Nonlinearity

PAPER

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To cite this article: Le Chen *et al* 2025 *Nonlinearity* **38** 055026

IOP Publishing | London Mathematical Society Nonlinearity

Nonlinearity **38** (2025) 055026 (25pp) https://doi.org/10.1088/1361-6544/add3b2

Global solution for superlinear stochastic heat equation on $\mathbb{R}^d$ under Osgood-type conditions

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Received 14 November 2023; revised 18 March 2025
Accepted for publication 2 May 2025
Published 12 May 2025

Recommended by Dr Jonathan Touboul

Abstract
We study the *stochastic heat equation* on $\mathbb{R}^d$ subject to a centered Gaussian noise that is white in time and colored in space.The drift term is assumed to satisfy an Osgood-type condition and the diffusion coefficient may have certain related growth. We show that there exists random field solution which do not explode in finite time. This complements and improves upon recent results on blow-up of solutions to stochastic partial differential equations.

Keywords: global solution, stochastic heat equation, reaction-diffusion, dalang's condition, superlinear growth, osgood-type conditions.

Mathematics Subject Classification numbers: 60H15

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1. Introduction

In this paper, we study the following superlinear *stochastic heat equation* (SHE) on $\mathbb{R}^d$:

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = \dfrac{1}{2}\Delta u(t,x) + b(u(t,x)) + \sigma(u(t,x))\dot{W}(t,x), \quad t>0,\ x\in\mathbb{R}^d, \\ u(0,\cdot) = u_0(\cdot), \end{cases} \tag{1.1}$$

where both b and σ are locally Lipschitz continuous, vanish at zero, and may have superlinear growth at infinity. The noise $\dot{W}$ is a centered Gaussian noise which is white in time and colored in space. Its covariance structure can be formally described by

$$\mathbb{E}\left[\dot{W}(s,y)\dot{W}(t,x)\right] = \delta_0(t-s)f(x-y), \tag{1.2}$$

where δ_0 is the Dirac delta measure and f is a nonnegative and nonnegative-definite tempered measure on $\mathbb{R}^d$. The case when $f=\delta_0$ refers to the space-time white noise. Note that the notation $f(x-y)$ is convenient when f has a density, and we use f itself to refer to its density. See remark 2.1 for the rigorous definition of the covariance structure and our conventions.

We note that f induces an inner product

$$\langle\varphi,\psi\rangle_H = \iint_{\mathbb{R}^{2d}} \varphi(y)\psi(z)f(y-z)\,\mathrm{d}y\mathrm{d}z \tag{1.3}$$

for $\varphi,\psi\in C_c^\infty(\mathbb{R}^d)$, i.e. smooth functions with compact supports in $\mathbb{R}^d$, and define the Hilbert space H to be the completion of $C_c^\infty(\mathbb{R}^d)$ under this inner product.

The solution to (1.1) is understood as an adapted, jointly measurable random field satisfying the following stochastic integral equation (see, e.g. definition 1.1 of [3] for the complete definition of the random field solution):

$$\begin{aligned} u(t,x) = (p_t * u_0)(x) &+ \int_0^t\int_{\mathbb{R}^d} p_{t-s}(x-y)\,b(u(s,y))\,\mathrm{d}y\mathrm{d}s \\ &+ \int_0^t\int_{\mathbb{R}^d} p_{t-s}(x-y)\,\sigma(u(s,y))\,W(\mathrm{d}s,\mathrm{d}y), \end{aligned} \tag{1.4}$$

where $p_t(x) = (2\pi t)^{-d/2}\exp\left(-|x|^2/2t\right)$ is the heat kernel and the stochastic integral is the *Walsh integral* [6, 29]. When both b and σ are globally Lipschitz continuous, there exists a unique global solution. Here, a global solution means that $u(t,x)$ is finite almost surely for all $x\in\mathbb{R}^d$ and $t>0$. However, when b and σ are only locally Lipschitz continuous, as is the case in this paper, this may no longer hold. To clarify this, we introduce some notation. For $p\geqslant 1$, define

$$V_p := L^p\left(\mathbb{R}^d\right)\cap L^\infty\left(\mathbb{R}^d\right) \quad \text{and} \quad \|\cdot\|_{V_p} := \max\left(\|\cdot\|_{L^p(\mathbb{R}^d)}, \|\cdot\|_{L^\infty(\mathbb{R}^d)}\right), \tag{1.5a}$$

where

$$\|f\|_{L^p(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d} |f(x)|^p\,\mathrm{d}x\right)^{1/p} \quad \text{and} \quad \|f\|_{L^\infty(\mathbb{R}^d)} := \operatorname{ess\,sup}_{x\in\mathbb{R}^d}|f(x)|. \tag{1.5b}$$

In this paper, a solution is said to be *global* if, almost surely, $\|u(t,\cdot)\|_{V_p}<\infty$ for all $t>0$. Conversely, if there exists a time $t>0$ such that $\|u(t,\cdot)\|_{V_p}=\infty$ with positive probability, the solution is said to *blow up in finite time*. By employing a truncation procedure, a local solution can always be uniquely constructed. This procedure will be utilized in the proof of the main theorem presented in this work.

For the one-dimensional deterministic ordinary differential equation

$$\frac{\mathrm{d}v}{\mathrm{d}t} = b(v(t)) \quad \text{with } b \geqslant 0 \text{ and } v(0) = c > 0, \tag{1.6}$$

the *Osgood condition* [20] characterizes finite-time explosion. Solutions explode in finite time if and only if the following *finite Osgood condition* holds:

$$T_{\text{blow-up}} := \int_c^\infty \frac{1}{b(u)} \mathrm{d}u < +\infty. \tag{1.7}$$

In fact, the above quantity is precisely the explosion time. This can be seen by rewriting (1.6) as follows $t = \int_0^t \frac{1}{b(u(s))} \mathrm{d}u(s)$. After a change of variable, which is one-to-one because $b(u) > 0$, we obtain

$$t = \int_{v(0)=c}^{v(t)} \frac{1}{b(s)} \mathrm{d}s.$$

This shows that (1.7) is both necessary and sufficient for blow-up of solutions to (1.6). The Osgood condition does not fully characterize finite-time explosion for deterministic partial differential equations as demonstrated by the famous example provided by Fujita [13].

Finite-time explosion for *stochastic partial differential equations* with superlinear b and σ have only recently gained some attention. The case of a bounded spatial domain has received more attention. Bonder and Groisman [9] demonstrated that if b satisfies (1.7), then a one-dimensional SHE with additive space-time white noise, on a bounded spatial domain, will always explode. Bonder and Groisman's result was generalized to higher spatial dimensions and more general stochastic noises by Foondun and Nualart [11]. Foondun and Nualart showed that if σ is bounded away from 0 and ∞ in the sense that $0 < c \leqslant \sigma(u) \leqslant C < \infty$, then analogous to deterministic ordinary differential equations, the Osgood condition (1.7) on b characterizes finite-time explosion for the SHE with additive noise on bounded domains in any spatial dimension. In other words, the SHE with bounded initial data will explode in finite time if and only if b satisfies (1.7).

Dalang, Khoshnevisan, and Zhang [8] first investigated the case where both σ and b can grow superlinearly, and they established that global solutions exist if b grows no faster than $u \log u$ and σ grow slower than $u(\log u)^{1/4}$. Salins [23] demonstrated that if b does not satisfy condition (1.7), i.e. b satisfies the following *infinite Osgood condition* holds:

$$\int_c^\infty \frac{1}{b(u)} \mathrm{d}u = +\infty \quad \text{for all } c > 0, \tag{1.8}$$

then to guarantee the existence of global solutions (i.e. solutions for all time), one can allow σ to grow superlinearly as long as it satisfies an appropriate Osgood-type assumption. Shang and Zhang [28] studied a superlinear stochastic heat Equation on a bounded domain driven by a Brownian motion (namely, space-independent white noise).

Finite-time explosion for the superlinear stochastic wave Equations has been investigated in [12] and [16], which proved sufficient conditions for finite-time explosion and those for global solutions, respectively. In both works, the compact support property, which is inherited from the fundamental solution of the wave equation, plays a crucial role.

The question of finite-time explosion for the stochastic heat Equation on unbounded spatial domains is more complicated because solutions to the SHE can be unbounded in space in the sense that $\mathbb{P}(\sup_x |u(t,x)| = +\infty) = 1$ for every $t > 0$. Shang and Zhang [27] showed that if b grows like $u\log(u)$ and if σ is bounded and Lipschitz, then there exist global solutions to the SHE on $\mathbb{R}$. Salins [22] established that under the non-explosive Osgood condition on b (1.8), which permits growth rates faster than $u\log u$, global solutions exist when $\sigma \equiv 1$. Conversely, Foondun, Khoshnevisan, and Nualart [10] proved that if b satisfies the explosive Osgood condition (1.7) and σ is bounded away from both 0 and ∞, then the solution explodes 'instantaneously and everywhere,' meaning that local solutions cannot exist for any positive amount of time.

Chen and Huang [4] investigated the existence of global solutions to the SHE defined on an unbounded spatial domain under assumptions that guarantee that spatial supremum $\sup_{x\in\mathbb{R}^d} u(t,x)$ remains finite. Specifically, they asusme that $b(0) = 0$, $\sigma(0) = 0$ and that the initial data $u_0 \in L^p(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ for some $p > 2$. We impose these same restrictions in the current paper. The current paper is a major improvement over the results of [4], examining scenarios where solutions remain spatially bounded, while allowing for more general assumptions that accommodate faster growth of the superlinear b and σ terms.

In our proof, we employ a stopping-time argument originally introduced by Salins [23]. However, while Salins' work focuses on equations defined on bounded domains, our study considers the case where the spatial domain is the unbounded space $\mathbb{R}^d$. To analyze the behavior of these systems on unbounded domains, we need our stopping times to be based on the V_p norm, defined in (1.5), which keeps track of both the spatial $L^\infty(\mathbb{R}^d)$ and $L^p(\mathbb{R}^d)$ norms. By using exponential moment estimates instead of L^p moment estimates, we can improve upon the results of [23] and allow extra growth in σ. This improvement can also be applied to the bounded domain setting.

