



# Hölder regularity for the nonlinear stochastic time-fractional slow & fast diffusion equations on $\mathbb{R}^d$

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Received: 4 May 2021 / Revised: 20 February 2022 / Accepted: 28 February 2022  
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## Abstract

In this paper, we study the space-time Hölder continuity of the solution to the following nonlinear time-fractional slow and fast diffusion equation:

$$\left(\partial^\beta + \frac{\nu}{2}(-\Delta)^{\alpha/2}\right)u(t, x) = I_t^\gamma [\sigma(u(t, x)) \dot{W}(t, x)], \quad t > 0, x \in \mathbb{R}^d,$$

where  $\dot{W}$  is the space-time white noise,  $\alpha \in (0, 2]$ ,  $\beta \in (0, 2)$ ,  $\gamma \geq 0$  and  $\nu > 0$ . The existence/uniqueness of a random field solution has been obtained in [9] under the condition that  $2(\beta + \gamma) - 1 - d\beta/\alpha > 0$ . The Hölder regularity of the solution has been obtained in the same reference, but only for the case  $\beta + \gamma \leq 1$ . In this paper, we use the idea from the *local fractional derivative* to establish the Hölder regularity of the solution for all possible cases  $\beta \in (0, 2)$ , which in particular recovers the special case in [9] when  $\beta \in (0, 1 - \gamma]$ . As a rather surprising consequence, when  $\gamma = 0$ ,  $\alpha = 2$  and  $\beta$  is close to 2, the space and time Hölder exponents are both to  $1-$ , which is different from the known Hölder exponents for the stochastic wave equation which are  $(1/2)-$ .

**Keywords** Nonlinear stochastic time-fractional slow and fast diffusion equations (primary) · Local fractional derivative · Fractional Taylor theorem · Hölder continuity · Mittag-Leffler function · Fox-H function

**Mathematics Subject Classification** 60H15 (primary) · 60G60 · 26A33

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## 1 Introduction

In this article, we study the Hölder continuity of the solution of the following nonlinear stochastic time-fractional slow and fast diffusion equations:

$$\begin{cases} \left( \partial^\beta + \frac{\nu}{2}(-\Delta)^{\alpha/2} \right) u(t, x) = I_t^\gamma [\sigma(u(t, x)) \dot{W}(t, x)], & t > 0, x \in \mathbb{R}^d, \\ u(0, \cdot) = \mu & \text{if } \beta \in (0, 1], \\ u(0, \cdot) = \mu_0, \quad \frac{\partial}{\partial t} u(0, \cdot) = \mu_1 & \text{if } \beta \in (1, 2), \end{cases} \quad (1.1)$$

where  $\nu > 0$  is the diffusion parameter, the function  $\sigma : \mathbb{R} \mapsto \mathbb{R}$  is globally Lipschitz continuous, and  $\dot{W}$  denotes the space-time white noise. In this equation, there are three fractional operators, among which  $(-\Delta)^{\alpha/2}$ ,  $\alpha \in (0, 2]$ , refers to the standard *fractional Laplacian* in the spatial variable;  $\partial^\beta$  denotes the *Caputo fractional derivative*

$$\partial^\beta f(t) := \begin{cases} \frac{1}{\Gamma(m-\beta)} \int_0^t d\tau \frac{f^{(m)}(\tau)}{(t-\tau)^{\beta+1-m}} & \text{if } m-1 < \beta < m, \\ \frac{d^m}{dt^m} f(t) & \text{if } \beta = m, \end{cases} \quad (1.2)$$

and  $I_{s+}^\gamma$  refers to the (left-sided) *Riemann-Liouville fractional integral* of order  $\gamma > 0$ :

$$(I_{s+}^\gamma f)(t) := \frac{1}{\Gamma(\gamma)} \int_s^t (t-r)^{\gamma-1} f(r) dr, \quad \text{for } t > s.$$

The term  $I_t^\gamma [\sigma(u(t, x)) \dot{W}(t, x)]$  in (1.1) refers to the more cumbersome notation

$$(I_{0+}^\gamma [\sigma(u(\cdot, x)) \dot{W}(\cdot, x)])(t).$$

The existence and uniqueness of a random field solution has been established in [9] under the *Dalang's condition*:  $d < \Theta$ , which is equivalent to

$$\rho > 0 \quad \text{and} \quad d < 2\alpha, \quad (1.3)$$

where the constants  $\rho$  and  $\Theta$  are defined as follows:

$$\rho := 2(\beta + \gamma) - 1 - d\beta/\alpha \quad \text{and} \quad \Theta := 2\alpha + \frac{\alpha}{\beta} \min(2\gamma - 1, 0). \quad (1.4)$$

The stochastic partial differential equations (SPDE's) with time-fractional differential operators have received many attentions recently, which have been studied in one way or another using various tools from stochastic analysis; see [2, 8–10, 15, 20, 21]. This current paper follows the same setup as [9], where the existence/uniqueness of the solution to (1.1), moment Lyapunov exponent, sample-path Hölder regularity,

etc., have been studied. In particular, the sample-path regularity was obtained under the following additional condition:

$$\beta + \gamma \leq 1, \tag{1.5}$$

which in particular excludes the case when  $\beta \in (1, 2)$ . The question is whether one can get rid of this additional requirement.

By carefully examining the corresponding proof in [9], one finds that the completely monotonic property [24] of the *two-parameter Mittag-Leffler function* has been applied in a crucial way. This complete monotonic property, applied to  $E_{\beta, \beta+\gamma}(-x)$ ,  $x \geq 0$ , says that

$$\begin{aligned} x \in [0, \infty) \mapsto E_{\beta, \beta+\gamma}(-x) \text{ is complete monotonic} \\ \iff 0 < \beta \leq \min(\beta + \gamma, 1) \\ \iff 0 < \beta < 1 \text{ and } \gamma \geq 0, \end{aligned} \tag{1.6}$$

which in particular restricts the results in [9] only to the cases when  $\beta \in (0, 1]$ . Moreover, the proof in [9] requires an even stronger condition, namely, (1.5), than those on the far right side of (1.6). It is well known that when  $\beta \in (1, 2)$ , the Mittag-Leffler function has finite (odd) number of negative real zeros (see, e.g., Section 18.1 of [13]), or the function of interest in (1.6) is in general oscillatory when  $\beta \in (1, 2)$ , which, together with the fact that one needs to bound the Mittag-Leffler functions of the form in (1.6) from below in the proof of [2], poses a significant challenge to extend the method used in [9] to cover the cases without the extra restriction (1.5). Instead, inspired by the *local fractional derivatives* (LFD), we manage to obtain the desired regularity results without condition (1.5), which covers the corresponding results obtained in [9] as a special case. The arguments are much more involved, as one may expect, than those in [9].

In order to state our main result, let us first recall that for a given subset  $D \subseteq [0, \infty) \times \mathbb{R}^d$  and positive constants  $\beta_1$  and  $\beta_2$ ,  $C_{\beta_1, \beta_2}(D)$  denotes the set of (*locally Hölder continuous functions over D*), that is, any function  $v : [0, \infty) \times \mathbb{R}^d \mapsto \mathbb{R}$  with the property that for each compact set  $K \subseteq D$ , there is a finite constant  $C$  such that for all  $(t, x)$  and  $(s, y) \in K$ ,

$$|v(t, x) - v(s, y)| \leq C (|t - s|^{\beta_1} + |x - y|^{\beta_2}).$$

Denote  $C_{\beta_1-, \beta_2-}(D) := \cap_{\alpha_1 \in (0, \beta_1)} \cap_{\alpha_2 \in (0, \beta_2)} C_{\alpha_1, \alpha_2}(D)$ .

