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MOMENTS AND ASYMPTOTICS FOR A CLASS OF SPDES WITH SPACE-TIME WHITE NOISE

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ABSTRACT. In this article, we consider the nonlinear stochastic partial differential equation of fractional order in both space and time variables with constant initial condition:

$$\left(\partial_t^\beta + \frac{\nu}{2}\left(-\Delta\right)^{\alpha/2}\right) u(t,x) = \; I_t^\gamma \left[\lambda u(t,x)\dot{W}(t,x)\right] \quad t>0,\; x\in\mathbb{R}^d,$$

with constants $\lambda \neq 0$ and $\nu > 0$, where ∂_t^β is the *Caputo fractional derivative* of order $\beta \in (0,2]$, I_t^γ refers to the *Riemann-Liouville integral* of order $\gamma \geq 0$, and $(-\Delta)^{\alpha/2}$ is the standard *fractional/power of Laplacian* with $\alpha > 0$. We concentrate on the scenario where the noise $\dot{W}$ is the space-time white noise. The existence and uniqueness of solution in the Itô-Skorohod sense is obtained under Dalang's condition. We obtain explicit formulas for both the second moment and the second moment Lyapunov exponent. We derive the p-th moment upper bounds and find the matching lower bounds. Our results solve a large class of conjectures regarding the order of the p-th moment Lyapunov exponents. In particular, by letting $\beta = 2$, $\alpha = 2$, $\gamma = 0$, and $d = 1$, we confirm the following standing conjecture for the stochastic wave equation:

$$\frac{1}{t}\log \mathbb{E}[|u(t,x)|^p] \asymp p^{3/2}, \quad \text{for } p \geq 2 \text{ as } t \to \infty.$$

The method for the lower bounds is inspired by a recent work of Hu and Wang, where the authors focus on the space-time colored Gaussian noise case.

CONTENTS

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1. Introduction

Let $\dot{W}$ be a *space-time white noise*, namely, a centered Gaussian noise with covariance

$$\mathbb{E}[\dot{W}(t,x)\dot{W}(s,y)] = \delta(t-s)\delta(x-y), \tag{1.1}$$

where $\delta(\cdot)$ is the Dirac delta function. The following *stochastic heat equation* (SHE)

$$\text{(SHE)} \quad \begin{cases} \left(\dfrac{\partial}{\partial t} - \dfrac{\nu}{2}\dfrac{\partial^2}{\partial x^2}\right) u(t,x) = \lambda u(t,x)\dot{W}(t,x), & t>0, x\in\mathbb{R}, \\ u(0,\cdot) = u_0, \end{cases} \tag{1.2}$$

and *stochastic wave equation* (SWE)

$$\text{(SWE)} \quad \begin{cases} \left(\dfrac{\partial^2}{\partial t^2} - \dfrac{\nu}{2}\dfrac{\partial^2}{\partial x^2}\right) u(t,x) = \lambda u(t,x)\dot{W}(t,x), & t>0, x\in\mathbb{R}, \\ u(0,\cdot) = u_0, \quad \dfrac{\partial}{\partial t}u(0,\cdot) = u_1, \end{cases} \tag{1.3}$$

with $\lambda \neq 0$, $\nu > 0$, $u_0, u_1 \in \mathbb{R}$, are two canonical stochastic partial differential equations.

Denote the *p-th moment Lyapunov exponent* of $u(t,x)$ by

$$l(p) := \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\left[|u(t,x)|^p\right]. \tag{1.4}$$

The corresponding system is said to be *intermittent* if the function $p : [1,\infty) \mapsto l(p)/p$ is *strictly* increasing. For SHE (1.2), the intermittency property suggests that the solution $u(t,x)$ develops a few high peaks (see [24, 39]). With the help of *Feynman-Kac representation* for the moments of the solution to SHE (1.2), assuming $\nu = 1$ and $u_0 > 0$, Chen [19] obtained the precise value $l(p) = \frac{\lambda^4}{24}p(p^2-1)$ which yields the intermittency property. We refer to [5, 8, 19–21, 23, 34, 39, 51] and the references therein for more literature on the intermittency property of stochastic heat equations.

In contrast, much less is known for the hyperbolic counterpart—SWE (1.3). The lack of Feynman-Kac representation poses a notable difficulty; for instance, the existence of the limit in the definition (1.4) of $l(p)$ in general is not even clear, except that $l(2)$ (resp. $l(p)$ for $p \geq 2$) was obtained in [3, 7] (resp. [2]) for SWE with noise that is white in time (resp. with noise that does not depend on time). Thus, instead of studying $l(p)$, we aim to provide bounds for lower and upper Lyapunov exponents:

$$C_1 p^{\theta_1} \leq \liminf_{t\to\infty} \frac{1}{t} \log \mathbb{E}\left[|u(t,x)|^p\right] \leq \limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}\left[|u(t,x)|^p\right] \leq C_2 p^{\theta_2}, \tag{1.5}$$

with the hope of matching the exponents (i.e. $\theta_1 = \theta_2$), which also suggests the presence of some type of intermittency property for the hyperbolic system.

To our best knowledge, only the following upper bound of the p-th moment Lyapunov exponents are known. In particular, Conus *et al* [22] proved that for some constant $C > 0$,

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}\left[|u(t,x)|^p\right] \le Cp^{3/2} \quad \text{for all } p \ge 2. \tag{1.6}$$

It has been long conjectured that the exponent $3/2$ in (1.6) is sharp (i.e., $\theta_1 = \theta_2 = 3/2$ in (1.5)); yet there lacks a rigorous proof. Dalang and Mueller [27] studied the three-dimensional SWE with a Gaussian noise that is white-in-time and colored-in-space with a bounded covariance function, namely,

$$\mathbb{E}\left[\dot{W}(t,x)\dot{W}(s,y)\right] = \delta(t-s)f(x-y), \tag{1.7}$$

where f is a nonnegative, nonnegative definite and bounded function. Using an earlier developed Feynman-Kac-type formula for moments in [28], they established the following large-time asymptotics:

$$C_1 p^{4/3} \le \liminf_{t\to\infty} \frac{1}{t} \log \mathbb{E}\left[|u(t,x)|^p\right] \le \limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}\left[|u(t,x)|^p\right] \le C_2 p^{4/3}, \tag{1.8}$$

for all $p \ge 2$ and $x \in \mathbb{R}^3$; see [27, Theorem 1.1]. To obtain the lower bound in (1.8), their arguments crucially depend on the property that one can find a small indicator function below $f(x)$ near the origin, i.e., $c\mathbb{1}_{\{|x|\le r\}} \le f(x)$ for all $x \in \mathbb{R}^3$. This requirement prevents the adaptation of their method to solve the space-time white noise case.

Recently, Hu and Wang [38] obtained the matching lower and upper p-th moment Lyapunov exponents for a wide range of SPDEs with space-time colored Gaussian noise

$$\mathbb{E}\left[\dot{W}(t,x)\dot{W}(s,y)\right] = \gamma(t-s)\Lambda(x-y). \tag{1.9}$$

Their proof relies heavily on the assumption $\gamma(t) \ge C|t|^{-\theta}$ and $\Lambda(x) \ge C|x|^{-\lambda}$ which does not hold for the white noise case. As a consequence, the small ball nondegeneracy property for Green's function (see Section 3.1 *ibid.*) which plays a key role in Hu-Wang's argument does not apply to the white noise case (see Remark 2.1 *ibid.*). To resolve this issue, a nondegeneracy property for the product of Green's functions is tailored specially for the white noise case (see Proposition 5.1). Moreover, in the white noise case, the balanced Feynman diagrams (see Definition 5.7) make the right contribution to the desired lower bound, while all admissible Feynman diagrams do in the colored noise case. Therefore, just like the SHE case, the SWE with space-time white noise needs a separate treatment.

One of the major contributions of this paper is to carry out such arguments and confirm the conjecture, in Proposition 5.10, about the moment asymptotics of SWE (1.3) by showing that if $u_0 > 0$ and $u_1 \ge 0$, then

(1.10a)

$$C_1\, p^{3/2} \le \liminf_{t\to\infty} \frac{1}{t} \log \mathbb{E}[|u(t,x)|^p] \le \limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}[|u(t,x)|^p] \le C_2\, p^{3/2}, \qquad p \ge 2,$$

(1.10b)

$$C_1\, t \le \liminf_{p\to\infty} \frac{1}{p^{3/2}} \log \mathbb{E}[|u(t,x)|^p] \le \limsup_{p\to\infty} \frac{1}{p^{3/2}} \log \mathbb{E}[|u(t,x)|^p] \le C_2\, t, \quad t > 0,$$

where

$$C_1 := \frac{|\lambda|}{48\sqrt{2}\, e^{3/2} (2\nu)^{1/4}} \qquad \text{and} \qquad C_2 := \frac{\sqrt{2}|\lambda|}{(2\nu)^{1/4}}. \tag{1.11}$$

Note that the above constants C_1 and C_2 are not optimal. For related results in both parabolic and hyperbolic settings, please see Appendix A.

It turns out that the method we use to resolve the above conjecture can be applied to a much wider class of stochastic partial differential equations (SPDEs). Indeed, in this paper, we will study the following stochastic fractional diffusion equation with both SHE (1.2) and SWE (1.3) as two special cases:

$$\begin{cases} \left(\partial_t^\beta + \frac{\nu}{2}(-\Delta)^{\alpha/2}\right) u(t,x) = I_t^\gamma\left[\lambda u(t,x)\dot{W}(t,x)\right], & t>0,\ x\in\mathbb{R}^d,\\ u(0,\cdot)=u_0, & \text{if } \beta\in(0,1],\\ u(0,\cdot)=u_0, \quad \dfrac{\partial}{\partial t}u(0,\cdot)=u_1, & \text{if } \beta\in(1,2], \end{cases} \tag{1.12}$$

with

$$\alpha>0, \quad \beta\in(0,2], \quad \gamma\ge 0, \quad \lambda\ne 0, \quad \nu>0, \quad u_0,\, u_1\in\mathbb{R},$$

where $\dot{W}$ is space-time white noise, $(-\Delta)^{\alpha/2}$ is the *standard* ($\alpha=2$), *fractional* ($0<\alpha<2$) or *power* ($\alpha>2$) of *Laplacian*. The symbol ∂_t^β denotes the *Caputo fractional differential* operator of order $\beta>0$:

$$\partial_t^\beta f(t) := \begin{cases} \dfrac{1}{\Gamma(n-\beta)}\displaystyle\int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\beta+1-n}}\mathrm{d}\tau, & \text{if } \beta\ne n,\\ \dfrac{\mathrm{d}^n}{\mathrm{d}t^n} f(t), & \text{if } \beta=n, \end{cases}$$

where $n=\lceil\beta\rceil$ is the smallest integer that is not smaller than β (i.e., $\lceil\cdot\rceil$ is the ceiling function), and $\Gamma(x)$ is the *Gamma function* (see Remark 2.7 for a brief recall). We use I_t^γ to refer to the *Riemann-Liouville integral* in the time variable to the right of zero I_{0+}^γ; see Definition 2.1.

The SPDE (1.12) is interpreted as the following integral equation:

$$u(t,x) = J_0(t,x) + \lambda\int_0^t\int_{\mathbb{R}^d} p(t-s,x-y)u(s,y)W(\mathrm{d}s,\mathrm{d}y), \tag{1.13}$$

where $J_0(t,x)$ is the solution to the homogeneous equation (see (3.3) and (3.4)), $p(t,x)$ is the underlying fundamental solution (see (3.2)), and the stochastic integral refers to the *Walsh* or *Skorohod integral.* Set the following four constants:

$$\begin{aligned} &\theta := 2(\beta+\gamma)-2-\frac{\beta d}{\alpha}, && t_p := p^{1+\frac{1}{1+\theta}}t,\\ &\Theta := (2\pi)^{-d}\int_{\mathbb{R}^d} E^2_{\beta,\beta+\gamma}(-\tfrac{1}{2}\nu|\xi|^\alpha)\mathrm{d}\xi, && \hat{t} := \Theta\,\Gamma(\theta+1)\,t^{\theta+1}, \end{aligned} \tag{1.14}$$

where the function $E_{a,b}(z)$ is the *Mittag-Leffler function* of two parameters (see, e.g., [41, Section 1.8]), i.e., for $a,b>0$,

$$E_{a,b}(z) := \sum_{k=0}^\infty \frac{z^k}{\Gamma(ak+b)}, \quad z\in\mathbb{C}. \tag{1.15}$$

We use the convention $E_a(\cdot) := E_{a,1}(\cdot)$.

We will prove in Theorem 3.3 that under *Dalang's condition*

$$\begin{cases} d < 2\alpha + \dfrac{\alpha}{\beta}\min\{2\gamma-1,0\}, & \text{if } \beta\in(0,2),\\ d<\alpha\min\{2,1+\gamma\}, & \text{if } \beta=2, \end{cases} \tag{1.16}$$

there exists a unique random field solution $u(t,x)$ with finite p-th moment for all $p \geq 2$, $t > 0$ and $x \in \mathbb{R}^d$. It is an easy exercise to check that Dalang's condition (1.16) implies that $\theta > -1$ and $\Theta < \infty$, so that all constants in (1.14) are well-defined. The aim of this paper is to establish Theorem 1.1, which gives the exact formula for the second moment and the sharp moment asymptotics terms of t and p, respectively.

Theorem 1.1. *Suppose that Dalang's condition* (1.16) *is satisfied and let* $u(t,x)$ *be the solution to* (1.12)*. Recall that the quantities* θ*,* Θ*,* t_p *and* $\hat{t}$ *are defined in* (1.14)*. Then the solution* $u(t,x)$ *is stationary in* x *and the* p*-th moment satisfies the following properties:*

(a) *When* $p = 2$*,*

$$\mathbb{E}\left[u^2(t,x)\right] = \begin{cases} u_0^2\, E_{\theta+1}\left(\lambda^2 \hat{t}\right) & \text{if } \beta \in (0,1], \\ u_0^2\, E_{\theta+1}\left(\lambda^2 \hat{t}\right) \\ +2u_0 u_1 t\, E_{\theta+1,2}\left(\lambda^2 \hat{t}\right) & \text{if } \beta \in (1,2], \\ +2u_1^2 t^2\, E_{\theta+1,3}\left(\lambda^2 \hat{t}\right) \end{cases} \tag{1.17}$$

for all $t > 0$ *and* $x \in \mathbb{R}^d$*. As a consequence,*

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\left[u(t,x)^2\right] = \left(\lambda^2 \Theta \Gamma(\theta+1)\right)^{\frac{1}{\theta+1}}, \quad \text{for all } x \in \mathbb{R}^d. \tag{1.18}$$

(b) *For any* $p \geq 2$*,*

$$||u(t,x)||_p^2 \leq \begin{cases} 2u_0^2\, E_{\theta+1}\left(8\lambda^2 p \hat{t}\right) & \text{if } \beta \in (0,1], \\ 2u_0^2\, E_{\theta+1}\left(8\lambda^2 p \hat{t}\right) \\ +4u_0 u_1 t\, E_{\theta+1,2}\left(8\lambda^2 p \hat{t}\right) & \text{if } \beta \in (1,2], \\ +4u_1^2 t^2\, E_{\theta+1,3}\left(8\lambda^2 p \hat{t}\right) \end{cases} \tag{1.19}$$

for all $t > 0$ *and* $x \in \mathbb{R}^d$*. As a consequence,*

$$\limsup_{pt^{\theta+1}\to\infty} \frac{1}{t_p} \log \mathbb{E}[|u(t,x)|^p] \leq \frac{1}{2}\left(8\lambda^2 \Theta \Gamma(\theta+1)\right)^{\frac{1}{\theta+1}}. \tag{1.20}$$

In particular, by freezing $p \geq 2$ *or* $t > 0$*, we have the following two asymptotics:*

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}[|u(t,x)|^p] \leq \frac{1}{2}\left(8\lambda^2 \Theta \Gamma(\theta+1)\right)^{\frac{1}{\theta+1}} p^{1+\frac{1}{\theta+1}}, \tag{1.21a}$$

$$\limsup_{p\to\infty} \frac{1}{p^{1+\frac{1}{\theta+1}}} \log \mathbb{E}[|u(t,x)|^p] \leq \frac{1}{2}\left(8\lambda^2 \Theta \Gamma(\theta+1)\right)^{\frac{1}{\theta+1}} t. \tag{1.21b}$$

(c) *If in addition, we have*
  (1) *either* $\beta \in (0,2)$ *and the fundamental function* $p(t,x)$ *is nonnegative or* $\alpha = \beta = 2$ *and* $\gamma = 0$*; and*
  (2) *the initial position* u_0 *is strictly positive and the initial velocity* u_1 *is nonnegative,*

there exists a positive constant C *such that for all* $x \in \mathbb{R}^d$*,*

$$\liminf_{pt^{\theta+1}\to\infty} \frac{1}{t_p} \log \mathbb{E}\left[|u(t,x)|^p\right] \geq C. \tag{1.22}$$

In particular, by freezing $p \geq 2$ or $t > 0$, with the same constant C as in (1.22)*, we have the following two asymptotics:*

$$\liminf_{t\to\infty} \frac{1}{t} \log \mathbb{E}\left[|u(t,x)|^p\right] \geq C p^{1+\frac{1}{\theta+1}}, \tag{1.23a}$$

$$\liminf_{p\to\infty} \frac{1}{p^{1+\frac{1}{\theta+1}}} \log \mathbb{E}\left[|u(t,x)|^p\right] \geq Ct. \tag{1.23b}$$

As applications of Theorem 1.1, in Appendix A we shall revisit some SPDEs which have attracted considerable attention in the literature and calculate sharp moment asymptotics for the solutions.

Remark 1.2. The motivation of studying the SPDE (1.12) with the Caputo fractional derivative ∂_t^β and the Riemann-Liouville integral I_t^γ can be summarized as follows.

(i) The Caputo fractional derivative ∂_t^β is a natural choice for the Cauchy problem with data prescribed at time zero; see, e.g., Chapters 1 and 3 of [31] for an in-depth discussion. Moreover, the parameter β ranges from 0 to 2 so that one can put SHE (1.2) and SWE (1.3) in a unified framework. It is known that there may exist some property jump for the solution $u(t,x)$ at $\beta = 1$ or 2, which is a consequence of the corresponding property jump for the fundamental solution $p(t,x)$ at $\beta = 1$ or 2. Our results explicitly demonstrate this phenomenon. Depending on the specific properties under consideration, such property jump may be pronounced or not at all. For example, for Dalang's condition (1.16), there is no property jump at $\beta = 1$, but one is present at $\beta = 2$. However, for the moment asymptotics, as can be seen from (1.21) and (1.23), there is no property jump at both $\beta = 1$ and 2. Note that the observation is consistent with the moment asymptotics when the noise is time-independent; see Theorem 1.7 and Example 2.7 of [10].

(ii) The Riemann-Liouville integral I_t^γ was introduced historically in the literature, e.g., with $\gamma = 1-\beta$ (for $\beta \in (0,1)$) in [44], $\gamma = \lceil \beta \rceil - \beta$ (for $\beta \in (0,2)$) in [6], $\gamma = 0$ in [13], and a general $\gamma \geq 0$ in [14]. This fractional integral provides some non-local effect (in time) and demonstrates a smoothing effect, as can be seen from the moment asymptotics (1.21) and (1.23). Mathematically, the incorporation of this fractional integral is both natural and does not introduce any additional difficulty in the analysis, which can be seen from the proof of Theorem 2.8. Moreover, such a fractional integral offers a slightly more versatile framework for potential mathematical modeling.

To conclude the introduction, we highlight some of the contributions of this paper:

(1) The conjecture on the moment asymptotics of SWE (1.3) is solved; see (1.10) and Example A.2.

(2) For the solutions of a large class of SPDEs, explicit representations (1.17) for the second moments and (1.18) for the second moment Lyapunov exponents are obtained; the asymptotic behavior of p-th moments is characterized sharply by the upper bounds (1.20) and the lower bounds (1.22). Regarding the quantities obtained in Theorem 1.1, as will be shown in Appendix A, some of them recover known results for SPDEs with some specific parameters $(\alpha, \beta, \gamma, d)$ in the literature, while, to our best knowledge, most of them (in particular the lower bounds for p-th moments) are new. Moreover, the quantities that characterize the asymptotics of

the solutions to (1.12) depend on the parameters $(\alpha, \beta, \gamma, d)$ in an interesting way (see also the figures in Appendix A for an illustration), which may relate to physical phenomena and desire further investigation.

(3) For the fundamental solution of (1.12), we extend the results in [14] from $\alpha \in (0, 2]$ to all $\alpha > 0$ (see Section 2.3). As a consequence, Dalang's condition (1.16) allows to consider SPDE (1.12) in high dimension d, if α is sufficiently big.

The paper is organized as follows: Some preliminaries on the fractional calculus, Mittag-Leffler functions, and the fundamental solutions under the settings of $\alpha > 0$, $\beta \in (0, 2]$ and $\gamma \geq 0$ are given in Section 2. Then in Section 3, we establish the existence and uniqueness of the solution in a slightly more general setting. The second moment formula and the p-th moment upper bounds are obtained in Section 4; while the lower bounds are derived in Section 5. Finally, in Appendix A we provide some examples with discussions, and in Appendix B we prove some technical lemmas that are used in the paper.