For the stochastic noise, one commonly assumes the following *Sanz-Solé-Sarrà condition* [24, 25], which is sometimes referred to as the *strengthened Dalang's condition* in the literature:

$$\Upsilon_\alpha := (2\pi)^{-d} \int_{\mathbb{R}^d} \frac{\widehat{f}(\mathrm{d}\xi)}{(1+|\xi|^2)^{1-\alpha}} < \infty, \quad \text{for some } 0 < \alpha < 1, \tag{1.9}$$

where we use $\widehat{g}(\xi)$ to denote the Fourier transform of, e.g. Schwarz test function g, namely, $\widehat{g}(\xi) := \mathcal{F}g(\xi) = \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}x\cdot\xi} g(x)\,\mathrm{d}x$, and $\widehat{f}$ is in the generalized sense, i.e.

$$\int_{\mathbb{R}^d} \widehat{g}(x) f(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} g(\xi)\widehat{f}(\mathrm{d}\xi)$$

for all Schwarz functions g. Note that $\widehat{f}$ is a nonnegative tempered measure; see remark 2.1. When $\alpha = 0$, it reduces to the weaker *Dalang's condition* [7]:

$$\Upsilon(\beta) := (2\pi)^{-d} \int_{\mathbb{R}^d} \frac{\widehat{f}(\mathrm{d}\xi)}{\beta + |\xi|^2} < +\infty \quad \text{for some and hence for all } \beta > 0. \tag{1.10}$$

Note that condition (1.10) is a sufficient condition for the existence and uniqueness of the mild solution in $L^2(\Omega)$ when b and σ are globally Lipschitz continuous and the initial condition is bounded. It becomes a necessary and sufficient condition when σ is constant; see [7]. In order to get slightly stronger results, instead of condition (1.9), we make the following slightly weaker assumption on the noise (see lemma 2.2):

Assumption 1.1. There exists $\alpha \in (0,1]$ such that

$$\limsup_{s\downarrow 0} s^{1-\alpha} \int_{\mathbb{R}^d} \mathrm{e}^{-s|\xi|^2} \widehat{f}(\mathrm{d}\xi) < +\infty. \tag{1.11}$$

When the noise is one-dimensional space-time white noise, $f = \delta_0$, assumption 1.1 is satisfied with $\alpha = 1/2$ but the strengthened Dalang's condition (1.9) is satisfied only for $\alpha \in (0, 1/2)$.

Regarding the drift term b and the diffusion coefficient σ, we make the following Osgood-type assumptions following [23].

Assumption 1.2. Assume the following:

1. Both b and σ are locally Lipschitz continuous;
2. $b(0) = 0$ and $\sigma(0) = 0$;
3. There exists a positive, increasing function $h : [0,\infty) \to [0,\infty)$ such that:
   (a) (Superlinear growth) $\mathbb{R}_+ \ni u \to u^{-1}h(u)$ is non-decreasing; with $\lim_{u\to 0} u^{-1}h(u) \geqslant \exp(1/\alpha)$;
   (b) (Osgood-type condition of the infinite type) $\int_1^\infty \frac{1}{h(u)} \mathrm{d}u = +\infty$;
   (c) For all $u \in \mathbb{R}$, $|b(u)| \leqslant h(|u|)$;
   (d) For all $u \in \mathbb{R}$, it holds that

$$|\sigma(u)| \leqslant |u|^{1-\alpha/2} (h(|u|))^{\alpha/2} \left(\log\left(\frac{h(|u|)}{|u|}\right)\right)^{-1/2}; \tag{1.12}$$

where the constant α in parts (a) and (d) is given in assumption 1.1.

Remark 1.3. In the bounded domain setting [23], Salins assumed that $|\sigma(u)| \leqslant |u|^{1-\gamma}(h(|u|))^{\gamma}$ for some $\gamma < \alpha/2$ (where that paper uses the notation $1 - \eta = \alpha$). Our proof is based on exponential tail estimates; see lemma 2.6 below. The condition on σ, given by (1.12), allows for a faster growth rate for σ than that in [23]. The arguments based on the exponential tail estimates can also be applied in the finite domain setting.

Remark 1.4. The assumption $\lim_{u\to 0} \frac{h(u)}{u} \geqslant \exp(1/\alpha)$ is useful for our analysis, but it can be relaxed to $\lim_{u\to 0} \frac{h(u)}{u} > 0$. If there is a superlinear function $h : [0,\infty) \to [0,\infty)$ that satisfies Assumption 1.2(b)–(c) and $u \mapsto \frac{h(u)}{u}$ is increasing, then it is always possible to build a larger h that satisfies the bound $\lim_{u\to 0} \frac{h(u)}{u} \geqslant \exp(1/\alpha)$. Specifically, for large $C > 1$, $\tilde{h}(u) := Ch(u)$, will continue to dominate b and σ in the appropriate way, and $\lim_{u\to 0} \frac{\tilde{h}(u)}{u} = C \lim_{h\to 0} \frac{h(u)}{u}$.

For $p \geqslant 1$, recall that the space V_p is defined in (1.5). To guarantee that solutions to (1.1) remain bounded in space, we make the following assumption on the initial data.

Assumption 1.5. The initial data $u_0 \in V_p$ for some $p \geqslant 2$.

Remark 1.6. We observe that, $V_p \subset V_q$ when $q \geqslant p \geqslant 1$, and the assumption $p \geqslant 2$ will be made below to enable the use of the Burkholder–Davis–Gundy inequality. See also lemma 2.3 below.

The aim of this present paper is to prove the following theorem:

Theorem 1.7. *Suppose that the noise satisfies assumption 1.1 for some $\alpha \in (0,1]$, the initial condition satisfies assumption 1.5 for some $p \geqslant (2+d)/\alpha$, and that $b(\cdot)$ and $\sigma(\cdot)$ satisfy assumption 1.2. Then, we have the following:*

1. *There exists a unique mild solution $u(t,x)$ to (1.1) for all $(t,x) \in (0,+\infty) \times \mathbb{R}^d$.*
2. *Moreover, if f satisfies the strengthened Dalang's condition (1.9) for some $\alpha \in (0,1]$, then the solution $u(t,x)$ is Hölder continuous: $u \in C^{\alpha/2-,\alpha-}\left((0,T] \times \mathbb{R}^d\right)$ a.s. where $C^{\alpha_1-,\alpha_2-}(D)$ denotes the Hölder continuous function on the space-time domain D with exponents $\alpha_1 - \epsilon$ and $\alpha_2 - \epsilon$ in time and space, respectively, for every small $\epsilon > 0$.*

When both b and σ are globally Lipschitz continuous, the conditions that $b(0) = 0, \sigma(0) = 0$ and $u_0 \in V_p$ guarantee that the solution to (1.1) remains in V_p almost surely. This follows from the results in [7]. This allows us to perform a localization procedure on the solution. We can prove that the solution cannot explode in finite time by proving that the sequence of hitting times

$$\tau_n := \inf\left\{t > 0 : \|u(t,\cdot)\|_{V_p} \geqslant 3^n\right\} \tag{1.13}$$

have the property that

$$\mathbb{P}\left(\sup_n \tau_n = \infty\right) = 1. \tag{1.14}$$

Our main result—theorem 1.7—provides the optimal condition on the drift term b, which can be seen from the following theorem:

Theorem 1.8. *Let $b : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz continuous, nondecreasing and convex function that vanishes at zero ($b(0) = 0$). Suppose that $\sigma(\cdot)$ is globally Lipschitz continuous. Moreover, we assume that σ vanishes at zero and is bounded, namely, $\sigma(0) = 0$ and $\sup_{u\in\mathbb{R}} |\sigma(u)| \leqslant K$. Under the noise assumption—assumption 1.1, if b is non-negative and increasing, and satisfies the finite Osgood condition (1.7), then for any $p \geqslant 2$, there exists some non-negative initial condition $u_0(\cdot) \in V_p$ to (1.1) such that solutions to (1.1) explode in finite time with positive probability in the sense that*

$$\mathbb{P}\left(\sup_n \tau_n < \infty\right) > 0.$$

where the hitting times τ_n are defined in (1.13).

Remark 1.9. Theorem 1.8 might hold true in certain cases without the condition that σ is bounded. However, having this restriction simplifies its proof and is sufficient to demonstrate the optimality of our condition on $b(\cdot)$. Determining the optimal condition on the diffusion coefficient σ is still an open problem, which will be left for further study. Our setting of theorem 1.8 is different than the settings investigated by Bonder and Groisman [9], Foondun and Nualart [11], and Foondun, Khoshnevisan, and Nualart [10] because we require that $\sigma(0) = 0$. In all of these other works, the authors assumed that σ was uniformly bounded away from 0, and they can show that explosion occurs with probability one. Because σ is degenerate near zero, we do not expect that explosion should occur with probability one. Instead we show that there exist initial data from which explosion is possible with positive probability.

Remark 1.10. Our assumptions of theorem 1.8 allow σ to be identically zero, so that (1.1) becomes deterministic. If we choose $b(u) = u^p$ with $p \geqslant 1 + 2/d$, then Fujita showed that there exist initial data for which solutions will not explode [13]. See also section 20 of [21] for more information on this and other blow-up results for non-linear PDEs. In Fujita's example, there also exist initial data that cause explosion. theorem 1.8 for the stochastic setting claims the existence of an initial profile from which explosion happens with positive probability. In our proof of positive probability explosion we take the initial data to be a large constant multiplied by a heat kernel. It will not be possible to fully describe the sets of initial data from which explosion is possible without additional assumptions on the local behavior of f and σ near the degenerate points of σ. We leave this characterization for future work.

In order to introduce some examples, we use the following notation for the repeated logarithm function: $\log_1(u) := \log(u)$ and for $k \geqslant 2$, $\log_k(u) := \log\left(\log_{k-1} u\right)$.

Example 1.11. Examples of superlinear Osgood-type h functions include $h(u) = u\log(u)$, $h(u) = u\log(u)\log_2(u)$, $h(u) = u\log(u)\log_2(u)\log_3(u)$, and so on, as listed in the first column of the tables in table Tb:Bdd(a). In particular, we have two special cases:

- Since $\int_{\mathbb{R}^d} e^{-s|\xi|^2} d\xi = (2\pi)^d \int_{\mathbb{R}^d} p_s^2(y) dy = (\pi/s)^{d/2}$, we see that when $d = 1$, assumption 1.1 is satisfied with $\alpha = 1/2$. Therefore, in case of the spatial dimension one and $\dot{W}$ is space-time white noise, we can take $\alpha = 1/2$ in (1.12). In particular, we have the concrete examples listed in table 1(b).
- If $f(0) < \infty$, or equivalently, $\int_{\mathbb{R}^d} \widehat{f}(\xi) d\xi < +\infty$, then $\alpha = 1$. In this case, the growth rates for σ and some typical h are listed in Table 1.

In general, for $\alpha \in (0, 1]$ given in assumption 1.1, the function h with repeated logarithms and the corresponding growth bound for σ are listed in table 1(c).

In [4], it is demonstrated that there exists a global solution when $b(u)$ grows as fast as $u\log(u)$ and $\sigma(u)$ grows as fast as $u(\log u)^{\alpha/2}$. This paper utilizes a method motivated by [8], which studied the setting of a bounded one-dimensional domain. Our new result is stronger than the previous ones because we can allow b to grow faster than $u\log u$ as long as b is dominated by an Osgood-type h. We believe that the methods used in [4] cannot easily be extended to deal with b growing faster than $u\log u$.