**Theorem 1** *Let  $u(t, x)$  be the solution to (1.1) under Dalang’s condition (1.3) and assume that the initial conditions  $\mu, \mu_0$  and  $\mu_1$  are such that*

$$\sup_{(s, x) \in [0, t] \times \mathbb{R}^d} |J_0(s, x)| < \infty, \quad \text{for all } t \geq 0, \tag{1.7}$$

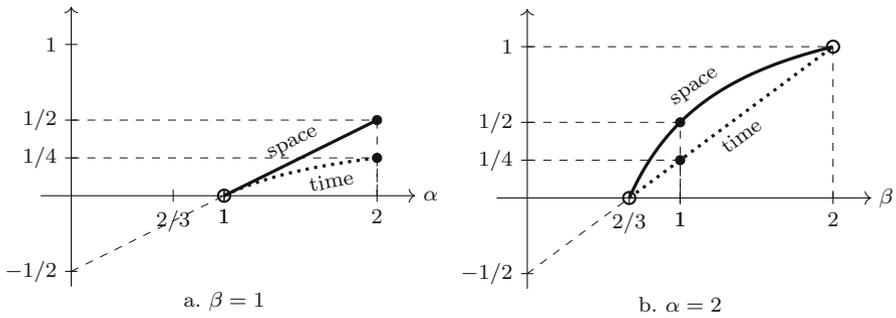


Fig. 1 The exponents for the case when  $\gamma = 0$  and  $d = 1$

where  $J_0(t, x)$  refers to the solution to the homogeneous equation of (1.1) (see (2.4)). Then

$$u(\cdot, \cdot) \in C_{\frac{1}{2} \min(\rho, 2)-, \frac{1}{2} \min(\rho-d, 2)-} \left( (0, \infty) \times \mathbb{R}^d \right), \quad \text{a.s.} \quad (1.8)$$

**Remark 1** Under the extra condition  $\beta + \gamma \leq 1$ , we see that  $\rho \leq 1 - d\beta/\alpha < 1$  and hence, the time exponent  $\frac{1}{2} \min(\rho, 2)$  reduces to  $\rho/2$ , which recovers the result in [9]; see (1.17) or Theorem 3.3, *ibid*. When  $\beta = 1, \gamma = 0$  and  $d = 1$ , we have a *stochastic heat equation with a fractional Laplacian*. In this case, the exponents recover the known results  $([\frac{1}{2} - \frac{1}{2\alpha}]-, \frac{\alpha-1}{2}-)$ ; see, e.g., [6, Proposition 4.4] for a reference and Figure 1-a for an illustration.

**Remark 2** When  $\gamma = 0, \alpha = 2$  and  $d = 1$ , the results in (1.8) reduce to

$$u(\cdot, \cdot) \in C_{(\frac{3}{4}\beta - \frac{1}{2})-, (\frac{3}{2} - \frac{1}{\beta})-} \left( (0, \infty) \times \mathbb{R}^d \right), \quad \text{a.s.};$$

see Figure 1-b for the time and space exponents. It is clear that when  $\beta = 1$ , our results recovers the standard exponents  $(1/4-, 1/2-)$  for the *stochastic heat equation* (SHE); see, e.g., [4]. However, when  $\beta = 2$ , it is known that the *stochastic wave equation* (SWE) has the exponents  $(1/2-, 1/2-)$ ; see, e.g., Part (6) of Theorem 4.2.1 in [3]. On the other hand, our results require  $\beta$  to be strictly smaller than 2. But when  $\beta$  is very close to 2, our results suggest the exponents to be  $(1-, 1-)$ , which is rather unexpected. As shown in [9], the upper bounds for the Lyapunov exponents vary from those of SHE to SWE when changing  $\beta$  from 1 to 2 and moreover, when tuning the other parameters, all known results can be recovered; see Section 1.3 of [9]. By this observation, one would expect to see that when  $\beta$  varies from 1 to 2-, even though  $\beta$  has to be strictly smaller than 2, the Hölder exponents should change from the ones for SHE  $(1/4-, 1/2-)$  to those for SWE  $(1/2-, 1/2-)$ , instead of  $(1-, 1-)$ . We cannot explain why there exists such a phase transition of the Hölder regularity from  $\beta < 2$  to  $\beta = 2$ , but it is consistent with the fact that the fundamental solution  $x \neq 0 \mapsto Y(t, x)$  (see Section 2 for the notation) with  $t > 0$  fixed degenerates from a smooth function

to a piece-wise continuous function, and it is also consistent with the regularization effect of  $\beta$ ; see Example 1.1 of [9].

**Remark 3** In Theorem 1, for simplicity, we have assumed that the initial conditions satisfy condition (1.7). This assumption includes the  $L^\infty(\mathbb{R}^d)$ -initial conditions, such as the constant initial data. Since we do not make any assumptions on the regularity of the initial conditions, the statement in (1.8) has to exclude explicitly the case when  $t = 0$ . Moreover, with some more efforts, one should be able to extend these results to include the *rough initial conditions*, such as the Dirac delta initial data, as studied in [2, 4, 6, 7, 9]. We leave this extension to the interested readers.

The proof of this theorem relies on a crucial estimate given in Proposition 2 below. In Section 2, we introduce some notation and preliminaries including the statement of Proposition 2 itself, based on which we streamline the proof of Theorem 1. Then we prepare some auxiliary results in Sections 3 and 4, before we finally prove Proposition 2 in Section 5.

## 2 Notation, preliminaries, and proof of Theorem 1

We use  $\lceil x \rceil$  and  $\lfloor x \rfloor$  to denote the conventional ceiling and floor functions, respectively. Recall that (see [9]) the fundamental solutions to (1.1) consist of the following three functions:

$$Z_{\alpha,\beta,d}(t, x) := \pi^{-\frac{d}{2}} t^{\lceil \beta \rceil - 1} |x|^{-d} H_{2,3}^{2,1} \left( \frac{|x|^\alpha}{2^{\alpha-1} \nu t^\beta} \mid \begin{matrix} (1,1), (\lceil \beta \rceil, \beta) \\ (d/2, \alpha/2), (1,1), (1, \alpha/2) \end{matrix} \right), \tag{2.1}$$

$$Z^*_{\alpha,\beta,d}(t, x) := \pi^{-\frac{d}{2}} |x|^{-d} H_{2,3}^{2,1} \left( \frac{|x|^\alpha}{2^{\alpha-1} \nu t^\beta} \mid \begin{matrix} (1,1), (1, \beta) \\ (d/2, \alpha/2), (1,1), (1, \alpha/2) \end{matrix} \right), \tag{2.2}$$

$$Y_{\alpha,\beta,\gamma,d}(t, x) := \pi^{-\frac{d}{2}} |x|^{-d} t^{\beta+\gamma-1} H_{2,3}^{2,1} \left( \frac{|x|^\alpha}{2^{\alpha-1} \nu t^\beta} \mid \begin{matrix} (1,1), (\beta+\gamma, \beta) \\ (d/2, \alpha/2), (1,1), (1, \alpha/2) \end{matrix} \right), \tag{2.3}$$

where  $H_{2,3}^{2,1}(* \mid \cdot)$  denotes the *Fox H-function* (see, e.g., [16]). For simplicity, in the following we will simply write these functions as  $Z$ ,  $Z^*$ , and  $Y$ . The solution to (1.1) is understood as the *mild solution*

$$u(t, x) = J_0(t, x) + \iint_{[0,t] \times \mathbb{R}^d} Y(t-s, x-y) \sigma(u(s, y)) W(ds, dy),$$

with the stochastic integral being the *Walsh integral* [26], and  $J_0(t, x)$  is the solution to the homogeneous equation, namely,

$$J_0(t, x) = \begin{cases} (Z(t, \cdot) * \mu)(x) & \text{if } \beta \in (0, 1], \\ (Z^*(t, \cdot) * \mu_0)(x) + (Z(t, \cdot) * \mu_1)(x) & \text{if } \beta \in (1, 2). \end{cases} \tag{2.4}$$

SPDE's of the above form have been widely studied; see, e.g., [5, 12, 14] and references therein.

