\frac{2ab - \alpha b + 2ar - a}{2a} \right)$$

and plugging in the value for B proves (6.1). ◻

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7 Appendix

Proof of Lemma 3.4 In this proof, $\mu(\mathrm{d}x) = \varphi(x)\mathrm{d}x = C_{\alpha,d}|x|^{-(d-\alpha)}\mathrm{d}x$. By the change of variables $x' = (\nu/2)^{1/a}\, x$ and $y' = (\nu/2)^{1/a}\, y$, we see that

$$\rho_{\nu,a}\left(|\cdot|^{-\alpha}\right) = \sup_{\|f\|_{L^2(\mathbb{R}^d)}=1} \int_{\mathbb{R}^d} \left[\int_{\mathbb{R}^d} \frac{f(x+y)f(y)}{\sqrt{1+\frac{\nu}{2}|x+y|^a}\sqrt{1+\frac{\nu}{2}|y|^a}}\mathrm{d}y\right]^2 \mu(\mathrm{d}x)$$

$$= \left(\frac{\nu}{2}\right)^{-\alpha/a} \sup_{\|f\|_{L^2(\mathbb{R}^d)}=1} \int_{\mathbb{R}^d} \left[\int_{\mathbb{R}^d} \frac{f\left(\left(\frac{\nu}{2}\right)^{-1/a}(x+y)\right) f\left(\left(\frac{\nu}{2}\right)^{-1/a} y\right)}{\sqrt{1+|x+y|^a}\sqrt{1+|y|^a}} \left(\frac{\nu}{2}\right)^{-d/a} \mathrm{d}y\right]^2 \varphi(x)\mathrm{d}x.$$

By setting $f^*(x) = (\nu/2)^{-d/(2a)} f\left((\nu/2)^{-1/a}\, x\right)$, we see that

$$\begin{aligned}
\rho_{\nu,a}\left(|\cdot|^{-\alpha}\right) &= \left(\frac{\nu}{2}\right)^{-\alpha/a} \sup_{\|f\|_{L^2(\mathbb{R}^d)}=1} \int_{\mathbb{R}^d} \left[\int_{\mathbb{R}^d} \frac{f^*(x+y)\, f^*(y)}{\sqrt{1+|x+y|^a}\sqrt{1+|y|^a}}\mathrm{d}y\right]^2 \varphi(x)\mathrm{d}x \\
&= \left(\frac{\nu}{2}\right)^{-\alpha/a} \sup_{\|f^*\|_2=1} \int_{\mathbb{R}^d} \left[\int_{\mathbb{R}^d} \frac{f^*(x+y)\, f^*(y)}{\sqrt{1+|x+y|^a}\sqrt{1+|y|^a}}\mathrm{d}y\right]^2 \varphi(x)\mathrm{d}x \\
&= \left(\frac{\nu}{2}\right)^{-\alpha/a} \rho_{2,a}\left(|\cdot|^{-\alpha}\right),
\end{aligned}$$

where the second equality is due to the fact that $\int_{\mathbb{R}^d} f(x)^2\mathrm{d}x = \int_{\mathbb{R}^d} f^*(x)^2\mathrm{d}x$. Then an application of (3.16) proves (3.18).

Similarly, for (3.21), by change of variables $\xi'_{\sigma(j)} = (\nu/2)^{1/a}\, \xi_{\sigma(j)}$, we see that

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{n} \log\left[\frac{1}{(n!)^2} \int_{(\mathbb{R}^d)^n} \left(\sum_{\sigma\in\Sigma_n} \prod_{k=1}^n \frac{1}{1+\frac{\nu}{2}|\sum_{j=k}^n \xi_{\sigma(j)}|^a}\right)^2 \mu(\mathrm{d}\vec{\xi}\,)\right] \\
&= \lim_{n\to\infty} \frac{1}{n} \log\left[\frac{1}{(n!)^2} \left(\frac{\nu}{2}\right)^{-n\alpha/a} \int_{(\mathbb{R}^d)^n} \left[\sum_{\sigma\in\Sigma_n} \prod_{k=1}^n \frac{1}{1+\left|\sum_{j=k}^n \xi_{\sigma(j)}\right|^a}\right]^2 \mu(\mathrm{d}\vec{\xi})\right] \\
&= \log\left[\left(\frac{\nu}{2}\right)^{-\alpha/a}\right] + \lim_{n\to\infty} \frac{1}{n} \log\left[\frac{1}{(n!)^2} \int_{(\mathbb{R}^d)^n} \left[\sum_{\sigma\in\Sigma_n} \prod_{k=1}^n \frac{1}{1+\left|\sum_{j=k}^n \xi_{\sigma(j)}\right|^a}\right]^2 \mu(\mathrm{d}\vec{\xi})\right] \\
&= \log\left(\left(\frac{\nu}{2}\right)^{-\alpha/a}\right) + \log\left(\rho_{2,a}\left(|\cdot|^{-\alpha}\right)\right) = \log\left(\rho_{\nu,a}\left(|\cdot|^{-\alpha}\right)\right),
\end{aligned}$$

where we have applied (3.15) and (3.18). This completes the proof of Lemma 3.4. $\square$