Interestingly, there remains a specific scenario where the method used in [4] yields a stronger result than theorem 1.7 of the current paper. Specifically, when b grows no faster than $u\log u$ and σ grows as $u(\log u)^{\alpha/2}$, the main result of [4] can be used to prove that solutions never explode. Based on our current strategy of using stopping time arguments and exponential estimates, we can let σ grow like (see table 1(c) with $K = 1$)

$$u\left[\log(u)\right]^{\alpha/2}\left[\log_2(u)\right]^{-1/2}.$$

However, if $\alpha < 1$, then we cannot achieve $u(\log u)^{\alpha/2}$ growth for σ. This suggests that both approaches to this problem are useful. In the case where $\alpha = 1$, our result allows for b and σ that both grow faster than those in [4].

This paper is organized as follows. In section 2, we introduce some notation and establish some technical results. Our main result—theorem 1.7—is proved in section 3. theorem 1.8 is proved in section 4. Finally, we recall the Burkholder–Davis–Gundy inequality for the martingale in Banach space in the appendix.

Table 1. The growth rate of σ is listed in the second column. The h's in the first column are typical examples that satisfy the Osgood condition (see part 3-(b) of assumption 1.2). The growth rate of σ is listed in the second column.

(a) The case when the noise has a bounded correlation function, i.e. $f(0) < \infty$: $\alpha = 1$.	
$h(u) \sim$	$\sigma(u)$ can grow as fast as
$u\log(u)$	$u(\log u)^{1/4}(\log_2(u))^{-1/2}$
$u\log(u)\log_2(u)$	$u(\log u)^{1/4}(\log_2(u))^{-1/4}$
$u\log(u)\log_2(u)\log_3(u)$	$u(\log u)^{1/4}(\log_2(u))^{-1/4}(\log_3(u))^{1/4}$
(b) The case when $d = 1$ and the noise is the space-time white noise: $\alpha = 1/2$.	
$h(u) \sim$	$\sigma(u)$ can grow as fast as
$u\log(u)$	$u(\log u)^{1/2}(\log_2(u))^{-1/2}$
$u\log(u)\log_2(u)$	$u(\log u)^{1/2}$
$u\log(u)\log_2(u)\log_3(u)$	$u(\log u)^{1/2}(\log_3(u))^{1/2}$
(c) The general case: $\alpha \in (0,1]$ with $K \geqslant 1$.	
$h(u) \sim$	$\sigma(u)$ can grow as fast as
$u\prod_{k=1}^{K}\log_k(u)$	$u(\log_2(u))^{-1/2}\prod_{k=1}^{K}(\log_k(u))^{\alpha/2}$

2. Some preliminaries

In the following, $\|\cdot\|_{L^p}$ refers to $\|\cdot\|_{L^p(\mathbb{R}^d)}$ with $p \in [1,\infty]$ and $\|X\|_p := \mathbb{E}\left(|X|^p\right)^{1/p}$.

Remark 2.1. Let Φ and Ψ be Schwarz test functions on $\mathbb{R}^d$, and let ϕ and $\psi \in C_c^\infty(\mathbb{R})$, namely, smooth and compactly supported test functions on $\mathbb{R}$. The covariance of the noise is given by

$$\begin{aligned}&\mathbb{E}\left[\int_0^\infty\int_{\mathbb{R}^d}\phi(s)\Phi(x)W(\mathrm{d}s,\mathrm{d}x)\int_0^\infty\int_{\mathbb{R}^d}\psi(s)\Psi(y)W(\mathrm{d}s,\mathrm{d}y)\right]\\&=\left(\int_0^\infty\phi(s)\psi(s)\,\mathrm{d}s\right)\times\langle\Phi,\Psi*f\rangle_{L^2(\mathbb{R}^d)}.\end{aligned} \tag{2.1}$$

where '$*$' denotes the convolution in the spatial variable and $\langle g,h\rangle_{L^2(\mathbb{R}^d)} := \int_{\mathbb{R}^d} g(x)h(x)\mathrm{d}x$. By the Bochner–Schwarz theorem, the covariance operator is well-defined provided that f is a non negative-definite tempered measure on $\mathbb{R}^d$. In that case, Plancherel's theorem implies that the Fourier transform of f is a nonnegative tempered measure on $\mathbb{R}^d$ and the $L^2(\mathbb{R}^d)$ inner product in (2.1) can be written as

$$\langle\Phi,\Psi*f\rangle_{L^2(\mathbb{R}^d)} = (2\pi)^{-d}\int_{\mathbb{R}^d}\widehat{\Phi}(\xi)\,\overline{\widehat{\Psi}(\xi)}\widehat{f}(\mathrm{d}\xi)\,, \tag{2.2}$$

where the bar denotes complex conjugation. When f has a density, the $L^2(\mathbb{R}^d)$ inner product in (2.1) is more conveniently written as

$$\langle\Phi,\Psi*f\rangle_{L^2(\mathbb{R}^d)} = \iint_{\mathbb{R}^{2d}}\Phi(x)f(x-y)\Psi(y)\,\mathrm{d}x\mathrm{d}y. \tag{2.3}$$

Similarly, when $\widehat{f}$ has a density, $\widehat{f}(\mathrm{d}\xi)$ in (2.2) is often written as $\widehat{f}(\xi)\mathrm{d}\xi$. Throughout this paper, for readability, we use the notation $\widehat{f}(\xi)\mathrm{d}\xi$ instead of $\widehat{f}(\mathrm{d}\xi)$. In other words, $\widehat{f}(\xi)\mathrm{d}\xi$ should be understood as $\widehat{f}(\mathrm{d}\xi)$. Similarly, the convolution in the form of the righ-hand side of (2.3) should be understood as the left-hand side of the same equation.

Lemma 2.2. *If f satisfies the strengthened Dalang's condition (1.9) with some $\alpha \in (0,1)$, then it satisfies assumption 1.1 with the same α.*

Proof. Notice that $\sup_{s>0} s^{1-\alpha}\mathrm{e}^{-s|\xi|2} = C_\alpha|\xi|^{-2(1-\alpha)}$ with $C_\alpha = (1-\alpha)^{1-\alpha}\mathrm{e}^{-(1-\alpha)}$. Hence,

$$\int_{\mathbb{R}^d} s^{1-\alpha}\mathrm{e}^{-s|\xi|^2}\widehat{f}(\mathrm{d}\xi) = \left(\int_{|\xi|\leqslant 1} + \int_{|\xi|>1}\right) s^{1-\alpha}\mathrm{e}^{-s|\xi|^2}\widehat{f}(\mathrm{d}\xi)$$
$$\leqslant s^{1-\alpha}\widehat{f}(B_1) + C_\alpha\int_{|\xi|>1}|\xi|^{-2(1-\alpha)}\widehat{f}(\mathrm{d}\xi) \leqslant s^{1-\alpha}\widehat{f}(B_1) + CC_\alpha\Upsilon_\alpha,$$

for some constant $C>0$, where B_1 is the unit ball in $\mathbb{R}^d$. Since $\widehat{f}$ is a tempered measure; see remark 2.1, $\widehat{f}(B_1) < \infty$. Taking $\limsup_{s\downarrow 0}$ on both sides of the above inequality proves the lemma. □

We will often use the following simple but useful property of V_p, which shows that the space V_p is monotone in p.

Lemma 2.3. *If $1 \leqslant p \leqslant r < \infty$, then $V_p \subseteq V_r \subseteq L^r(\mathbb{R}^d)$ and for any $v \in V_p$,*

$$\|v\|_{L^r} \leqslant \|v\|_{V_r} \leqslant \|v\|_{V_p}. \tag{2.4}$$

Proof. The proof is straightforward. Let $r \in [p,+\infty)$. Then notice that $\|v\|_{L^r}^r = \int_{\mathbb{R}^d}|v(x)|^r\mathrm{d}x$, which is less than $\left(\int_{\mathbb{R}^d}|v(x)|^p\mathrm{d}x\right)\sup_{x\in\mathbb{R}^d}|v(x)|^{r-p} = \|v\|_{L^p}^p\|v\|_{L^\infty}^{r-p} \leqslant \|v\|_{V_p}^r$. □

Lemma 2.4. *Assume that b and σ in (1.1) satisfy assumption 1.2. For any $p \geqslant 1$, if $v \in V_p$, then the compositions $f(v) \in V_p$ and $\sigma(v) \in V_p$. Moreover,*

$$\|b(v)\|_{V_p} \leqslant h\left(\|v\|_{V_p}\right) \quad \text{and} \quad \|\sigma(v)\|_{V_p} \leqslant \|v\|_{V_p}^{1-\alpha/2}h\left(\|v\|_{V_p}\right)^{\alpha/2}\left(\log\left(\frac{h\left(\|v\|_{V_p}\right)}{\|v\|_{V_p}}\right)\right)^{-1/2}. \tag{2.5}$$

Proof. We first consider the case when $p=\infty$. We claim that

$$\|b(v)\|_{L^\infty} \leqslant h(\|v\|_{L^\infty}) \quad \text{and}$$
$$\|\sigma(v)\|_{L^\infty} \leqslant \|v\|_{L^\infty}^{1-\alpha/2}h(\|v\|_{L^\infty})^{\alpha/2}\left(\log\left(\frac{h(\|v\|_{L^\infty})}{\|v\|_{L^\infty}}\right)\right)^{-1/2}. \tag{2.6}$$

Since $|b(v(x))| \leqslant |h(v(x))|$ for all $x \geqslant 0$, the first inequality in (2.6) is proved by taking supremum on both sides of this inequality proves. As for the second inequality in (2.6), denote $g(x) := \frac{h(x)}{x}$. Part 3 (a) of assumption 1.2 says that $g(x)$ is nondecreasing for $x \geqslant 0$ and $g(x) \geqslant \exp(1/\alpha)$. Let

$$G(u) := \frac{u^{\alpha/2}}{\sqrt{\log u}} \quad \text{and} \quad F(x) := G(g(x)) = \frac{g(x)^{\alpha/2}}{\sqrt{\log(g(x))}}.$$

$F(x)$, as a composition of the two nondecreasing functions $g(x)$ and $G(u)$, is again nondecreasing for all $x \geqslant 0$. This proves the second inequality of (2.6). Note that the nondecreasing property of G is due to the following elementary calculation:

$$G'(u) = \frac{u^{-1+\alpha/2}}{2\alpha\,[\log(u)]^{3/2}}\left(\log(u) - \frac{1}{\alpha}\right), \quad \text{provided that } u \geqslant \exp(1/\alpha).$$