Notation. Let $W = \left\{W_t(A) : A \in \mathcal{B}_b\left(\mathbb{R}^d\right), t \geq 0\right\}$ be space-time white noise defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathcal{B}_b(\mathbb{R}^d)$ is the collection of Borel sets with finite Lebesgue measure. Let

$$\mathcal{F}_t := \sigma(W_s(A) : 0 \leq s \leq t, A \in \mathcal{B}_b(\mathbb{R}^d)) \vee \mathcal{N} \qquad (t \geq 0)$$

be the natural filtration augmented by the σ-field $\mathcal{N}$ generated by all $\mathbb{P}$–null sets in $\mathcal{F}$. We use $||\cdot||_p$ to denote the $L^p(\Omega)$–norm. The Fourier transform $\mathcal{F}g = \widehat{g}$ of a function $g \in L^1(\mathbb{R}^d)$ is given by

$$\mathcal{F}g(\xi) = \widehat{g}(\xi) := \int_{\mathbb{R}^d} g(x) e^{-ix\cdot\xi} \mathrm{d}x. \tag{1.24}$$

We use

$$B_r(x) := \left\{ y \in \mathbb{R}^d : \; |y - x| = \sqrt{(y_1 - x_1)^2 + \cdots + (y_d - x_d)^2} < r \right\}$$

to denote an open ball centered at $x \in \mathbb{R}^d$ with radius r. For $a \in \mathbb{R}$, $\lceil a \rceil$ (resp. $\lfloor a \rfloor$) is the smallest (resp. largest) integer that is not smaller (resp. larger) than a, i.e., the ceiling (resp. floor) function. We use the convention $\mathbb{N} = \{1, 2, \dots\}$.

2. Preliminaries

2.1. **Fractional calculus and Mittag-Leffler functions.** In this subsection, we provide some preliminaries on fractional integrals and derivatives in the sense of Riemann-Liouville and we also recall Caputo fractional derivatives. We refer to [41, 46] for details.

Let $\alpha \geq 0$ be a constant and $[a, b]$ be a finite interval on $\mathbb{R}$. Let $f(x)$ be a complex-valued function defined on $[a, b]$. We only recall the left-sided integrals/derivatives which will be used in this article, and the right-sided case is similar and thus omitted.

Definition 2.1. The *Riemann-Liouville integral* $I^\alpha_{a+} f$ of order $\alpha > 0$ is defined by

$$(I^\alpha_{a+} f)(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x - t)^{1-\alpha}} \mathrm{d}t, \quad x \in [a, b]. \tag{2.1}$$

Definition 2.2. The *Riemann-Liouville derivative* $D^\alpha_{a+}f$ of order $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}$ is defined by

$$\left(D^\alpha_{a+}f\right)(x) := \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(I^{n-\alpha}_{a+}f\right)(x) = \frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\int_a^x \frac{f(t)}{(x-t)^{\alpha-n+1}}\mathrm{d}t, \quad n = \lceil\alpha\rceil,$$

and when $\alpha = n \in \mathbb{N}$, $\left(D^\alpha_{a+}f\right)(x) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}f(x)$. We use the convention that $D^\alpha_{a+}f := I^{-\alpha}_{a+}f$, when $\alpha < 0$.

For $1 \le p \le \infty$, we denote by $L^p(a,b)$ the set of complex-valued functions f on $[a,b]$ with finite L^p-norm $\|f\|_p$, where

$$\|f\|_p = \begin{cases} \left(\int_a^b |f(x)|^p dt\right)^{\frac{1}{p}}, & 1 \le p < \infty, \\ \operatorname*{ess\,sup}_{a\le x\le b}|f(x)|, & p = \infty. \end{cases}$$

Lemma 2.3 (Property 2.2 on p. 74 of [41])**.** *For $\alpha > \beta > 0$ and $f(x) \in L^p(a,b), 1 \le p \le \infty$, we have*

$$\left(D^\beta_{a+}I^\alpha_{a+}f\right)(x) = I^{\alpha-\beta}_{a+}f(x), \quad \textit{for } x \in [a,b] \textit{ almost everywhere.}$$

Lemma 2.4 (Property 2.5 on p. 81 of [41])**.** *For $\alpha, \beta > 0$, we have*

$$\left(I^\alpha_{0+}t^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}x^{\beta+\alpha-1} \quad \textit{and} \quad \left(D^\alpha_{0+}t^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}x^{\beta-\alpha-1}.$$

Definition 2.5 ((2.4.1) on p. 91 of [41])**.** The *Caputo fractional derivative* of order α on $[a,b]$ can be defined via the *Riemann-Liouville derivative* as follows,

$$\left({}^C D^\alpha_{a+}f\right)(x) := \left(D^\alpha_{a+}\left[f(\cdot) - \sum_{k=0}^{\lceil\alpha\rceil-1}\frac{f^{(k)}(a)}{k!}(\cdot - a)^k\right]\right)(x), \quad x \in [a,b]. \tag{2.2}$$

We are ready to recall the formulas of the solutions to Cauchy problems for differential equations with the Caputo fractional derivatives. For $\gamma \in [0,1)$, we define the *weighted space* $C_\gamma[a,b]$ *of continuous functions* as follows,

$$C_\gamma[a,b] := \left\{f(x) : (x-a)^\gamma f(x) \in C[a,b]\right\}.$$

Consider the following Cauchy Problem, for $\lambda \in \mathbb{R}, n \in \mathbb{N}$ and $n-1 < \beta < n$,

$$\begin{cases} ({}^C D^\beta_{0+}f)(x) - \lambda f(x) = y(x), & x \in [0,b], \\ f^{(k)}(0) = b_k, & b_k \in \mathbb{R} \text{ for } k = 0,1,\dots,n-1. \end{cases} \tag{2.3}$$

We suppose that $y(x) \in C_\gamma[0,b]$ with $0 \le \gamma < 1$ and $\gamma \le \beta$. Then (2.3) has a unique solution given by (see [41, (4.1.62)]):

$$f(x) = \sum_{j=0}^{n-1} b_j x^j E_{\beta,j+1}(\lambda x^\beta) + \int_0^x (x-t)^{\beta-1}E_{\beta,\beta}(\lambda(x-t)^\beta)y(t)\mathrm{d}t, \tag{2.4}$$

where $E_{a,b}(z)$ is the Mittag-Leffler function; see (1.15). One may get more explicit expressions for special values of a and b, which will be used in this paper:
(2.5)

$$E_{1/2}(z) = 2e^{z^2}\Phi\left(\sqrt{2}z\right), \quad E_1(z) = e^z, \quad E_2(z) = \cosh(\sqrt{z}), \quad E_{2,2}(z) = \frac{\sinh(\sqrt{z})}{\sqrt{z}},$$

where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{x^2}{2}} \mathrm{d}x$ is the cumulative distribution function of standard normal distribution. Another formula that will be useful in this paper is

$$E_{\alpha,\beta}\left(|z|\right) - \frac{1}{\Gamma(\beta)} = |z| E_{\alpha,\alpha+\beta}\left(|z|\right), \tag{2.6}$$

which can be obtained immediately using the definition of the Mittag-Leffler function in (1.15); see also (1.8.38) on p.45 of [41]. The asymptotic behavior of the Mittag-Leffler function along the positive and negative real lines plays an important role in the paper, which has been summarized in Lemma 2.6:

Lemma 2.6. *If all $a > 0$ and $b \in \mathbb{C}$, we have that*

- *if $a < 2$, as $z \to +\infty$,*

$$E_{a,b}(z) = \frac{1}{a} z^{(1-b)/a} \exp(z^{1/a}) - \frac{1}{\Gamma(b-a)} \frac{1}{z} + o\left(z^{-1}\right);$$

- *if $a \geq 2$, as $z \to +\infty$,*

$$\begin{aligned} E_{a,b}(z) = \frac{1}{a} \sum_{n \in \mathbb{Z}: |n| \leq a/4} & \left(z^{1/a} \exp\left[\frac{2n\pi i}{a}\right]\right)^{1-b} \exp\left[\exp\left(\frac{2n\pi i}{a}\right) z^{1/a}\right] \\ & - \frac{1}{\Gamma(b-a)} \frac{1}{z} + o\left(z^{-1}\right); \end{aligned}$$

- *if $a < 2$, as $z \to -\infty$,*

$$E_{a,b}(z) = -\frac{1}{\Gamma\left(b-a\right)} \frac{1}{z} + o\left(z^{-1}\right);$$

- *if $a = 2$, as $z \to -\infty$,*

$$E_{a,b}(z) = |z|^{(1-b)/2} \cos\left(\sqrt{|z|} + \frac{(1-b)\pi}{2}\right) - \frac{1}{\Gamma(b-2)} \frac{1}{z} + o(z^{-1}).$$

In particular, for all $C > 0$,

$$\lim_{t \to \infty} \frac{1}{t} \log E_{a,b}\left(C t^a\right) = C^{\frac{1}{a}}.$$

Proof. The case when $z \to \infty$ is derived from 1.8.27 (resp. 1.8.29) of [41] when $a < 2$ (resp. $a \geq 2$). The case when $a < 2$ (resp. $a = 2$) and $z \to -\infty$ is a consequence of 1.8.28 (resp. 1.8.31) (*ibid.*). When $a < 2$, the statement for the limit is a direct consequence of the asymptotics at $+\infty$. When $a \geq 2$, denoting $z = Ct^a$, we have

$$\begin{aligned} \lim_{t\to\infty} \frac{1}{t} \log E_{a,b}(z) &= \lim_{t\to\infty} \frac{1}{t} \log \sum_{n \in \mathbb{Z}: |n| \leq a/4} \left(z^{\frac{1}{a}} \exp\left[\frac{2n\pi i}{a}\right]\right)^{1-b} \exp\left[\exp\left(\frac{2n\pi i}{a}\right) z^{\frac{1}{a}}\right] \\ &= \lim_{t\to\infty} \frac{1}{t} \log \sum_{n \in \mathbb{Z}: |n| \leq a/4} z^{\frac{1-b}{a}} \exp\left[z^{\frac{1}{a}} \cos \frac{2n\pi}{a}\right] \exp\left[i \left(\frac{(1-b)2n\pi}{a} + z^{\frac{1}{a}} \sin\left(\frac{2n\pi}{a}\right)\right)\right] \\ &= \lim_{t\to\infty} \frac{1}{t} \log \left(z^{\frac{1-b}{a}} \exp\left(z^{\frac{1}{a}}\right)\right) = C^{\frac{1}{a}}. \end{aligned}$$

□

Remark 2.7. We recall some basic facts for the Gamma function. For complex numbers with strictly positive real part, the Gamma function is defined via a convergent improper integral:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \mathrm{d}t, \quad \mathrm{Re}(z) > 0.$$

Then, Gamma function is defined as the analytic continuation of this integral function which is a holomorphic function in the whole complex plane except non-positive integers where the function has simple poles. The Gamma function has no zeros, so the reciprocal gamma function $\frac{1}{\Gamma(z)}$ is an entire function which has the following infinite product expansion:

$$\frac{1}{\Gamma(z)} = \lim_{n\to\infty} \frac{z(z+1)\cdots(z+n)}{n! n^z}, \quad z \in \mathbb{C}.$$

In particular, we will use the following fact (see, e.g., [45, 5.2.1 on p. 136])

$$\frac{1}{\Gamma(z)} \equiv 0, \quad \text{for } z = 0, -1, -2, \dots . \tag{2.7}$$

and will need the *reflection formula* for the Gamma function (see, e.g., [45, 5.5.3 on p. 138]), namely,

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}, \quad z \neq 0, \pm 1, \dots . \tag{2.8}$$

2.2. **Fox H-function.** The *Fox H-function* plays a critical role in expressing the fundamental solutions to our equations. It is a generalization of the *Meijer G-function* (see Chapter 16 of [45]). Detailed properties of the Fox H-function can be found in many books including Chapters 1 & 2 of [40], Section 1.12 of [41], Section 8.2 of [47]. Its applications to statistics and astrophysics can be found in [43]. See also the book [32] for the applications of the Fox H-function to the pseudo-differential equations of parabolic type. In this part, we give a brief account of this special function.

Let m, n, p, q be integers such that $0 \le m \le q$ and $0 \le n \le p$. Let $a_i, b_j \in \mathbb{C}$ and $\alpha_i, \beta_j \in \mathbb{R}_+$ for $i = 1, \cdots, p$ and $j = 1, \cdots, q$. Denote

$$a^* \coloneqq \sum_{i=1}^n \alpha_i - \sum_{i=n+1}^p \alpha_i + \sum_{j=1}^m \beta_j - \sum_{j=m+1}^q \beta_j, \tag{2.9}$$

$$\Delta \coloneqq \sum_{j=1}^q \beta_j - \sum_{i=1}^p \alpha_i, \tag{2.10}$$

$$\delta \coloneqq \prod_{i=1}^p \alpha_i^{-\alpha_i} \prod_{j=1}^q \beta_j^{\beta_i}, \tag{2.11}$$

$$\mu \coloneqq \sum_{j=1}^q b_j - \sum_{i=1}^p a_i + \frac{p-q}{2}. \tag{2.12}$$

Now we consider the following ratio of the Gamma functions:

$$\mathcal{H}_{p,q}^{m,n}(s) := \frac{\prod_{i=1}^{n} \Gamma\left(1-a_i-\alpha_i s\right)}{\prod_{i=n+1}^{p} \Gamma\left(a_j+\alpha_i s\right)} \times \frac{\prod_{j=1}^{m} \Gamma\left(b_j+\beta_j s\right)}{\prod_{j=m+1}^{q} \Gamma\left(1-b_j-\beta_j s\right)}. \tag{2.13}$$

Assume that the two sets of poles of $\mathcal{H}_{p,q}^{m,n}(s)$ in (2.13) (see Remark 2.7 for the poles of the Gamma function) do not overlap, i.e.,

$$\left\{b_{j\ell} = \frac{-b_j-\ell}{\beta_j},\;\; \ell=0,1,\cdots\right\} \bigcap \left\{a_{ik} = \frac{1-a_i+k}{\alpha_i},\;\; k=0,1,\cdots\right\} = \emptyset, \tag{2.14}$$

and let $\mathcal{L}$ be any contour that separates these two sets of poles. Then the following *Mellin-Barnes integral*, named as the *Fox H-function* [35],

$$\frac{1}{2\pi i}\int_{\mathcal{L}} \mathcal{H}_{p,q}^{m,n}(s) z^{-s} \mathrm{d}s =: H_{p,q}^{m,n}\left(z \,\middle|\, \begin{matrix} (a_1,\alpha_1),\cdots,(a_p,\alpha_p) \\ (b_1,\beta_1),\cdots,(b_q,\beta_q) \end{matrix}\right), \tag{2.15}$$

is well defined in many cases. To be more precise, it is analytic with respect to z in the sector

$$|\arg(z)| < a^* \times \frac{\pi}{2} \quad \text{provided} \quad a^* > 0.$$

In the case $a^* \le 0$, we do not have the sector analyticity, but if $\Delta = 0$, it is still analytic for both $|z| > \delta$ and $0 < |z| < \delta$. At $|z| = \delta$, the function is well defined when $\mathrm{Re}(\mu) < -1$. See Theorems 1.1 and 1.2 of [40] for the precise statement and [12] for a diagram exposition.

2.3. **Fundamental solutions.** The fundamental solutions to (1.12) in the case when $d=1$, $\beta=1$, $\gamma=0$, and $\alpha \in 2\mathbb{N}$ have been studied in [42] and [36]. In [29], Debbi studied the fundamental solutions to (1.12) when $d=1$, $\beta=1$, $\gamma=0$, and $\alpha \in (1,\infty)\setminus\mathbb{N}$ and then, with Dozzi [30], they studied the corresponding SPDEs with space-time white noise. This part can be viewed as a generalization of their results to a class of more general of SPDEs. The Fox H-function, introduced in Section 2.2, allows us to study the fundamental solutions to (1.12) with much more general parameters in a unified way.

Theorem 2.8 generalizes Theorem 4.1 of [14] from $\alpha \in (0,2]$ and $\beta \in (0,2)$ to the case $\alpha > 0$ and $\beta \in (0,2]$. The statement of the theorem remains almost the same except the conditions on α and β. The proof also follows the same lines of arguments as those in [14]; one may also check the proof of Theorem 3.1 in [13] for the case when $\gamma = 0$. The case when $\beta = 2$ is new. For the readers' convenience, we state the theorem and present its proof below to indicate why the ranges of α and β can be extended.

Theorem 2.8. *For $\alpha \in (0,\infty)$, $\beta \in (0,2]$, and $\gamma \ge 0$, the solution to*

(2.16)
$$\begin{cases} \left(\partial_t^\beta + \frac{\nu}{2}(-\Delta)^{\alpha/2}\right) u(t,x) = I_t^\gamma\left[f(t,x)\right], & t>0,\; x\in\mathbb{R}^d, \\ \left.\frac{\partial^k}{\partial t^k} u(t,x)\right|_{t=0} = u_k(x), & 0 \le k \le \lceil \beta \rceil - 1,\;\; x \in \mathbb{R}^d, \end{cases}$$

is

$$u(t,x) = J_0(t,x) + \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}y\, f(s,y)\, {}_tD_{0+}^{\lceil\beta\rceil-\beta-\gamma} Z(t-s,x-y), \tag{2.17}$$

where ${}_tD_{0+}^{\lceil\beta\rceil-\beta-\gamma}$ *denotes the Riemann-Liouville derivative* $D_{0+}^{\lceil\beta\rceil-\beta-\gamma}$ *acting on the time variable,*

$$J_0(t,x) := \sum_{k=0}^{\lceil\beta\rceil-1} \int_{\mathbb{R}^d} u_k(y) \partial_t^{\lceil\beta\rceil-1-k} Z(t,x-y)\mathrm{d}y \tag{2.18}$$

is the solution to the homogeneous equation and $Z(t,x) := Z_{\alpha,\beta,d}(t,x)$ *is the corresponding fundamental solution. If we denote*

$$Y(t,x) := Y_{\alpha,\beta,\gamma,d}(t,x) = {}_tD_{0+}^{\lceil\beta\rceil-\beta-\gamma} Z_{\alpha,\beta,d}(t,x),$$
$$Z^*(t,x) := Z^*_{\alpha,\beta,d}(t,x) = \frac{\partial}{\partial t} Z_{\alpha,\beta,d}(t,x), \quad \text{if } \beta \in (1,2],$$

then we have the following Fourier transforms:

$$\mathcal{F}Z_{\alpha,\beta,d}(t,\cdot)(\xi) = t^{\lceil\beta\rceil-1} E_{\beta,\lceil\beta\rceil}(-\tfrac{1}{2}\nu t^\beta |\xi|^\alpha), \tag{2.19}$$
$$\mathcal{F}Y_{\alpha,\beta,\gamma,d}(t,\cdot)(\xi) = t^{\beta+\gamma-1} E_{\beta,\beta+\gamma}(-\tfrac{1}{2}\nu t^\beta |\xi|^\alpha), \tag{2.20}$$
$$\mathcal{F}Z^*_{\alpha,\beta,d}(t,\cdot)(\xi) = t^{k} E_{\beta,k+1}(-\tfrac{1}{2}\nu t^\beta |\xi|^\alpha), \quad \text{if } \beta \in (1,2]. \tag{2.21}$$

Moreover, when $\beta \in (0,2)$*, we have the following explicit expressions:*

$$Z(t,x) = \pi^{-\frac{d}{2}} t^{\lceil\beta\rceil-1} |x|^{-d} H_{2,3}^{2,1}\left(\frac{|x|^\alpha}{2^{\alpha-1}\nu t^\beta} \,\middle|\, \begin{matrix} (1,1),\, (\lceil\beta\rceil,\beta) \\ (d/2,\alpha/2),\, (1,1),\, (1,\alpha/2) \end{matrix} \right), \tag{2.22}$$

$$Y(t,x) = \pi^{-\frac{d}{2}} |x|^{-d} t^{\beta+\gamma-1} H_{2,3}^{2,1}\left(\frac{|x|^\alpha}{2^{\alpha-1}\nu t^\beta} \,\middle|\, \begin{matrix} (1,1),\, (\beta+\gamma,\beta) \\ (d/2,\alpha/2),\, (1,1),\, (1,\alpha/2) \end{matrix} \right), \tag{2.23}$$

and, if $\beta \in (1,2)$*,*

$$Z^*(t,x) = \pi^{-\frac{d}{2}} |x|^{-d} H_{2,3}^{2,1}\left(\frac{|x|^\alpha}{2^{\alpha-1}\nu t^\beta} \,\middle|\, \begin{matrix} (1,1),\, (1,\beta) \\ (d/2,\alpha/2),\, (1,1),\, (1,\alpha/2) \end{matrix} \right), \tag{2.24}$$

where $H_{2,3}^{2,1}\left(\cdots \,\middle|\, \begin{smallmatrix} \cdots \\ \cdots \end{smallmatrix}\right)$ *refers to the Fox* H*-function* [40].

