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Two-point Correlation Function and Feynman-Kac Formula for the Stochastic Heat Equation

Le Chen[1] **· Yaozhong Hu**[1] **· David Nualart**[1]

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Abstract In this paper, we obtain an explicit formula for the two-point correlation function for the solutions to the stochastic heat equation on $\mathbb{R}$. The bounds for p-th moments proved in Chen and Dalang (Ann. Probab. 2015) are simplified. We validate the Feynman-Kac formula for the p-point correlation function of the solutions to this equation with measure-valued initial data.

Keywords Stochastic heat equation · Two-point correlation function · Feynman-Kac formula · Brownian local time · Malliavin calculus

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✉ Le Chen
chenle02@gmail.com; chenle@ku.edu

Yaozhong Hu
yhu@ku.edu

David Nualart
nualart@ku.edu

[1] Department of Mathematics, University of Kansas, Lawrence, KS 66045, USA

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1 Introduction

Consider the following stochastic heat equation

$$\begin{cases} \left(\frac{\partial}{\partial t}-\frac{\nu}{2}\frac{\partial^2}{\partial x^2}\right)u(t,x)=\rho(u(t,x))\,\dot{W}(t,x),\ x\in\mathbb{R},\ t>0, \\ u(0,\cdot)=\mu(\cdot)\ , \end{cases} \tag{1.1}$$

where $\dot{W}$ is a space-time white noise and $\rho:\mathbb{R}\to\mathbb{R}$ is a globally Lipschitz function. The initial data μ is a signed Borel measure, which we assume belongs to the set

$$\mathcal{M}_H(\mathbb{R}):=\left\{\text{signed Borel measures } \mu, \text{ s.t. } \int_{\mathbb{R}} e^{-ax^2}|\mu|(\mathrm{d}x)<+\infty,\ \text{for all } a>0\right\}.$$

In the above, we denote $|\mu|:=\mu_+ + \mu_-$, where $\mu=\mu_+-\mu_-$ and $\mu_\pm$ are the two non-negative Borel measures with disjoint support that provide the Jordan decomposition of μ. The set $\mathcal{M}_H(\mathbb{R})$ can be equivalently characterized by the condition that

$$(|\mu|*G_\nu(t,\cdot))(x)=\int_{\mathbb{R}} G_\nu(t,x-y)|\mu|(\mathrm{d}y)<+\infty\ , \quad \text{for all } t>0 \text{ and } x\in\mathbb{R}, \tag{1.2}$$

where $*$ denotes the convolution in the space variable and $G_\nu(t,x)$ is the one-dimensional heat kernel function

$$G_\nu(t,x):=\frac{1}{\sqrt{2\pi\nu t}}\exp\left\{-\frac{x^2}{2\nu t}\right\},\quad t>0,\ x\in\mathbb{R}\,.$$

The initial condition $u(0,\cdot)=\mu(\cdot)$ is understood as $\lim_{t\downarrow 0}u(t,\cdot)=\mu(\cdot)$ in the sense of distribution (we identify a measure as a distribution in the usual sense; see [7, Theorem 1.7],). Denote

$$J_0(t,x):=(\mu*G_\nu(t,\cdot))(x)=\int_{\mathbb{R}} G_\nu(t,x-y)\mu(\mathrm{d}y).$$

Define the kernel function

$$\mathcal{K}(t,x)=\mathcal{K}(t,x;\nu,\lambda):=G_{\frac{\nu}{2}}(t,x)\cdot\left(\frac{\lambda^2}{\sqrt{4\pi\nu t}}+\frac{\lambda^4}{2\nu}e^{\frac{\lambda^4 t}{4\nu}}\Phi\left(\lambda^2\sqrt{\frac{t}{2\nu}}\right)\right), \tag{1.3}$$

where $\Phi(x)=\int_{-\infty}^x(2\pi)^{-1/2}e^{-y^2/2}\mathrm{d}y$. Some functions related to $\Phi(x)$ are the error functions $\mathrm{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-y^2}\mathrm{d}y$ and $\mathrm{erfc}(x)=1-\mathrm{erf}(x)$. Note that $\Phi(x)=\mathrm{erfc}(-x/\sqrt{2})/2$.