The interesting part of the proof is showing that the $L^p(\mathbb{R}^d)$ norm is bounded. To this end, observe that

$$\|b(v)\|_{L^p}^p = \int_{\mathbb{R}^d} |b(v(x))|^p \mathrm{d}x \leqslant \int_{\mathbb{R}^d} |v(x)|^p \left(\frac{h(|v(x)|)}{|v(x)|}\right)^p \mathrm{d}x \leqslant \|v\|_{L^p}^p \sup_{x\in\mathbb{R}^d}\left(\frac{h(|v(x)|)}{|v(x)|}\right)^p,$$

where the first inequality is obtained by using the bound on b given in assumption 1.2. From the assumption that $v \mapsto \frac{h(v)}{v}$ is increasing, the above display is bounded by

$$\leqslant \|v\|_{L^p}^p \left(\frac{h(\|v\|_{L^\infty})}{\|v\|_{L^\infty}}\right)^p \leqslant \|v\|_{V_p}^p \left(\frac{h\left(\|v\|_{V_p}\right)}{\|v\|_{V_p}}\right)^p. \tag{2.7}$$

Therefore, we can conclude that $\|b(v)\|_{L^p} \leqslant h(\|v\|_{V_p})$. Combining this with the first relation in (2.6) proves the first inequality in (2.5). The argument for the case of σ is similar and one needs to use assumption 1.2 and the following inequality:

$$\|\sigma(v)\|_{L^p}^p \leqslant \int_{\mathbb{R}^d} |v(x)|^p \left(\frac{h(|v(x)|)}{|v(x)|}\right)^{\alpha p/2}\left(\log\left(\frac{h(|v(x)|)}{|v(x)|}\right)\right)^{-p/2} \mathrm{d}x. \tag{2.8}$$

The rest of the arguments are the same as those for b. □

Lemma 2.5. *(1) Let $p \geqslant 1$. If, for some deterministic constants $T, M > 0$, the random field $\Psi : \Omega \times \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ satisfies $\sup_{t\in[0,T]} \|\Psi(t,\cdot)\|_{V_p} \leqslant M < \infty$, almost surely, then*

$$\left\| \int_0^t \int_{\mathbb{R}^d} p_{t-s}(\cdot - y)\,\Psi(s,y)\,\mathrm{d}y\mathrm{d}s \right\|_{V_p} \leqslant tM, \quad \text{almost surely.} \tag{2.9}$$

(2) Let $\Phi : \Omega \times \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ be an adapted and jointly measurable random field. Suppose that $p \geqslant 2$, and for some $\alpha \in (0,1]$, assumption 1.1 is satisfied. If for any $T > 0$, there exists a constant $M > 0$ such that $\sup_{t\in[0,T]} \|\Phi(t,\cdot)\|_{V_p} \leqslant M$ a.s. then for all $k > \max\{(2+d)/\alpha, p\}$, there exists a constant $C > 0$ depending only on (d,α,p), but not on (T,M,k), such that

$$\mathbb{E}\left(\sup_{t\in[0,T]} \left\| \int_0^t \int_{\mathbb{R}^d} p_{t-s}(\cdot - y)\,\Phi(s,y)\,W(\mathrm{d}s,\mathrm{d}y) \right\|_{V_p}^k\right) \leqslant C^k k^{k/2} T^{(\alpha k - d)/2}\left(1 + T^{d/2}\right) M^k. \tag{2.10}$$

Proof. Part (1) is obtained by an application of the Minkowski inequality. Part (2) will be proved in three steps. Denote the stochastic integral by $Z(t,x)$. By the factorization lemma (see [5, section 5.3.1]), for $\beta \in (0,\alpha)$,

$$Z(t,x) = \frac{\sin(\beta\pi/2)}{\pi}\int_0^t\int_{\mathbb{R}^d}(t-r)^{-1+\beta/2}\,Y(r,z)\,p_{t-r}(x-z)\,\mathrm{d}z\mathrm{d}r, \tag{2.11}$$

$$Y(r,z) = \int_0^r\int_{\mathbb{R}^d}(r-s)^{-\beta/2}\,p_{r-s}(z-y)\,\Phi(s,y)\,W(\mathrm{d}s,\mathrm{d}y). \tag{2.12}$$

In the following, we use C to denote a generic constant that does not depend on T, M, k and p, whose value may change at each appearance.

Step I. In this step, we will show that for all $k > \max((2+d)/\beta, p)$, $p \geqslant 2$, and $T > 0$, it holds that

$$\mathbb{E}\left(\sup_{(t,x)\in[0,T]\times\mathbb{R}^d}|Z(t,x)|^k\right) \leqslant C^k k^{k/2} M^k T^{(\alpha k-d)/2}. \tag{2.13}$$

By Hölder inequality with exponents k and $\frac{k}{k-1}$, for arbitrary $t > 0$ and $x \in \mathbb{R}^d$,

$$|Z(t,x)|^k \leqslant C^k\left(\int_0^t\int_{\mathbb{R}^d}(t-s)^{\frac{(\beta/2-1)k}{k-1}}\,|p_{t-s}(x-y)|^{\frac{k}{k-1}}\mathrm{d}y\mathrm{d}s\right)^{k-1}\int_0^t\int_{\mathbb{R}^d}|Y(s,y)|^k\mathrm{d}y\mathrm{d}s.$$

We use the fact that

$$|p_{t-s}(x-y)|^{\frac{k}{k-1}} = p_{t-s}(x-y)\,|p_{t-s}(x-y)|^{\frac{1}{k-1}} \leqslant C(t-s)^{-\frac{d}{2(k-1)}}\,p_{t-s}(x-y),$$

along with the fact that $p_{t-s}(\cdot)$ is a density to bound the above expression by

$$|Z(t,x)|^k \leqslant \left(\int_0^t (t-s)^{\frac{(\beta/2-1)k}{k-1}-\frac{d}{2(k-1)}}\,\mathrm{d}s\right)^{k-1}\int_0^t\int_{\mathbb{R}^d}|Y(s,y)|^k\mathrm{d}y\mathrm{d}s.$$

Notice that

$$\frac{(\beta/2-1)k}{k-1} - \frac{d}{2(k-1)} > -1 \quad\Longleftrightarrow\quad k > \frac{2+d}{\beta}.$$

Hence, if we choose $k > (2+d)/\beta$, then the first integral is finite and

$$|Z(t,x)|^k \leqslant C t^{\frac{\beta k}{2}-\frac{d}{2}-1}\int_0^t\int_{\mathbb{R}^d}|Y(s,y)|^k\mathrm{d}y\mathrm{d}s. \tag{2.14}$$

Hence,

$$\mathbb{E}\left(\sup_{(t,x)\in[0,T]\times\mathbb{R}^d}|Z(t,x)|^k\right) \leqslant C T^{\frac{\beta k}{2}-\frac{d}{2}-1}\int_0^T\mathrm{d}r\int_{\mathbb{R}^d}\mathrm{d}z\,\mathbb{E}\left(|Y(r,z)|^k\right).$$

Now, for $Y(r,z)$, by the Burkholder–Davis–Gundy inequality, we see that

$$\|Y(r,z)\|_k^2 \leqslant 8k\int_0^r\mathrm{d}s\iint_{\mathbb{R}^{2d}}\mathrm{d}y\mathrm{d}y'\;(r-s)^{-\beta}f(y-y')\;p_{r-s}(y)\,\|\Phi(s,z-y)\|_k \times p_{r-s}(y')\,\|\Phi(s,z-y')\|_k. \tag{2.15}$$

Then by the Minkowski inequality, we see that

$$\begin{aligned}\int_{\mathbb{R}^d}\mathbb{E}\left(|Y(r,z)|^k\right)\mathrm{d}z \leqslant & C^k k^{k/2}\int_{\mathbb{R}^d}\left(\int_0^r \mathrm{d}s\iint_{\mathbb{R}^{2d}}\mathrm{d}y\mathrm{d}y'\ (r-s)^{-\beta}f(y-y')\right.\\ &\left.\times p_{r-s}(y)p_{r-s}(y')\,\|\Phi(s,z-y)\|_k\,\|\Phi(s,z-y')\|_k\right)^{k/2}\mathrm{d}z\\ \leqslant & C^k k^{k/2}\left(\int_0^r \mathrm{d}s\iint_{\mathbb{R}^{2d}}\mathrm{d}y\mathrm{d}y'\ (r-s)^{-\beta}f(y-y')\right.\\ &\left.\times p_{r-s}(y)p_{r-s}(y')\left\|\|\Phi(s,\cdot-y)\|_k\|\Phi(s,\cdot-y')\|_k\right\|_{L^{k/2}}\right)^{k/2}\\ \leqslant & C^k k^{k/2}\left(\int_0^r \mathrm{d}s\iint_{\mathbb{R}^{2d}}\mathrm{d}y\mathrm{d}y'\ (r-s)^{-\beta}f(y-y')\right.\\ &\left.\times p_{r-s}(y)p_{r-s}(y')\left\|\|\Phi(s,\cdot)\|_k\right\|_{L^k}^2\right)^{k/2},\end{aligned}$$

where in the last inequality we applied the Hölder inequality. If $k\geqslant p$, then we can use the Fubini theorem and the assumption on $\Phi(\cdot,\circ)$ to obtain that

$$\begin{aligned}\left\|\|\Phi(s,\cdot)\|_k\right\|_{L^k}^k &= \mathbb{E}\left(\int_{\mathbb{R}^d}|\Phi(s,z)|^k\mathrm{d}z\right)\\ &\leqslant \mathbb{E}\left(\left(\int_{\mathbb{R}^d}|\Phi(s,z)|^p\mathrm{d}z\right)\|\Phi(s)\|_{L^\infty}^{k-p}\right)\\ &\leqslant \mathbb{E}\left(\sup_{s\in[0,T]}\|\Phi(s,\cdot)\|_{V_p}^k\right)\leqslant M^k,\end{aligned}$$

for all $s\in[0,T]$, Hence,

$$\int_{\mathbb{R}^d}\mathbb{E}\left(|Y(r,z)|^k\right)\mathrm{d}z \leqslant C^k k^{k/2}M^k\left(\int_0^r \mathrm{d}s\iint_{\mathbb{R}^{2d}}\mathrm{d}y\mathrm{d}y'\ (r-s)^{-\beta}f(y-y')p_{r-s}(y)p_{r-s}(y')\right)^{k/2}.$$

By the Plancherel theorem, we see that

$$\begin{aligned}&\int_0^r \mathrm{d}s\iint_{\mathbb{R}^{2d}}\mathrm{d}y\mathrm{d}y'\ (r-s)^{-\beta}f(y-y')p_{r-s}(y)p_{r-s}(y')\\ &\quad=(2\pi)^{-d}\int_0^r\mathrm{d}s\ (r-s)^{-\beta}\int_{\mathbb{R}^d}\mathrm{e}^{-(r-s)|\xi|^2}\widehat{f}(\xi)\,\mathrm{d}\xi\\ &\quad=(2\pi)^{-d}\int_0^r\mathrm{d}s\,s^{-\beta}\int_{\mathbb{R}^d}\mathrm{e}^{-s|\xi|^2}\widehat{f}(\xi)\,\mathrm{d}\xi.\end{aligned}$$

By assumption 1.1 and the fact that the function $s\to\int_{\mathbb{R}^d}\mathrm{e}^{-s|\xi|^2}\widehat{f}(\xi)\mathrm{d}\xi$ is non-increasing, we see that for some universal constant $C>0$,

$$\int_{\mathbb{R}^d}\mathrm{e}^{-s|\xi|^2}\widehat{f}(\xi)\,\mathrm{d}\xi\leqslant Cs^{-(1-\alpha)},\quad \text{for all } s>0.$$

Hence,

$$\int_0^r \mathrm{d}s\iint_{\mathbb{R}^{2d}}\mathrm{d}y\mathrm{d}y'\ (r-s)^{-\beta}f(y-y')p_{r-s}(y)p_{r-s}(y')\leqslant C\int_0^r s^{-\beta+\alpha-1}\mathrm{d}s = Cr^{\alpha-\beta}. \tag{2.16}$$

Thus, we have that

$$\int_{\mathbb{R}^d} \mathbb{E}\left(|Y(r,z)|^k\right) \mathrm{d}z \leqslant C^k k^{k/2} M^k r^{(\alpha-\beta)k/2}. \tag{2.17}$$

Finally, if $k > \max((2+d)/\beta, p)$, by putting (2.17) back into (2.14), we prove the claim in (2.13).