Proof. The proof follows a standard argument using the Fourier and Laplace transforms in the space and time variables, respectively, which are denoted by $\widehat{f}$ and $\widetilde{g}$. Let us apply the Fourier transform to (2.16) first to obtain

$$\begin{cases} \partial_t^\beta \widehat{u}(t,\xi) + \dfrac{\nu}{2}|\xi|^\alpha \widehat{u}(t,\xi) = I_t^\gamma \left[\widehat{f}(t,\xi)\right], & \xi \in \mathbb{R}^d, \\ \dfrac{\partial^k}{\partial t^k}\widehat{u}(t,\xi)\bigg|_{t=0} = \widehat{u}_k(\xi), & 0 \le k \le \lceil\beta\rceil - 1,\, \xi \in \mathbb{R}^d. \end{cases}$$

Apply the Laplace transform on the Caputo derivative using [31, Theorem 7.1 on p. 134]:

$$\mathcal{L}\left[\partial_t^\beta\, \widehat{u}(t,\xi)\right](s) = s^\beta\, \widetilde{\widehat{u}}(s,\xi) - \sum_{k=0}^{\lceil\beta\rceil-1} s^{\beta-1-k}\, \widehat{u}_k(\xi).$$

On the other hand, it is known that (see, e.g., [48, (7.14) on p. 140])

$$\mathcal{L}I_t^\gamma\left[\widehat{f}(t,\xi)\right] = s^{-\gamma}\widetilde{\widehat{f}}(s,\xi), \quad \Re(\gamma) > 0.$$

Thus,

$$\widetilde{\widehat{u}}(s,\xi) = \left(s^\beta + \frac{\nu}{2}|\xi|^\alpha\right)^{-1} \left[\sum_{k=0}^{\lceil\beta\rceil-1} s^{\beta-1-k}\,\widehat{u}_k(\xi) + s^{-\gamma}\widetilde{\widehat{f}}(s,\xi)\right].$$

Notice that (see, e.g., [46, (1.80) on p. 21])

$$\mathcal{L}\left[t^{\beta-1}E_{\alpha,\beta}(-\lambda t^\alpha)\right](s) = \frac{s^{\alpha-\beta}}{s^\alpha+\lambda}, \quad \text{for } \Re(s) > |\lambda|^{1/\alpha}.$$

Hence,

$$\widehat{u}(t,\xi) = \sum_{k=0}^{\lceil\beta\rceil-1} t^k E_{\beta,k+1}\left(-\frac{\nu}{2}|\xi|^\alpha t^\beta\right)\widehat{u}_k(\xi) + \int_0^t \mathrm{d}\tau\, \tau^{\beta+\gamma-1}E_{\beta,\beta+\gamma}\left(-\frac{\nu}{2}|\xi|^\alpha\tau^\beta\right)\widehat{f}(t-\tau,\xi).$$

Now if we denote

$$U(t,\xi) := t^{\lceil\beta\rceil-1}E_{\beta,\lceil\beta\rceil}\left(-\frac{\nu}{2}|\xi|^\alpha t^\beta\right), \tag{2.25}$$

using the fact that ${}_tD_{0+}^\gamma = \frac{\mathrm{d}^\gamma}{\mathrm{d}t^\gamma}$ when $\gamma\in\mathbb{Z}$ and for all $\gamma\in\mathbb{R}$ (see [46, (1.82) on p. 21])

$${}_tD_{0+}^\gamma\left(t^{\beta-1}E_{\alpha,\beta}(\lambda t^\alpha)\right) = t^{\beta-\gamma-1}E_{\alpha,\beta-\gamma}(\lambda t^\alpha),$$

we see that

$$\widehat{u}(t,\xi) = \sum_{k=0}^{\lceil\beta\rceil-1}\left(\frac{\mathrm{d}^k}{\mathrm{d}t^k}U(t,\xi)\right)\widehat{u}_{\lceil\beta\rceil-1-k}(\xi) + \int_0^t \mathrm{d}\tau\,\left({}_tD_{0+}^{\lceil\beta\rceil-\beta-\gamma}U(\tau,\xi)\right)\widehat{f}(t-\tau,\xi).$$

It remains to prove the expressions in (2.22) – (2.24) under the assumption that $\beta\in(0,2)$. A key observation is that for $Z_{\alpha,\beta,d}(t,x)$ defined in (2.22), its Fourier transform is given by $U(t,\xi)$ in (2.25), namely,

$$\mathcal{F}Z_{\alpha,\beta,d}(t,\cdot)(\xi) = U(t,\xi), \quad \text{for all } \alpha>0,\ \beta\in(0,2), \text{ and } d\ge 1. \tag{2.26}$$

Indeed, (2.26) is proved in Lemma 4.2 of [13], but only for the case of $\alpha\in(0,2]$. Here we claim that the restriction of $\alpha\in(0,2]$ is not necessary. In the proof of this lemma, one needs to consider two cases separately: $d=1$ and $d\ge 2$. In the case of $d=1$, the conditions we need are

$$\frac{2-\beta}{\alpha} > 0 \quad\text{and}\quad 1\wedge\alpha>0.$$

For the second case – $d\ge 2$, the proof is a direct application of Corollary 2.5.1 of [40], where one needs to verify the following conditions:

Conditions in [40]	the corresponding conditions in our setting
$a^*>0$	$2-\beta>0$
(2.6.8)	$\min(\alpha,d)>0$
(2.6.9)	$d>1$
(2.6.10)	$d>1$

Apparently, the above two conditions hold for all $\alpha > 0$ and $\beta \in (0,2)$. Hence, Lemma 4.2 of [13] is true for all $\alpha > 0$ and $\beta \in (0,2)$. This proves both (2.19) and (2.22). Once one obtains the expressions for $Z_{\alpha,\beta,d}(t,x)$ and $\mathcal{F}Z_{\alpha,\beta,d}(t,\cdot)(\xi)$, it is routine to obtain the corresponding expressions of their fractional or integer derivatives/integrals; see [13] for more details. This completes the proof of Theorem 2.8. □

Remark 2.9. For the case $\beta = 2$, the expression in (2.19) can be simplified using the fourth expression in (2.5).

Remark 2.10. The corresponding parameters (a^*, Δ, δ) in (2.9), (2.10), and (2.11) for the Fox H-functions $Z(1,x)$, $Z^*(1,x)$ and $Y(1,x)$ are all the same:

$$a^* = 2-\beta, \quad \Delta = \alpha - \beta, \quad \text{and} \quad \delta = 2^{-\alpha}\left(2^{\alpha/2}\alpha^{\alpha/2} + \alpha^{\alpha}\right)\beta^{-\beta}.$$

The corresponding μ parameters (2.12) are equal to, respectively,

$$\frac{1}{2}(-2\lceil\beta\rceil + d + 1), \quad \frac{d-1}{2}, \quad \text{and} \quad \frac{1}{2}(-2\beta - 2\gamma + d + 1).$$

Remark 2.11. When $\beta \in (0,2)$, we have the condition $a^* = 2-\beta > 0$. Then thanks to part (iii) of Theorem 1.2 in [40], the Fox H-functions $Z(1,x)$, $Z^*(1,x)$ and $Y(1,x)$ are nontrivial analytic functions for $x \neq 0$, where the expressions can be obtained by using residue calculus. By the identity theorem (see [1, Theorem 3.2.6]) and the fact that analytic functions $Z(1,x)$, $Z^*(1,x)$ and $Y(1,x)$ are not identical to zero, the support of analytic functions $Z(1,x)$, $Z^*(1,x)$ and $Y(1,x)$ is the whole pace. In this context, we have omitted the verification of condition (1.1.6) from the same reference, which confirms the absence of overlapping poles. These conditions have been verified in [13]. For an in-depth discussion on checking these conditions, we refer the interested reader to the symbolic computational tools and the corresponding documentation in [12]. It should be noted that these functions potentially exhibit singular behavior at $x = 0$. The comprehensive analysis of their asymptotic properties near zero has been given in Lemma 4.3 and Remark 4.4 of [14].

3. Existence and uniqueness of the solution for the nonlinear equation

In this section, we shall establish the well-posedness of (1.12) by working under slightly more general settings as follows. For $\alpha > 0$, $\beta \in (0,2]$, $\gamma \geq 0$, $\nu > 0$, and $\dot{W}$ as in (1.12), consider

(3.1)
$$\begin{cases} \left(\partial_t^\beta + \frac{\nu}{2}(-\Delta)^{\alpha/2}\right)u(t,x) = I_t^\gamma\left[\rho\left(u(t,x)\right)\dot{W}(t,x)\right], & t>0,\ x\in\mathbb{R}^d, \\ u(0,x) = u_0(x), & x\in\mathbb{R}^d, \text{ if } \beta\in(0,1], \\ u(0,x) = u_0(x), \quad \dfrac{\partial}{\partial t}u(0,x) = u_1(x), & x\in\mathbb{R}^d, \text{ if } \beta\in(1,2], \end{cases}$$

where ρ is Lipschitz continuous and $u_0,\ u_1 \in L^\infty\left(\mathbb{R}^d\right)$. The fundamental solutions for (3.1), as well as (1.12), consist of three components: $Z_{\alpha,\beta,d}(t,x)$, $Z^*_{\alpha,\beta,d}(t,x)$ and $Y_{\alpha,\beta,\gamma,d}(t,x)$, which have been studied in [14, Theorem 4.1] for the case when $\beta \in (0,2)$ and $\alpha \in (0,2]$. The more general setting, namely, the case when $\alpha > 0$

and $\beta \in (0,2]$, is proved in Theorem 2.8. Throughout the rest of the article, we will write

$$p(t,x) := Y_{\alpha,\beta,\gamma,d}(t,x), \tag{3.2}$$

whose Fourier transform is given in (2.20).

The solution to the homogeneous equation of (3.1) is given by

$$\begin{aligned} &J_0(t,x) \\ &\quad = \begin{cases} \int_{\mathbb{R}^d} Z_{\alpha,\beta,d}(t,x-y)u_0(y)\mathrm{d}y, & \text{if } \beta \in (0,1], \\ \int_{\mathbb{R}^d} Z^*_{\alpha,\beta,d}(t,x-y)u_0(y)\mathrm{d}y + \int_{\mathbb{R}^d} Z_{\alpha,\beta,d}(t,x-y)u_1(y)\mathrm{d}y, & \text{if } \beta \in (1,2]. \end{cases} \end{aligned} \tag{3.3}$$

When the initial conditions u_0 and u_1 are two constants, then by (2.19) and (2.21), $J_0(t,x)$ does not depend on x and hence is denoted by $J_0(t)$ later on:

$$J_0(t) = \begin{cases} u_0 \mathcal{F} Z_{\alpha,\beta,d}(t,\cdot)(0) = u_0, & \text{if } \beta \in (0,1], \\ u_0 \mathcal{F} Z^*_{\alpha,\beta,d}(t,\cdot)(0) + u_1 \mathcal{F} Z_{\alpha,\beta,d}(t,\cdot)(0) = u_0 + u_1 t, & \text{if } \beta \in (1,2]. \end{cases} \tag{3.4}$$

Definition 3.1. A process $u = \left\{u(t,x) : t \ge 0, x \in \mathbb{R}^d\right\}$ is called a *random field solution* to (3.1) if it is adapted to the filtration $\{\mathcal{F}_t\}_{t\ge 0}$, jointly measurable with respect to $\mathcal{B}\left((0,\infty)\times\mathbb{R}^d\right)\times\mathcal{F}$, square integrable in the sense that

$$\int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}y\, p(t-s,x-y)^2 \mathbb{E}\left[\rho\left(u(s,y)\right)^2\right] < \infty, \quad \text{for all } t>0 \text{ and } x \in \mathbb{R}^d,$$

and satisfies the following integral equation a.s.

$$u(t,x) = J_0(t,x) + \int_0^t \int_{\mathbb{R}^d} p(t-s,x-y)\rho\left(u(s,y)\right) W(\mathrm{d}s,\mathrm{d}y), \tag{3.5}$$

for all $t>0$ and $x \in \mathbb{R}^d$, where $J_0(t,x)$ is given by (3.3) and the stochastic integral on the right-hand side is the *Walsh integral* [50].

The existence and uniqueness of the mild solution to (3.1) with the bounded initial conditions are well covered by the classical Dalang-Walsh theory; see [25, 26, 50]. In that theory, *Dalang's condition* usually refers to some simplified, but still equivalent, conditions to

$$\int_0^t \int_{\mathbb{R}^d} p(s,y)^2 \mathrm{d}y\mathrm{d}s < \infty, \quad \text{for all } t > 0. \tag{3.6}$$

Note that condition (3.6) is the necessary and sufficient condition for the existence and uniqueness of a global solution for the corresponding linear equation, i.e., the case when $\rho(u) \equiv 1$ in (3.1). Lemma 3.2 finds out the explicit form of Dalang's condition for (1.12) and (3.1), which extends Lemma 5.3 of [14] from the case $\alpha \in (0,2]$ and $\beta \in (0,2)$ to the case $\alpha > 0$ and $\beta \in (0,2]$.

Lemma 3.2 (Dalang's condition)**.** *For the SPDE* (3.1)*, Dalang's condition* (3.6) *is equivalent to* (1.16)*.*

Proof. By (2.20) and the Parseval-Plancherel identity, we have

$$\begin{aligned}\int_{\mathbb{R}^d} |p(s,x)|^2 \mathrm{d}x &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} |\widehat{p}(s,\xi)|^2 \mathrm{d}\xi \\ &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} s^{2(\beta+\gamma)-2} E^2_{\beta,\beta+\gamma}\left(-\tfrac{1}{2}\nu|\xi|^\alpha s^\beta\right)\mathrm{d}\xi \\ &= s^{2(\beta+\gamma)-2-\beta d/\alpha}\frac{1}{(2\pi)^d}(2^{-1}\nu)^{-d/\alpha}\int_{\mathbb{R}^d} E^2_{\beta,\beta+\gamma}\left(-|\eta|^\alpha\right)\mathrm{d}\eta, \qquad (3.7)\end{aligned}$$

where in the last step we have used the change of variable $\xi = (2^{-1}\nu s^\beta)^{-1/\alpha}\eta$. Then clearly, Dalang's condition (3.6) is equivalent to

$$\begin{cases} 2(\beta+\gamma)-2-\frac{\beta}{\alpha}d > -1 \Longleftrightarrow d < 2\alpha + \frac{\alpha}{\beta}(2\gamma-1), \\ \int_{\mathbb{R}^d} E^2_{\beta,\beta+\gamma}(-|\xi|^\alpha)\mathrm{d}\xi < \infty. \end{cases} \qquad (3.8)$$

To characterize the second condition in (3.8), noting that $E^2_{\beta,\beta+\gamma}(-|\cdot|)$ is locally integrable, it suffices to know the asymptotic behavior of $E^2_{\beta,\beta+\gamma}(-|\xi|^\alpha)$ as $|\xi|\to\infty$. By Lemma 2.6, as $|\xi|\to\infty$

(3.9)

$$E_{\beta,\beta+\gamma}(-|\xi|^\alpha) = \begin{cases} \dfrac{1}{\Gamma(\gamma)|\xi|^\alpha} + o(|\xi|^{-\alpha}), & \beta\in(0,2), \\ \dfrac{\cos\left(\sqrt{|\xi|^\alpha}-\pi(\gamma+1)/2\right)}{|\xi|^{\alpha(1+\gamma)/2}} + \dfrac{1}{\Gamma(\gamma)|\xi|^\alpha} + o\left(|\xi|^{-\alpha}\right), & \beta = 2. \end{cases}$$

Then, for $\beta\in(0,2)$, condition (3.8) is equivalent to

$$\begin{cases} d < 2\alpha + \frac{\alpha}{\beta}(2\gamma-1) \text{ and } d < 2\alpha, & \text{if } \gamma > 0, \\ d < 2\alpha - \frac{\alpha}{\beta}, & \text{if } \gamma = 0, \end{cases} \iff d < 2\alpha + \frac{\alpha}{\beta}\min\{2\gamma-1, 0\},$$

where the equivalence for the case $\gamma > 0$ is straightforward by (3.9) since $\frac{1}{\Gamma(\gamma)} \neq 0$, and for the case $\gamma = 0$, the equivalence follows from the facts that the first condition $d < 2\alpha - \frac{\alpha}{\beta}$ in (3.8) implies $\int_{\mathbb{R}^d} E^2_{\beta,\beta+\gamma}(-|\xi|^\alpha)\mathrm{d}\xi < \infty$ by (3.9) and that $\frac{1}{\Gamma(\gamma)} = 0$ by (2.7).

For the case $\beta = 2$, by (3.9), we have as $|\xi|\to\infty$,

$$\begin{aligned} E^2_{\beta,\beta+\gamma}(-|\xi|^\alpha) =& \frac{\cos^2\left(\sqrt{|\xi|^\alpha}-\pi(\gamma+1)/2\right)}{|\xi|^{\alpha(1+\gamma)}} + \frac{1}{\Gamma^2(\gamma)|\xi|^{2\alpha}} \\ &+ 2\frac{\cos\left(\sqrt{|\xi|^\alpha}-\pi(\gamma+1)/2\right)}{\Gamma(\gamma)|\xi|^{\alpha(3+\gamma)/2}} + o\left(|\xi|^{-\alpha\min(\frac{3+\gamma}{2},2)}\right). \end{aligned}$$

Thus, if $\gamma > 0$, the second condition in (3.8) is equivalent to, for any $\varepsilon > 0$,

$$\int_{|\xi|>\varepsilon} \frac{\cos^2\left(\sqrt{|\xi|^\alpha}-\pi(\gamma+1)/2\right)}{|\xi|^{\alpha(1+\gamma)}}\mathrm{d}\xi < \infty \quad \text{and} \quad \int_{|\xi|>\varepsilon}\frac{1}{|\xi|^{2\alpha}}\mathrm{d}\xi < \infty,$$

where the first condition is equivalent to $\alpha(1+\gamma) > d$ by Lemma B.1, and the second one is $2\alpha > d$. If $\gamma = 0$, the second condition in (3.8) is equivalent to $\alpha > d$.

Therefore, when $\beta = 2$, we have that (3.8) is equivalent to

$$\begin{cases} d<\alpha(\gamma+\frac{3}{2}) \quad \text{and} \quad d<\alpha\min\{2,1+\gamma\}, & \text{if } \gamma>0, \\ d<\alpha(\gamma+\frac{3}{2}) \quad \text{and} \quad d<\alpha, & \text{if } \gamma=0, \end{cases} \iff \quad d<\alpha\min\{2,1+\gamma\}.$$

This completes the proof of Lemma 3.2. □

Under Dalang's condition, it is routine (see, e.g., Theorem 13 of [26] or the proof of Theorem 2.4 of [9]) to establish Theorem 3.3 regarding the existence and uniqueness of the solution to (3.1), the proof of which will be left for the interested readers.

Theorem 3.3. *Under Dalang's condition* (1.16)*, if the initial conditions are bounded, namely, u_0 and $u_1 \in L^\infty(\mathbb{R}^d)$, then there exists a unique (in the sense of versions) random field solution $u(t,x)$ to* (3.1)*, which is $L^2(\Omega)$-continuous with bounded p-th moments on any finite time interval:*

$$\sup_{0\le t\le T}\sup_{x\in\mathbb{R}^d}\mathbb{E}\left[|u(t,x)|^p\right] < \infty, \quad \text{for all } p\ge 2 \text{ and } T>0. \tag{3.10}$$

Before the end of this section, we make some remarks:

Remark 3.4 (Rough initial data). The main focus of this paper is the exact moment formula with constant initial condition. Theorem 3.3 presents the existence and uniqueness of the solution in a slightly more general setting, which still falls in the classical Dalang-Walsh theory. For the measure-valued initial conditions, such as the Dirac delta initial condition, more efforts are needed and property (3.10) no longer holds; see [7–9, 14, 17].

Remark 3.5 (Hölder regularity). In [14] and [11], the space-time Hölder regularity of the solution to (1.12) has been obtained (in the case of $\alpha \in (0,2]$ and $\beta \in (0,2)$). It is an interesting open problem to extend the Hölder regularity results in [14] and [11] to the more general setting, namely, $\alpha > 0$ and $\beta \in (0,2]$.

Remark 3.6 (Second moment comparison for nonlinear SPDEs). Let $u(t,x)$ be the solution to (3.1) as stated in Theorem 3.3. Suppose that ρ is Lipschitz continuous and satisfies the following cone condition with some constants $0 \le \underline{\lambda} \le \overline{\lambda}$:

$$\underline{\lambda}|x| \le |\rho(x)| \le \overline{\lambda}|x|, \quad \text{for all } x \in \mathbb{R}.$$

Then by denoting the right-hand side of (1.17) by $f_\lambda(t)$, the moment formula in Theorem 1.1 can be extended directly to this case by the following moment comparison principle for the second moment:

$$f_{\underline{\lambda}}(t) \le \mathbb{E}\left[u(t,x)^2\right] \le f_{\overline{\lambda}}(t). \tag{3.11}$$

When the noise is white in time but colored in space (see (1.7)), the moment comparison principle (for $p \ge 2$) or more generally the stochastic comparison principle becomes much more involved and the parabolic nature of the equation will play an important role. Hence, one can in principle only handle the case when $\beta = 1$. One may check the work along this line in [16–18]. However, for the space-time white noise case, the second moment comparison as in (3.11) comes for free. Note that when the noise is colored in time (see (1.9)), to the best of our knowledge, one can only handle the linear case, namely, $\rho(u) = \lambda u$. In this case, the moment comparison principle can be easily established by comparing the moments chaos by chaos.