When $\rho(u)=\lambda u$, the following moment formula is proved in [6]

$$\mathbb{E}\left(u(t,x)^2\right)=J_0^2(t,x)+(J_0^2\star\mathcal{K})(t,x), \tag{1.4}$$

where "$\star$" denotes the convolution in both space and time variables, that is,

$$(J_0^2\star\mathcal{K})(t,x):=\int_0^t\mathrm{d}s\int_{\mathbb{R}}\mathrm{d}y\ J_0^2(s,y)\mathcal{K}(t-s,x-y). \tag{1.5}$$

As for the two-point correlation function, define

$$\begin{aligned}\mathcal{I}(t,x_1,\tau,x_2;\nu,\lambda):=\lambda^2\textstyle\int_0^t\mathrm{d}r\int_{\mathbb{R}}\mathrm{d}z\left[J_0^2(r,z)+\left(J_0^2(\cdot,\circ)\star\mathcal{K}(\cdot,\circ;\nu,\lambda)\right)(r,z)\right]\\ \times G_\nu(t-r,x_1-z)G_\nu(\tau-r,x_2-z).\end{aligned}$$

Then by [6, (2.26)], for all $\tau\geq t>0$ and $x_1,x_2\in\mathbb{R}$,

$$\mathbb{E}\left[u(t,x_1)u(\tau,x_2)\right]=J_0(t,x_1)J_0(\tau,x_2)+\mathcal{I}(t,x_1,\tau,x_2;\nu,\lambda)\ . \tag{1.6}$$

The first goal of this note is to simplify the moment formulas (1.4) and (1.6) for the case $t = \tau$. Note that the terms $(J_0^2 \star \mathcal{K})(t, x)$ and $\mathcal{I}(\dots)$ involve four and six integrals, respectively. We will reduce these integrals into only two integrals: two convolutions of the initial data with respect to a kernel function.

It is well known that if the initial data is a function, then the moments of the solution to (1.1) with $\rho(u) = \lambda u$ admit a Feynman-Kac representation; see [10]. Suppose that $\mu(\mathrm{d}x) = u_0(x)\mathrm{d}x$ where u_0 is a bounded measurable function. This representation says that for all $x_i \in \mathbb{R}, i = 1, \dots, n,$

$$\mathbb{E}\left[\prod_{i=1}^{n} u(t, x_i)\right] = \mathbb{E}^{\mathbf{B}}\left[\prod_{i=1}^{n} u_0\left(x_i + B_t^i\right) \exp\left(\lambda^2 \sum_{1\le i<j\le n} \int_0^t \delta_{x_j - x_i}\left(B_s^i - B_s^j\right) \mathrm{d}s\right)\right], \quad (1.7)$$

where $\{B_t^i, t \geq 0\}$, $i = 1, \dots, n$, are i.i.d. standard Brownian motions on $\mathbb{R}$, and the expectation is with respect to all these Brownian motions.

On one hand, direct evaluation of this expectation when $n = 2$ is not easy since it involves the joint law of a standard Brownian motion B_t and its local time L_t^a at an arbitrary level $a \in \mathbb{R}$. To the best of our knowledge, we are not aware of any references for this joint law expect the case where $a = 0$. We will derive this joint distribution and then give an alternative and more probabilistic proof of the formula for the two-point correlation function.