Proof of (4.9) Starting from (3.5), by the change of variables $t_i' = t_i/c$ and the scaling property in (3.7), we have that

$$
\begin{aligned}
\mathcal{F} f_n(\cdot, 0, ct)(\xi_1, \cdots, \xi_n) &= \int_{[0,ct]^n_<} \overline{\prod_{k=1}^n \mathcal{F}G(t_{k+1} - t_k, \cdot)\left(\sum_{j=1}^k \xi_j\right)} \mathrm{d}\vec{t} \\
&= \int_{[0,t]^n_<} \overline{\prod_{k=1}^n \mathcal{F}G\left(c(t_{k+1} - t_k), \cdot\right)\left(\sum_{j=1}^k \xi_j\right)} c^n \mathrm{d}\vec{t} \\
&= \int_{[0,t]^n_<} \overline{\prod_{k=1}^n c^{b+r-1} \mathcal{F}G\left(t_{k+1} - t_k, \cdot\right)\left(c^{b/a}\sum_{j=1}^k \xi_j\right)} c^n \mathrm{d}\vec{t}
\end{aligned}
$$

where in the last line we applied (3.6). Now,

$$
\begin{aligned}
\mathcal{F} f_n(\cdot, 0, ct)(\xi_1, \cdots, \xi_n) &= \int_{[0,t]^n_<} \overline{\prod_{k=1}^n c^{b+r-1} \mathcal{F}G\left(t_{k+1} - t_k, \cdot\right)\left(c^{b/a}\sum_{j=1}^k \xi_j\right)} c^n \mathrm{d}\vec{t} \\
&= c^{n(b+r)} \mathcal{F} f_n\left(\cdot, 0, t\right)\left(c^{b/a}\xi_1, \cdots, c^{b/a}\xi_n\right),
\end{aligned}
$$

from which we see that

$$
\begin{aligned}
&\int_0^\infty e^{-t} \left\| \widetilde{f}_n(\cdot, 0, t) \right\|^2_{\mathcal{H}^{\otimes n}} \mathrm{d}t = \int_0^\infty e^{-t} \int_{\mathbb{R}^{nd}} \left|\mathcal{F}\widetilde{f}_n(\cdot, 0; t)(\xi_1, \cdots, \xi_n)\right|^2 \mu(\mathrm{d}\vec{\xi}\,)\mathrm{d}t \\
&= \int_0^\infty e^{-2t} \int_{\mathbb{R}^{nd}} \left|\mathcal{F}\widetilde{f}_n(\cdot, 0; 2t)(\xi_1, \cdots, \xi_n)\right|^2 \mu(\mathrm{d}\vec{\xi}\,)\, 2\, \mathrm{d}t \\
&= 2^{2n(b+r)} \int_0^\infty 2e^{-2t} \int_{\mathbb{R}^{nd}} \left|\mathcal{F}\widetilde{f}_n(\cdot, 0, t)(2^{b/a}\xi_1, \cdots, 2^{b/a}\xi_n)\right|^2 \mu(\mathrm{d}\vec{\xi}\,)\mathrm{d}t
\end{aligned}
$$

where in the last line we have applied (3.7). Now,

$$
\begin{aligned}
&\int_0^\infty e^{-t} \left\| \widetilde{f}_n(\cdot, 0, t) \right\|^2_{\mathcal{H}^{\otimes n}} \mathrm{d}t = 2^{2n(b+r)} \int_0^\infty 2e^{-2t} \\
&\qquad \int_{\mathbb{R}^{nd}} \left|\mathcal{F}\widetilde{f}_n(\cdot, 0, t)(2^{b/a}\xi_1, \cdots, 2^{b/a}\xi_n)\right|^2 \mu(\mathrm{d}\vec{\xi}\,)\mathrm{d}t \\
&\quad = 2^{2n(b+r)} \int_0^\infty 2e^{-2t} \int_{\mathbb{R}^{nd}} \left|\mathcal{F}\widetilde{f}_n(\cdot, 0; t)(\xi_1, \cdots, \xi_n)\right|^2 2^{\frac{-nbd}{a}} 2^{\frac{nb(d-\alpha)}{a}} \mu(\mathrm{d}\vec{\xi}\,)\mathrm{d}t
\end{aligned}
$$

where the last line follows from a change of variables and recalling that $\mu(\mathrm{d}\vec{\xi}\,)$ in (3.17). Lastly,

$$
\begin{aligned}
&\int_0^\infty e^{-t} \left\| \widetilde{f}_n(\cdot, 0, t) \right\|^2_{\mathcal{H}^{\otimes n}} \mathrm{d}t = 2^{2n(b+r)} \int_0^\infty 2e^{-2t} \\
&\qquad \int_{\mathbb{R}^{nd}} \left|\mathcal{F}\widetilde{f}_n(\cdot, 0; t)(\xi_1, \cdots, \xi_n)\right|^2 2^{\frac{-nbd}{a}} 2^{\frac{nb(d-\alpha)}{a}} \mu(\mathrm{d}\vec{\xi}\,)\mathrm{d}t
\end{aligned}
$$

$$= 2^{n(2(b+r)-b\alpha/a)} \int_0^\infty 2e^{-2t} \int_{\mathbb{R}^{nd}} \left|\mathcal{F}\widetilde{f}_n(\cdot, 0; t)(\xi_1, \cdots, \xi_n)\right|^2 \mu(\mathrm{d}\vec{\xi}\,)\mathrm{d}t$$
$$= \frac{2^{n(2(b+r)-b\alpha/a)}}{(n!)^2} \int_0^\infty 2e^{-2t} \int_{\mathbb{R}^{nd}} H_n(t, \vec{x})^2 \mathrm{d}\vec{x}\mathrm{d}t,$$

which proves (4.9). □

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