Step II. In this step, we will show that for all $k > \max((2+d)/\beta, 2)$, $p \geqslant 2$, and $T > 0$, it holds that

$$\mathbb{E}\left(\sup_{t\in[0,T]} \|Z(t,\cdot)\|_{L^p}^k\right) \leqslant C^k k^{k/2} M^k T^{\alpha k/2}. \tag{2.18}$$

By the Minkowski inequality and the Hölder inequality, we see that for $t \in [0,T]$,

$$\begin{aligned}
\|Z(t,\cdot)\|_{L^p} &\leqslant \int_0^t \mathrm{d}r\, (t-r)^{-1+\beta/2} \int_{\mathbb{R}^d} \mathrm{d}z\, p_{t-r}(z) \, \|Y(r,\cdot - z)\|_{L^p} \\
&= \int_0^t (t-r)^{-1+\beta/2} \|Y(r,\cdot)\|_{L^p} \mathrm{d}r \\
&\leqslant \left(\int_0^t (t-r)^{\frac{k}{k-1}(-1+\beta/2)} \mathrm{d}r\right)^{\frac{k-1}{k}} \left(\int_0^t \|Y(r,\cdot)\|_{L^p}^k \mathrm{d}r\right)^{1/k}.
\end{aligned}$$

If $k > (2+d)/\beta$, the above dr–integral is finite and hence,

$$\mathbb{E}\left(\sup_{t\in[0,T]} \|Z(t,\cdot)\|_{L^p}^k\right) \leqslant C^k T^{-1+\beta k/2} \int_0^T \mathbb{E}\left(\|Y(r,\cdot)\|_{L^p}^k\right) \mathrm{d}r. \tag{2.19}$$

To estimate $\mathbb{E}\left(\|Y(r,\cdot)\|_{L^p}^k\right)$, if we assume that $k,p \geqslant 2$, then we can apply the BDG inequality in lemma A.5 to get

$$\begin{aligned}
&\mathbb{E}\left(\|Y(r,\cdot)\|_{L^p}^k\right) \\
&\leqslant C^k k^{\frac{k}{2}} \mathbb{E}\left[\left(\int_0^t \left[\int_{\mathbb{R}^d} \left(\iint_{\mathbb{R}^{2d}} (t-s)^{-\beta} p_{t-s}(x-y) p_{t-s}(x-y') \Phi(s,y) \Phi(s,y') f(y-y') \mathrm{d}y \mathrm{d}y'\right)^{\frac{p}{2}} \mathrm{d}x\right]^{\frac{2}{p}} \mathrm{d}s\right)^{k/2}\right] \\
&= C^k k^{\frac{k}{2}} \mathbb{E}\left[\left(\int_0^t (t-s)^{-\beta} \left[\int_{\mathbb{R}^d} \left(\iint_{\mathbb{R}^{2d}} p_{t-s}(y) p_{t-s}(y') \Phi(s,x-y) \Phi(s,x-y') f(y-y') \mathrm{d}y \mathrm{d}y'\right)^{\frac{p}{2}} \mathrm{d}x\right]^{\frac{2}{p}} \mathrm{d}s\right)^{k/2}\right] \\
&\leqslant C^k k^{\frac{k}{2}} \mathbb{E}\left[\left(\int_0^t \iint_{\mathbb{R}^{2d}} (t-s)^{-\beta} p_{t-s}(y) p_{t-s}(y') \|\Phi(s,\cdot - y) \Phi(s,\cdot - y')\|_{L^{p/2}} f(y-y') \mathrm{d}y \mathrm{d}y' \mathrm{d}s\right)^{k/2}\right] \\
&\leqslant C^k k^{\frac{k}{2}} \mathbb{E}\left[\left(\int_0^t \iint_{\mathbb{R}^{2d}} (t-s)^{-\beta} p_{t-s}(y) p_{t-s}(y') \|\Phi(s,\cdot)\|_{L^p}^2 f(y-y') \mathrm{d}y \mathrm{d}y' \mathrm{d}s\right)^{k/2}\right].
\end{aligned}$$

Then based on the assumption that $\|\Phi(s,\cdot)\|_{L^p} \leqslant M$ a.s. and thanks to (2.16), we see that

$$\mathbb{E}\left(\|Y(r,\cdot)\|_{L^p}^k\right) \leqslant C^k k^{k/2} M^k r^{(\alpha-\beta)k/2}.$$

Combining the above estimate with (2.19) proves (2.18).

Step III. Finally, combining the results from the previous two steps shows that if $k > \max((2+d)/\beta, p)$ and $p \geqslant 2$, then for all $T > 0$,

$$\begin{aligned}\mathbb{E}\left(\sup_{t\in[0,T]}\|Z(t,\cdot)\|_{V_p}^k\right) &\leqslant \mathbb{E}\left(\sup_{t\in[0,T]}\left(\|Z(t,\cdot)\|_{L^p}^k + \|Z(t,\cdot)\|_{L^\infty}^k\right)\right)\\ &\leqslant C^k\mathbb{E}\left(\sup_{t\in[0,T]}\|Z(t,\cdot)\|_{L^p}^k\right) + C^k\mathbb{E}\left(\sup_{t\in[0,T]}\|Z(t,\cdot)\|_{L^\infty}^k\right)\\ &\leqslant C^k k^{k/2} M^k\left(T^{(\alpha k-d)/2} + T^{\alpha k/2}\right).\end{aligned}$$

This completes the proof of lemma 2.5. □

We next give the exponential estimates for the stochastic integral in the previous lemma, which will be used in the proof of our main theorem. The space-time white noise case has been considered by Athreya, *et al* [1]; see also [2, 15, 17].

Lemma 2.6 (exponential estimates). *Assume that $\Phi : \Omega \times \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ be an adapted and jointly measurable random field and $p \geqslant 2$. If assumption 1.1 holds for some $\alpha \in (0,1]$, and if for any $T \geqslant 0$, there exists a constant $M = M(T,p) \geqslant 0$ such that*

$$\sup_{t\in[0,T]}\|\Phi(t,\cdot)\|_{V_p} \leqslant M, \quad \text{a.s.},$$

then, there exists a constant $C > 0$ independent of M and T such that for any $\delta > 0$,

$$\mathbb{P}\left(\sup_{t\in[0,T]}\left\|\int_0^t\int_{\mathbb{R}^d} G(t-s,x-y)\,\Phi(s,y)\,W(\mathrm{d}s,\mathrm{d}y)\right\|_{V_p} > \delta\right) \leqslant C\left(1+T^{-d/2}\right)\mathrm{e}^{-C\delta^2 M^{-2}T^{-\alpha}}. \tag{2.20}$$

Proof. Denote $Z(t,x) := \int_0^t\int_{\mathbb{R}^d} G(t-s,x-y)\Phi(s,y)W(\mathrm{d}s,\mathrm{d}y)$. Fix an arbitrary $\lambda > 0$. From Taylor series, we see that

$$\mathbb{E}\left[\exp\left(\lambda\sup_{t\in[0,T]}\|Z(t,\cdot)\|_{V_p}\right)\right] = \sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\mathbb{E}\left[\sup_{t\in[0,T]}\|Z(t,\cdot)\|_{V_p}^k\right].$$

We claim that

$$\mathbb{E}\left[\exp\left(\lambda\sup_{t\in[0,T]}\|Z(t,\cdot)\|_{V_p}\right)\right] \leqslant \left(1+T^{d/2}\right)T^{-d/2}\sum_{k=0}^{\infty}\frac{\lambda^k C^k M^k T^{k\alpha/2} k^{k/2}}{k!}. \tag{2.21}$$

Indeed, the above inequality (2.21) follows from the moment estimates in part (2) of lemma 2.5 for $k > \max(p,(d+2)/\alpha)$. If $k \leqslant \max(p,(d+2)/\alpha)$, one can use the Jensen inequality and then pick the leading constant big enough. This proves the claim in (2.21).

In order to transform the summation in (2.21) into an exponential form, we apply Stirling's approximation $k! \sim (k/e)^k\sqrt{2\pi k}$ to see that

$$\frac{k^{k/2}}{k!} \times \Gamma(k/2+1) \sim \frac{1}{\sqrt{2}}\left(\sqrt{e/2}\right)^k,$$

which implies that for some universal constant $\Theta > 0$,

$$\frac{k^{k/2}}{k!} \leqslant \frac{\Theta^{k+1}}{\Gamma(k/2+1)}, \quad \text{for all } k = 0, 1, 2, \cdots.$$

Therefore, from (2.21), we see that

$$\begin{aligned}\mathbb{E}\left[\exp\left(\lambda \sup_{t\in[0,T]} \|Z(t,\cdot)\|_{V_p}\right)\right] &\leqslant \left(1+T^{-d/2}\right)\Theta \sum_{k=0}^{\infty} \frac{\Theta^k C^k \lambda^k M^k T^{k\alpha/2}}{\Gamma(k/2+1)} \\ &= \left(1+T^{-d/2}\right)\Theta \exp\left(\Theta^2 C^2 \lambda^2 M^2 T^{\alpha}\right)\left[1+\mathrm{Erf}\left(\Theta C \lambda M T^{\alpha}\right)\right],\end{aligned}$$

where $\mathrm{Erf}(\cdot)$ is the error function and the equality can be found, e.g. in formula 7.2.6 in [18]. Notice that $\mathrm{Erf}(x) \leqslant 1$. Hence, we have that

$$\mathbb{E}\left[\exp\left(\lambda \sup_{t\in[0,T]} \|Z(t,\cdot)\|_{V_p}\right)\right] \leqslant 2\Theta\left(1+T^{-d/2}\right)\exp\left(\Theta^2 C^2 \lambda^2 M^2 T^{\alpha}\right), \tag{2.22}$$