On the other hand, when the initial condition is a measure the meaning of (1.7) is not clear. Another aim of this paper is to make sense of (1.7) for initial data in $\mathcal{M}_H(\mathbb{R})$ using Malliavin calculus. More precisely, we show that $\prod_{i=1}^{n} u_0\left(x_i + B_t^i\right)$ belongs to the Meyer-Watanabe space $\mathbb{D}^{-\alpha,p}$ of Wiener distributions for any $p > 1$ and $\alpha > n(1 - 1/p)$, and the exponential factor in (1.7)

$$Y_n := \exp\left(\lambda^2 \sum_{1\le i<j\le n} \int_0^t \delta_{x_j - x_i}\left(B_s^i - B_s^j\right) \mathrm{d}s\right) \quad (1.8)$$

belongs to $\mathbb{D}^{\alpha,p}$ for any $p > 1$ and $\alpha < \frac{1}{2}$. Then, we can choosing p such that $n(1-1/p) < \alpha < 1/2$, we can write

$$\mathbb{E}\left[\prod_{i=1}^{n} u(t, x_i)\right] = \left\langle \prod_{i=1}^{n} u_0\left(x_i + B_t^i\right), Y_n \right\rangle, \quad (1.9)$$

where $\langle \cdot, \cdot \rangle$ denotes the duality between $\mathbb{D}^{-\alpha,p}$ and $\mathbb{D}^{\alpha,q}$, if $1/p + 1/q = 1$.

When $\rho(u)$ in (1.1) is nonlinear but satisfies the global Lipschitz condition, the explicit formula for the moment of the solution is impossible. We obtain an upper bound for the p-moment of the solution and a lower bound for the second moment. The idea is to compare them with the ones in the linear case.

We first state our main results in Section 2. The result for the two-point correlation function, Theorem 2.1, is proved in Section 3. In Section 4, the joint law of (B_t, L_t^a), Theorem 2.8, is proved. In Section 5, we give the alternative proof of our two-point correlation formula for function-valued initial data. Finally, by proving Theorems 2.10 and 2.11 in Sections 6.1 and 6.2, respectively, we make sense of (1.7) for measure-valued initial data.

One may think to extend the above frame to two spatial dimensions. However, since the equation has no classical solution for space time white noise, a renormalization procedure may be needed even in linear case ([4]).

Throughout of the paper, $||\cdot||_p$ denotes $L^p(\Omega)$ norm.

2 Main Results

2.1 Formulas for Two-Point Correlation Function

Theorem 2.1 *Suppose that $\mu \in \mathcal{M}_H(\mathbb{R})$. If $\rho(u) = \lambda u$, then for all $t > 0$ and $x_1, x_2 \in \mathbb{R}$,*

$$\mathbb{E}\left[u(t,x_1)u(t,x_2)\right] = \iint_{\mathbb{R}^2} \mu(\mathrm{d}z_1)\mu(\mathrm{d}z_2)\, \mathcal{K}^*(t, x_1 - z_1, x_2 - z_2, x_1 - x_2; \lambda), \tag{2.1}$$

or

$$\begin{aligned}\mathbb{E}\left[u(t,x_1)u(t,x_2)\right] = &\quad J_0(t,x_1)J_0(t,x_2)\\ &+ \iint_{\mathbb{R}^2} \mu(\mathrm{d}z_1)\mu(\mathrm{d}z_2)\, \mathcal{K}^\dagger(t, x_1 - z_1, x_2 - z_2, x_1 - x_2; \lambda),\end{aligned} \tag{2.2}$$

where

$$\begin{aligned}\mathcal{K}^\dagger(t,z_1,z_2,y;\lambda) =& \frac{\lambda^2}{2\nu} G_{\nu/2}\left(t, \frac{z_1+z_2}{2}\right) \exp\left(\frac{\lambda^2}{4\nu}\left[\lambda^2 t - 2(|y| + |y-(z_1-z_2)|)\right]\right)\\ &\times \Phi\left(\frac{\lambda^2 t - (|y| + |y-(z_1-z_2)|)}{\sqrt{2\nu t}}\right).\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}\mathcal{K}^*(t,z_1,z_2,y;\lambda) &= G_\nu(t,z_1)G_\nu(t,z_2) + \mathcal{K}^\dagger(t,z_1,z_2,y;\lambda)\\ &= G_{\nu/2}\left(t, \frac{z_1+z_2}{2}\right)\Bigg[G_{2\nu}(t, z_1 - z_2)\\ &\quad + \frac{\lambda^2}{2\nu}\exp\left(\frac{\lambda^2}{4\nu}\left[\lambda^2 t - 2(|y| + |y-(z_1-z_2)|)\right]\right)\\ &\quad \times \Phi\left(\frac{\lambda^2 t - (|y| + |y-(z_1-z_2)|)}{\sqrt{2\nu t}}\right)\Bigg].\end{aligned} \tag{2.4}$$