Remark 3.7 (Wiener chaos expansion). When $\rho(u) = \lambda u$, instead of using Dalang-Walsh theory, one can equivalently establish the solution to (1.12) using the Wiener chaos expansion specified as follows: Set $u_0(t,x) = J_0(t)$ (see (3.4)) and for $n \geq 1$,

$$\begin{aligned} & u_n(t,x) \\ & = J_0(t) + \sum_{k=1}^{n} \int_{[0,t]^k} \int_{\mathbb{R}^{kd}} g_k(s_1,\dots,s_k,x_1,\dots,x_k;t,x) W(\mathrm{d}s_1,\mathrm{d}x_1)\cdots W(\mathrm{d}s_k,\mathrm{d}x_k), \end{aligned}$$

where

$$\begin{aligned} & g_k(s_1,\dots,s_k,x_1,\dots,x_k;t,x) \\ & = \lambda^k p(t-s_k,x-x_k)p(s_k-s_{k-1},x_k-x_{k-1})\cdots p(s_2-s_1,x_2-x_1)J_0(s_1)\mathbb{1}_{\{0<s_1<\cdots<s_k<t\}} \\ & = \lambda^k \prod_{r=1}^{k} p(s_{r+1}-s_r,x_{r+1}-x_r)J_0(s_1)\mathbb{1}_{\{0<s_1<\cdots<s_k<t\}}, \end{aligned} \tag{3.12}$$

where we use the convention $s_{k+1} = t$ and $x_{k+1} = x$. Then, the mild solution has the following so-called Wiener chaos representation:

$$u(t,x) = J_0(t) + \sum_{k=1}^{\infty} I_k(f_k(\cdot;t,x)), \tag{3.13}$$

where $f_k(\cdot;t,x)$ is the symmetrization of $g_k(\cdot;t,x)$ given by, denoting by $\mathcal{P}_k$ the set of all permutations of $\{1,\dots,k\}$,

$$f_k(s_1,\dots,s_k,x_1,\dots,x_k;t,x) = \frac{1}{k!}\sum_{\sigma\in\mathcal{P}_k} g_k(s_{\sigma(1)},\dots,s_{\sigma(k)},x_{\sigma(1)},\dots,x_{\sigma(k)};t,x) \tag{3.14}$$

and $I_k(f_k(\cdot;t,x))$ denotes the k-th multiple Wiener-Itô integral. We refer the interested readers to [37] for more details.

4. Second moment formula and upper bounds for the p-th moments

In this section, we shall prove parts (a) and (b) of Theorem 1.1.

Proof of part (a) *of Theorem* 1.1. By the Itô-Walsh isometry, we have

$$\mathbb{E}\left[u^2(t,x)\right] = J_0^2(t) + \lambda^2 \int_0^t \int_{\mathbb{R}^d} p^2(t-s,x-y)\mathbb{E}\left[u^2(s,y)\right]\mathrm{d}s\mathrm{d}y,$$

where $J_0(t)$ is given by (3.4). Note that due to the choice of the constant initial conditions, the solution to the homogeneous equation does not depend on x, i.e., $J_0(t,x) = J_0(t)$. Hence, through a standard Picard iteration, one can show that the second moment $\mathbb{E}\left(u(t,x)^2\right)$ does not depend on x. Let $\eta(t) = \mathbb{E}\left(u(t,x)^2\right)$. Invoking (3.7), the above equation can be written as

$$\eta(t) = J_0^2(t) + \lambda^2\Theta \int_0^t (t-s)^\theta \eta(s)\mathrm{d}s, \tag{4.1}$$

where θ and Θ are given in (1.14). Now we solve the fractional integral equation (4.1) for $\beta \in (0,1]$ and for $\beta \in (1,2]$ separately.

Case 1. When $\beta \in (0,1]$, we have $J_0(t) = u_0$ by (3.4) and thus (4.1) is equivalent to

$$\begin{cases} \left(D_{0+}^{\theta+1}\eta\right)(t) = \lambda^2\Theta\Gamma(\theta+1)\eta(t) + (D_{0+}^{\theta+1}u_0^2)(t), \\ \eta(0) = u_0^2 \quad \text{and} \quad \eta^{(k)}(0) = 0, \quad \text{for } k = 1,2,\cdots,\lceil\theta\rceil, \end{cases}$$

where $D^{\theta+1}$ is Riemann-Liouville derivative given in Definition 2.2. Using the Caputo fractional derivative given in Definition 2.5, it can be written as

$$\begin{cases} \left({}^C D_{0+}^{\theta+1}\eta\right)(t) = \lambda^2\Theta\Gamma(\theta+1)\eta(t). \\ \eta(0) = u_0^2 \quad \text{and} \quad \eta^{(k)}(0) = 0, \quad \text{for } k = 1,2,\ldots,\lceil\theta\rceil, \end{cases}$$

of which the solution is directly given by (2.4):

$$\eta(t) = u_0^2 E_{\theta+1}\left(\lambda^2\hat{t}\,\right).$$

This proves the first part of (1.17).

Case 2. When $\beta \in (1,2]$, we have $J_0(t) = u_0 + u_1 t$ by (3.4) and (4.1) now is

$$\eta(t) = u_0^2 + 2u_0u_1t + u_1^2t^2 + \lambda^2\Theta\int_0^t (t-s)^\theta\eta(s)\mathrm{d}s. \tag{4.2}$$

Let $f(t) := 2u_0u_1t + u_1^2t^2$ then (4.1) can be written as

$$\begin{cases} (D_{0+}^{\theta+1}\eta)(t) = \lambda^2\Theta\Gamma(\theta+1)\eta(t) + (D_{0+}^{\theta+1}[u_0^2 + f(\cdot)])(t), \\ \eta(0) = u_0^2, \eta^{(1)}(0) = 2u_0u_1, \eta^{(2)}(0) = 2u_1^2, \\ \eta^{(k)}(0) = 0, \quad \text{for } k = 3,\ldots,\lceil\theta\rceil. \end{cases} \tag{4.3}$$

In order to apply the formula (2.4), we will transform (4.3) into a Caputo fractional differential equation. When $\theta+1 \in (0,1)$, by (2.2), we can write (4.3) as

$$\begin{cases} {}^C D_{0+}^{\theta+1}\eta(t) = \lambda^2\Theta\Gamma(\theta+1)\eta(t) + (D_{0+}^{\theta+1}f)(t). \\ \eta(0) = u_0^2. \end{cases}$$

The solution now follows directly from (2.4):

(4.4)

$$\eta(t) = u_0^2E_{\theta+1}\left(\lambda^2\hat{t}\,\right) + \int_0^t (t-s)^\theta E_{\theta+1,\theta+1}\left(\lambda^2\Theta\Gamma(\theta+1)(t-s)^{\theta+1}\right)(D_{0+}^{\theta+1}f)(s)\mathrm{d}s.$$

For the integral on the right-hand side, by (1.15) and Lemma 2.3 we have

$$\begin{aligned} &\int_0^t (t-s)^\theta E_{\theta+1,\theta+1}\left(\lambda^2\Theta\Gamma(\theta+1)(t-s)^{\theta+1}\right)(D_{0+}^{\theta+1}f)(s)\mathrm{d}s \\ &\quad= \int_0^t (t-s)^\theta \sum_{k=0}^\infty \frac{(\lambda^2\Theta\Gamma(\theta+1))^k}{\Gamma((k+1)(\theta+1))}(t-s)^{k(\theta+1)}(D_{0+}^{\theta+1}f)(s)\mathrm{d}s \\ &\quad= \sum_{k=0}^\infty (\lambda^2\Theta\Gamma(\theta+1))^k \left(I_{0+}^{(\theta+1)(k+1)}D_{0+}^{\theta+1}f\right)(t) \\ &\quad= \sum_{k=0}^\infty (\lambda^2\Theta\Gamma(\theta+1))^k \left(I_{0+}^{k(\theta+1)}f\right)(t). \end{aligned}$$

The term $\left(I_{0+}^{k(\theta+1)} f\right)(t) = \left(I_{0+}^{k(\theta+1)}(2u_0u_1s + u_1^2s^2)\right)(t)$ can be computed explicitly noting that Lemma 2.4 yields
(4.5)

$$\left(I_{0+}^{k(\theta+1)} s\right)(t) = \frac{t^{k(\theta+1)+1}}{\Gamma(k(\theta+1)+2)} \quad \text{and} \quad \left(I_{0+}^{k(\theta+1)} s^2\right)(t) = \frac{2t^{k(\theta+1)+2}}{\Gamma(k(\theta+1)+3)}.$$

Combining (4.4)–(4.5) and applying (1.15), we arrive at

$$\eta(t) = u_0^2 E_{\theta+1}\left(\lambda^2 \hat{t}\,\right) + 2u_0u_1 t E_{\theta+1,2}\left(\lambda^2 \hat{t}\,\right) + 2u_1^2 t^2 E_{\theta+1,3}\left(\lambda^2 \hat{t}\,\right).$$

This proves the second part of (1.17) for $\theta + 1 \in (0,1)$. For the other two cases $\theta+1 \in [1,2)$ and $\theta+1 \geq 2$, one can calculate in a similar way and prove the desired result. Finally, (1.18) is a direct consequence of Lemma 2.6. This completes the proof of part (a) of Theorem 1.1.

□

Remark 4.1 (Another approach). Alternatively, one can also solve (4.1) directly by an application of Lemma B.2 as follows:

$$\eta(t) = J_0^2(t) + \int_0^t J_0^2(s) K(t-s) \mathrm{d}s,$$

where $J_0(t)$ is given in (3.4) and the resolvent kernel function $K(\cdot)$ is given by

$$K(t) = \lambda^2 \Theta \Gamma(\theta+1) t^\theta E_{\theta+1,\theta+1}\left(\lambda^2 \hat{t}\,\right).$$

Thus we have, denoting $A = \lambda^2 \Theta \Gamma(\theta+1)$,

$$\eta(t) = J_0^2(t) + A \int_0^t J_0^2(s)(t-s)^\theta E_{\theta+1,\theta+1}\left(A(t-s)^{\theta+1}\right) \mathrm{d}s.$$

When $J_0(t) = u_0$, we have by the definition (1.15) of $E_{a,b}$,

$$\begin{aligned}
\eta(t) &= u_0^2 + u_0^2 \sum_{k=0}^\infty \frac{A^{k+1}}{\Gamma((\theta+1)(k+1))} \int_0^t (t-s)^{(\theta+1)k+\theta} \mathrm{d}s \\
&= u_0^2 + u_0^2 \sum_{k=0}^\infty \frac{A^{k+1} t^{(\theta+1)(k+1)}}{\Gamma((\theta+1)(k+1)+1)} = u_0^2 + u_0^2 \sum_{k=1}^\infty \frac{A^k t^{(\theta+1)k}}{\Gamma\left((\theta+1)k+1\right)} \\
&= u_0^2 E_{\theta+1}\left(\lambda^2 \hat{t}\,\right).
\end{aligned}$$

This proves the equality of (1.17) for $\beta \in (0,1]$. Applying similar computations to the case $J_0(t) = u_0 + u_1 t$, we can justify the second part of (1.17) for $\beta \in (1,2]$. Indeed, the p-th moment upper bounds will be obtained using this approach in the next proof.

Proof of part (b) *of Theorem* 1.1. Fix an arbitrary $p \geq 2$. By (3.5) we have

$$\|u(t,x)\|_p \leq |J_0(t)| + \left(\mathbb{E}\left[\left|\int_0^t \int_{\mathbb{R}^d} p(t-s,x-y)\lambda u(s,y) W(\mathrm{d}s,\mathrm{d}y)\right|^p\right]\right)^{1/p}.$$

Applying the Burkholder-Davis-Gundy inequality, we have

$$\|u(t,x)\|_p \leq |J_0(t)| + C_p \left(\mathbb{E}\left[\left(\int_0^t \int_{\mathbb{R}^d} p^2(t-s,x-y)\lambda^2 u^2(s,y) \mathrm{d}s\mathrm{d}y\right)^{p/2}\right]\right)^{1/p},$$

where C_p is the universal constant in the Burkholder-Davis-Gundy inequality satisfying $C_p \in (0, 2\sqrt{p})$ and $C_p = (2 + o(1))\sqrt{p}$ as $p \to \infty$ (see [4,23]). By Minkowski's inequality, we get

$$\|u(t,x)\|_p \le |J_0(t)| + 2\sqrt{p}\left(\int_0^t \int_{\mathbb{R}^d} \lambda^2 p^2(t-s, x-y) \|u(s,y)\|_p^2 \mathrm{d}s\mathrm{d}y\right)^{1/2}.$$

Denote $\psi(t) = \|u(t,x)\|_p^2$, which does not depend on x since the solution is stationary. Recall the definitions of θ and Θ in (1.14). Hence, the above integral inequality can be rewritten as

$$\psi(t) \le 2\left(J_0^2(t) + 4p\lambda^2\Theta \int_0^t (t-s)^\theta \psi(s)\mathrm{d}s\right).$$

Applying Lemma B.2, we have

$$\psi(t) \le 2J_0^2(t) + 2\int_0^t J_0^2(s) K(t-s)\mathrm{d}s,$$

where

$$K(t) = 8p\lambda^2\Theta\Gamma(\theta+1)t^\theta E_{\theta+1,\theta+1}\left(8p\lambda^2\hat{t}\,\right).$$

Then, one can apply the same computations as those in Remark 4.1 to simplify the above ds integral in order to obtain (1.19). Finally, (1.21a) and (1.21b) follow from Lemma 2.6 directly. This proves part (b) of Theorem 1.1. □

5. Lower bounds for the p-th moments

Compared with the upper bound for the p-th moment, the computation for the lower bound is more involved. The methodology used in this section is inspired by the recent work of Hu and Wang [38]. Some ideas are originated from Dalang and Mueller [27].

5.1. **Nondegeneracy and positivity of the fundamental functions.** In Proposition 5.1, we prove a nondegeneracy property of the fundamental solutions, which is tailored specially for the spatial white noise. Conditions for the fundamental solutions to be nonnegative are given in Remark 5.3.

Proposition 5.1. *For all $\varepsilon > 0$ and $c > 2\sqrt{2/\nu}$, assuming either*

(1) *the fundamental solution $p(\cdot, \circ)$ is nonnegative and $\beta \in (0,2)$, or*
(2) *$\alpha = \beta = 2$, $\gamma = 0$, and $d = 1$,*

we have

$$\int_{B_\varepsilon(x)} p(t, a-y)p(s, b-y)\mathrm{d}y \ge C\varepsilon^{-d}(ts)^{\beta+\gamma-1}, \tag{5.1}$$

for all $x \in \mathbb{R}^d$, $s,t \in \left[2\sqrt{2/\nu}\varepsilon^{\alpha/\beta},\, c\varepsilon^{\alpha/\beta}\right]$, and $a,b \in B_\varepsilon(x)$, where $C > 0$ is a constant independent of (a,b,s,t,x,ε).

Proof. Denote the integral in (5.1) by I. We first estimate I under condition (1). In this case, from (2.23), we see that

$$I = \int_{B_\varepsilon(x)} p(t,a-y)p(s,b-y)\mathrm{d}y = \int_{B_\varepsilon(x)} \pi^{-d/2}|y-a|^{-d} t^{\beta+\gamma-1} h\left(\frac{|y-a|^\alpha}{2^{\alpha-1}\nu t^\beta}\right)$$
$$\times \pi^{-d/2}|y-b|^{-d} s^{\beta+\gamma-1} h\left(\frac{|y-b|^\alpha}{2^{\alpha-1}\nu s^\beta}\right)\mathrm{d}y,$$

where
$$h(x) := H^{2,1}_{2,3}\left(x \,\middle|\, \begin{matrix}(1,1),\,(\beta+\gamma,\beta)\\(d/2,\alpha/2),\,(1,1),\,(1,\alpha/2)\end{matrix}\right).$$
Set $I := \pi^{-d}(ts)^{\beta+\gamma-1} I'$. Notice when $\beta \in (0,2)$, the fundamental solution $p(t,x)$ is a smooth function for $t>0$ and $x\neq 0$ and the support of $p(t,x)$ is the whole space; see Remark 2.11 for more details. Moreover, under the nonnegative assumption on $p(t,x)$, denoting $A := \left[2\sqrt{2/\nu},\, c\right]$ and using the change of variable $y=\varepsilon x'$, we have

$$\begin{aligned}
I' &\geq \inf_{\substack{a,b\in B_\varepsilon(x)\\ c_1,c_2\in A}} \int_{B_\varepsilon(x)} |y-a|^{-d} h\left(\frac{|y-a|^\alpha}{2^{\alpha-1}\nu c_1^\beta \varepsilon^\alpha}\right) \times |y-b|^{-d} h\left(\frac{|y-b|^\alpha}{2^{\alpha-1}\nu c_2^\beta \varepsilon^\alpha}\right) \mathrm{d}y\\
&= \inf_{\substack{a',b'\in B_\varepsilon(0)\\ c_1,c_2\in A}} \int_{B_\varepsilon(0)} |y-a'|^{-d} h\left(\frac{|y-a'|^\alpha}{2^{\alpha-1}\nu c_1^\beta \varepsilon^\alpha}\right) \times |y-b'|^{-d} h\left(\frac{|y-b'|^\alpha}{2^{\alpha-1}\nu c_2^\beta \varepsilon^\alpha}\right) \mathrm{d}y\\
&= \varepsilon^{-d} \inf_{\substack{a',b'\in B_\varepsilon(0)\\ c_1,c_2\in A}} \int_{B_1(0)} |x'-a'/\varepsilon|^{-d} h\left(\frac{|x'-a'/\varepsilon|^\alpha}{2^{\alpha-1}\nu c_1^\beta}\right)\\
&\quad \times |x'-b'/\varepsilon|^{-d} h\left(\frac{|x'-b'/\varepsilon|^\alpha}{2^{\alpha-1}\nu c_2^\beta}\right) \mathrm{d}x'\\
&\geq C\varepsilon^{-d},
\end{aligned}$$

for $s=c_2\varepsilon^{\alpha/\beta}, t=c_1\varepsilon^{\alpha/\beta} \in \left[2\sqrt{2/\nu}\varepsilon^{\alpha/\beta},\, c\varepsilon^{\alpha/\beta}\right]$, where

$$\begin{aligned}
C &= \inf_{\substack{a',b'\in B_\varepsilon(0)\\ c_1,c_2\in A}} \int_{B_1(0)} |x'-a'/\varepsilon|^{-d} h\left(\frac{|x'-a'/\varepsilon|^\alpha}{2^{\alpha-1}\nu c_1^\beta}\right)\\
&\quad \times |x'-b'/\varepsilon|^{-d} h\left(\frac{|x'-b'/\varepsilon|^\alpha}{2^{\alpha-1}\nu c_2^\beta}\right) \mathrm{d}x'\\
&= \inf_{\substack{a',b'\in B_1(0)\\ c_1,c_2\in A}} \int_{B_1(0)} |x'-a'|^{-d} h\left(\frac{|x'-a'|^\alpha}{2^{\alpha-1}\nu c_1^\beta}\right) \times |x'-b'|^{-d} h\left(\frac{|x'-b'|^\alpha}{2^{\alpha-1}\nu c_2^\beta}\right) \mathrm{d}x' > 0.
\end{aligned}$$

This proves (5.1) under condition (1).

Now we assume condition (2). It is well known that when $\alpha=\beta=2$, $\gamma=0$ and $d=1$,
$$p(t,x) = \frac{1}{\sqrt{2\nu}} \mathbb{1}_{\{|x|<\sqrt{\nu/2}\,t\}}.$$
For all $a,b,x'\in B_\varepsilon(x)$ and $2\sqrt{2/\nu}\,\varepsilon \leq s,t \leq c\varepsilon$, we have
$$\mathbb{1}_{\{|x'-a|<\sqrt{\nu/2}\,t\}} \mathbb{1}_{\{|x'-b|<\sqrt{\nu/2}\,s\}} \equiv 1.$$
Hence,
$$I = \int_{B_\varepsilon(x)} \frac{1}{2\nu} \mathbb{1}_{\{|x'-a|<\sqrt{\nu/2}\,t\}} \mathbb{1}_{\{|x'-b|<\sqrt{\nu/2}\,s\}} \mathrm{d}x' = \frac{\varepsilon}{\nu} \geq \frac{\varepsilon^{-1}}{\nu c^2} ts, \tag{5.2}$$
where the inequality holds since $0\leq s,t\leq c\,\varepsilon$. This completes the proof of Proposition 5.1. ∎

Remark 5.2. The nondegeneracy in (5.1) provides a lower bound for the integral of two Green's functions, while the small ball nondegeneracy in [38] concerns the integral of one single Green's function and does not fit in the proof of lower bounds for the white noise case (see Step 3 in the proof of Theorem 5.9). Moreover, we omit the term "small ball" since the radius of $B_\varepsilon(x)$ is not required to be sufficiently small.