In [9], Theorem 1 was proved in the case of  $\beta + \gamma \leq 1$  via the following continuity results on  $Y(t, x)$ :

**Proposition 1** (Proposition 5.4 of [9]). *Suppose  $\alpha \in (0, 2]$ ,  $\beta \in (0, 2)$  and  $\gamma \geq 0$ . Under Dalang's condition (1.3),  $Y(t, x)$  satisfies the following two properties:*

(i) *For all  $0 < \theta < (\Theta - d) \wedge 2$  and  $T > 0$ , there is some nonnegative constant  $C = C(\alpha, \beta, \gamma, \nu, \theta, T, d)$  such that for all  $t \in (0, T]$  and  $x, y \in \mathbb{R}^d$ ,*

$$\iint_{\mathbb{R}_+ \times \mathbb{R}^d} dr dz (Y(t-r, x-z) - Y(t-r, y-z))^2 \leq C |x-y|^\theta. \quad (2.5)$$

(ii) *If  $\beta \leq 1$  and  $\beta + \gamma \leq 1$ , then there is some nonnegative constant  $C = C(\alpha, \beta, \gamma, \nu, d)$  such that for all  $s, t \in (0, \infty)$  with  $s \leq t$ , and  $x \in \mathbb{R}^d$ ,*

$$\int_0^s dr \int_{\mathbb{R}^d} dz (Y(t-r, x-z) - Y(s-r, x-z))^2 \leq C(t-s)^{2(\beta+\gamma)-1-d\beta/\alpha}, \quad (2.6)$$

$$\int_s^t dr \int_{\mathbb{R}^d} dz Y^2(t-r, x-z) \leq C(t-s)^{2(\beta+\gamma)-1-d\beta/\alpha}. \quad (2.7)$$

The constraint  $\beta + \gamma \leq 1$  comes from part (ii). Hence, in order to prove our main result, we need only to establish a slightly weaker version of (2.6) as stated in the next proposition:

**Proposition 2** *Let  $\alpha \in (0, 2]$ ,  $\beta \in (0, 2)$ ,  $\gamma \geq 0$ ,  $0 \leq s \leq t \leq T < \infty$ ,  $x \in \mathbb{R}^d$ , and  $\rho$  be the constant defined in (1.4). Assume that Dalang's condition (1.3) holds. If  $\rho < 2$ , then there exists a nonnegative constant  $C = C(\alpha, \beta, \gamma, \nu, d, T)$  such that for all  $q \in (0, \rho)$ , it holds that*

$$\int_0^s dr \int_{\mathbb{R}^d} dz (Y(t-r, x-z) - Y(s-r, x-z))^2 \leq C(t-s)^q. \quad (2.8)$$

If  $\rho > 2$ , (2.8) is true with  $q$  replaced by 2.

The proof of this proposition is given in Section 5. Now we are ready to prove our main result.

**Proof** The proof follows the same arguments as those in the proof of Theorem 3.3 in [9], Theorem 1 is immediate via equations (2.5) and (2.7) in Proposition 1 and Proposition 2. See also the proof of Proposition 4.3 in [4] for a more detailed exposition of these arguments.  $\square$

Now let us introduce some more notation for this paper. For any  $\gamma_1, \gamma_2 \in \mathbb{R}$ , denote

$$C_{\gamma_1, \gamma_2} := \frac{2^{1-d+d/\alpha}}{\Gamma(d/2)\pi^{d/2}\nu^{d/\alpha}} \int_0^\infty u^{d-1} E_{\beta, \beta+\gamma_1}(-u^\alpha) E_{\beta, \beta+\gamma_2}(-u^\alpha) du \quad (2.9a)$$

$$C_\gamma := C_{\gamma,\gamma}, \tag{2.9b}$$

$$C_{\gamma_1,\gamma_2}^* := \frac{2^{1-d+d/\alpha}}{\Gamma(d/2)\pi^{d/2}\nu^{d/\alpha}} \int_0^\infty u^{d-1} |E_{\beta,\beta+\gamma_1}(-u^\alpha)E_{\beta,\beta+\gamma_2}(-u^\alpha)| du \tag{2.9c}$$

$$C_\gamma^* := C_{\gamma,\gamma}^*, \tag{2.9d}$$

where  $E_{\beta,\beta+\gamma}(\cdot)$  is the two-parameter Mittag-Leffler function and these constants are well defined provided that  $d < 2\alpha$  (see Dalang’s condition (1.3)) thanks to the following asymptote property (see, e.g., Theorem 1.6 of [22]):

$$|E_{\beta,\zeta}(z)| \leq \frac{A}{1+|z|} \quad \text{for } \beta \in (0, 2), \zeta \in \mathbb{R}, |\arg(z)| = \pi. \tag{2.10}$$

It is clear that  $C_{\gamma_1,\gamma_2}^* > 0$  and  $C_\gamma^* = C_\gamma > 0$ , but  $C_{\gamma_1,\gamma_2}$  may take negative values. As shown in [9, Lemma 5.5], using the Fourier transform of  $Y_{\alpha,\beta,\gamma,d}(t, \cdot)$ , namely,

$$\mathcal{F}Y_{\alpha,\beta,\gamma,d}(t, \cdot)(\xi) = t^{\beta+\gamma-1} E_{\beta,\beta+\gamma}(-2^{-1}\nu t^\beta |\xi|^\alpha), \tag{2.11}$$

and the Plancherel theorem, we have

$$\int_{\mathbb{R}^d} Y_{\alpha,\beta,\gamma,d}^2(t, x) dx = C_\gamma t^{2(\beta+\gamma-1)-d\beta/\alpha}, \quad \text{for all } t > 0. \tag{2.12}$$

Following the proof of part (ii) of Proposition 5.4 in [9] or by (2.12), we have that

$$\begin{aligned} & \int_0^s dr \int_{\mathbb{R}^d} dz [Y(t-r, x-z) - Y(s-r, x-z)]^2 \\ &= \frac{C_\gamma}{\rho} [t^\rho - (t-s)^\rho + s^\rho] - 2h_s(t), \end{aligned}$$

where

$$\begin{aligned} h_s(t) &:= \frac{1}{(2\pi)^d} \int_0^s dr \int_{\mathbb{R}^d} d\xi (t-r)^{\beta+\gamma-1} E_{\beta,\beta+\gamma}(-2^{-1}\nu(t-r)^\beta |\xi|^\alpha) \\ &\times (s-r)^{\beta+\gamma-1} E_{\beta,\beta+\gamma}(-2^{-1}\nu(s-r)^\beta |\xi|^\alpha). \end{aligned} \tag{2.13}$$

For all  $t, s \geq r, \xi \in \mathbb{R}^d$  and  $\delta \in \mathbb{R}$ , denote

$$\begin{aligned} \mathfrak{E}(t, \delta, r, \xi) &:= \frac{1}{(2\pi)^d} (t-r)^{\delta-1} E_{\beta,\delta}(-2^{-1}\nu(t-r)^\beta |\xi|^\alpha) \\ &\times (s-r)^{\beta+\gamma-1} E_{\beta,\beta+\gamma}(-2^{-1}\nu(s-r)^\beta |\xi|^\alpha). \end{aligned} \tag{2.14}$$

To obtain the estimation in Proposition 2, it boils down to the Hölder continuity of  $[s, T] \ni t \mapsto h_s(t)$  at  $t = s$ . Note that it is more convenient to consider  $h_s(t)$  as a function of  $t$  with  $s$  fixed than the other way round. We will investigate Hölder

continuity of  $h_s(t)$  at  $t = s$  via its *local fractional derivative* (LFD) in Section 4, which will be used to prove Proposition 2 in Section 5.

Finally, we will need the following two lemmas:

**Lemma 1** (Formula 23 of Table 9.1 on p. 173 of [23]). *For  $\text{Re}(\beta) > 0, \text{Re}(\alpha) > 0, q \in \mathbb{R}$ , and  $\lambda > 0$ , it holds that*

$$\left( I_{r+}^q \left[ (\cdot - r)^{\beta-1} E_{\alpha,\beta} (\lambda(\cdot - r)^\alpha) \right] \right) (t) = (t - r)^{\beta+q-1} E_{\alpha,\beta+q} (\lambda(t - r)^\alpha) \quad (2.15)$$

for  $t > r$ , where we use the convention that  $I_{r+}^q = D_{r+}^{-q}$  for  $q < 0$ ; see (4.2) below for the notation.

**Remark 4** When  $q$  is a negative integer,  $I_{r+}^q$  becomes the conventional derivative, which ensures that (2.15) also holds for  $\beta < 0$ .

**Lemma 2** *Let  $1 \leq d < 2\alpha, \beta \in (0, 2), \delta, \delta' \in \mathbb{R}$ . Then for all  $t, s > 0$ ,*

$$\int_{\mathbb{R}^d} \left| E_{\beta,\beta+\delta} \left( -\frac{1}{2} \nu t^\beta |x|^\alpha \right) \times E_{\beta,\beta+\delta'} \left( -\frac{1}{2} \nu s^\beta |x|^\alpha \right) \right| dx \leq C_{\delta,\delta'} [t \cdot s]^{-\frac{\beta}{2\alpha}d}.$$

The proof of this lemma follows immediately from the arguments of [9, Lemma 5.5] with an application of the Cauchy-Schwarz inequality; note that the condition  $\beta \in (0, 2)$  is required due to (2.10).