Finally, we can derive the exponential tail estimates by the Chebyshev inequality: for any $\lambda > 0$,

$$\begin{aligned}&\mathbb{P}\left(\sup_{t\in[0,T]} \|Z(t,\cdot)\|_{V_p} > \delta\right) \\ &\quad = \mathbb{P}\left(\sup_{t\in[0,T]} \exp\left(\lambda \|Z(t,\cdot)\|_{V_p}\right) > \exp(\lambda\delta)\right) \\ &\quad \leqslant \exp(-\lambda\delta)\,\mathbb{E}\left[\exp\left(\lambda \sup_{t\in[0,T]} \|Z(t,\cdot)\|_{V_p}\right)\right] \\ &\quad \leqslant 2\Theta\left(1+T^{-d/2}\right)\exp\left(\Theta^2 C^2 \lambda^2 M^2 T^{\alpha} - \lambda\delta\right) \\ &\quad = 2\Theta\left(1+T^{-d/2}\right)\exp\left(\Theta^2 C^2 M^2 T^{\alpha}\left(\lambda - \frac{\delta}{2\Theta^2 C^2 M^2 T^{\alpha}}\right)^2 - \frac{\delta^2}{4\Theta^2 C^2 M^2 T^{\alpha}}\right).\end{aligned}$$

By choosing $\lambda = \frac{\delta}{2\Theta^2 C^2 M^2 T^{\alpha}}$, we can conclude that

$$\mathbb{P}\left(\sup_{t\in[0,T]} \|Z(t,\cdot)\|_{V_p} > \delta\right) \leqslant 2\Theta\left(1+T^{-d/2}\right)\exp\left(-\frac{\delta^2}{4\Theta^2 C^2 M^2 T^{\alpha}}\right),$$

which proves lemma 2.6. □

In the next theorem, we generalize theorem 1.6 of [4] from the original Dalang condition (1.10) to the weaker condition—Assumption (1.1). Part (1) of theorem 2.1 originates from the moment formula in [3]. Part (2) of theorem 2.1 shows that if the initial condition $u_0 \in V_p$ with $p \geqslant 1$, then for the Equation (1.1) with both b and σ being globally Lipschitz and vanishing at zero, i.e. $b(0) = \sigma(0) = 0$, the solution $u(t,\cdot) \in V_p$ for any $t > 0$, a.s.

Theorem 2.7 (moment formulas under Lipschitz condition). *Assuming assumption 1.1, and that both b and σ are globally Lipschitz continuous with*

$$L_\sigma := \sup_{z\in\mathbb{R}} \frac{|\sigma(z)-\sigma(0)|}{|z|} \quad \text{and} \quad L_b := \sup_{z\in\mathbb{R}} \frac{|b(z)-b(0)|}{|z|}.$$

Then we have:

(1) *Suppose that u_0 is a rough initial condition, namely, u_0 is locally finite, signed measure such that $J_+(t,x) := (p_t * |u_0|)(x) < \infty$ for all $t>0$ and $x\in\mathbb{R}^d$, where $|u_0| = u_{0,+}+u_{0,-}$ and $u_0 = u_{0,+}-u_{0,-}$ is the Jordan decomposition. Then for any $p \geqslant 2$,*

$$\|u(t,x)\|_p \leqslant C(\tau + J_+(t,x)) \exp\left(Ct \max\left(p^{1/\alpha} L_\sigma^{2/\alpha}, L_b\right)\right), \tag{2.23}$$

where the constant C does not depend on (t,x,p,L_b,L_σ) and

$$\tau := \frac{|b(0)|}{L_b} \vee \frac{|\sigma(0)|}{L_\sigma}.$$

(2) *If $u_0 \in L^\infty(\mathbb{R}^d)\cap L^p(\mathbb{R}^d)$ and assume that $\sigma(0)=b(0)=0$, then for all $t>0$ and $p > \frac{2+d}{\alpha}$,*

$$\left\| \sup_{(s,x)\in[0,t]\times\mathbb{R}^d} u(s,x) \right\|_p \leqslant \|u_0\|_{L^\infty} + C\|u_0\|_{L^p}(L_b+L_\sigma)\exp\left(Ct\max\left(p^{1/\alpha}L_\sigma^{2/\alpha}, L_b\right)\right),$$

where the constant C does not depend on (t,x,p,L_b,L_σ).

Proof. Comparing the proofs of parts (b) and (c) of theorem 1.6 of [4], we see that one only needs to prove part (1) of the theorem, the proof of which follows a similar argument of that used in part (b) of Theorem 1.6 (*ibid.*). The proof of part (2) is similar that of part (c) of Theorem 1.6 (*ibid.*), and so it will not be repeated here. However, lemma 2.5 is used to handle the moment estimates. This allows us to work under assumption 1.1 instead of the stronger condition 1.9. Proceeding now to part (1), according to the proof of part (b) of Theorem 1.6 (*ibid.*),

$$\|u(t,x)\|_p \leqslant \sqrt{3} J_+(t,x) H_{8pL_\sigma^2, L_b^2}(t;1)^{1/2},$$

where the notation $H_{a,b}(t;1)$ is introduced in section 2.2 in [4]. An upper bound of $H_{a,b}(t;1)$ is given in lemma 2.1 in [4], i.e.

$$\begin{aligned}\limsup_{t\to\infty} \frac{1}{t}\log H_{a,b}(t;1) &\leqslant \inf\left\{\beta>0 : a\Upsilon(2\beta) + \frac{b}{2\beta^2} < \frac{1}{2}\right\} \\ &\leqslant \max\left(\inf\left\{\beta>0 : a\Upsilon(2\beta) < \frac{1}{4}\right\}, \inf\left\{\beta>0 : \frac{b}{2\beta^2} < \frac{1}{4}\right\}\right).\end{aligned}$$

For the first argument in the above maximum, we want to find β such that

$$\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \frac{\widehat{f}(\xi)}{2\beta + |\xi|^2} \mathrm{d}\xi < \frac{1}{4a}.$$

Notice that

$$\begin{aligned}\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\frac{\widehat{f}(\xi)}{2\beta+|\xi|^2}\mathrm{d}\xi &= \frac{1}{(2\pi)^d}\int_0^\infty\int_{\mathbb{R}^d}\mathrm{e}^{-2\beta s}\mathrm{e}^{-s|\xi|^2}\widehat{f}(\xi)\,\mathrm{d}\xi\,\mathrm{d}s\\ &= \frac{1}{(2\pi)^d}\int_0^{1/\beta}\int_{\mathbb{R}^d}\mathrm{e}^{-2\beta s}\mathrm{e}^{-s|\xi|^2}\widehat{f}(\xi)\,\mathrm{d}\xi\,\mathrm{d}s\\ &\quad+\frac{1}{(2\pi)^d}\int_{1/\beta}^\infty\int_{\mathbb{R}^d}\mathrm{e}^{-2\beta s}\mathrm{e}^{-s|\xi|^2}\widehat{f}(\xi)\,\mathrm{d}\xi\,\mathrm{d}s\\ &=: I_1+I_2.\end{aligned}$$

According to assumption 1.1,

$$I_1 \leqslant C\int_0^{1/\beta} s^{\alpha-1}\mathrm{d}s = \frac{C}{\beta^\alpha},$$

and similarly,

$$I_2 = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\frac{\mathrm{e}^{-\left(2\beta+|\xi|^2\right)\frac{1}{\beta}}}{2\beta+|\xi|^2}\widehat{f}(\xi)\,\mathrm{d}\xi \leqslant \frac{C}{2\beta}\int_{\mathbb{R}^d}\mathrm{e}^{-|\xi|^2\times\frac{1}{\beta}}\widehat{f}(\xi)\,\mathrm{d}\xi \leqslant \frac{C}{\beta}\left(\frac{1}{\beta}\right)^{\alpha-1} = \frac{C}{\beta^\alpha}.$$

Therefore,

$$\max\left(\inf\left\{\beta>0: a\Upsilon(2\beta)<\frac{1}{4}\right\},\inf\left\{\beta>0:\frac{b}{2\beta^2}<\frac{1}{4}\right\}\right)\leqslant C\max\left(a^{1/\alpha},b^{1/2}\right)<\infty.$$

Finally, replacing a and b by $8pL_\sigma^2$ and L_b^2, respectively, proves part (1). □

3. Proof of theorem 1.7

Now we are ready to prove the main result—theorem 1.7.

Proof of theorem 1.7. The proof follows the same strategy as that in [23]. First we define the cutoff functions for b and σ:

$$b_n(u) := \begin{cases} b(-3^n) & \text{if } u<-3^n\\ b(u) & \text{if } |u|\leqslant 3^n\\ b(3^n) & \text{if } u>3^n\end{cases} \quad\text{and}\quad \sigma_n(u) := \begin{cases} \sigma(-3^n) & \text{if } u<-3^n\\ \sigma(u) & \text{if } |u|\leqslant 3^n\\ \sigma(3^n) & \text{if } u>3^n\end{cases}, \quad\text{respectively.}$$

Since both $b_n(\cdot)$ and $\sigma_n(\cdot)$ are globally Lipschitz continuous, by part (2) of theorem 2.1 with b and σ are replaced by b_n and σ_n, for $p>(2+d)/\alpha$, there is a unique solution solving

$$\begin{aligned}u_n(t,x) &= \int_{\mathbb{R}^d}p_t(x-y)u_0(y)\,\mathrm{d}y+\int_0^t\int_{\mathbb{R}^d}p_{t-s}(x-y)\,b_n(u_n(s,y))\,\mathrm{d}y\mathrm{d}s\\ &\quad+\int_0^t\int_{\mathbb{R}^d}p_{t-s}(x-y)\,\sigma_n(u_n(s,y))\,W(\mathrm{d}s,\mathrm{d}y)\end{aligned}$$

with $\|u_n(t,\cdot)\|_{V_p}<\infty$. Denote the following sequence of stopping times

$$\tau_n := \inf\left\{t>0:\ \|u_n(t,\cdot)\|_{V_p}>3^n\right\}. \tag{3.1}$$

It is easy to check that the solutions are *consistent* in the sense that $u_n(t,x) = u_m(t,x)$ for all $t < \tau_n$ whenever $n < m$. We can define a *local mild solution* to (1.1) by setting

$$u(t,x) := u_n(t,x) \quad \text{when } t < \tau_n.$$

This local mild solution will exist until the explosion time $\tau_\infty := \sup_n \tau_n$. A local solution is called a *global solution* if $\tau_\infty = \infty$ with probability one.