Remark 5.3 (Nonnegativity of fundamental solutions). The nonnegativity of Green's functions associated with (1.12) was first proved in [13] for the case $\gamma = 0$, and was later extended in [14, Theorem 4.6] to allow $\gamma \geq 0$; see also Remark 1.2 of [10]. It is known that the Green's function is nonnegative in the following three cases:

$$\begin{cases} (1) \quad \alpha \in (0,2],\ \beta \in (0,1],\ \gamma \geq 0,\ d \geq 1; \\ (2) \quad 1 < \beta < \alpha \leq 2,\ \gamma > 0,\ 1 \leq d \leq 3; \\ (3) \quad 1 < \beta = \alpha < 2,\ \gamma > \dfrac{d+3}{2} - \beta,\ 1 \leq d \leq 3. \end{cases} \tag{5.3}$$

5.2. **Feynman diagram formula.** In this part, we recall the Feynman Diagram formula, which is useful to compute the expectation of products of multiple Wiener-Itô integrals. We refer interested readers to Section 5.3 of [37] for more details about the multiple Wiener-Itô integrals.

On the lattice $\mathbb{Z}^2$, we use (k,ℓ) to denote a vertex, and an ordered pair $[(k_1,\ell_1),(k_2,\ell_2)]$ to denote a directed edge pointing from (k_1,ℓ_1) to (k_2,ℓ_2).

Definition 5.4. Let $p \geq 1$ and $\vec{n} = (n_1,\dots,n_p) \in \mathbb{N}^p$ with $|\vec{n}| = n_1 + \cdots + n_p$ be given. A *Feynman diagram* is a directed graph $\mathcal{D} = (V,E)$ consisting of the set of all vertices

$$V = \Big\{ (k,\ell):\ 1 \leq k \leq p,\ 1 \leq \ell \leq n_k \Big\}$$

and a set E of directed edges satisfying $k_1 < k_2$ if $[(k_1,\ell_1),(k_2,\ell_2)] \in E$. A Feynman diagram $\mathcal{D} = (V,E)$ is called *admissible* if each vertex is associated with one and only one edge. The set of all admissible diagrams is denoted by $\mathbb{D} = \mathbb{D}_{\vec{n}}$.

We shall provide a formula for $\mathbb{E}\big[I_{n_1}(h_1)\dots I_{n_p}(h_p)\big]$ for square integrable functions

$$h_i : \left(\mathbb{R}_+ \times \mathbb{R}^d\right)^{n_i} \to \mathbb{R}, \quad i = 1,\dots,p, \tag{5.4}$$

where $I_{n_i}(h_i)$ refers to the n_i-th multiple Wiener-Itô integral. In particular, given an admissible Feynman diagram $\mathcal{D} \in \mathbb{D}_{\vec{n}}$, for h_i given in (5.4), denote

$$\begin{aligned} F_{\mathcal{D}}(h_1,\dots,h_p) = &\int_{\mathbb{R}_+^{|\vec{n}|}} \int_{\mathbb{R}^{d|\vec{n}|}} \mathrm{d}\mathbf{t}\mathrm{d}\mathbf{x} \prod_{i=1}^{p} h_i\left(t_{(i,1)}, x_{(i,1)}; \dots; t_{(i,n_i)}, x_{(i,n_i)}\right) \\ &\times \prod_{[(k_1,\ell_1),(k_2,\ell_2)] \in E(\mathcal{D})} \delta(t_{(k_1,\ell_1)} - t_{(k_2,\ell_2)}) \delta(x_{(k_1,\ell_1)} - x_{(k_2,\ell_2)}), \end{aligned} \tag{5.5}$$

where we use the notation $\mathrm{d}\mathbf{t}\mathrm{d}\mathbf{x} = \prod_{i=1}^p \prod_{r_i=1}^{n_i} \mathrm{d}t_{(i,r_i)} \mathrm{d}x_{(i,r_i)}$. Then we have (see [38, Theorem 5.3] and [37, Theorems 5.7 and 5.8]),

$$\mathbb{E}\big[I_{n_1}(h_1)\dots I_{n_p}(h_p)\big] = \sum_{\mathcal{D} \in \mathbb{D}_{\vec{n}}} F_{\mathcal{D}}(h_1,\dots,h_p). \tag{5.6}$$

A direct consequence of (5.6) is that $\mathbb{E}\big[I_{n_1}(h_1)\dots I_{n_p}(h_p)\big]=0$ if $|\vec{n}|=n_1+\dots+n_p$ is an odd integer, since the number of vertices in an admissible diagram must be even.

In particular, for any $t>0$ and $x\in\mathbb{R}^d$, considering the multiple Wiener-Itô integrals $I_k\left(f_k\left(\cdot;t,x\right)\right)$ in the chaos expansion (3.13) for the solution $u(t,x)$ with f_k given in (3.14) which is a symmetrization of g_k in (3.12), we have the following result (see [38, Theorem 5.4]):

Lemma 5.5. *Let $p\ge 1$ and $\vec{n}=(n_1,\dots,n_p)\in\mathbb{N}^p$ be given. Fix arbitrary $t>0$ and $x_1,\dots,x_p\in\mathbb{R}^d$. Recall that $f_n(\cdot;t,x)$ and $g_n(\cdot;t,x)$ be given in* (3.14) *and* (3.12)*, respectively. Then*

$$
\begin{aligned}
\mathbb{E}\left[\prod_{\ell=1}^{p} I_{n_\ell}\left(f_{n_\ell}\left(\cdot;t,x_\ell\right)\right)\right] &= \sum_{\mathcal{D}\in\mathbb{D}_{\vec{n}}} F_{\mathcal{D}}\left(g_{n_1}\left(\cdot;t,x_1\right),\dots,g_{n_p}\left(\cdot;t,x_p\right)\right)\\
&= \sum_{\mathcal{D}\in\mathbb{D}_{\vec{n}}}\int_{[0,t]^{|\vec{n}|}}\int_{\mathbb{R}^{d|\vec{n}|}} \mathrm{d}\mathbf{t}\mathrm{d}\mathbf{x}\,\lambda^{|\vec{n}|}\\
&\quad\times\left(\prod_{[(k_1,l_1),(k_2,l_2)]\in E(\mathcal{D})}\delta(t_{(k_1,l_1)}-t_{(k_2,l_2)})\delta(x_{(k_1,l_1)}-x_{(k_2,l_2)})\right)\\
&\quad\times\left(\prod_{i=1}^{p} J_0\left(t_{(i,1)}\right)\mathbb{1}_{\{0<t_{(i,1)}<\dots<t_{(i,n_i)}<t\}}\prod_{r_i=1}^{n_i} p\big(t_{(i,r_i+1)}-t_{(i,r_i)},x_{(i,r_i+1)}-x_{(i,r_i)}\big)\right),
\end{aligned}
\tag{5.7}
$$

where we have used the convention that

$$
\left(t_{(i,n_i+1)},\,x_{(i,n_i+1)}\right)=(t,x_i)\quad \textit{for all } i=1,\cdots p. \tag{5.8}
$$

Example 5.6. Let $\mathcal{D}_1$ (resp. $\mathcal{D}_2$) refer to the red (resp. blue) admissible Feynman diagram in Figure 5.1. Under the setting of Lemma 5.5, let

$$
F_i=F_{\mathcal{D}_i}\left(g_1\left(\cdot;t,x\right),g_2\left(\cdot;t,x\right),g_2\left(\cdot;t,x\right),g_3\left(\cdot;t,x\right)\right)\quad i=1,2.
$$

Then we claim that $F_1=0$ because its integrand contains the following factor:

$$
\mathbb{1}_{\{0<t_{(3,1)}<t_{(3,2)}<t\}}\mathbb{1}_{\{0<t_{(3,2)}<t_{(3,1)}<t_{(2,2)}<t\}}
$$

which is identically equal to zero. Hence, due to the delta potentials and the simplex conditions in (5.7), edges starting from one column should not cross with each other. This is the case for F_2:

$$\begin{aligned}
F_2 = & \int_{[0,t]^4} \mathrm{d}t_{(1,1)}\mathrm{d}t_{(2,1)}\mathrm{d}t_{(2,2)}\mathrm{d}t_{(3,2)} \int_{\mathbb{R}^{4d}} \mathrm{d}x_{(1,1)}\mathrm{d}x_{(2,1)}\mathrm{d}x_{(2,2)}\mathrm{d}x_{(3,2)} \\
& \times J_0\left(t_{(1,1)}\right) \mathbb{1}_{\{0<t_{(1,1)}<t\}} p_{t-t_{(1,1)}}\left(x-x_{(1,1)}\right) \\
& \times J_0\left(t_{(2,1)}\right) \mathbb{1}_{\{0<t_{(2,1)}<t_{(2,2)}<t\}} p_{t-t_{(2,2)}}\left(x-x_{(2,2)}\right) p_{t_{(2,2)}-t_{(2,1)}}\left(x_{(2,2)}-x_{(2,1)}\right) \\
& \times J_0\left(t_{(1,1)}\right) \mathbb{1}_{\{0<t_{(1,1)}<t_{(3,2)}<t\}} p_{t-t_{(3,2)}}\left(x-x_{(3,2)}\right) p_{t_{(3,2)}-t_{(1,1)}}\left(x_{(3,2)}-x_{(1,1)}\right) \\
& \times J_0\left(t_{(2,1)}\right) \mathbb{1}_{\{0<t_{(2,1)}<t_{(2,2)}<t_{(3,2)}<t\}} \\
& \times p_{t-t_{(3,2)}}\left(x-x_{(3,2)}\right) p_{t_{(3,2)}-t_{(2,2)}}\left(x_{(3,2)}-x_{(2,2)}\right) p_{t_{(2,2)}-t_{(2,1)}}\left(x_{(2,2)}-x_{(2,1)}\right) \\
= & \int_{[0,t]^4} \mathrm{d}t_{(1,1)}\mathrm{d}t_{(2,1)}\mathrm{d}t_{(2,2)}\mathrm{d}t_{(3,2)} \int_{\mathbb{R}^{4d}} \mathrm{d}x_{(1,1)}\mathrm{d}x_{(2,1)}\mathrm{d}x_{(2,2)}\mathrm{d}x_{(3,2)} \\
& \times J_0^2\left(t_{(1,1)}\right) \mathbb{1}_{\{0<t_{(1,1)}<t_{(3,2)}<t\}} p_{t-t_{(1,1)}}\left(x-x_{(1,1)}\right) p_{t_{(3,2)}-t_{(1,1)}}\left(x_{(3,2)}-x_{(1,1)}\right) \\
& \times J_0^2\left(t_{(2,1)}\right) \mathbb{1}_{\{0<t_{(2,1)}<t_{(2,2)}<t_{(3,2)}<t\}} p_{t-t_{(2,2)}}\left(x-x_{(2,2)}\right) \\
& \times p^2_{t-t_{(3,2)}}\left(x-x_{(3,2)}\right) p_{t_{(3,2)}-t_{(2,2)}}\left(x_{(3,2)}-x_{(2,2)}\right) p^2_{t_{(2,2)}-t_{(2,1)}}\left(x_{(2,2)}-x_{(2,1)}\right).
\end{aligned}$$

Note that the original 2×8-multiple integral has been collapsed to the above 2×4-multiple integral. The remaining variables are the roots of all edges in $E(\mathcal{D})$.

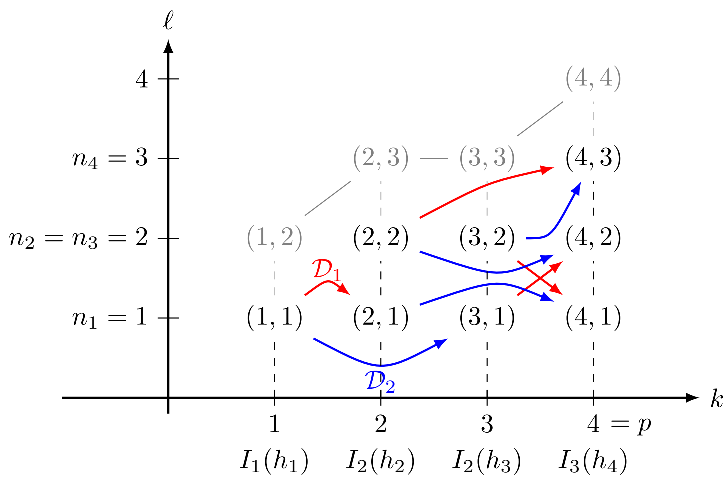


FIGURE 5.1. Two admissible (red $\mathcal{D}_1$ and blue $\mathcal{D}_2$) Feynman diagrams for the case when $p = 4$, $\vec{n} = (1,2,2,3)$ and $|\vec{n}| = 8$; see Example 5.6. Convention (5.8) applies at the gray vertices in the settings of Lemma 5.5.

Definition 5.7. For any $m \in \mathbb{N}$ and $p \in 2\mathbb{N}$, we say that $\vec{n} = (n_1, \ldots, n_p)$ is a *balanced partition* of $2m$ if

(1) $|\vec{n}| = 2m$;
(2) $n_i \in \{m_p, m_p + 1\}$ for all $i = 1, \ldots, p$, where $m_p := \lfloor 2m/p \rfloor$;
(3) $n_1 + \cdots + n_{p/2} = m$;
(4) $r_p \in [0, p)$ is the remainder of $2m/p$.

Moreover, under this setting, an admissible Feynman diagram $\mathcal{D} = (V, E)$ is called a *balanced diagram* provided

$$[(k_1, \ell_1), (k_2, \ell_2)] \in E(\mathcal{D}) \quad \Longrightarrow \quad \ell_1 = \ell_2 \quad \text{and} \quad k_1 \le p/2 < k_2.$$

The set of all balanced diagrams is denoted by $\mathbb{D}_{\vec{n}}^{=}$. It is clear that $\mathbb{D}_{\vec{n}}^{=} \subset \mathbb{D}_{\vec{n}}$.

It is straightforward to show the existence of a balanced partition, which is however not unique in general. Let us check a few examples:

Example 5.8.
(1) In Figure 5.1, we have $p = m = 4$. The partition $\vec{n} = (1, 2, 2, 3)$ is not a balanced partition. Indeed, for this example, the only balanced partition is $\vec{n} = (2, 2, 2, 2)$.
(2) If $p = 4$ and $m = 3$, the following partitions are all balanced:

$$(1, 2, 2, 1)\,, \quad (2, 1, 2, 1)\,, \quad (1, 2, 1, 2)\,.$$

However, $(1, 1, 2, 2)$ is not balanced.
(3) If $p = 6$ and $m = 7$, it is easy to check that $\vec{n} = (3, 2, 2, 2, 3, 2)$ is a balanced partition, upon which a balanced diagram is given; see Figure 5.2.

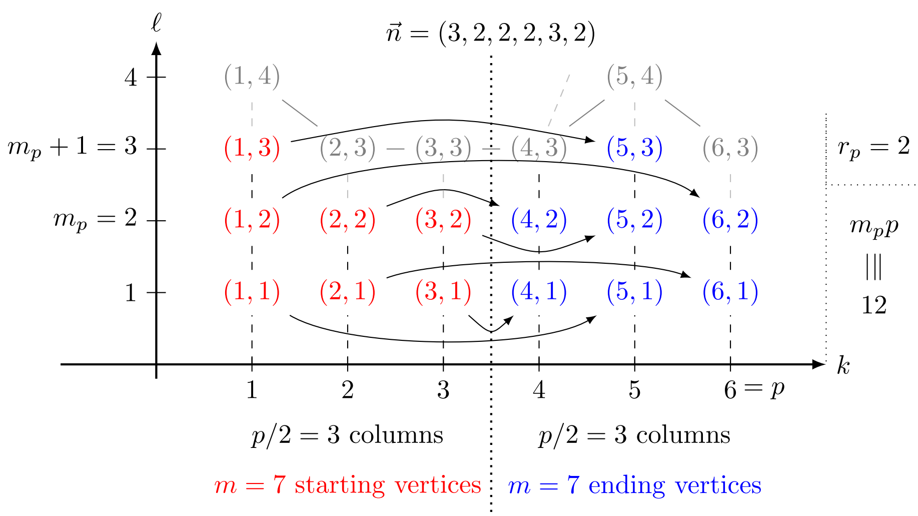


FIGURE 5.2. One example of the balanced partition in case of $m = 7$ and $p = 6$ with a balanced diagram (all edges are horizontal starting from the left half of the vertices pointing to the right half). The grayed-out vertices correspond to the convention (5.8).

5.3. **Proof of the lower bounds.** In this subsection, we prove part (c) of Theorem 1.1, i.e., we derive a lower bound for $\mathbb{E}\left[|u(t, x)|^p\right]$ which is consistent with the upper bound obtained in part (b) of Theorem 1.1. We restate part (c) of Theorem 1.1 in Theorem 5.9:

Theorem 5.9. *Assume that*

(1) *either $\beta \in (0, 2)$ and the fundamental function $p(t, x)$ is nonnegative or $\alpha = \beta = 2$ and $\gamma = 0$;*

(2) *the initial position* u_0 *is strictly positive and the initial velocity* u_1 *is nonnegative;*
(3) *Dalang's condition* (1.16) *is satisfied.*

Then we have for all $t>0$, $x\in\mathbb{R}^d$, *and* $p\ge 2$ *such that* $t\,p^{\frac{1}{\theta+1}}$ *is sufficiently large (recall that* θ *is given in* (1.14)*), there exists a constant* c *that does not depend on* (t,x,p) *such that*

$$\mathbb{E}\left[|u(t,x)|^p\right]\ge u_0^p\exp\left(c|\lambda|^{\frac{2}{\theta+1}}p^{1+\frac{1}{\theta+1}}t\right). \tag{5.9}$$

Proof. The proof is based on the Feynman diagram formula for the p-th moments and the non-degeneracy property of the Green's function – Proposition 5.1, which is inspired by [38, Theorem 3.6]. In the following, we will first consider the case when $p\ge 2$ is an even integer. The extension to the general case $p\ge 2$ will be done at the end of the proof. Now choose an arbitrary $p\in 2\mathbb{N}$ and let $t>0$ and $x\in\mathbb{R}^d$ be fixed. By (3.13), we have

$$\begin{aligned}
\mathbb{E}\left[|u(t,x)|^p\right]&=\mathbb{E}\left[\prod_{j=1}^p\sum_{n_j=0}^\infty I_{n_j}\left(f_{n_j}(\cdot,t,x)\right)\right]\\
&=\sum_{n_1=0}^\infty\cdots\sum_{n_p=0}^\infty\mathbb{E}\Big[I_{n_1}(f_{n_1}(\cdot,t,x))\dots I_{n_p}(f_{n_p}(\cdot,t,x))\Big]\\
&=\sum_{m=0}^\infty\sum_{\substack{\vec{n}\in\mathbb{N}^p\\|\vec{n}|=2m}}\sum_{\mathcal{D}\in\mathbb{D}_{\vec{n}}}F_{\mathcal{D}}\left(g_{n_1}(\cdot;t,x),\dots,g_{n_p}(\cdot;t,x)\right),
\end{aligned}\tag{5.10}$$

where we have used the convention that $I_0(f_0(\cdot,t,x))=J_0(t)$. We will find the lower bound in three steps:

Step 1. We first take care of the three summations in (5.10):

(i) in the first summation of (5.10), we require $m\ge p/2$;
(ii) in the second summation of (5.10), we only consider the balanced partitions of $2m$;
(iii) in the third summation of (5.10), we restrict us to the balanced diagrams $\mathbb{D}_{\vec{n}}^{=}$.

Applying the Feynman diagram formula in Lemma 5.5 and noting that $\inf_{s\in[0,t]}J_0(s)\ge u_0$ (see (3.4)), we have

$$\mathbb{E}\left[|u(t,x)|^p\right]\ge u_0^p\sum_{m=p/2}^\infty\sum_{\substack{\vec{n}\in\mathbb{N}^p\\|\vec{n}|=2m\\\vec{n}\text{ is balanced}}}\sum_{\mathcal{D}\in\mathbb{D}_{\vec{n}}^{=}}\lambda^{2m}\,I_0, \tag{5.11}$$

with

(5.12)

$$\begin{aligned}
I_0:=\int_{[0,t]^{2m}}\int_{\mathbb{R}^{2md}}&\left(\prod_{[(k_1,l_1),(k_2,l_2)]\in E(\mathcal{D})}\delta\left(t_{(k_1,l_1)}-t_{(k_2,l_2)}\right)\delta\left(x_{(k_1,l_1)}-x_{(k_2,l_2)}\right)\right)\\
&\times\prod_{i=1}^p\mathbb{1}_{\left\{0<t_{(i,1)}<\dots<t_{(i,n_i)}<t\right\}}\prod_{r_i=1}^{n_i}p\Big(t_{(i,r_i+1)}-t_{(i,r_i)},x_{(i,r_i+1)}-x_{(i,r_i)}\Big)\mathrm{d}\mathbf{t}\mathrm{d}\mathbf{x},
\end{aligned}$$

where we have used the assumption that $p(t,x)$ is nonnegative and the convention (5.8).