### 3 Continuity of $h_s^{(n)}(t)$

Throughout of the rest of the article, unless specified otherwise, let  $q$  and  $n$  be given by

$$0 \leq n < q < \rho \leq n + 1 \quad \text{with } n \in \mathbb{Z}_+. \quad (3.1)$$

The aim of this section is to show that  $h_s^{(n)}(t)$ , defined in (2.13), is continuous for  $t \in [s, T]$  and compute  $h'_s(t)$  and  $h''_s(t)$  at  $t = s$ , in case of  $\rho > 1$  and  $\rho > 2$ , respectively.

**Lemma 3** *Assuming (3.1), for any  $0 < s < T < \infty$ , it holds that  $h_s^{(n)}(\cdot) \in C([s, T])$ . Moreover,*

$$h_s^{(n)}(t) \Big|_{t=s} = \begin{cases} C_\gamma \frac{1}{2} s^{\rho-1} & \text{with } n = 1 \text{ when } \rho > 1, \\ \left[ C_{\gamma,\gamma-1} - \frac{C_{\gamma-1}}{\rho-2} \right] s^{\rho-2} & \text{with } n = 2 \text{ when } \rho > 2, \end{cases}$$

where  $C_\gamma$  and  $C_{\gamma_1,\gamma_2}$  are defined in (2.9).

**Proof** In the proof, we will use  $C$  to denote a generic constant that may change its value at each appearance. When  $\rho > n \geq 0$ , we first claim that one can switch the double integrals and differentiation in order to have that

$$h_s^{(n)}(t) = \int_0^s dr \int_{\mathbb{R}^d} d\xi \frac{d^n}{dt^n} \mathfrak{E}(t, \beta + \gamma, r, \xi), \quad \text{for all } t \in [s, T]. \tag{3.2}$$

Indeed, apply Lemma 1 with  $q = -n$  (see Remark 4) to see that,

$$\begin{aligned} \left| \frac{d^n}{dt^n} \mathfrak{E}(t, \beta + \gamma, r, \xi) \right| &= |\mathfrak{E}(t, \beta + \gamma - n, r, \xi)| \\ &\leq C \frac{1}{1 + 2^{-1}v(t-r)^\beta |\xi|^\alpha} \times \frac{1}{1 + 2^{-1}v(s-r)^\beta |\xi|^\alpha} \\ &\leq C \left(1 + 2^{-1}v(s-r)^\beta |\xi|^\alpha\right)^{-2}, \end{aligned}$$

where we have applied the asymptotic property (2.10). Notice that the upper bound does not depend on  $t$  and is  $d\xi$  integrable:

$$\int_{\mathbb{R}^d} \left(1 + 2^{-1}v(s-r)^\beta |\xi|^\alpha\right)^{-2} d\xi = C(s-r)^{-\frac{d\beta}{\alpha}} \int_0^\infty \frac{u^{d-1}}{(1+u^\alpha)^2} du,$$

which is finite because  $d < 2\alpha$ . Therefore, one can apply Fubini’s theorem to see that for all  $t \in [s, T]$ ,

$$\int_{\mathbb{R}^d} d\xi \frac{d^n}{dt^n} \mathfrak{E}(t, \beta + \gamma, r, \xi) = \frac{d^n}{dt^n} \int_{\mathbb{R}^d} d\xi \mathfrak{E}(t, \beta + \gamma, r, \xi). \tag{3.3}$$

Moreover, notice that, by the Cauchy-Schwarz inequality for the  $d\xi$ -integral,

$$\begin{aligned} \left| \frac{d^n}{dt^n} \int_{\mathbb{R}^d} d\xi \mathfrak{E}(t, \beta + \gamma, r, \xi) \right| &\leq \int_{\mathbb{R}^d} d\xi |\mathfrak{E}(t, \beta + \gamma - n, r, \xi)| \\ &\leq \frac{1}{(2\pi)^d} (t-r)^{\beta+\gamma-n-1} (s-r)^{\beta+\gamma-1} \\ &\quad \times \left( \int_{\mathbb{R}^d} d\xi \left| E_{\beta, \beta+\gamma-n} \left( -2^{-1}v(t-r)^\beta |\xi|^\alpha \right) \right| \right)^{1/2} \\ &\quad \times \left( \int_{\mathbb{R}^d} d\xi \left| E_{\beta, \beta+\gamma} \left( -2^{-1}v(s-r)^\beta |\xi|^\alpha \right) \right| \right)^{1/2} \\ &= C_* (t-r)^{\beta+\gamma-n-1-\frac{d\beta}{2\alpha}} (s-r)^{\beta+\gamma-1-\frac{d\beta}{2\alpha}}, \end{aligned}$$

where the last equality is obtained through change of variables followed by applications of (2.9), and  $C_* := \sqrt{C_{\gamma-n} C_\gamma}$ . We claim that

$$\beta + \gamma - n - 1 - \frac{d\beta}{2\alpha} \leq 0. \tag{3.4}$$

Otherwise, we will have  $\beta + \gamma - 1 - \frac{d\beta}{2\alpha} > 0$  as well, which implies that

$$0 < \left( \beta + \gamma - 1 - \frac{d\beta}{2\alpha} \right) + \left( \beta + \gamma - n - 1 - \frac{d\beta}{2\alpha} \right) = \rho - n - 1,$$

which contradicts (3.1). Thanks to (3.4), we can replace  $t$  by  $s$  to see that

$$\left| \frac{d^n}{dt^n} \int_{\mathbb{R}^d} d\xi \mathfrak{E}(t, \beta + \gamma, r, \xi) \right| \leq C_*(s - r)^{2(\beta+\gamma)-n-2-\frac{d\beta}{\alpha}} = C_*(s - r)^{\rho-n-1},$$

with the right-hand side being both independent of  $t$  and integrable with respect to  $dr$ :

$$\int_0^s C_*(s - r)^{\rho-n-1} dr = \frac{C_*}{\rho - n} s^{\rho-n},$$

thanks to the fact that  $\rho > n$ . Therefore, one can apply Fubini’s theorem again to see that, for all  $t \in [s, T]$ ,

$$\int_0^s dr \frac{d^n}{dt^n} \int_{\mathbb{R}^d} d\xi \mathfrak{E}(t, \beta + \gamma, r, \xi) = \frac{d^n}{dt^n} \int_0^s dr \int_{\mathbb{R}^d} d\xi \mathfrak{E}(t, \beta + \gamma, r, \xi). \tag{3.5}$$

A combination of (3.3) and (3.5) proves (3.2). Moreover, the bounds obtained from the above two Cases I and II prove the following useful bound

$$\int_0^s dr \int_{\mathbb{R}^d} d\xi \left| \frac{d^n}{dt^n} \mathfrak{E}(t, \beta + \gamma, r, \xi) \right| \leq \frac{C_*}{\rho - n} s^{\rho-n} < \infty. \tag{3.6}$$

Another direct consequence of (3.2) is that  $t \mapsto h_s^{(n)}(t)$  is a continuous function on  $[s, T]$ .

Now we are ready to calculate  $h'_s(t)|_{t=s}$  and  $h''_s(t)|_{t=s}$  by passing the derivatives all the way inside the double integrals. In case of  $\rho > 1$ , we apply Lemma 1 to (3.2) for  $n = 1$  to see that

$$h'_s(t) \Big|_{t=s} = \int_{\mathbb{R}^d} d\xi \int_0^s dr \mathfrak{E}(t, \beta + \gamma - 1, r, \xi).$$

Thanks to Lemma 1, one can integrate  $dr$  to see that

$$\begin{aligned} h'_s(t) \Big|_{t=s} &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{1}{2} \left[ s^{\beta+\gamma-1} E_{\beta, \beta+\gamma} \left( -2^{-1} \nu(s)^\beta |\xi|^\alpha \right) \right]^2 d\xi \\ &= \frac{C_\gamma}{2} s^{2(\beta+\gamma-1)-\frac{d\beta}{\alpha}} = \frac{C_\gamma}{2} s^{\rho-1}. \end{aligned}$$

If  $\rho > 2$ , apply Lemma 1 to (3.2) for  $n = 2$  and integrate  $h''(t)$  by parts, we have

$$h''_s(t) \Big|_{t=s} = \int_{\mathbb{R}^d} d\xi \mathfrak{E}(s, \beta + \gamma - 1, r, \xi) \Big|_{r=s}^{r=0}$$

$$\begin{aligned}
 & - \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} d\xi \left[ (s-r)^{\beta+\gamma-2} E_{\beta, \beta+\gamma-1} \left( -2^{-1} v(s-r)^\beta |\xi|^\alpha \right) \right]^2 \Big|_{r=s}^{r=0} \\
 & = C_{\gamma, \gamma-1} s^{2(\beta+\gamma)-3-d\beta/\alpha} - C_{\gamma-1} \frac{1}{\rho-2} s^{2(\beta+\gamma)-3-d\beta/\alpha} \\
 & = \left[ C_{\gamma, \gamma-1} - \frac{C_{\gamma-1}}{\rho-2} \right] s^{\rho-2},
 \end{aligned}$$

where both constants  $C_{\gamma-1}$  and  $C_{\gamma, \gamma-1}$  are defined in (2.9). □