In particular,

$$||u(t,x)||_2^2 = \iint_{\mathbb{R}^2} \mu(\mathrm{d}z_1)\mu(\mathrm{d}z_2)\, \mathcal{K}^*(t, x - z_1, x - z_2, 0; \lambda). \tag{2.5}$$

In the following, we will use the convention that for any pair (w_1, w_2) of variables,

$$\bar{w} := (w_1 + w_2)/2 \quad \text{and} \quad \Delta w := w_2 - w_1. \tag{2.6}$$

Remark 2.2 Formulas (2.1) and (2.2) are in the convolution form. One can also write them in the following inner product form:

$$\mathbb{E}\left[u(t,x_1)u(t,x_2)\right] = \iint_{\mathbb{R}^2} \mu(\mathrm{d}z_1)\mu(\mathrm{d}z_2)\, K^*\left(t, x_1, x_2, z_1, z_2; \lambda\right), \tag{2.7}$$

or

$$\mathbb{E}\left[u(t,x_1)u(t,x_2)\right] = J_0(t,x_1)J_0(t,x_2) + \iint_{\mathbb{R}^2} \mu(\mathrm{d}z_1)\mu(\mathrm{d}z_2)\, K^\dagger\left(t, x_1, x_2, z_1, z_2; \lambda\right), \tag{2.8}$$

where

$$K^{\dagger}(t, x_1, x_2, z_1, z_2; \lambda) = \frac{\lambda^2}{2\nu} G_{\nu/2}(t, \bar{x} - \bar{z}) \exp\left(-\frac{\lambda^2(|\Delta x| + |\Delta z|)}{2\nu} + \frac{\lambda^4 t}{4\nu}\right) \times \Phi\left(\frac{\lambda^2\sqrt{t}}{\sqrt{2\nu}} - \frac{|\Delta x| + |\Delta z|}{\sqrt{2\nu t}}\right), \tag{2.9}$$

and

$$\begin{aligned} K^{*}(t, x_1, x_2, z_1, z_2) &= G_{\nu}(t, x_1 - z_1) G_{\nu}(t, x_2 - z_2) + K^{\dagger}(t, x_1, x_2, z_1, z_2; \lambda) \\ &= G_{\nu/2}(t, \bar{x} - \bar{z}) \Bigg[G_{2\nu}(t, \Delta x - \Delta z) + \frac{\lambda^2}{2\nu} \exp\left(-\frac{\lambda^2(|\Delta x| + |\Delta z|)}{2\nu} + \frac{\lambda^4 t}{4\nu}\right) \\ &\quad \times \Phi\left(\frac{\lambda^2\sqrt{t}}{\sqrt{2\nu}} - \frac{|\Delta x| + |\Delta z|}{\sqrt{2\nu t}}\right)\Bigg]. \end{aligned} \tag{2.10}$$

Example 2.3 (Delta initial data) When $\mu = \delta_0$, then

$$||u(t, x)||_2^2 = \mathcal{K}^{*}(t, x, x, 0; \lambda) = \lambda^{-2}\, \mathcal{K}(t, x),$$

and

$$\begin{aligned} \mathbb{E}\left[u(t, x_1)u(t, x_2)\right] &= \mathcal{K}^{*}(t, x_1, x_2, x_1 - x_2; \lambda) \\ &= G_{\nu}(t, x_1)G_{\nu}(t, x_2) + \mathcal{K}^{\dagger}(t, x_1, x_2, x_1 - x_2; \lambda) \\ &= G_{\nu}(t, x_1)G_{\nu}(t, x_2) + \frac{\lambda^2}{2\nu} G_{\frac{\nu}{2}}\left(t, \frac{x_1 + x_2}{2}\right) \\ &\quad \times \exp\left(\frac{\lambda^2(\lambda^2 t - 2|x_1 - x_2|)}{4\nu}\right) \Phi\left(\frac{\lambda^2 t - |x_1 - x_2|}{\sqrt{2\nu t}}\right). \end{aligned}$$

This recovers the results in [6, Corollary 2.8].