One may check Figure 5.2 for some examples of the selected Feynman diagrams. Recall that $m_p = \lfloor 2m/p \rfloor$ and r_p is the remainder of $2m/p$ (see Definition 5.7), namely,

$$2m = m_p \times p + r_p, \quad \text{with } 0 \le r_p < p.$$

It is easy to see that r_p has to be an even integer. With these restrictions, for each fixed $m \ge p/2$, one can check that the total number of diagrams satisfying (ii) and (iii) is at least $((p/2)!)^{m_p} \times (r_p/2)!$.

Step 2. Now we proceed to shrink the integral region for the d$\mathbf{t}$-integral of I_0 in (5.12) as follows: Denote $L = \frac{t}{m_p+1}$, $t_i = \frac{(2i-1)t}{2(m_p+1)}$, $a_i = t_i - L/4$ and $b_i = t_i + L/4$ for $i = 1, \ldots, m_p + 1$. Let $I_i = [a_i, b_i]$. Then these intervals I_i are disjoint with length $L/2$; See Figure 5.3 for an illustration.

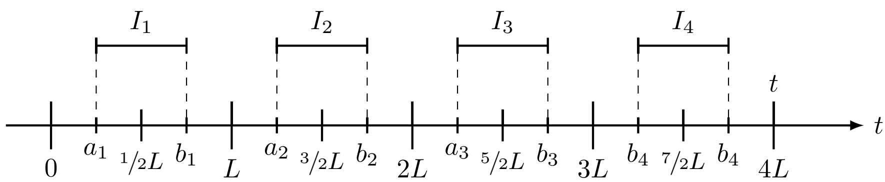


FIGURE 5.3. Some illustrations for Step 2 in the proof of Theorem 5.9 with $m_p = 4$

For the integral with respect to time variables in (5.12), we only integrate on the region where for each $i \in \{1, \ldots, p\}$, $t_{(i,r_i)}$ is in I_{r_i} for $1 \le r_i \le n_i$, and hence

$$\frac{t}{2(m_p+1)} = \frac{1}{2}L \le t_{(i,r_i+1)} - t_{(i,r_i)} \le \frac{3}{2}L = \frac{3t}{2(m_p+1)}. \tag{5.13}$$

Then, for each integer $m \ge p/2$, by choosing

$$\varepsilon := \left(\sqrt{\frac{\nu}{2}}\frac{pt}{16m}\right)^{\beta/\alpha}, \tag{5.14}$$

we have that

$$t_{(i,r_i+1)} - t_{(i,r_i)} \in \left[2\sqrt{\frac{2}{\nu}}\,\varepsilon^{\alpha/\beta}, 12\sqrt{\frac{2}{\nu}}\varepsilon^{\alpha/\beta}\right], \quad i = 1, \ldots, p. \tag{5.15}$$

Step 3. Now we study the spatial integral portion of I_0 in (5.12), which is equal to

$$\int_{\mathbb{R}^{2md}} \mathrm{d}\mathbf{x} \prod_{i=1}^{p} \prod_{r_i=1}^{n_i} p\big(t_{(i,r_i+1)} - t_{(i,r_i)}, x_{(i,r_i+1)} - x_{(i,r_i)}\big)$$
$$\times \left(\prod_{[(k_1,l_1),(k_2,l_2)] \in E(\mathcal{D})} \delta(x_{(k_1,l_1)} - x_{(k_2,l_2)})\right).$$

It is bounded from below if one replaces the integral region $\mathbb{R}^{2md}$ by $\big(B_\varepsilon^2(x)\big)^m$ for any $\varepsilon > 0$. In particular, by Step 2, we see that $\mathbf{t}$ satisfies $t_{(i,r_i)} \in I_{r_i}$ for

$1 \le r_i \le n_i, 1 \le i \le p$, i.e., (5.15) holds true. Hence, we can apply Proposition 5.1 with ε given in (5.14) and $c = 12\sqrt{2/\nu}$ to bound the above integral from below as follows:

$$\geq C^m\,(pt/m)^{-\beta md/\alpha}\prod_{i=1}^{p}\prod_{r_i=1}^{n_i}|t_{(i,r_i+1)}-t_{(i,r_i)}|^{\beta+\gamma-1}. \tag{5.16}$$

Note that here we have used the convention (5.8) with the critical choice of all x_i being the same. Therefore, we can find a lower bound of I_0 in (5.12) with only time integral:

(5.17)

$$\begin{aligned} I_0 \geq C^m\,(pt/m)^{-\beta md/\alpha}\int_{[0,t]^{2m}}\mathrm{d}\mathbf{t}\prod_{i=1}^{p}\prod_{r_i=1}^{n_i}|t_{(i,r_i+1)}-t_{(i,r_i)}|^{\beta+\gamma-1}\mathbb{I}_{\{t_{(i,r_i)}\in I_{r_i}\}} \\ \times\left(\prod_{[(k_1,l_1),(k_2,l_2)]\in E(\mathcal{D})}\delta(t_{(k_1,l_1)}-t_{(k_2,l_2)})\right). \end{aligned}$$

Step 4. Finally, we will carry out the remaining $\mathrm{d}\mathbf{t}$-integral in (5.17) and complete the proof. We will use C to denote a generic constant that does not depend on (t,p,m) and whose value may change at each appearance. Now denote the integral in (5.17) by I, which can be bounded from below as follows:

$$\begin{aligned} I \geq\; & C^m L^{2m(\beta+\gamma-1)}\int_{[0,t]^{2m}}\mathrm{d}\mathbf{t}\prod_{i=1}^{p}\prod_{r_i=1}^{n_i}\mathbb{I}_{\{t_{(i,r_i)}\in I_{r_i}\}} \\ & \times\left(\prod_{[(k_1,l_1),(k_2,l_2)]\in E(\mathcal{D})}\delta(t_{(k_1,l_1)}-t_{(k_2,l_2)})\right) \\ =\; & C^m L^{2m(\beta+\gamma-1)}\left(\frac{L}{2}\right)^m = C^m\left(\frac{t}{m_p}\right)^{m(2\beta+2\gamma-1)}. \end{aligned}$$

Replace the space-time integral in (5.11) by the above lower bound, together with the factor in front of the integral in (5.17), to see that

$$\begin{aligned} \mathbb{E}\left[|u(t,x)|^p\right] \geq\; & u_0^p\sum_{m\geq p/2}\sum_{\substack{\vec{n}\in\mathbb{N}^p\\ |\vec{n}|=2m\\ \vec{n}\text{ is balanced}}}\sum_{\mathcal{D}\in\mathbb{D}_{\vec{n}}^{=}} C^m\lambda^{2m}\left(\frac{pt}{m}\right)^{-\beta dm/\alpha}\left(\frac{t}{m_p}\right)^{m(2\beta+2\gamma-1)} \\ \geq\; & u_0^p\sum_{m\geq p/2}C^m\lambda^{2m}\left(\frac{pt}{m}\right)^{-\beta dm/\alpha}\left(\frac{t}{m_p}\right)^{m(2\beta+2\gamma-1)}((p/2)!)^{m_p}\times(r_p/2)!, \end{aligned} \tag{5.18}$$

where we have used the fact that there are at least $((p/2)!)^{m_p}\times(r_p/2)!$ terms in the double summations.

Thanks to the following well known bounds to the Gamma function, which is related to the Stirling formula (see, e.g., 5.1.10 on p. 141 of [45])

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n < n! < 2\sqrt{2\pi n}\left(\frac{n}{e}\right)^n, \quad \text{for all } n\geq 1, \tag{5.19}$$

we see that up to a constant, one can replace $n!$ by $\sqrt{2\pi n}\,(n/e)^n$. Hence, by (5.19) and the fact $(r_p/p)^{r_p/2}\geq e^{-p/2e}\geq C^m$ for $m\geq p/2$ if we choose $C\leq e^{-1/e}$, we have for $m \geq p/2$,

$$((p/2)!)^{m_p} \cdot (r_p/2)! \geq C^m \, (p)^{(p\, m_p)/2} \cdot (p)^{r_p/2} \, (r_p/p)^{r_p/2}$$
$$\geq C^m p^{\frac{p \times m_p + r_p}{2}} = C^m p^m.$$

Then bound t/m_p in (5.18) from below by $pt/(2m)$ to see that

$$\mathbb{E}\left[|u(t,x)|^p\right] \geq u_0^p \sum_{m \geq p/2} C^m \lambda^{2m} \left(\frac{pt}{m}\right)^{-\beta d m/\alpha} \left(\frac{pt}{2m}\right)^{m(2\beta+2\gamma-1)} p^m$$
$$\geq u_0^p \sum_{m \geq p/2} \left(\frac{\left(C|\lambda|^{\frac{2}{\theta+1}} p^{1+\frac{1}{\theta+1}} t\right)^m}{m!}\right)^{(\theta+1)}. \tag{5.20}$$

Let $n := C|\lambda|^{\frac{2}{\theta+1}} p^{1+\frac{1}{\theta+1}} t$. If $p^{\frac{1}{\theta+1}} t$ is sufficiently large such that $p^{\frac{1}{\theta+1}} t \geq \frac{1}{2} C^{-1} |\lambda|^{-\frac{2}{\theta+1}}$, clearly we have $n \geq p/2$. Noting $\theta + 1 > 0$, for $p \in 2\mathbb{N}$, we get

$$\mathbb{E}\left[|u(t,x)|^p\right] \geq u_0^p \sum_{m \geq p/2} \left(\frac{n^m}{m!}\right)^{(\theta+1)} \geq u_0^p \sum_{m \geq n} \left(\frac{n^m}{m!}\right)^{(\theta+1)}$$
$$\geq u_0^p \exp\left(c|\lambda|^{\frac{2}{\theta+1}} p^{1+\frac{1}{\theta+1}} t\right),$$

where the third inequality follows from Lemma B.4.

Finally, for a real number $p \geq 2$, let p^* be the biggest even integer that is not greater than p. By Jensen's inequality, we have

$$\mathbb{E}\left[|u(t,x)|^p\right] \geq \left(\mathbb{E}[|u(t,x)|^{p*}]\right)^{p/p^*} \geq u_0^p \exp\left((1/2)^{\frac{1}{\theta+1}} c|\lambda|^{\frac{2}{\theta+1}} p^{1+\frac{1}{\theta+1}} t\right).$$

This completes the proof of Theorem 5.9. □

For the wave equation (1.3), a more careful computation in the proof of Theorem 5.9 together with (1.20) provides a relatively precise estimation as given in Proposition 5.10:

Proposition 5.10. *For the wave equation* (1.3) *with constant initial conditions, we have that*

$$C_1 \leq \liminf_{pt^2 \to \infty} t_p^{-1} \log \mathbb{E}\left[|u(t,x)|^p\right] \leq \limsup_{pt^2 \to \infty} t_p^{-1} \log \mathbb{E}\left[|u(t,x)|^p\right] \leq C_2,$$

where C_1 and C_2 are given in (1.11)*, and the lower bound holds provided additionally that $u_0 > 0$ and $u_1 \geq 0$. In particular, by freezing t or p, one obtains the inequalities in* (1.10).

Proof. In the SWE case where $\alpha = \beta = 2, \gamma = 0$, (5.14) and (5.15) become

$$\varepsilon := \sqrt{\frac{\nu}{2} \frac{pt}{16m}} \quad \text{and} \quad t_{(i,r_i+1)} - t_{(i,r_i)} \in \left[2\sqrt{\frac{2}{\nu}}\, \varepsilon, 12\sqrt{\frac{2}{\nu}} \varepsilon\right],$$
$$i = 1, \ldots, p, \quad \text{respectively.}$$

By (5.2), the constant C in (5.1) is equal to $2^{-1}12^{-2}$. Hence, C in both (5.16) and (5.17) is equal to $3^{-2}(2\nu)^{-1/2}$. Noting (5.13), the integral in (5.17), denoted by I, can be bounded from below by

$$I \geq (L/2)^{2m} \int_{[0,t]^{2m}} \mathrm{d}\mathbf{t} \prod_{i=1}^{p} \prod_{r_i=1}^{n_i} \mathbb{I}_{\{t_{(i,r_i)} \in I_{r_i}\}} \left(\prod_{[(k_1,l_1),(k_2,l_2)] \in E(\mathcal{D})} \delta(t_{(k_1,l_1)} - t_{(k_2,l_2)}) \right)$$

$$= 4^{-m} L^{2m} \left(\frac{L}{2}\right)^{m} \geq 64^{-m} \left(\frac{t}{m_p}\right)^{3m},$$

where the last inequality holds since for $m \geq p/2$, $\frac{L}{2} = \frac{t}{2(m_p+1)} \geq \frac{t}{4m_p}$. Replacing the term I_0 in (5.11) by the above lower bound, together with the factor in front of the integral in (5.17), we see that, for $p \in 2\mathbb{N}$,

(5.21)

$$\mathbb{E}\left[|u(t,x)|^p\right] \geq u_0^p \sum_{m \geq p/2} \sum_{\substack{\vec{n} \in \mathbb{N}^p \\ |\vec{n}| = 2m \\ \vec{n} \text{ is balanced}}} \sum_{\mathcal{D} \in \mathbb{D}_{\vec{n}}^{=}} \left(\frac{1}{24^2\sqrt{2\nu}}\right)^m \lambda^{2m} \left(\frac{pt}{m}\right)^{-m} \left(\frac{t}{m_p}\right)^{3m}$$

$$\geq u_0^p \sum_{m \geq p/2} \left(\frac{1}{24^2\sqrt{2\nu}}\right)^m \lambda^{2m} \left(\frac{pt}{m}\right)^{-m} \left(\frac{t}{m_p}\right)^{3m} ((p/2)!)^{m_p} \times (r_p/2)!.$$

By (5.19) and the fact that $(r_p/p)^{r_p/2} \geq e^{-p/(2e)}$, we have that

$$((p/2)!)^{m_p} \times (r_p/2)! \geq (2e)^{-m} (p)^{(p\, m_p)/2} \times (p)^{r_p/2} (r_p/p)^{r_p/2}$$
$$\geq (2e)^{-m} e^{-p/(2e)} p^{(p \times m_p + r_p)/2} = (2e)^{-m} e^{-p/(2e)} p^m.$$

Then because $\frac{t}{m_p} \geq \frac{pt}{2m}$, the estimate in (5.21) becomes

$$\mathbb{E}\left[|u(t,x)|^p\right] \geq \left(u_0\, e^{-1/(2e)}\right)^p \sum_{m \geq p/2} \left(\frac{1}{2 \times 24^2 e\sqrt{2\nu}}\right)^m \lambda^{2m} \left(\frac{pt}{m}\right)^{-m} \left(\frac{t}{m_p}\right)^{3m} p^m$$

$$\geq \left(u_0\, e^{-1/(2e)}\right)^p \sum_{m \geq p/2} \left(\frac{1}{96^2 e\sqrt{2\nu}}\right)^m \lambda^{2m} \left(\frac{pt}{m}\right)^{2m} p^m$$

$$\geq \left(u_0\, e^{-1/(2e)}\right)^p \sum_{m \geq p/2} \left(\frac{\left(C|\lambda| p^{3/2} t\right)^m}{m!}\right)^2,$$

where the last inequality is obtained through $m^m \leq e^m m!$ and

$$C = 96^{-1} e^{-3/2} (2\nu)^{-1/4}.$$

Let $n \coloneqq C|\lambda|p^{3/2}t$. If $p^{1/2}t$ is sufficiently large, we must have $p/2 \leq n$. Hence, for $p \in 2\mathbb{N}$,

$$\mathbb{E}\left[|u(t,x)|^p\right] \geq \left(u_0\, e^{-1/(2e)}\right)^p \sum_{m \geq n} \left(\frac{n^m}{m!}\right)^2 \geq \left(u_0\, e^{-1/(2e)}\right)^p \exp\left(c\nu^{-1/4} |\lambda| p^{3/2} t\right),$$

where the last inequality follows from Lemma B.4 with c being any positive number strictly less than $2C = 48^{-1} e^{-3/2} 2^{-1/4}$. Similarly as in the end of the proof of Theorem 5.9, for a real number $p \geq 2$ with p^* being the biggest even integer that is not greater than p, we have

$$\mathbb{E}\left[|u(t,x)|^p\right] \geq \left(\mathbb{E}[|u(t,x)|^{p^*}]\right)^{p/p^*} \geq \left(u_0\, e^{-1/(2e)}\right)^p \exp\left((1/2)^{1/2} c\nu^{-1/4} |\lambda| p^{3/2} t\right).$$

The upper bound C_2 follows directly from (1.20) and Example A.2. This completes the proof of Proposition 5.10. □

Appendix A. Examples and discussions

In this section, we give some concrete examples for the main result Theorem 1.1. We will use C_1 and C_2 to denote generic constants which are independent of t and p.

Example A.1 (SHE). When $\alpha = 2$, $\beta = 1$, $\gamma = 0$ and $d = 1$, equation (1.12) reduces to SHE (1.2). In this case, Dalang's condition (1.16) is satisfied and

$$\theta = -1/2, \quad \Theta = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-\nu|\xi|^2}\mathrm{d}\xi = \frac{1}{\sqrt{4\pi\nu}}, \quad \hat{t} = \frac{\sqrt{t}}{\sqrt{4\nu}}, \quad \text{and} \quad t_p = p^3 t.$$

(1) *Second moment formula:* The second moment formula (1.17) reduces to

$$\mathbb{E}[u^2(t,x)] = u_0^2\, E_{1/2}\left(\frac{\lambda^2}{\sqrt{4\nu}}t^{1/2}\right) = 2u_0^2\, e^{\frac{\lambda^4 t}{4\nu}}\,\Phi\left(\frac{\lambda^2 t^{1/2}}{\sqrt{2\nu}}\right), \tag{A.1}$$

where we have applied (2.5). This formula recovers the one obtained in [9] as a special case; see Corollary 2.5. *ibid.*

(2) *Second moment Lyapunov exponent:* From (A.1), we immediately see that

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}[u(t,x)^2] = \frac{\lambda^4}{4\nu}. \tag{A.2}$$

Results obtained by Balan and Song [3] also reduce to this special case with the exact second moment Lyapunov exponent being equal to $1/4$ (where $\lambda = 1$ and $\nu = 1$); see Remark 1.6 *ibid.*

(3) *Moment asymptotics:* Because the heat kernel is nonnegative, we can combine the asymptotics in (1.21) and (1.23) to conclude that

(A.3a)

$$C_1 p^3 \le \liminf_{t\to\infty}\frac{1}{t}\log\mathbb{E}[|u(t,x)|^p] \le \limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}[|u(t,x)|^p] \le C_2 p^3, \qquad p\ge 2,$$

(A.3b)

$$C_1\, t \le \liminf_{p\to\infty}\frac{1}{p^3}\log\mathbb{E}[|u(t,x)|^p] \le \limsup_{p\to\infty} p^{-3}\log\mathbb{E}[|u(t,x)|^p] \le C_2\, t, \qquad t>0.$$

These asymptotics are consistent with the exact asymptotics obtained by X. Chen; see [19, Theorem 1.1, Remark 3.1].

Example A.2 (SWE). When $\alpha = 2$, $\beta = 2$, $\gamma = 0$ and $d = 1$, equation (1.12) reduces to SWE (1.3). In this case, $J_0(t) = u_0 + u_1 t$, Dalang's condition (1.16) is satisfied, and

$$\theta = 1, \quad \Theta = \frac{1}{\pi}\int_0^\infty \frac{\sin\left(\sqrt{\nu/2}\,\xi\right)^2}{(\nu/2)\,\xi^2}\mathrm{d}\xi = \frac{1}{\sqrt{2\nu}}, \quad \hat{t} = \frac{t^2}{\sqrt{2\nu}}, \quad \text{and} \quad t_p = p^{3/2}t,$$

where the first equality in the equation of Θ follows from (1.14), (1.15), and the fact that

$$\begin{aligned} E_{2,2}\left(-2^{-1}\nu|\xi|^2\right) &= \sum_{k=0}^{\infty} \frac{(-1)^k \left(2^{-1}\nu|\xi|^2\right)^k}{\Gamma(2k+2)} = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\sqrt{\nu/2}|\xi|\right)^{2k}}{(2k+1)!} \\ &= \frac{1}{\sqrt{\nu/2}|\xi|} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\sqrt{\nu/2}|\xi|\right)^{2k+1}}{(2k+1)!} = \frac{\sin\left(\sqrt{\nu/2}|\xi|\right)}{\sqrt{\nu/2}|\xi|}, \end{aligned}$$

and the second equality in the equation of Θ is obtained via Lemma B.3.