### 4 Hölder continuity of $t \mapsto h_s^{(n)}(t)$ via local fractional derivative

In this section, we will calculate the *local fractional derivative* (LFD) of  $h_s(t)$  at  $t = s$ , and then prove the following Proposition 3 about the Hölder continuity of  $h_s(t)$  at  $t = s$ :

**Proposition 3** For all  $t \in [s, \infty)$ , for  $n$  and  $q$  defined in (3.1), it holds that

$$h_s(t) - \sum_{k=0}^n \frac{h_s^{(k)}(s)}{k!} (t-s)^k = o[(t-s)^q].$$

**Proof** The proof of this proposition is an application of the Fractional Taylor theorem (see Lemma 5) to  $h_s(t)$  with all conditions verified in Lemma 7 below. □

In the following, we first give a brief overview of about LFD in Section 4.1 below.

#### 4.1 Local fractional derivative

Recall that for  $-\infty < s < T < \infty$  and  $q \geq 0$ , the (left-sided) *Riemann-Liouville fractional integral* of order  $q$  of  $f \in L^1([s, T]; \mathbb{R})$  is defined by (see, e.g., [23,(2.17), p.33])

$$(I_{s+}^q f)(t) := \frac{1}{\Gamma(q)} \int_s^t f(y)(t-y)^{q-1} dy, \quad t \in (s, T), \tag{4.1}$$

and the corresponding (left-sided) fractional derivative of order  $q$  is defined formally by (see, e.g., [23,(2.22), p.35])

$$(D_{s+}^q f)(t) := \left( \frac{d}{dt} \right)^{n+1} \left( I_{s+}^{n+1-q} f \right)(t) \tag{4.2}$$

$$= \frac{1}{\Gamma(n+1-q)} \left( \frac{d}{dt} \right)^{n+1} \int_s^t f(y)(t-y)^{n-q} dy. \tag{4.3}$$

When  $q = n$ , the above definition gives  $f^{(n)}(t)$  (if  $f^{(n)}(t)$  exists) on  $(s, T)$ .

**Definition 1** For arbitrary  $q > 0$ , the (left-sided) *local fractional derivative* (LFD) of  $f(t)$  of order  $q$  at  $t = s$  is defined as, provided the limit exists,

$$ID^q f(s) := \lim_{t \rightarrow s^+} \left( D_{s^+}^q \left[ f(\cdot) - \sum_{k=0}^n \frac{f^{(k)}(s)}{k!} (\cdot - s)^k \right] \right) (t), \quad 0 < n < q \leq n + 1. \tag{4.4}$$

Clearly, when  $q \in \mathbb{Z}_+$ ,  $ID^q f(s) = f^{(q)}(s)$ . Kolwankar and Gangal [18, 19] introduced the above LFD of order  $q \in (0, 1)$  to investigate the regularity of non-differentiable functions. Recall that the (left-sided) *Caputo fractional derivative* is defined as (see, e.g., [17,(2.4.1), p.91])

$$(\partial_{s^+}^q f)(t) := \left( D_{s^+}^q \left[ f(\cdot) - \sum_{k=0}^n \frac{f^{(k)}(s)}{k!} (\cdot - s)^k \right] \right) (t), \quad n = [q] - 1, \tag{4.5}$$

whereas the notation  $\partial^\beta$  in (1.2) is a short-hand notation for  $\partial_{0+}^\beta$ .

Comparing (4.4) and (4.5), one sees that the LFD is nothing but the limit of Caputo fractional derivative, namely,  $ID^q f(s) = \lim_{t \rightarrow s} (\partial_{s^+}^q f)(t)$ . When  $f^{(n)}(t)$  is absolutely continuous on  $[s, T]$ , both the Riemann-Liouville derivative and the Caputo fractional derivative are well defined *a.e.* on  $[s, T]$  with the Caputo derivative having a shorter form after  $[q]$  times of integration-by-parts (see [23, Theorem 2.2, p.39] or [17, Theorem 2.1, p.92]):

$$(\partial_{s^+}^q f)(t) = \frac{1}{\Gamma(n + 1 - q)} \int_s^t \frac{f^{(n+1)}(\tau)}{(t - \tau)^{q-n}} d\tau, \quad n = [q] - 1. \tag{4.6}$$

In particular, when  $s = 0$ , this reduces to the form (1.2).

However, in this paper, for  $0 \leq n < q < n + 1$ , the function  $h_s^{(n)}(t)$  is in general not absolutely continuous over  $[s, T]$ , or equivalently,  $h_s^{(n+1)}(t)$  may not be well defined. When computing  $(\partial_{s^+}^q h_s)(t)$ , one cannot blindly apply the expression (4.6). We do not presume that  $(\partial_{s^+}^q h_s)(t)$  is well defined until it is shown in Lemma 7. We will need the following composition property of Caputo derivative without assuming  $f^{(n)}(t)$  to be absolutely continuous, which shows that the LFD of  $f(t)$  of order  $q \in \mathbb{R}_+ \setminus \mathbb{Z}_+$  is also the  $(q - n)$ -th order LFD of  $f^{(n)}(t)$ , where  $n = [q]$ .

**Lemma 4** Suppose  $0 < n < q < n + 1$  and  $T > 0$ . Let  $f^{(n-1)}(t)$  be absolutely continuous on  $[s, T]$ . If either  $(\partial_{s^+}^q f)(t)$  or  $(\partial_{s^+}^{q-n} f^{(n)})(t)$  exists on  $(s, T)$ , then  $(\partial_{s^+}^q f)(t) = (\partial_{s^+}^{q-n} f^{(n)})(t)$  on  $(s, T)$ .

**Proof** Fix an arbitrary  $t \in (s, T)$ . Starting from the definition of the Caputo derivative (4.5), via integration-by-parts, we see that

$$(\partial_{s^+}^q f)(t)$$

$$\begin{aligned}
 &= \frac{1}{\Gamma(n+1-q)} \left(\frac{d}{dt}\right)^{n+1} \left\{ -\frac{(t-y)^{n+1-q}}{n+1-q} \left[ f(y) - \sum_{k=0}^n \frac{f^{(k)}(s)}{k!} (y-s)^k \right] \right\} \Big|_{y=s}^t \\
 &\quad + \int_s^t \frac{(t-y)^{n+1-q}}{n+1-q} \frac{d}{dy} \left[ f(y) - \sum_{k=0}^n \frac{f^{(k)}(s)}{k!} (y-s)^k \right] dy \\
 &= \frac{1}{\Gamma(n+1-q)} \left(\frac{d}{dt}\right)^{n+1} \int_s^t \frac{(t-y)^{n+1-q}}{n+1-q} \left[ f'(y) - \sum_{k=1}^n \frac{f^{(k)}(s)}{(k-1)!} (y-s)^{k-1} \right] dy \\
 &= \frac{1}{\Gamma(n+1-q)} \left(\frac{d}{dt}\right)^n \int_s^t (t-y)^{n-q} \left[ f'(y) - \sum_{k=1}^n \frac{f^{(k)}(s)}{(k-1)!} (y-s)^{k-1} \right] dy.
 \end{aligned}$$

Repeat the above integration-by-parts  $n$  times to see that

$$(\partial_{s+}^q f)(t) = \frac{1}{\Gamma(n+1-q)} \frac{d}{dt} \int_s^t (t-y)^{n-q} [f^{(n)}(y) - f^{(n)}(s)] dy, \tag{4.7}$$

which is nothing but what we need to prove, namely,  $(\partial_{s+}^q f)(t) = (\partial_{s+}^{q-n} f^{(n)})(t)$ .  $\square$

**Remark 5** If  $f^{(n)}$  is absolutely continuous on  $[s, T]$  as is assumed in Theorem 2.1 of [17], one can apply one more time this integration-by-parts to arrive at (4.6). Here, we will postpone the determination of the well-posedness of  $(\partial_{s+}^q f^{(n)})(t)$  latter.

The following *Fractional Taylor theorem* is crucial for the estimation of  $h_s(t)$ .