We build the deterministic sequence

$$a_n := \min\left\{\frac{\Theta 3^{n+1}}{h(3^{n+1})}, \frac{1}{n}\right\}, \tag{3.2}$$

with the constant $\Theta \in (0,1/3)$ to be determined later. Just like in [23], the Osgood condition $\int_1^\infty \frac{1}{h(u)} du = +\infty$ guarantees that

$$\sum_{n=1}^{\infty} a_n = +\infty.$$

Our goal is to show that the tripling times are bounded below by this deterministic sequence $\tau_{n+1} - \tau_n \geqslant a_n$ for all large n, which implies that there is a global solution. To this end, we derive the following moment estimates.

Claim: There exist constants $C > 0$ and $q > 1$, both independent of n, such that

$$\mathbb{P}(\tau_{n+1} - \tau_n < a_n) \leqslant C n^{-q}, \quad \text{for all } n \in \mathbb{N}. \tag{3.3}$$

Indeed, as mentioned previously, each τ_n is well-defined and the solution $u(\tau_n, \cdot) \in V_p$. Therefore, we can restart the process at time τ_n. For all $t > 0$ and $x \in \mathbb{R}^d$, define

$$\begin{aligned}
U_n(t,x) &:= \int_{\mathbb{R}^d} p_t(x-y)\, u(\tau_n, y)\, \mathrm{d}y, \\
I_n(t,x) &:= \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y)\, b(u(\tau_n + s, y))\, 1_{\{s \in [0, \tau_{n+1} - \tau_n]\}} \mathrm{d}y \mathrm{d}s, \\
Z_n(t,x) &:= \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y)\, \sigma(u(\tau_n + s, y))\, 1_{\{s \in [0, \tau_{n+1} - \tau_n]\}} W((\tau_n + \mathrm{d}s), \mathrm{d}y).
\end{aligned}$$

Then for all $t \in [0, \tau_{n+1} - \tau_n]$,

$$u(\tau_n + t, x) = U_n(t,x) + I_n(t,x) + Z_n(t,x) \tag{3.4}$$

Furthermore, the presence of the indicator function $1_{\{s \in [0, \tau_{n+1} - \tau_n]\}}$ in the definitions of $I_n(t,x)$ and $Z_n(t,x)$ guarantees that the integrands are bounded in V_p–norm.

Because $p_t(\cdot)$ is a probability density, it follows from Young's inequality for convolutions that for any $t > 0$,

$$\|U_n(t,\cdot)\|_{V_p} \leqslant \|u(\tau_n, \cdot)\|_{V_p} = 3^n. \tag{3.5}$$

Because of the definition of the stopping time τ_n in (3.1) and lemma 2.4, we see that

$$\left\| b(u(\tau_n + s, y))\, 1_{\{s \in [0, \tau_{n+1} - \tau_n]\}} \right\|_{V_p} \leqslant h\left(3^{n+1}\right). \tag{3.6}$$

Therefore, lemma 2.5 with $M = h(3^{n+1})$ guarantees that for $t \in [0, \tau_{n+1} - \tau_n]$,

$$\|I_n(t,\cdot)\|_{V_p} \leqslant t h\left(3^{n+1}\right). \tag{3.7}$$

In particular, if $t \in [0, a_n \wedge (\tau_{n+1} - \tau_n)]$, then by the definition of a_n in (3.2), we have that

$$|I_n(t,\cdot)| \leqslant a_n h\left(3^{n+1}\right) \leqslant 3^n. \tag{3.8}$$

The event $\{\tau_{n+1} - \tau_n < a_n\}$ can only occur if $\|u(\tau_n + t, \cdot)\|_{V_p} > 3^{n+1}$ for some $t \in (0, a_n)$. But because $\|U_n(t,\cdot)\|_{V_p}$ and $\|I_n(t,\cdot)\|_{V_p}$ are each less than 3^n if $t \in [0, a_n \wedge (\tau_{n+1} - \tau_n)]$, the V_p–norm can only triple in this short amount of time if the stochastic term satisfies

$$\sup_{t \in [0, a_{n+1} \wedge (\tau_{n+1} - \tau_n)]} \|Z_n(t,\cdot)\|_{V_p} > 3^n.$$

Because of the definition of the stopping time τ_n, (3.1) and lemma 2.4, the V_p–norm of the integrand of the stochastic integral is bounded with probability one by

$$\left\|\sigma\left(u\left(\tau_n + s, y\right)\right) 1_{\{s \in [0, \tau_{n+1} - \tau_n]\}}\right\|_{V_p} \leqslant 3^{(n+1)(1-\alpha/2)} \left(h\left(3^{n+1}\right)\right)^{\alpha/2} \left(\log\left(\frac{h\left(3^{n+1}\right)}{3^{n+1}}\right)\right)^{-1/2} \tag{3.9}$$

Using the exponential estimate (2.20) with

$$T = a_n, \quad \delta = 3^n, \quad \text{and} \quad M = 3^{(n+1)(1-\alpha/2)} \left[h\left(3^{n+1}\right)\right]^{\alpha/2} \left(\log\left(\frac{h\left(3^{n+1}\right)}{3^{n+1}}\right)\right)^{-1/2},$$

we have that

$$\begin{aligned}
\mathbb{P}\left(\tau_{n+1} - \tau_n < a_n\right) &\leqslant \mathbb{P}\left(\sup_{t \in [0, a_n]} \|Z_n(t,\cdot)\|_{V_p} \geqslant 3^n\right) \\
&\leqslant C\left(1 + a_n^{-d/2}\right) \exp\left(-\frac{C 3^{2(n+1)} \log\left(\frac{h\left(3^{n+1}\right)}{3^{n+1}}\right)}{3^{2(n+1)} \left(\frac{h(3^{n+1})}{3^{n+1}}\right)^{\alpha} a_n^{\alpha}}\right) \\
&\leqslant C\left(1 + a_n^{-d/2}\right) \exp\left(-C\Theta^{-\alpha} \left|\log\left(a_n/\Theta\right)\right|\right) \\
&\leqslant C a_n^{C\Theta^{-\alpha} - d/2}.
\end{aligned}$$

The second-to-last inequality in the above display is a consequence of the definition of a_n and (3.2), which guarantees that $a_n \leqslant \frac{\Theta 3^{n+1}}{h(3^{n+1})}$. Now set $q := C\Theta^{-\alpha} - d/2$ and choose $\Theta \in (0, 1/3)$ small enough so that $q > 1$. Then we obtain that

$$\mathbb{P}\left(\tau_{n+1} - \tau_n < a_n\right) \leqslant C a_n^q. \tag{3.10}$$

From the definition of a_n in (3.2), we also know that $a_n \leqslant 1/n$. Therefore,

$$\mathbb{P}\left(\tau_{n+1} - \tau_n < a_n\right) \leqslant C n^{-q}. \tag{3.11}$$

This proves the claim in (3.3).

Finally, we can prove the main result. From the claim in (3.3),

$$\sum_{n=1}^{\infty} \mathbb{P}(\tau_{n+1} - \tau_n < a_n) \leqslant C \sum_{n=1}^{\infty} n^{-q} < +\infty. \tag{3.12}$$

By the Borel-Cantelli Lemma, with probability one there exists a random $N(\omega)$ such that for all $n \geqslant N(\omega)$, $\tau_{n+1} - \tau_n \geq a_n$. Because $\sum a_n = +\infty$ this implies that

$$\mathbb{P}\left(\sup_n \tau_n = +\infty\right) = 1 \tag{3.13}$$

proving that the solutions cannot explode in finite time. This completes the proof of theorem 1.7. □

4. An explosion example—the proof of theorem 1.8

Proof of theorem 1.8. We will prove this theorem via contradiction. Fix an arbitrary $p \geqslant 2$. We assume that the conclusion is false, namely, for all $u_0 \in V_p$,

$$\|u(t,\cdot)\|_{V_p} < \infty, \quad \text{a.s. for all } t > 0, \tag{4.1}$$

and seek a contradiction. For this purpose, it suffices to consider the initial condition of the following form, where $p_1(x)$ is the heat kernel from (1.4),

$$u_0(x) = \Theta p_1(x), \quad \text{for some } \Theta > 0. \tag{4.2}$$

It is clear that $u_0 \in V_p$. The proof consists of the following two steps.

Step 1. Let $t \in (0,1)$. Multiply $p_{1-t}(x)$ on both sides of (1.4) and integrate over x to obtain

$$Y_t = Y_0 + D_t + M_t \quad \text{a.s. for all } t \in (0,1), \tag{4.3}$$

where

$$\begin{aligned}
Y_t &:= \int_{\mathbb{R}^d} u(t,x) p_{1-t}(x) \mathrm{d}x \quad \text{with} \quad Y_0 = (p_1 * u_0)(0) = \Theta (4\pi)^{-d/2}, \\
D_t &:= \int_0^t \int_{\mathbb{R}^d} p_{1-s}(y) b(u(s,y)) \mathrm{d}s\mathrm{d}y, \quad \text{and} \\
M_t &:= \int_0^t \int_{\mathbb{R}^d} p_{1-s}(y) \sigma(u(s,y)) W(\mathrm{d}s,\mathrm{d}y).
\end{aligned}$$

In this step, we claim that under (4.1), there exists $\Theta_0 > 0$ such that

$$\mathbb{P}\left(Y_{1/2} \geqslant 2L\right) > 0, \quad \text{for all } L > 0 \text{ and } \Theta \geqslant \Theta_0. \tag{4.4}$$

The boundedness assumption on σ ensures that M_t is a martingale. Assumption in (4.1) guarantees that Y_t is well defined since

$$0 \leqslant Y_t \leqslant \|u(t,\cdot)\|_{L^\infty} \leqslant \|u(t,\cdot)\|_{V_p} < \infty, \quad \text{a.s. for all } t \in (0,1). \tag{4.5}$$

Note that the nonnegativity of Y_t comes from the comparison principle (see [3, 14] and references therein), which requires conditions such as $\sigma(0) = 0$ and $b(0) = 0$. However, we will show that $X_t := \mathbb{E}(Y_t)$ will blow up at $t = 1/2$ provided that Θ is large enough, which then implies the claim in (4.4).

It remains to show the blow-up of X_t. By treating $p_{1-s}(y)\mathrm{d}y d\mathbb{P}$ as a probability measure on $\mathbb{R}^d \times \Omega$, we can apply Jensen's inequality to see that

$$\mathbb{E}(D_t) \geqslant \int_0^t \mathrm{d}s\, b\left(\mathbb{E}\left[\int_{\mathbb{R}^d} \mathrm{d}y\, p_{1-s}(y)\, u(s,y)\right]\right) = \int_0^t \mathrm{d}s\, b\left(\mathbb{E}[Y_s]\right),$$

from which we obtain the following integral inequality

$$X_t \geqslant \Theta(4\pi)^{-d/2} + \int_0^t b(X_s)\,\mathrm{d}s.$$

Hence, by the finite Osgood condition (1.7), for some $Y_0 > 0$, X_t blows up in finite time. By increasing the value of Θ whenever necessary, one can ensure that $X_{1/2} = \infty$. This completes the proof of the claim in (4.4).