(1) *Second moment formula:* The second moment formula (1.17) becomes

$$\mathbb{E}\left[u^2(t,x)\right] = u_0^2 E_2\left(\frac{\lambda^2 t^2}{\sqrt{2\nu}}\right) + 2u_0 u_1 t E_{2,2}\left(\frac{\lambda^2 t^2}{\sqrt{2\nu}}\right) + 2u_1^2 t^2 E_{2,3}\left(\frac{\lambda^2 t^2}{\sqrt{2\nu}}\right).$$

Now using (2.6) and the special cases in (2.5), we see that

$$\begin{aligned} \mathbb{E}\left[u^2(t,x)\right] = -\frac{2^{3/2}\nu^{1/2}u_1^2}{\lambda^2} + \left(u_0^2 + \frac{2^{3/2}\nu^{1/2}u_1^2}{\lambda^2}\right)\cosh\left(\frac{|\lambda|t}{(2\nu)^{1/4}}\right) \\ + \frac{2^{5/4}\nu^{1/4}u_0u_1}{|\lambda|}\sinh\left(\frac{|\lambda|t}{(2\nu)^{1/4}}\right), \end{aligned} \tag{A.4}$$

which recovers [7, Corollary 1.1].[1]

(2) *Second moment Lyapunov exponent:* From (A.4), we immediately see that

$$\lim_{t\to\infty} \frac{1}{t}\log \mathbb{E}[u(t,x)^2] = \frac{|\lambda|}{(2\nu)^{1/4}}, \tag{A.5}$$

which has also been obtained by Balan and Song in [3, Remark 1.6].

(3) *Moment asymptotics:* Since the fundamental solution is nonnegative, combining the asymptotics in (1.21) and (1.23) shows (1.10); see Proposition 5.10 for more details. The upper bound in the large-time asymptotics (1.10a) is consistent with [22, Theorem 3.1] and [7, Theorem 2.7].

Example A.3 (SFHE)**.** When $\alpha > 0$, $\beta = 1$, $\gamma = 0$ and $d = 1$, equation (1.12) becomes the following one-dimensional stochastic fractional heat equation:

$$\text{(SFHE)} \quad \begin{cases} \left(\dfrac{\partial}{\partial t} + \dfrac{\nu}{2}(-\Delta)^{\alpha/2}\right) u(t,x) = \lambda u(t,x)\dot{W}(t,x), & t>0, x\in\mathbb{R}, \\ u(0,\cdot) = u_0. \end{cases} \tag{A.6}$$

[1] There is a typo in the paper [7] where the fundamental solution for the wave kernel should be $\frac{1}{2\kappa}\mathbb{1}_{[-\kappa t,\kappa t]}(x)$ instead of $\frac{1}{2}\mathbb{1}_{[-\kappa t,\kappa t]}(x)$; see the equation after (1.2) *ibid.* If one sets $\kappa = 1$ *ibid.* or equivalently sets $\nu = 2$ in the current paper, the results should coincide.

In this case, Dalang's condition (1.16) becomes $\alpha > d = 1$. For $\alpha > 1$, we have

$$\begin{aligned}
\theta =& -\frac{1}{\alpha},\\
\Theta_{\alpha,\nu} =& \frac{1}{2\pi}\int_{\mathbb{R}} e^{-\nu|\xi|^\alpha}\mathrm{d}\xi = \frac{\Gamma\left(1+1/\alpha\right)}{\nu^{1/\alpha}\pi},\\
\hat{t} =& \frac{\Gamma\left(1-1/\alpha\right)\Gamma\left(1+1/\alpha\right)}{\nu^{1/\alpha}\pi} t^{1-1/\alpha} = \Big(\nu^{1/\alpha}\alpha\sin\left(\pi/\alpha\right)\Big)^{-1} t^{1-1/\alpha},\\
t_p =& p^{1+\alpha/(\alpha-1)} t,
\end{aligned}$$

where in computing $\hat{t}$ we have use the reflection formula (2.8).

(1) *Second moment formula:* The second moment formula (1.17) reduces to

$$\mathbb{E}[u^2(t,x)] = u_0^2\, E_{1-1/\alpha}\left(\frac{\lambda^2}{\nu^{1/\alpha}\alpha\sin\left(\pi/\alpha\right)} t^{1-1/\alpha}\right). \tag{A.7}$$

In [9], this equation with $\alpha \in (1,2]$ has been studied with a nonhomogeneous initial condition.

(2) *Second moment Lyapunov exponent:* From (A.7), we immediately see that

$$\lim_{t\to\infty}\frac{1}{t}\mathbb{E}[|u(t,x)|^2] = \left(\frac{\lambda^2}{\nu^{1/\alpha}\alpha\sin\left(\pi/\alpha\right)}\right)^{\alpha/(\alpha-1)}, \qquad \alpha > 1; \tag{A.8}$$

see Figure A.1 for a plot of this expression as a function of α.

(3) *Moment asymptotics:* If $\alpha \in (1,2]$, the fundamental solution is nonnegative (see Remark 5.3), then the asymptotics in (1.21) and (1.23) reduce to

(A.9a)

$$C_1 p^{\frac{2\alpha-1}{\alpha-1}} \le \liminf_{t\to\infty}\frac{\log\mathbb{E}[|u(t,x)|^p]}{t} \le \limsup_{t\to\infty}\frac{\log\mathbb{E}[|u(t,x)|^p]}{t} \le C_2 p^{\frac{2\alpha-1}{\alpha-1}}, \quad p \ge 2,$$

(A.9b)

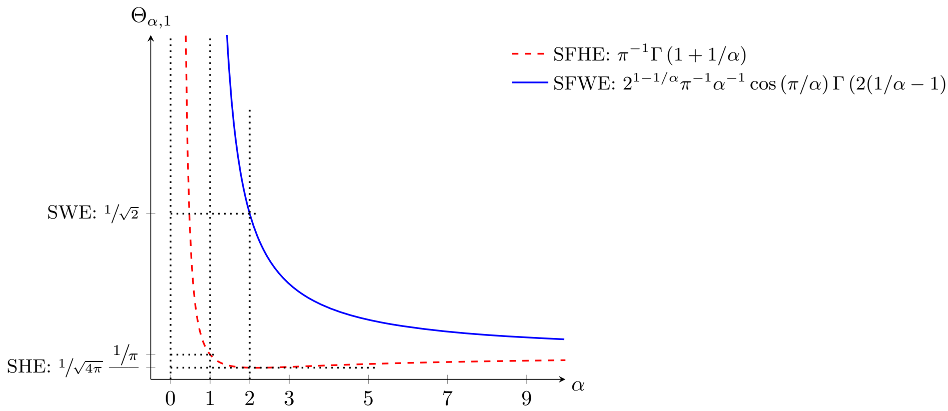
$$C_1\, t \le \liminf_{p\to\infty}\frac{\log\mathbb{E}[|u(t,x)|^p]}{p^{\frac{2\alpha-1}{\alpha-1}}} \le \limsup_{p\to\infty}\frac{\log\mathbb{E}[|u(t,x)|^p]}{p^{\frac{2\alpha-1}{\alpha-1}}} \le C_2\, t, \qquad t > 0.$$

The upper bound in the large-time asymptotics (A.9a) is consistent with [9, Theorem 3.4]. In [20, Theorem 1.1], Chen *et al* obtained the exact large-time asymptotics when the noise is colored in the sense of (1.9). Note also that only the lower bounds in (A.9a) and (A.9b) require the nonnegativity of the fundamental solution. The upper bounds still hold true for all $\alpha > 1$.

Example A.4 (SFWE)**.** For the stochastic fractional wave equation

(A.10)

$$\text{(SFWE)}\quad \begin{cases} \left(\dfrac{\partial^2}{\partial t^2} + \dfrac{\nu}{2}\left(-\Delta\right)^{\alpha/2}\right) u(t,x) = \lambda u(t,x)\dot{W}(t,x), & t > 0, x \in \mathbb{R},\\ u(0,\cdot) = u_0, \quad \dfrac{\partial}{\partial t}u(0,\cdot) = u_1, \end{cases}$$



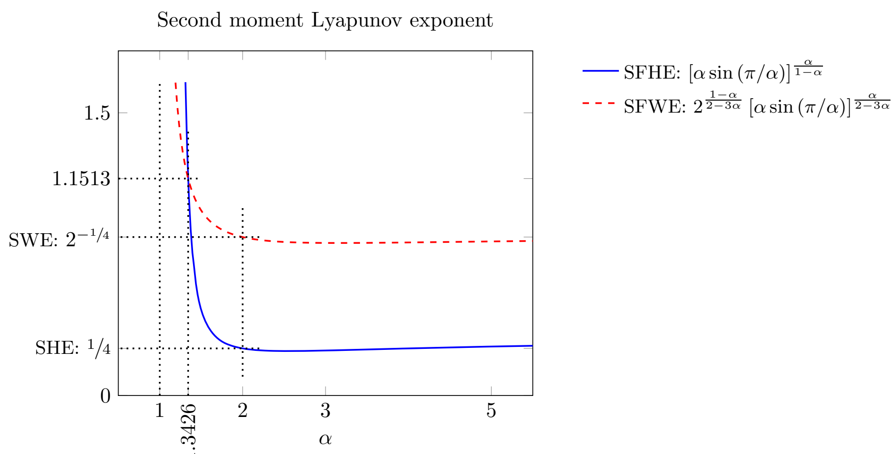


FIGURE A.1. Plots of both $\Theta_{\alpha,\nu}$ and the second moment Lyapunov exponents as functions of $\alpha \in (1,\infty)$ with $\nu = \lambda = 1$ for both SFHE in Example A.3 and SFWE in Example A.4. For the second moment Lyapunov exponents, two curves intersect at $(1.3426, 1.1513)$ via some numerical solver.

i.e., $\alpha > 0$, $\beta = 2$, $\gamma = 0$ and $d = 1$, Dalang's condition (1.16) becomes $\alpha > 1$, and the quantities in (1.14) reduce to

$$
\begin{aligned}
\theta =& 2(1-1/\alpha),\\
\Theta_{\alpha,\nu} =& \frac{1}{\pi}\int_0^\infty \frac{\sin^2\left(\sqrt{\nu/2}\,\xi^{\alpha/2}\right)}{(\nu/2)\,\xi^\alpha}\mathrm{d}\xi = \frac{2^{2-1/\alpha}\cos(\pi/\alpha)\,\Gamma\left(2\left(1/\alpha-1\right)\right)}{\nu^{1/\alpha}\pi\alpha},\\
\hat{t} =& \frac{2^{2-1/\alpha}\cos(\pi/\alpha)\,\Gamma(3-2/\alpha)\,\Gamma(2/\alpha-2)}{\nu^{1/\alpha}\pi\alpha} t^{3-2/\alpha}\\
=& \frac{2^{2-1/\alpha}\cos(\pi/\alpha)}{\nu^{1/\alpha}\sin(2\pi/\alpha)\,\alpha} t^{3-2/\alpha} = \frac{2^{1-1/\alpha}}{\nu^{1/\alpha}\sin(\pi/\alpha)\,\alpha} t^{3-2/\alpha},\\
t_p =& p^{1+\alpha/(3\alpha-2)} t,
\end{aligned}
$$

where we have applied Lemma B.3 and the reflection formula (2.8) in computing Θ and $\hat{t}$, respectively.

(1) *Second moment formula:* By (1.17), the second moment formula is

$$
\begin{aligned}
\mathbb{E}\left[u^2(t,x)\right] = & u_0^2\, E_{3-2/\alpha}\left(\frac{2^{1-1/\alpha}\lambda^2}{\nu^{1/\alpha}\sin(\pi/\alpha)\,\alpha}t^{3-2/\alpha}\right) \\
& + 2u_0u_1 t\, E_{3-2/\alpha,2}\left(\frac{2^{1-1/\alpha}\lambda^2}{\nu^{1/\alpha}\sin(\pi/\alpha)\,\alpha}t^{3-2/\alpha}\right) \\
& + 2u_1^2 t^2\, E_{3-2/\alpha,3}\left(\frac{2^{1-1/\alpha}\lambda^2}{\nu^{1/\alpha}\sin(\pi/\alpha)\,\alpha}t^{3-2/\alpha}\right).
\end{aligned} \tag{A.11}
$$

(2) *Second moment Lyapunov exponent:* From (A.11), we immediately see that

$$
\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}\left[u(t,x)^2\right] = \left(\frac{2^{1-1/\alpha}\lambda^2}{\nu^{1/\alpha}\sin(\pi/\alpha)\,\alpha}\right)^{\alpha/(3\alpha-2)}, \qquad \alpha>1; \tag{A.12}
$$

see Figure A.1 for a plot of this expression as a function of α.

(3) *Moment asymptotics:* The asymptotics in (1.21) shows that

$$
\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}[|u(t,x)|^p] \le C_2 p^{\frac{4\alpha-2}{3\alpha-2}}, \qquad p\ge 2, \tag{A.13a}
$$

$$
\limsup_{p\to\infty} p^{-\frac{4\alpha-2}{3\alpha-2}}\log\mathbb{E}[|u(t,x)|^p] \le C_2\, t, \qquad t>0. \tag{A.13b}
$$

The large-time asymptotics in (A.13a) is consistent with Proposition 4.1 of [49]. Since we don't know if the fundamental solution is nonnegative, we cannot apply the lower asymptotics in (1.23). To the best of our knowledge, formulas (A.11) and (A.12) and the limit (A.13) are new.

Example A.5. The following one-parameter family of SPDEs

$$
\begin{cases}
\left(\partial_t^\beta - \dfrac{\nu}{2}\dfrac{\partial^2}{\partial x^2}\right)u(t,x) = I_t^{\lceil\beta\rceil-\beta}\left[\lambda u(t,x)\dot{W}(t,x)\right], & t>0, x\in\mathbb{R}, \\
u(0,\cdot)=u_0, & \text{if } \beta\in(0,1], \\
u(0,\cdot)=u_0, \quad \dfrac{\partial}{\partial t}u(0,\cdot)=u_1, & \text{if } \beta\in(1,2)
\end{cases} \tag{A.14}
$$

has been studied in [6]. This is the case when $d=1$, $\alpha=2$, $\beta\in(0,2)$ and $\gamma=\lceil\beta\rceil-\beta$ and the upper bound of the large-time asymptotics was obtained (*ibid.*). It can be easily checked that Dalang's condition (1.16) holds true for all $\beta\in(0,2)$ in this case. The quantities in (1.14) reduce to

$$
\theta = 2\left(\lceil\beta\rceil-1\right)-\beta/2, \qquad t_p = p^{1+\frac{2}{4\lceil\beta\rceil-2-\beta}}t,
$$

$$
\Theta_{\beta,\nu} = \frac{1}{\pi}\int_0^\infty E_{\beta,\lceil\beta\rceil}^2\left(-\frac{\nu}{2}\xi^2\right)\mathrm{d}\xi, \qquad \hat{t} = \Theta_{\beta,\nu}\Gamma\left(2\lceil\beta\rceil-1-\beta/2\right)t^{2\lceil\beta\rceil-1-\beta/2},
$$

and hence we have the following:

(1) *Second moment formula:* By (1.17), the second moment formula is

(A.15)
$$\mathbb{E}\left[u^2(t,x)\right] = \begin{cases} u_0^2\, E_{1-\beta/2}\left(\lambda^2\Theta_{\beta,\nu}\Gamma\left(1-\beta/2\right)t^{1-\beta/2}\right) & \text{if } \beta\in(0,1], \\ u_0^2\, E_{3-\beta/2}\left(\lambda^2\Theta_{\beta,\nu}\Gamma\left(3-\beta/2\right)t^{3-\beta/2}\right) & \\ +2u_0u_1\, t\, E_{3-\beta/2,2}\left(\lambda^2\Theta_{\beta,\nu}\Gamma\left(3-\beta/2\right)t^{3-\beta/2}\right) & \text{if } \beta\in(1,2). \\ +2u_1^2\, t^2\, E_{3-\beta/2,3}\left(\lambda^2\Theta_{\beta,\nu}\Gamma\left(3-\beta/2\right)t^{3-\beta/2}\right) & \end{cases}$$

(2) *Second moment Lyapunov exponent:* From (A.11), we see that

(A.16)
$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}\left[u(t,x)^2\right] = \left(\lambda^2\Theta_{\beta,\nu}\Gamma\left(2\lceil\beta\rceil-1-\beta/2\right)\right)^{\frac{2}{4\lceil\beta\rceil-2-\beta}}.$$

(3) *Moment asymptotics:* Since the fundamental solution in this case is nonnegative (see Remark 5.3), we can combine the asymptotics in both (1.21) and (1.23) to see that

(A.17a)
$$C_1 p^{\frac{4\lceil\beta\rceil-\beta}{4\lceil\beta\rceil-2-\beta}} \le \liminf_{t\to\infty}\frac{\log\mathbb{E}[|u(t,x)|^p]}{t} \le \limsup_{t\to\infty}\frac{\log\mathbb{E}[|u(t,x)|^p]}{t} \le C_2 p^{\frac{4\lceil\beta\rceil-\beta}{4\lceil\beta\rceil-2-\beta}}, \quad p\ge 2,$$

(A.17b)
$$C_1 t \le \liminf_{p\to\infty}\frac{\log\mathbb{E}[|u(t,x)|^p]}{p^{\frac{4\lceil\beta\rceil-\beta}{4\lceil\beta\rceil-2-\beta}}} \le \limsup_{p\to\infty}\frac{\log\mathbb{E}[|u(t,x)|^p]}{p^{\frac{4\lceil\beta\rceil-\beta}{4\lceil\beta\rceil-2-\beta}}} \le C_2 t, \quad t>0.$$

The upper bound for the large-time asymptotics in (A.17a) recovers the results obtained in [6]; see Theorems 3.5 and 3.6 (*ibid.*). In particular, when $\beta\in(0,1]$, Mijena and Nane [44, Theorem 2] obtained the same upper bound as in (A.17a). Except the upper bound in (A.17a), all the rest results in this example are new.

In Figure A.2, we plot the graphs of θ, $\Theta_{\beta,\nu}$, $1+1/(1+\theta)$, and the second moment Lyapunov exponent as functions of β with λ and ν being set to 1 and 2, respectively.

Example A.6. Mijena and Nane [44] studied the case when $\beta\in(0,1]$, $\alpha\in(0,2]$, $\gamma=1-\beta$, namely,

(A.18)
$$\begin{cases}\left(\partial_t^\beta+\frac{\nu}{2}\left(-\Delta\right)^{\alpha/2}\right)u(t,x) = I_t^{1-\beta}\left[\lambda u(t,x)\dot{W}(t,x)\right], & t>0, x\in\mathbb{R}^d,\\ u(0,\cdot)=u_0, \end{cases}$$

under the condition

(A.19)
$$d<\alpha\,\min\left(2,\beta^{-1}\right).$$

Note that condition (A.19) is the same as (1.16) under this specific setting. In [44], the upper bound of the large-time exponent (1.21a) was obtained; see Theorem 2 *ibid.* Since the fundamental solution in this case is nonnegative (see Remark 5.3), we can apply Theorem 1.1 to have exact formulas for both the second moment and the second moment Lyapunov exponent, and to have matching lower bounds for the moment asymptotics. To be more precise, in this case we have

$$\theta=-\beta d/\alpha, \qquad t_p = p^{\frac{2\alpha-\beta d}{\alpha-\beta d}}t,$$

$$\Theta_{\alpha,\beta,d,\nu}=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}E_\beta^2\left(-\frac{\nu}{2}|\xi|^\alpha\right)\mathrm{d}\xi, \qquad \hat{t}=\Theta_{\alpha,\beta,d,\nu}\Gamma\left(1-\beta d/\alpha\right)t^{1-\beta d/\alpha},$$

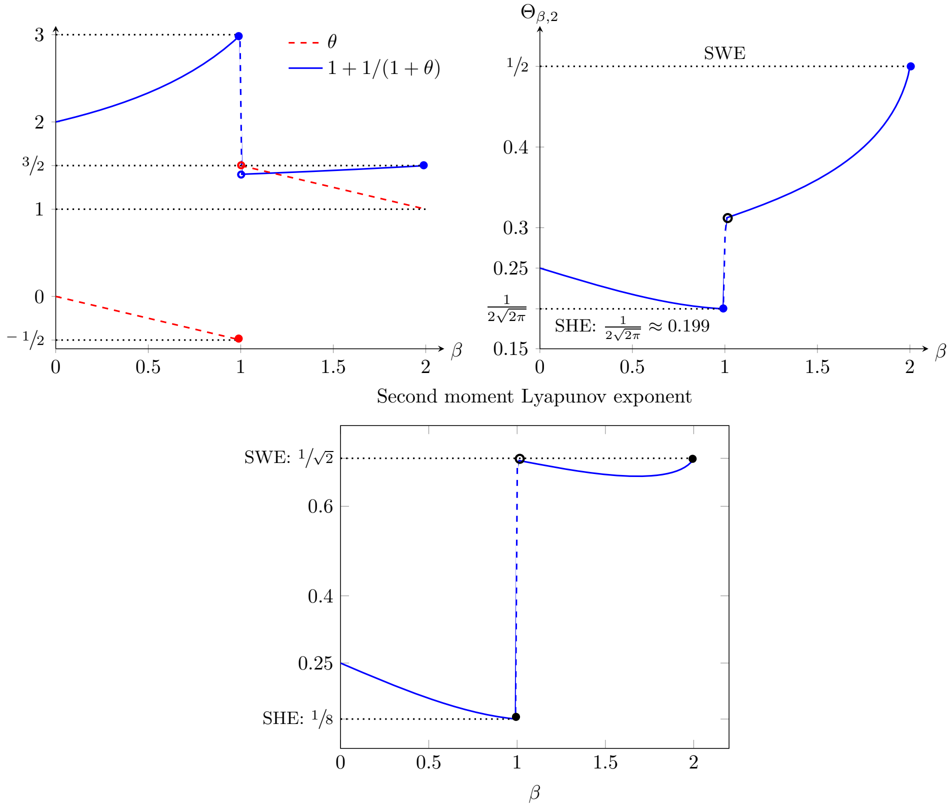


FIGURE A.2. Plots of the quantities in Example A.5 with $\lambda = 1$ and $\nu = 2$. For all these graphs, at the jump point $\beta = 1$, one needs to take the left limit.

and hence we have the following results:

(1) *Second moment formula:*

$$\mathbb{E}\left[u^2(t,x)\right] = u_0^2\, E_{1-\beta d/\alpha}\left(\lambda^2 \Theta_{\alpha,\beta,d,\nu}\, \Gamma\left(1-\beta d/\alpha\right) t^{1-\beta d/\alpha}\right). \tag{A.20}$$

(2) *Second moment Lyapunov exponent:*

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\left[u(t,x)^2\right] = \left(\lambda^2 \Theta_{\alpha,\beta,d,\nu}\, \Gamma\left(1-\beta d/\alpha\right)\right)^{\frac{\alpha}{\alpha-\beta d}}. \tag{A.21}$$

(3) *Moment asymptotics:*

(A.22a)

$$C_1 p^{\frac{2\alpha-\beta d}{\alpha-\beta d}} \le \liminf_{t\to\infty} \frac{\log \mathbb{E}[|u(t,x)|^p]}{t} \le \limsup_{t\to\infty} \frac{\log \mathbb{E}[|u(t,x)|^p]}{t} \le C_2 p^{\frac{2\alpha-\beta d}{\alpha-\beta d}}, \quad p \ge 2,$$

(A.22b)

$$C_1\, t \le \liminf_{p\to\infty} \frac{\log \mathbb{E}[|u(t,x)|^p]}{p^{\frac{2\alpha-\beta d}{\alpha-\beta d}}} \le \limsup_{p\to\infty} \frac{\log \mathbb{E}[|u(t,x)|^p]}{p^{\frac{2\alpha-\beta d}{\alpha-\beta d}}} \le C_2\, t, \qquad t > 0.$$

Note that except the two lower bounds in (A.22a) and (A.22b) require $\alpha \in (0,2]$, all the rest formulas/upper bounds in this example hold true for all $\alpha > 0$. In particular, this would allow higher dimensions for large α; see (A.19).