**Lemma 5** (Fractional Taylor theorem). *Let  $f$  be a real-valued function. Assume that for some  $q > 0$  and  $s \in \mathbb{R}$ , the following two conditions hold:*

1.  $f^{(\lceil q \rceil - 1)}(t)$  be right-continuous at  $s$ , namely,

$$\lim_{t \rightarrow s+} f^{(\lceil q \rceil - 1)}(t) = f^{(\lceil q \rceil - 1)}(s);$$

2.  $ID^q f(s)$  exists;

then as  $t \downarrow s$ ,

$$f(t) - \sum_{k=0}^{\lceil q \rceil - 1} \frac{f^{(k)}(s)}{k!} (t-s)^k = \frac{ID^q f(s)}{\Gamma(q+1)} (t-s)^q + o[(t-s)^q]. \tag{4.8}$$

**Remark 6** This lemma is a rephrasing of [1, Theorems 4 and 5], which has been both tailored for our application and generalized from  $q \in (0, 1)$  to all  $q > 0$ . To the best of our knowledge, Lemma 5 also appeared, with stronger or equivalent conditions, in the following references: In the case of  $n = 0$ , [18] proved it under a stronger condition that  $(\partial_{s+}^q f)(t)$  is absolutely continuous; Proposition 2 of [11] in fact needs that  $(I_{s+}^{1-q} f)(t)$  is absolutely continuous. For completeness and also for the readers' convenience, we reproduce below the proof of this lemma with slight more details.

**Proof of Lemma 5** When  $q \geq 1$  is an integer, (4.8) reduces to the classical Taylor's expansion. In the following, we will assume that  $q \notin \mathbb{Z}$ .

Let us first consider the case  $0 < q < 1$ . Since  $f(t)$  is right continuous at  $t = s$ , there exists some  $T > s$  such  $f \in C([s, T]) \subseteq L^1([s, T])$ . Therefore, for all  $t \in [s, T]$ ,

$$\begin{aligned} f(t) - f(s) &= \frac{d}{dt} \left( I_{s+}^1 [f(\cdot) - f(s)] \right) (t) = \frac{d}{dt} \left( I_{s+}^q \circ I_{s+}^{1-q} [f(\cdot) - f(s)] \right) (t) \\ &= \frac{d}{dt} \frac{1}{\Gamma(q)} \int_s^t \left( I_{s+}^{1-q} [f(\cdot) - f(s)] \right) (y) (t - y)^{q-1} dy, \end{aligned} \quad (4.9)$$

where the first and the second equalities are due to the fundamental theorem of calculus and Lemma 2.3 on p. 73 of [17], respectively, both applied to continuous functions. Now in order to apply the integration-by-parts formula to the above dy-integral, we need to show that

$$\text{the function } y \mapsto \left( I_{s+}^{1-q} [f(\cdot) - f(s)] \right) (y) \text{ is absolutely continuous on } [s, T]. \quad (4.10)$$

Indeed, by (4.5),

$$\frac{d}{dy} \left( I_{s+}^{1-q} [f(\cdot) - f(s)] \right) (y) = (\partial_{s+}^q f)(y),$$

where the right-hand side is well defined, at least at a small right neighborhood of  $s$ , thanks to the condition that  $D^q f(s)$  exists. Hence, by moving  $T$  towards  $s$  whenever necessary, we see that  $y \mapsto \left( I_{s+}^{1-q} [f(\cdot) - f(s)] \right) (y)$  is is differential *everywhere* on  $[s, T]$  with its derivative equal to  $(D_{s+}^q [f(\cdot) - f(s)])(y)$  or, equivalently,  $(\partial_{s+}^q f)(y)$  (see (4.5)). Again, the existence of  $D^q f(s)$  guarantees that

$$(\partial_{s+}^q f)(\cdot) \in L^\infty([s, T]) \cap L^1([s, T]). \quad (4.11)$$

This proves (4.10). Hence, we can apply the integration-by-parts to see that for all  $y \in [s, T]$ , the right-hand side of (4.9) is equal to

$$\begin{aligned} &\frac{d}{dt} \frac{1}{\Gamma(q)} \left[ -\frac{1}{q} (t - y)^q \times \left( I_{s+}^{1-q} [f(\cdot) - f(s)] \right) (y) \right]_{y=s}^{y=t} \\ &\quad + \int_s^t (\partial_{s+}^q f)(y) \times \frac{1}{q} (t - y)^q dy \\ &= \frac{d}{dt} \frac{1}{\Gamma(q)} \int_s^t (\partial_{s+}^q f)(y) \times \frac{1}{q} (t - y)^q dy, \end{aligned}$$

where the first term in the bracket is equal to zero because  $f \in C([s, T]) \subseteq L^1([s, T])$ . Thanks to the boundedness of the integrand (see (4.11)), one can switch the  $dt$  deriva-

tive and the dy integral to see that

$$f(t) - f(s) = (I_{s+}^q \circ D_{s+}^q [f(\cdot) - f(s)])(t), \quad \text{for all } t \in [s, T], \quad (4.12)$$

where the fractional integral is well defined thanks to (4.11). Finally, the definition of LFD in (4.4) implies that

$$\begin{aligned} (I_{s+}^q \circ D_{s+}^q [f(\cdot) - f(s)])(t) &= (I_{s+}^q [\mathbb{ID}^q f(\cdot) + o(1)])(t) \\ &= \frac{\mathbb{ID}^q f(s)}{\Gamma(q + 1)}(t - s)^q + o[(t - s)^q]. \end{aligned} \quad (4.13)$$

Combining (4.12) and (4.13) proves (4.8) for the case when  $q \in (0, 1)$ .

It remains to prove the case when  $q > 1$  and  $q \notin \mathbb{Z}$ . Notice that by Lemma 4, the existence of  $\mathbb{ID}^q f(s)$  is equivalent to the existence of  $\mathbb{ID}^{q-[q]} f^{([q])}(s)$ . Now we can apply the case of  $0 < q < 1$  to  $f^{([q])}(t)$  to see that

$$f^{([q])}(t) - f^{([q])}(s) = \frac{\mathbb{ID}^{q-[q]} f^{([q])}(s)}{\Gamma(q - [q] + 1)}(t - s)^{q-[q]} + o((t - s)^{q-[q]}). \quad (4.14)$$

Finally, the case is proved by applying the  $I_{s+}^{[q]}$  on both sides of (4.14). This completes the proof of Lemma 5. □

### 4.2 Local fractional derivative of $h_s^{(n)}(t)$

By Lemma 5 and Lemma 4, to prove Proposition 3 we only need to show  $\mathbb{ID}^{q-n} h_s^{(n)}(t) = 0$  for  $0 \leq n < q < n - 1$ . We will apply Fubini's Theorem to show the interchangeability of local fractional operator  $\mathbb{ID}^q$  and the integral operator  $\int_0^s \int_{\mathbb{R}^d}$  in  $h_s^{(n)}(t)$  in the following two lemmas.

**Lemma 6** *Assuming (3.1), for all  $T \in (s, \infty)$ , it holds that*

$$t \mapsto \left( I_{s+}^{n+1-q} \left[ h_s^{(n)}(\cdot) - h_s^{(n)}(s) \right] \right) (t) \in C([s, T]) \quad (4.15)$$

and for all  $t \in [s, T]$ ,

$$\begin{aligned} \left( I_{s+}^{n+1-q} [h_s^{(n)}(\cdot) - h_s^{(n)}(s)] \right) (t) &= \int_0^s dr \int_{\mathbb{R}^d} d\xi \\ &\times \left( I_{s+}^{n+1-q} [\mathfrak{E}(\cdot, \beta + \gamma - n, r, \xi) - \mathfrak{E}(s, \beta + \gamma - n, r, \xi)] \right) (t). \end{aligned} \quad (4.16)$$

**Proof** The statement of (4.15) is due to the fact that  $h_s^{(n)}(t) \in C([s, T])$ ; see Lemma 3. As for (4.16), there are two fractional integrals. The second one is trivially true since since  $h_s^{(n)}(s)$  does not depend on  $t$ . For the first fractional integral, we can switch the

order of integrations because, by (3.6),

$$\left( I_{s+}^{n+1-q} \int_0^s dr \int_{\mathbb{R}^d} d\xi |\mathfrak{E}(\cdot, \beta + \gamma - n, r, \xi)| \right) (t) < \left( I_{s+}^{n+1-q} C_T \right) (t) < C'_T < \infty.$$