Remark 2.4 Note that the function $m_2(t, x_1, x_2) = K^{*}(t, x_1, x_2, 0, 0)$, or equivalently $m_2(t, x_1, x_2) = \mathcal{K}^{*}(t, x_1, x_2, x_1 - x_2)$, solves the following parabolic equation (see e.g., [5, Theorem 3.2 on p. 46])

$$\begin{cases} \dfrac{\partial}{\partial t} m_2(t, x_1, x_2) = H_2(\nu, \lambda)\, m_2(t, x_1, x_2), & t > 0, x_1, x_2 \in \mathbb{R}, \\ m_2(0, x_1, x_2) = \delta_0(x_1)\delta_0(x_2), \end{cases}$$

where the operator

$$H_2(\nu, \lambda) = \frac{\nu}{2}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) + \lambda\delta_0(x_1 - x_2)$$

is the *2-particle Schrödinger operator* (see [5]). We remark that this operator resembles the perturbations of Laplace operator in $\mathbb{R}^d$ by point interactions, namely,

$$H_{d,\lambda,y} = -\Delta_d + \lambda\delta_y(\cdot), \quad \text{for } \lambda \in \mathbb{R} \text{ and } y \in \mathbb{R}^d,$$

which have been studied intensively in the literature; see, e.g., [2, 3]. The main difference between these two operators is the interaction term. The interaction measure in $H_2(\nu, \lambda)$ concentrates on the diagonal line $x_1 = x_1$, while that in $H_{2,\lambda,y}$ concentrates on a single point y.

Example 2.5 (Lebesgue's initial measure) When $\mu(\mathrm{d}x) = \mathrm{d}x$, then from (2.2) or (2.8),

$$\begin{aligned}\mathbb{E}\left[u(t,x_1)u(t,x_2)\right] &= 1 + \iint_{\mathbb{R}^2} \mathrm{d}z_1\mathrm{d}z_2\, \mathcal{K}^{\dagger}(t, z_1, z_2, x_1 - x_2; \lambda)\\ &= 1 + \frac{\lambda^2}{2\nu}\int_{\mathbb{R}} \mathrm{d}z \exp\left(\frac{\lambda^2}{4\nu}\left[\lambda^2 t - 2(|x_1 - x_2| + |x_1 - x_2 - z|)\right]\right)\\ &\quad\times \Phi\left(\frac{\lambda^2 t - (|x_1 - x_2| + |x_1 - x_2 - z)|)}{\sqrt{2\nu t}}\right)\\ &= 2e^{\frac{\lambda^4 t - 2\lambda^2 |x_1 - x_2|}{4\nu}}\,\Phi\left(\frac{\lambda^2 t - |x_1 - x_2|}{\sqrt{2\nu t}}\right) + 2\Phi\left(\frac{|x_1 - x_2|}{\sqrt{2\nu t}}\right) - 1, \quad (2.11)\end{aligned}$$

and in particular,

$$||u(t,x)||_2^2 = 2e^{\frac{\lambda^4 t}{4\nu}}\,\Phi\left(\frac{\lambda^2\sqrt{t}}{\sqrt{2\nu}}\right). \qquad (2.12)$$

These two formulas (2.11) and (2.12) recover the results in [6, Corollary 2.5]. Note that the equality in (2.11) can be established through integration by parts.