Example A.7. In this example, we study the following one-parameter family of SPDEs with SHE (1.2) (resp. SWE (1.3)) being a special (resp. limiting) case:

$$\left(\partial_t^\beta - \frac{\nu}{2}\frac{\partial^2}{\partial x^2}\right) u(t,x) = \lambda u(t,x)\dot{W}(t,x), \quad t>0,\ x\in\mathbb{R},\ \beta\in(0,2)\,, \tag{A.23}$$

with the same initial condition as SHE (1.2) (resp. SWE (1.3)) when $\beta \in (0,1]$ (resp. $\beta \in (1,2)$). This is the case when $\alpha = 2$, $\beta \in (0,2)$, $\gamma = 0$ and $d = 1$. Dalang's condition (1.16) reduces to

$$\beta > 2/3,$$

and quantities in (1.14) become

$$\theta := -2 + 3\beta/2, \qquad t_p := p^{3\beta/(3\beta-2)} t,$$

$$\Theta_{\beta,\nu} := \pi^{-1}\int_0^\infty E_{\beta,\beta}^2(-\tfrac{1}{2}\nu\xi^2)\mathrm{d}\xi, \qquad \hat{t} := \Theta_{\beta,\nu}\,\Gamma\left(-1+3\beta/2\right) t^{-1+3\beta/2}.$$

Note that the fundamental solution is nonnegative (see Remark 5.3). Here we summarize the properties of the solution to (A.23) as follows:

(1) *Second moment formula:*

(A.24)

$$\mathbb{E}\left[u^2(t,x)\right] = \begin{cases} u_0^2\, E_{-1+3\beta/2}\left(\lambda^2\Theta_{\beta,\nu}\,\Gamma\left(-1+3\beta/2\right) t^{-1+3\beta/2}\right) & \text{if } \beta\in(0,1], \\ \begin{aligned} & u_0^2\, E_{-1+3\beta/2}\left(\lambda^2\Theta_{\beta,\nu}\,\Gamma\left(-1+3\beta/2\right) t^{-1+3\beta/2}\right) \\ & +2u_0u_1\, t\, E_{-1+3\beta/2,2}\left(\lambda^2\Theta_{\beta,\nu}\,\Gamma\left(-1+3\beta/2\right) t^{-1+3\beta/2}\right) \\ & +2u_1^2\, t^2\, E_{-1+3\beta/2,3}\left(\lambda^2\Theta_{\beta,\nu}\,\Gamma\left(-1+3\beta/2\right) t^{-1+3\beta/2}\right) \end{aligned} & \text{if } \beta\in(1,2). \end{cases}$$

(2) *Second moment Lyapunov exponent:*

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}\left[u(t,x)^2\right] = \left(\lambda^2\Theta_{\beta,\nu}\,\Gamma\left(-1+3\beta/2\right)\right)^{2/(3\beta-2)}. \tag{A.25}$$

(3) *Moment asymptotics:*

(A.26a)

$$C_1 p^{\frac{3\beta}{3\beta-2}} \le \liminf_{t\to\infty}\frac{\log\mathbb{E}[|u(t,x)|^p]}{t} \le \limsup_{t\to\infty}\frac{\log\mathbb{E}[|u(t,x)|^p]}{t} \le C_2 p^{\frac{3\beta}{3\beta-2}}, \quad p\ge 2,$$

(A.26b)

$$C_1\, t \le \liminf_{p\to\infty}\frac{\log\mathbb{E}[|u(t,x)|^p]}{p^{3\beta/(3\beta-2)}} \le \limsup_{p\to\infty}\frac{\log\mathbb{E}[|u(t,x)|^p]}{p^{3\beta/(3\beta-2)}} \le C_2\, t, \qquad t>0.$$

Thanks to (2.5), all the above quantities when $\beta \to 2$ converge to the corresponding ones in Example A.2 for SWE (1.3); see Figure A.3 for some numerical simulations.

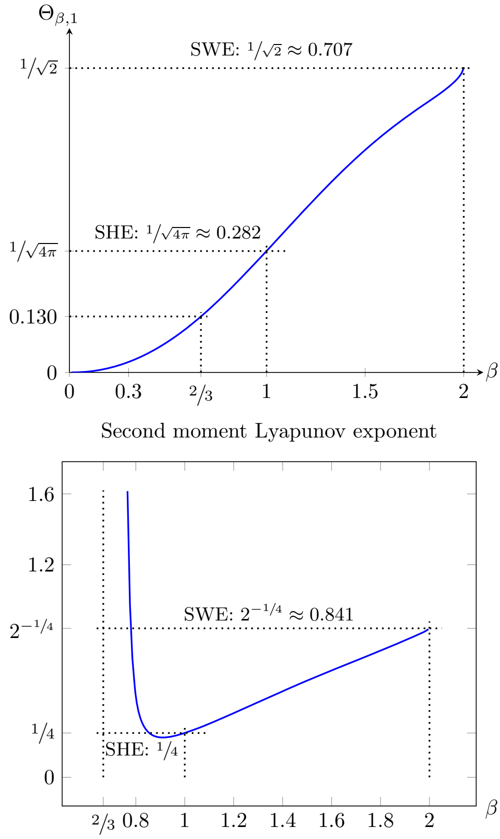


FIGURE A.3. Plots of $\Theta_{\beta,\nu}$ and the second moment Lyapunov exponent in Example A.7 with $\lambda = 1$ and $\nu = 1$

APPENDIX B. SOME MISCELLANEOUS LEMMAS

In this section, we provide some technical lemmas. Lemma B.1 will be used to prove Dalang's condition (1.16) in Theorem 3.3.

Lemma B.1. *For all* $\varepsilon > 0$, a, $b > 0$, *and* $c \in \mathbb{R}$, *it holds that*

$$\int_{B_\varepsilon^c(0)} \frac{\cos^2(|x|^a + c)}{|x|^b} \mathrm{d}x < \infty \quad \textit{if and only if } b > d. \tag{B.1}$$

Proof. We only consider the case $d \geq 2$ while the case $d = 1$ is similar but easier. Denote the integral in (B.1) by I. Since the integrand is radial,

$$I = C \int_\varepsilon^\infty \frac{\cos^2(r^a + c)}{r^b} r^{d-1} \mathrm{d}r, \quad \text{with } C = \frac{2\pi^{d/2}}{\Gamma(d/2)}.$$

Clearly, $b > d$ is a sufficient condition for (B.1). To get the necessity, observe that

$$\begin{aligned} I &= \frac{C}{a}\int_{\varepsilon^a}^{\infty} \cos^2(s+c)s^{-\frac{1}{a}(b-d)-1}\mathrm{d}s \\ &\geq \frac{C}{a}\sum_{n=N}^{\infty}\int_{n\pi-c}^{\left(n+\frac{1}{4}\right)\pi-c} \cos^2(s+c)s^{-(b-d)/a-1}\mathrm{d}s \\ &\geq \frac{C\pi^{-(b-d)/a}}{8a}\sum_{n=N+1}^{\infty} n^{-(b-d)/a-1}, \end{aligned}$$

where $N = N(\varepsilon^a, c)$ is a finite positive integer. The series on the right-hand side is convergent if and only if $b > d$ and thus $b > d$ is also necessary for (B.1). □

Lemma B.2 is a convolution-type Gronwall lemma, which was proved in Lemma A.2 of [15] for $\theta \in (-1, 0)$. But indeed, the same proof can be extended directly to all $\theta > -1$. One can use Lemma B.2 to obtain the moment formulas in Theorem 1.1 as pointed out in Remark 4.1.

Lemma B.2. *Suppose that $\theta > -1$, $\kappa > 0$ and that $g(\cdot) : \mathbb{R}_+ \to \mathbb{R}$ is a locally integrable function. If f satisfies*

$$f(t) = g(t) + \kappa\int_0^t (t-s)^{\theta} f(s)\mathrm{d}s, \quad \text{for } t \geq 0, \tag{B.2}$$

then

$$f(t) = g(t) + \int_0^t g(s)K(t-s)\mathrm{d}s, \tag{B.3}$$

with $K(t) = \kappa\Gamma(\theta+1)t^{\theta}E_{\theta+1,\theta+1}(\kappa\Gamma(\theta+1)t^{\theta+1})$. Moreover, if we further assume $g(\cdot) \geq 0$ and the equality in (B.2) *is replaced by $\leq$ (resp. $\geq$), then the equality in* (B.3) *is replaced by $\leq$ (resp. $\geq$) accordingly.*

Lemma B.3 will be used to obtain the explicit second moment formula for stochastic wave equation (i.e., $\beta = 2$) in Examples A.2 and A.4.

Lemma B.3. *For $\alpha > 1$ and $b > 0$, it holds that*

$$\begin{aligned} &\int_0^{\infty} \frac{\sin^2\left(b\,\xi^{\frac{\alpha}{2}}\right)}{\xi^{\alpha}}\mathrm{d}\xi \\ &\quad = \begin{cases} 2^{2\left(1-\frac{1}{\alpha}\right)}\alpha^{-1}\cos\left(\frac{\pi}{\alpha}\right)\Gamma\left(2\left(\frac{1}{\alpha}-1\right)\right)b^{2-\frac{2}{\alpha}} & \text{if } \alpha \in (1,2)\cup(2,\infty), \\ \frac{1}{2}\,b\pi, & \text{if } \alpha = 2. \end{cases} \end{aligned}$$

Proof. Denote the integral by I. By change of variable $z = \xi^{\alpha/2}$, we see that

$$I = \frac{2}{\alpha}\int_0^{\infty}\frac{\sin^2(bz)}{z^{3-2/\alpha}}\mathrm{d}z.$$

Let $f(x) = 1\!\!1_{[-b,b]}(x)$ and $g(x)$ be an even function defined as, for $x > 0$,

$$g(x) = \frac{\pi}{4\Gamma\left(2(1-1/\alpha)\right)\sin\left(\pi/\alpha\right)}\left[(x+b)^{1-2/\alpha} + |b-x|^{1-2/\alpha}\mathrm{sgn}(b-x)\right].$$

Now we compute the Fourier transforms for these two functions. It is clear that

$$\widehat{f}(\xi) = \frac{2\sin(b\,\xi)}{\xi}.$$

Let $h(\xi) = |\xi|^{-2(1-1/\alpha)} \sin(b\,|\xi|)$. By (2) on p.19 of [33], we see that

$$\mathcal{F}^{-1}h(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\xi} h(\xi)\mathrm{d}\xi = \frac{1}{\pi}\int_0^\infty h(\xi)\cos(x\xi)\mathrm{d}\xi = \frac{1}{\pi} g(x),$$

under the following condition:

$$\left|\frac{2}{\alpha} - 1\right| < 1 \quad \iff \quad \alpha > 1.$$

Hence, when $\alpha > 1$, we have $\widehat{g}(\xi) = \pi h(\xi)$. Then by the Plancherel theorem,

$$\int_{\mathbb{R}} f(x)g(x)\mathrm{d}x = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{f}(\xi)\overline{\widehat{g}(\xi)}\mathrm{d}\xi = \frac{1}{\pi}\int_0^\infty \frac{2\sin(b\,\xi)}{\xi}\pi\xi^{-2(1-1/\alpha)}\sin(b\,\xi)\,\mathrm{d}\xi = \alpha I.$$

On the other hand,

$$\begin{aligned}
\int_{\mathbb{R}} f(x)g(x)\mathrm{d}x =& 2\int_0^b g(x)\mathrm{d}x \\
=& \frac{\pi}{2\Gamma(2(1-1/\alpha))\sin(\pi/\alpha)}\int_0^b \left[(x+b)^{1-2/\alpha} + (b-x)^{1-2/\alpha}\right]\mathrm{d}x \\
=& \frac{\pi 2^{1-2/\alpha} b^{2-2/\alpha}}{\Gamma(2-2/\alpha)(2-2/\alpha)\sin(\pi/\alpha)} \\
=& \frac{\pi 2^{1-2/\alpha} b^{2-2/\alpha}}{\Gamma(3-2/\alpha)\sin(\pi/\alpha)}.
\end{aligned}$$

Hence, if $\alpha = 2$, the above expression becomes $b\pi$. This proves the lemma for the case $\alpha = 2$. Now if $\alpha \neq 2$, we have

$$\begin{aligned}
\int_{\mathbb{R}} f(x)g(x)\mathrm{d}x =& \frac{\pi 2^{1-2/\alpha} b^{2-2/\alpha}}{\Gamma(3-2/\alpha)\sin(\pi/\alpha)} \times \frac{\Gamma(2(1/\alpha-1))}{\Gamma(2(1/\alpha-1))} \\
=& \frac{2^{1-2/\alpha} b^{2-2/\alpha}\Gamma(2(1/\alpha-1))}{\sin(\pi/\alpha)}\sin(2\pi/\alpha) \\
=& 2^{2-2/\alpha} b^{2-2/\alpha}\Gamma(2(1/\alpha-1))\cos(\pi/\alpha),
\end{aligned}$$

where we have applied the reflection formula for Gamma function (2.8) in the second equality. This completes the proof of Lemma B.3. □

For two sequences of positive numbers $a_n, b_n, n \in \mathbb{N}$, we write

$$a_n \sim b_n \text{ if } \lim_{n\to\infty} a_n/b_n = 1.$$

Lemma B.4 will be used to calculate the lower bounds of moments in Theorem 5.9.

Lemma B.4.

(1) *As $n \to \infty$, we have*

$$\int_n^\infty t^n e^{-t}\mathrm{d}t \sim \frac{n^n}{e^n}\sqrt{\frac{\pi n}{2}} \quad \textit{and} \tag{B.4}$$

$$\sum_{m=0}^{n-1}\frac{n^m}{m!} \sim \sum_{m=n}^{2n-1}\frac{n^m}{m!} \sim \frac{1}{2}e^n. \tag{B.5}$$

(2) *Given $\alpha > 0$, for each constant $c < \alpha$, we have*

$$\sum_{m=n}^{\infty} \left(\frac{n^m}{m!}\right)^{\alpha} \geq \exp(cn), \quad \textit{for } n \textit{ sufficiently large.}$$

Proof. (1) Denote the integral in (B.4) by I_n. By change of variable $x = \frac{t}{\sqrt{n}} - \sqrt{n}$, we see that

$$I_n = \frac{n^n\sqrt{n}}{e^n} \int_0^{\infty} e^{-\sqrt{n}x} \left(1 + \frac{x}{\sqrt{n}}\right)^n \mathrm{d}x.$$

Notice that $g(x) := n \log\left(1 + \frac{x}{\sqrt{n}}\right) - \sqrt{n}x + x - \log(1 + x)$ satisfies the following properties:

$$g(0) = 0 \quad \text{and} \quad g'(x) = \frac{(\sqrt{n} - 1)x^2}{(1 + x)(\sqrt{n} + x)} \geq 0 \quad \text{for all } x \geq 0 \text{ and } n \geq 1,$$

from which we see that $\left(1 + \frac{x}{\sqrt{n}}\right)^n e^{-\sqrt{n}x} \leq (1+x)e^{-x}$ for all $x \geq 0$ and $n \geq 1$ with the upper bound being integrable. Hence, by the dominated convergence theorem and L'Hospital's rule, we conclude that

$$\lim_{n\to\infty} \frac{I_n\, e^n}{n^n\sqrt{n}} = \int_0^{\infty} \lim_{n\to\infty} e^{-\sqrt{n}x} \left(1 + \frac{x}{\sqrt{n}}\right)^n \mathrm{d}x = \int_0^{\infty} e^{-x^2/2}\mathrm{d}x = \sqrt{\frac{\pi}{2}},$$

which proves (B.4).

To prove (B.5), it suffices to show $R_n(n) \sim \frac{1}{2}\, e^n$ and $\lim_{n\to\infty} e^{-n} R_{2n}(n) = 0$, where $R_k(x)$ is the remainder function for the Taylor expansion of e^x:

$$R_k(x) = \sum_{m=k}^{\infty} \frac{x^m}{m!} = \int_0^x \frac{(x - t)^k}{k!} e^t \mathrm{d}t.$$

For $R_n(n)$, by change of variable we get

$$R_n(n) = \int_0^n \frac{(n - t)^n}{n!} e^t \mathrm{d}t = \frac{e^n}{n!} \int_0^n x^n e^{-x} \mathrm{d}x.$$

By Stirling's formula $n! \sim \sqrt{2\pi n}e^{-n}n^n$ and (B.4), we can show

$$\text{(B.6)} \qquad \int_0^n x^n e^{-x} \mathrm{d}x = \left(\Gamma(n + 1) - \int_n^{\infty} x^n e^{-x} \mathrm{d}x\right) \sim \frac{1}{2} n!.$$

For $R_{2n}(n)$, we have

$$R_{2n}(n) = \int_0^n \frac{(n - t)^{2n}}{(2n)!} e^t \mathrm{d}t \leq \frac{n^n}{(2n)!} \int_0^n (n - t)^n e^t \mathrm{d}t = \frac{e^n n^n}{(2n)!} \int_0^n x^n e^{-x} \mathrm{d}x.$$

Thus, by (B.6), we have

$$\lim_{n\to\infty} \frac{R_{2n}(n)}{e^n} \leq \lim_{n\to\infty} \frac{1}{2} \frac{n^n(n!)}{(2n)!} = \lim_{n\to\infty} \frac{1}{2} \frac{n^{2n}e^{-n}\sqrt{2\pi n}}{(2n)^{2n}e^{-2n}\sqrt{4\pi n}} = 0,$$

which proves (B.5).

(2) The desired result follows directly from the fact

$$\sum_{m=n}^{\infty}\left(\frac{n^m}{m!}\right)^{\alpha} \geq \sum_{m=n}^{2n-1}\left(\frac{n^m}{m!}\right)^{\alpha} \geq \begin{cases} \left(\displaystyle\sum_{m=n}^{2n-1}\frac{n^m}{m!}\right)^{\alpha}, & \text{if } \alpha \in (0,1], \\ n^{1-\alpha}\left(\displaystyle\sum_{m=n}^{2n-1}\frac{n^m}{m!}\right)^{\alpha}, & \text{if } \alpha > 1. \end{cases}$$

Then an application of (B.5) proves part (2). □

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