This proves the lemma. □

**Lemma 7** *Assuming (3.1), for all  $T \in (s, \infty)$ , it holds that  $(\partial_{s+}^{q-n} h_s^{(n)}) (t) \in C([s, T])$  and*

$$\left( \partial_{s+}^{q-n} h_s^{(n)} \right) (t) = \int_0^s dr \int_{\mathbb{R}^d} d\xi \left( \partial_{s+}^{q-n} \mathfrak{E}(\cdot, \beta + \gamma - n, r, \xi) \right) (t), \tag{4.17}$$

for all  $t \in [s, T]$ . Consequently,

$$ID^q h_s(s) = ID^{q-n} h_s^{(n)}(s) = 0, \quad \text{for all } s > 0. \tag{4.18}$$

**Proof** We will prove this lemma in the following three steps:

*Step 1.* We will first verify the condition of Fubini’s theorem is satisfied, so that (4.17) is valid a.e. on  $[s, T]$ . By the definition of the Caputo derivative (4.5) and Lemma 6,

$$\begin{aligned} & \left( \partial_{s+}^{q-n} h_s^{(n)} \right) (t) \\ &= \frac{d}{dt} \left( I_{s+}^{n+1-q} \left[ h_s^{(n)}(\cdot) - h_s^{(n)}(s) \right] \right) (t) \\ &= \frac{d}{dt} \int_0^s dr \int_{\mathbb{R}^d} d\xi \left( I_{s+}^{n+1-q} \left[ \mathfrak{E}(\cdot, \beta + \gamma - n, r, \xi) - \mathfrak{E}(s, \beta + \gamma - n, r, \xi) \right] \right) (t). \end{aligned}$$

Noticing that

$$\begin{aligned} & \frac{d}{dt} \left( I_{s+}^{n+1-q} \left[ \mathfrak{E}(\cdot, \beta + \gamma - n, r, \xi) - \mathfrak{E}(s, \beta + \gamma - n, r, \xi) \right] \right) (t) \\ &= \left( \partial_{s+}^{q-n} \mathfrak{E}(\cdot, \beta + \gamma - n, r, \xi) \right) (t), \end{aligned}$$

(4.17) is valid a.e. on  $[s, T]$  once one can justify that the derivative of  $t$  can be passed through the double integrals. To switch the derivative and the integrals, one can carry out some similar, but much more involved, dominated convergence arguments in two steps as the proof of Lemma 3. Here we will use a slightly different, but more convenient, criterion — Theorem 4 of [25]. Indeed, notice that, by (4.6),

$$\begin{aligned} & \left( \partial_{s+}^{q-n} \mathfrak{E}(\cdot, \beta + \gamma - n, r, \xi) \right) (t) \\ &= \frac{1}{\Gamma(n+1-q)} \int_s^t (t-y)^{n-q} \mathfrak{E}(y, \beta + \gamma - n - 1, r, \xi) dy, \end{aligned}$$

In light of Theorem 4 of [25], it suffices to show that

$$J := \int_0^s dr \int_{\mathbb{R}^d} d\xi \int_s^t dy (t - y)^{n-q} |\mathfrak{E}(y, \beta + \gamma - n - 1, r, \xi)| < \infty.$$

An application of Lemma 2 shows that

$$J \leq C'_* \int_0^s dr \int_s^t dy (y - r)^{\frac{\rho-3}{2}-n} (t - y)^{n-q} (s - r)^{\frac{\rho-1}{2}}, \tag{4.19}$$

where  $C'_* := \sqrt{C_{\gamma-n-1} C_\gamma}$ . When  $t < 2s$ , we have

$$\begin{aligned} J &= C'_* \left( \int_0^{2s-t} dr + \int_{2s-t}^s dr \right) \int_s^t dy (y - r)^{\frac{\rho-3}{2}-n} (t - y)^{n-q} (s - r)^{\frac{\rho-1}{2}} \\ &=: C'_* (J_1 + J_2). \end{aligned}$$

We first consider  $J_1$ . Because  $\frac{t+r}{2} \leq s$ , we have that the following:

$$\begin{aligned} J_1 &\leq \int_0^{2s-t} dr \int_{\frac{t+r}{2}}^t dy (y - r)^{\frac{\rho-3}{2}-n} (t - y)^{n-q} (s - r)^{\frac{\rho-1}{2}} \\ &\leq \int_0^{2s-t} dr \int_{\frac{t+r}{2}}^t dy \left( \frac{t - r}{2} \right)^{\frac{\rho-3}{2}-n} (t - y)^{n-q} (s - r)^{\frac{\rho-1}{2}} \\ &= \frac{1}{n + 1 - q} \int_0^{2s-t} dr \left( \frac{t - r}{2} \right)^{\frac{\rho-1}{2}-q} (s - r)^{\frac{\rho-1}{2}} \\ &\leq \frac{1}{n + 1 - q} \int_0^{2s-t} dr \left( \frac{s - r}{2} \right)^{\frac{\rho-1}{2}-q} (s - r)^{\frac{\rho-1}{2}} \\ &\leq \frac{1}{n + 1 - q} \int_0^s dr \left( \frac{s - r}{2} \right)^{\frac{\rho-1}{2}-q} (s - r)^{\frac{\rho-1}{2}} \\ &= \frac{1}{n + 1 - q} 2^{q - \frac{\rho-1}{2}} \frac{1}{\rho - q} s^{\rho-q}, \end{aligned}$$

where the second inequality is due to  $\frac{t-r}{2} \leq y - r$  and the third inequality is thanks to  $\frac{\rho-1}{2} - q < 0$ , and recall from (3.1) that  $\rho - q \in (0, 1)$ . As for  $J_2$ , since  $\frac{t+r}{2} \geq s$ , we see that

$$\begin{aligned} J_2 &= \int_{2s-t}^s dr \left( \int_s^{\frac{t+r}{2}} dy + \int_{\frac{t+r}{2}}^t dy \right) (y - r)^{\frac{\rho-3}{2}-n} (t - y)^{n-q} (s - r)^{\frac{\rho-1}{2}} \\ &=: (J_{21} + J_{22}). \end{aligned}$$

Estimation for  $J_{22}$  is similar to  $J_1$ :

$$\begin{aligned} J_{22} &\leq \frac{1}{n+1-q} \int_{2s-t}^s dr \left(\frac{s-r}{2}\right)^{\frac{\rho-1}{2}-q} (s-r)^{\frac{\rho-1}{2}} \\ &\leq \frac{1}{n+1-q} \int_0^s dr \left(\frac{s-r}{2}\right)^{\frac{\rho-1}{2}-q} (s-r)^{\frac{\rho-1}{2}} \\ &= \frac{1}{n+1-q} 2^{q-\frac{\rho-1}{2}} \frac{1}{\rho-q} s^{\rho-q}. \end{aligned}$$

Finally, for  $J_{21}$ , we have that

$$\begin{aligned} J_{21} &= \int_{2s-t}^s dr \int_s^{\frac{t+r}{2}} dy (y-r)^{\frac{\rho-3}{2}-n} (t-y)^{n-q} (s-r)^{\frac{\rho-1}{2}} \\ &\leq \int_{2s-t}^s dr \left(\frac{t-r}{2}\right)^{n-q} (s-r)^{\frac{\rho-1}{2}} \int_s^{\frac{t+r}{2}} (y-r)^{\frac{\rho-3}{2}-n} dy \\ &= \frac{1}{n-\frac{\rho-1}{2}} \int_{2s-t}^s dr \left(\frac{t-r}{2}\right)^{n-q} (s-r)^{\frac{\rho-1}{2}} \left[ (s-r)^{\frac{\rho-1}{2}-n} - \left(\frac{t-r}{2}\right)^{\frac{\rho-1}{2}-n} \right]. \end{aligned}$$

where we have used the fact that  $n - (\rho - 1)/2 > 0$ . Because  $n - q < 0$  and  $(\rho - 1)/2 < n$ , we can replace  $(t - r)/2$  by  $(s - r)/2$  to see that

$$\begin{aligned} J_{21} &\leq C^\dagger \int_0^s (s-r)^{\rho-1-q} dr = \frac{C^\dagger}{\rho-q} s^{\rho-q}, \\ \text{with } C^\dagger &:= \frac{2}{2n-\rho+1} \left[ 2^{q-n} - 2^{q-\frac{\rho-1}{2}} \right]. \end{aligned}$$

Combining all these terms proves  $J < \infty$ . When  $t > 2s$ , we have  $\frac{t+r}{2} \geq s$  for all  $r \in (0, s)$ . Then  $J$  is dominated as  $J_2$  is in the case of  $t < 2s$ . Therefore, it is valid to switch the derivative and the double integrals to conclude that (4.17) holds *a.e.* on  $t \in [s, T]$ .