Step 2. Notice that Y_t in (4.3) can be equivalently written as

$$Y_t = Y_{1/2} + D_t^* + M_t^* \quad \text{a.s. for all } t \in (1/2, 1], \tag{4.6}$$

where the initial condition $Y_{1/2}$ is finite a.s. thanks to (4.5),

$$D_t^* := \int_{1/2}^t \int_{\mathbb{R}^d} p_{1-s}(y)\, b(u(s,y))\,\mathrm{d}s\mathrm{d}y \quad \text{and} \quad M_t^* := \int_{1/2}^t \int_{\mathbb{R}^d} p_{1-s}(y)\,\sigma(u(s,y))\, W(\mathrm{d}s,\mathrm{d}y).$$

Step 2-1. As in Step 1, the boundedness assumption on σ guarantees that $\{M_t^* : t \geqslant 1/2\}$ is a martingale. We claim that

$$\mathbb{P}\left(\inf_{t\in[1/2,1]} M_t^* \leqslant -L \,\middle|\, \mathcal{F}_{1/2}\right) \leqslant \exp\left(-\frac{L^2}{2C_f\|\sigma\|_{L^\infty}^2}\right), \quad \text{a.s. for all } L > 0, \tag{4.7}$$

where

$$C_f := \int_{1/2}^1 \mathrm{d}s \iint_{\mathbb{R}^{2d}} \mathrm{d}y_1\mathrm{d}y_2\, p_{1-s}(y_1)\, p_{1-s}(y_2) f(y_1 - y_2) < \infty. \tag{4.8}$$

First note that assumption 1.1 guarantees the finiteness of the constant C_f. Let $\lambda > 0$ be some constant to be chosen later. Since $\{\exp(-\lambda M_t^*) : t \geqslant 1/2\}$ is a submartingale, by Doob's submartingale inequality, we see that

$$\begin{aligned}
\mathbb{P}\left(\inf_{t\in[1/2,1]} M_t^* \leqslant -L \,\middle|\, \mathcal{F}_{1/2}\right) &= \mathbb{P}\left(\sup_{t\in[1/2,1]} \mathrm{e}^{-\lambda M_t^*} > \mathrm{e}^{\lambda L} \,\middle|\, \mathcal{F}_{1/2}\right) \leqslant e^{-\lambda L}\mathbb{E}\left[\mathrm{e}^{-\lambda M_1^*} \,\middle|\, \mathcal{F}_{1/2}\right] \\
&= \mathbb{E}\left(\exp\left\{-\lambda L + \frac{\lambda^2}{2}\int_{1/2}^1 \mathrm{d}s \iint_{\mathbb{R}^{2d}} \mathrm{d}y_1\mathrm{d}y_2 \right.\right. \\
&\qquad \left.\left. \times p_{1-s}(y_1)\, p_{1-s}(y_2)\,\sigma(u(s,y_1))\,\sigma(u(s,y_2)) f(y_1 - y_2)\right\} \,\middle|\, \mathcal{F}_{1/2}\right) \\
&\leqslant \exp\left(-\lambda L + \frac{1}{2} C_f \lambda^2 \|\sigma\|_{L^\infty}^2\right), \quad \text{a.s.,}
\end{aligned} \tag{4.9}$$

where in the last inequality we have used the fact that f is nonnegative. Optimizing the constant λ in (4.9) proves the claim in (4.7).

Step 2-2. Consider the following deterministic equation,

$$\widehat{Y}_t = L + \int_{1/2}^{t} b\left(\widehat{Y}_s\right) \mathrm{d}s, \quad \text{for } t \geqslant 1/2.$$

Since $b(\cdot)$ satisfies the finite Osgood condition (1.7), when $L > 0$ is large enough, the solution to the above equation will explode before time 1. Indeed, one can choose the smallest $L > 0$ such that $\int_L^\infty \frac{\mathrm{d}y}{b(y)} < 1/2$. In the following, we fix this constant L.

Next for all $t \in [1/2, 1]$, another application of Jensen's inequality with respect to the measure $p_{1-s}(y)\mathrm{d}y$ to in the term D_t^* in (4.5) shows that

$$Y_t \geqslant \left(Y_{1/2} + M_t^*\right) + \int_{1/2}^{t} b(Y_s)\,\mathrm{d}s, \quad \text{a.s. for all } t \in (1/2, 1].$$

Choose and fix arbitrary constant $\Theta > \Theta_0$. We claim that

$$\mathbb{P}(\Omega_L) > 0 \quad \text{with} \quad \Omega_L := \left\{Y_{1/2} + M_t^* \geqslant L : \text{ for all } t \in [1/2, 1]\right\}.$$

Indeed, by the claims in (4.3) and (4.6), we see that

$$\begin{aligned}
\mathbb{P}(\Omega_L) &\geqslant \mathbb{P}\left(\left\{Y_{1/2} \geqslant 2L\right\} \cap \left\{\inf_{t\in[1/2,1]} M_t^* > -L\right\}\right) \\
&= \mathbb{P}\left(Y_{1/2} \geqslant 2L\right) \mathbb{P}\left(\inf_{t\in[1/2,1]} M_t^* > -L \middle| Y_{1/2} \geqslant 2L\right) \\
&\geqslant \mathbb{P}\left(Y_{1/2} \geqslant 2L\right) \left(1 - \exp\left(-\frac{L^2}{2C_f \|\sigma\|_{L^\infty}^2}\right)\right) > 0.
\end{aligned}$$

Therefore,

$$Y_t \geqslant L + \int_{1/2}^{t} b(Y_s)\,\mathrm{d}s, \quad \text{a.s. on } \Omega_L \text{ for all } t \in [1/2, 1].$$

Since $b(\cdot)$ is nondecreasing, we see that $\widehat{Y}_t$ provides a sub-solution to Y_t in the sense that $Y_t \geqslant \widehat{Y}_t$ a.s. on Ω_L for all $t \in [1/2, 1]$. Hence, with positive probability, i.e. a.s. on Ω_L with $\mathbb{P}(\Omega_L) > 0$, $Y_1 \geqslant \widehat{Y}_1 = \infty$, which contradicts with (4.4). This completes the proof of theorem 1.8. ☐

Data availability statement

No new data were created or analysed in this study.

Acknowledgments

L C is partially supported by the NSF Grant DMS-2246850 and by a Collaboration Grant (#959981) from the Simons foundation. M S is partially supported by a Collaboration Grant from the Simon's foundation (#962543).

Appendix

In this appendix, we give a result about the Burkholder–Davis–Gundy inequality for the martingales taking values in a Banach space (typically $L^p(\mathbb{R}^d)$ in our setting). We begin by introducing some standard concepts, which can be found in, e.g. [19] or section 2.2 of [30].

Definition A.1 (definition 3.1 of [19]). A Banach space X is said to be *2-smooth* provided there exist an equivalent norm $\|\cdot\|$ and a constant $C \geqslant 2$ such that for all $x, y \in X$,

$$\|x+y\|^2 + \|x-y\|^2 \leqslant 2\|x\|^2 + C\|y\|^2 .$$

Definition A.2 (definition 2.3 of [19]). Let H be a separable Hilbert space and X a separable Banach space, $B \in L(H, X)$ be an operator from H to X, and ξ_i, $i = 1, 2, 3, \ldots$, be a sequence of independent standard Gaussian random variables, and $\{e_k : k = 1, 2, 3, \ldots\}$ be one orthonormal basis of H. Then B is called a *radonifying* operator if the series $\sum_{k=1}^{\infty} B(e_k)\xi_k$ converges in $L^2(\Omega, X)$. The space of radonifying operators are denoted by $\gamma(H, X)$, which is a Banach space with the *radonifying norm*

$$\|B\|_{\gamma} := \left(\mathbb{E} \left[\left\| \sum_{k=1}^{\infty} B(e_k)\,\xi_k \right\|_X^2 \right] \right)^{1/2} .$$

We have the following two facts (see, e.g. example 2.9 of [30] for details):

1. The Banach space $X = L^p(\mathbb{R}^d)$ for all $p \in [2, \infty)$ is 2-smooth and separable;
2. For any $B \in \gamma\left(H, L^p\left(\mathbb{R}^d\right)\right)$, it holds that

$$\|B\|_{\gamma} \leqslant C_p \left\| \, \|B\|_H \right\|_{L^p} . \tag{A.1}$$

The following result about BDG inequality for martingales with values in Banach space, is from theorem 1.1 in [26].

Theorem A.3. *Let X be a 2-smooth and separable Banach space with norm $\|\cdot\|_X$, W be a cylindrical Q-Wiener process (Q is the covariance operator) on a real separable Hilbert space H and $U = range\left(Q^{1/2}\right)$. Then, there exists a constant $\Pi < \infty$, depending only on $(X, \|\cdot\|_X)$, such that*

$$\left\| \sup_{0\leqslant t\leqslant \tau} \left\| \int_0^t \psi(s)\,\mathrm{d}W(s) \right\|_X \right\|_k \leqslant \Pi\sqrt{k} \left\| \left(\int_0^{\tau} \|\psi(s)\|_{\gamma}^2 \mathrm{d}s \right)^{1/2} \right\|_k , \quad \text{for all } k > 2, \tag{A.2}$$

where τ is a stopping time and ψ is any progressively measurable $\gamma(U, X)$-valued stochastic process satisfying

$$\int_0^t \|\psi(s)\|_{\gamma}^2 \mathrm{d}s < \infty \quad \text{for all } t \geqslant 0 \; a.s.$$

Since the Walsh integral can be written using the setup of H-valued process and the cylindrical Wiener process on H, we can apply theorem A.3 with $X = L^p(\mathbb{R}^d)$, $p \geqslant 2$, and combine (A.1) and (A.2) to get the following lemma:

Lemma A.4. *Let $p \geqslant 2$ be fixed and H be the Hilbert space introduced in (1.3). Assume that $\psi : \Omega \times \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be an adapted and jointly measurable random field such that*

1. *for each $(s,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, $\psi(s,\cdot,x) \in H$;*
2. *for each $s > 0$, $\|\psi(s,\cdot,\circ)\|_H \in L^p(\mathbb{R}^d)$.*

Then, for all $k > 2$ and $t > 0$, it holds that

$$\left\| \left\| \int_0^t \int_{\mathbb{R}^d} \psi(s,y,\circ)\, W(\mathrm{d}s,\mathrm{d}y) \right\|_{L^p} \right\|_k \leqslant C\sqrt{k} \left\| \left(\int_0^t \| \|\psi(s,\cdot,\circ)\|_H \|_{L^p}^2 \,\mathrm{d}s \right)^{1/2} \right\|_k, \tag{A.3}$$

where the constant C does not depend on k. Note that inequality (A.3) is nontrivial only when the right-hand side of the inequality is finite.

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