*Step 2.* Now we will show  $(\partial_{s+}^{q-n} h_s^{(n)})(t)$  is continuous on  $[s, T]$  by showing that

$$\lim_{t \rightarrow t_0} (\partial_{s+}^{q-n} h_s^{(n)})(t) = (\partial_{s+}^{q-n} h_s^{(n)})(t_0) \quad \text{for all } t_0 \in [s, T], \tag{4.20}$$

which then implies that (4.17) is valid for all  $t \in [s, T]$ . Notice that Step 1 proves that for *a.e.*  $t \in [s, T]$ ,

$$\begin{aligned} (\partial_{s+}^{q-n} h_s^{(n)})(t) &= C \int_{\mathbb{R}} dr \int_{\mathbb{R}} dy \int_{\mathbb{R}^d} d\xi \left( 1_{[0,s]}(r) 1_{[s,t]}(y) \right. \\ &\quad \left. \times \mathfrak{E}(y, \beta + \gamma - n - 1, r, \xi)(t - y)^{n-q} \right) \end{aligned} \tag{4.21}$$

with  $C = 1/\Gamma(n + 1 - q)$ , where one can choose whatever order of integrations thanks to Fubini's theorem. Hence, the proof of (4.20) amounts to passing the limit through three integrals.

First notice that one can switch the order of the limit  $t$  with the  $dr$ -integral because from the proof in Step 1, the integrand for the  $dr$ -integral is dominated by a function of  $(s - r)$  that does not depend on  $t$  and is integrable with respect to  $dr$ .

It remains to show that one can further pass the limit into the rest two integrals, namely,

$$\begin{aligned} \lim_{t \rightarrow t_0} \int_{\mathbb{R}} dy \int_{\mathbb{R}^d} d\xi \, 1_{[s,t]}(y) \mathfrak{E}(y, \beta + \gamma - n - 1, r, \xi)(t - y)^{n-q} \\ = \int_{\mathbb{R}} dy \int_{\mathbb{R}^d} d\xi \, 1_{[s,t_0]}(y) \mathfrak{E}(y, \beta + \gamma - n - 1, r, \xi)(t_0 - y)^{n-q}. \end{aligned} \tag{4.22}$$

In the following, let  $C$  be a general constant may vary from line to line. For  $t \in [s, T]$ , because  $n - q \in (-1, 0)$  and

$$\begin{aligned} \int_s^t (t - y)^{n-q} dy &= \frac{1}{n + 1 - q} (t - s)^{n+1-q} \leq \frac{1}{n + 1 - q} (T - s)^{n+1-q} \\ &= \int_s^T (T - y)^{n-q} dy, \end{aligned}$$

together with (2.10), we see that the integrand of the double integral can be bounded as

$$\begin{aligned} |1_{[s,t]}(y) \mathfrak{E}(y, \beta + \gamma - n - 1, r, \xi)(t - y)^{n-q}| \\ \leq C(s - r)^{2(\beta+\gamma)-n-3} (T - y)^{n-q} 1_{[s,T]}(y) (1 + 2^{-1} \nu (s - r)^\beta |\xi|^\alpha)^{-2} =: I(y, \xi). \end{aligned}$$

Notice that this upper bound  $I(y, \xi)$  does not depend on  $t$ . Moreover,

$$\begin{aligned} \int_{\mathbb{R}} dy \int_{\mathbb{R}^d} d\xi \, |I(y, \xi)| \\ \leq C(s - r)^{2(\beta+\gamma)-n-3} \int_s^T dy (T - y)^{n-q} \int_{\mathbb{R}^d} d\xi \, (1 + 2^{-1} \nu (s - r)^\beta |\xi|^\alpha)^{-2} \\ = C(s - r)^{2(\beta+\gamma)-n-3} (T - y)^{n+1-q} \int_{\mathbb{R}^d} d\xi \, (1 + 2^{-1} \nu (s - r)^\beta |\xi|^\alpha)^{-2} \\ < \infty. \end{aligned}$$

Then, an application of the dominated convergence proves (4.22). This proves (4.20). Therefore, (4.17) holds for all  $t \in [s, T]$ .

Step 3. Finally, for (4.18), the first equality holds trivially in case of  $n = 0$ , or by Lemma 4 otherwise. As for the second equality, by (4.4), (4.5) and Step 2, we can

push the limit of  $t$  inside the triple integrals of (4.21) to conclude that

$$\mathbb{D}^{q-n} h_s^{(n)}(s) = \lim_{t \rightarrow s_+} \left( \partial_{s_+}^{q-n} h_s^{(n)} \right) (t) = 0.$$

This completes the whole proof of Lemma 7.  $\square$

## 5 Proof of Proposition 2

Now we are ready to prove Proposition 2.

**Proof of Proposition 2** Denote the left-hand side of (2.8) by  $I$  and then

$$I = C_\gamma \int_0^s dr \left[ (t-r)^{2(\beta+\gamma-1)-d\beta/\alpha} + (s-r)^{2(\beta+\gamma-1)-d\beta/\alpha} \right] - 2h_s(t).$$

When  $\rho \leq 1$ , by Proposition 3, we have  $h_s(t) = h_s(s) + o[(t-s)^q]$  and so there is a constant  $K$ , such that  $|h_s(t) - h_s(s)| < K(t-s)^q$ . Therefore,

$$\begin{aligned} I &\leq \frac{C_\gamma}{\rho} [t^\rho - (t-s)^\rho + s^\rho] - 2[h_s(s) - K(t-s)^q] \\ &= \frac{C_\gamma}{\rho} [t^\rho - (t-s)^\rho + s^\rho - 2s^\rho] + 2K(t-s)^q \\ &= O[(t-s)^\rho] + 2K(t-s)^q, \end{aligned}$$

with  $2K(t-s)^q$  being the dominating term.

Similarly, when  $1 < \rho \leq 2$ , by Proposition 3 and Lemma 3,

$$\begin{aligned} I &\leq \frac{C_\gamma}{\rho} [t^\rho - (t-s)^\rho + s^\rho] - 2 \left[ h_s(s) + \frac{1}{2} C_\gamma s^{\rho-1} (t-s) - K(t-s)^q \right] \\ &= C_\gamma \left[ \frac{1}{\rho} (t^\rho - s^\rho) - s^{\rho-1} (t-s) \right] - \frac{C_\gamma}{\rho} (t-s)^\rho + 2 * K(t-s)^q \\ &= \frac{C_\gamma(\rho-1)s^{\rho-2}}{2} (t-s)^2 + O[(t-s)^3] + O[(t-s)^\rho] + 2K(t-s)^q, \end{aligned}$$

with  $2K(t-s)^q$  being the dominating term.

Finally, when  $\rho > 2$ ,

$$\begin{aligned} I &\leq \frac{C_\gamma}{\rho} [t^\rho - (t-s)^\rho + s^\rho] \\ &\quad - 2 \left[ h_s(s) + \frac{1}{2} C_\gamma s^{\rho-1} (t-s) + \frac{1}{2} h_s''(s) (t-s)^2 - K(t-s)^q \right] \\ &= C_\gamma \left[ \frac{1}{\rho} (t^\rho - s^\rho) - s^{\rho-1} (t-s) \right] - \frac{C_\gamma}{\rho} (t-s)^\rho - h_s''(s) (t-s)^2 + 2K(t-s)^q \end{aligned}$$

$$\begin{aligned}
&= \frac{C_\gamma(\rho-1)s^{\rho-2}}{2}(t-s)^2 + O\left[(t-s)^3\right] + O\left[(t-s)^\rho\right] - h_s''(s)(t-s)^2 \\
&\quad + 2K(t-s)^q \\
&= \left(\frac{C_\gamma(\rho-1)}{2} + \frac{C_{\gamma-1}}{\rho-2} - C_{\gamma,\gamma-1}\right)s^{\rho-2}(t-s)^2 + O\left[(t-s)^3\right] + O\left[(t-s)^\rho\right] \\
&\quad + 2K(t-s)^q,
\end{aligned}$$

where the second order term is the dominating term. This completes the proof of Proposition 2.  $\square$

## Declarations

**Conflict of interest** The authors declare that they have no conflict of interest.

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