Corollary 2.6 *Let L_t^x be the local time of the standard Brownian motion. Then for all $\lambda \in \mathbb{R}$, $t > 0$, and $x \in \mathbb{R}$,*

$$\mathbb{E}\left[\exp\left(\lambda^2 L_t^x\right)\right] = 2\, e^{\lambda^4 t/2 - \lambda^2 |x|}\,\Phi\left(\lambda^2\sqrt{t} - |x|/\sqrt{t}\right) + 2\Phi\left(|x|/\sqrt{t}\right) - 1.$$

In particular,

$$\mathbb{E}\left[\exp\left(\lambda^2 L_t^x\right)\right] \le 2\, e^{\lambda^4 t/2} + 1.$$

Proof Let $u(t,x)$ be a solution to (1.1) with $\rho(u) = \sqrt{2}\,\lambda u$, $\nu = 1$, and $u_0(x) \equiv 1$. By the Feynman-Kac formula (1.7) with $n = 2$, $x_1 = 0$, and $x_2 = x$,

$$\begin{aligned}\mathbb{E}[u(t/2,0)u(t/2,x)] &= \mathbb{E}\left[\exp\left(2\lambda^2\int_0^{t/2}\delta_x(B_s^1 - B_s^2)\mathrm{d}s\right)\right] = \mathbb{E}\left[\exp\left(2\lambda^2\int_0^{t/2}\delta_x(B'_{2s})\mathrm{d}s\right)\right]\\ &= \mathbb{E}\left[\exp\left(\lambda^2\int_0^{t}\delta_x(B'_r)\mathrm{d}r\right)\right] = \mathbb{E}\left[\exp\left(\lambda^2 L_t^x\right)\right],\end{aligned}$$

where B'_t is a standard Brownian motion. Then apply (2.11) with λ^2, t, and ν replaced by $2\lambda^2$, $t/2$, and 1, respectively. □

For the p-th moments, we have the following bound, which simplifies the expression in [6]. Denote

$$c_p = \begin{cases} 1 & \text{if } p = 2,\\ 2 & \text{if } p > 2.\end{cases}$$

Theorem 2.7 *Let $u(t,x)$ be a solution to (1.1) starting from $\mu \in \mathcal{M}_H(\mathbb{R})$. For all $t > 0$ and $x \in \mathbb{R}$, the following moment bounds hold:*

(1) *If $|\rho(x)| \le L_\rho\,|x|$ for all $x \in \mathbb{R}$, then for all $p \ge 2$,*

$$||u(t,x)||_p^2 \le \iint_{\mathbb{R}^2} |\mu|(\mathrm{d}z_1)|\mu|(\mathrm{d}z_2)\overline{\mathcal{K}}^*_{p,L_\rho}(t, x - z_1, x - z_2),$$

where

$$\overline{\mathcal{K}}^*_{p,L_\rho}(t,z_1,z_2)=c_p\,\mathcal{K}^*(t,z_1,z_2,0;c_p^2\sqrt{p/2}\,L_\rho);$$

(2) *If* $|\rho(x)|\ge l_\rho\,|x|$ *for all* $x\in\mathbb{R}$ *and if* $\mu\ge 0$, *then*

$$||u(t,x)||_2^2\ge\iint_{\mathbb{R}^2}\mu(\mathrm{d}z_1)\mu(\mathrm{d}z_2)\mathcal{K}^*(t,x-z_1,x-z_2,0;l_\rho).$$

Proof Part (2) is clear. As for (1), note that the upper bounds for both 2nd and p-th moments share similar forms (see [6, (2.21)]), but with different parameters. By the notation in [6], $a_{p,0}=\sqrt{c_p}$ and $z_p=c_p^{3/2}\sqrt{p/2}$. Then replacing the parameter λ in $\mathcal{K}^*$ by $a_{p,0}z_p\,L_\rho$ and multiplying it by a factor c_p, one passes from 2nd moment to p-th moment (see [6, Theorem 2.4]). □

For the alternative proof of Theorem 2.1, we will need the following joint density of the standard Brownian motion B_t and its local time L_t^a at a level $a\in\mathbb{R}$, which is by itself interesting. When $a=0$, it is well known (see, e.g., [8, Exercise 1, on p. 181]) that this law is

$$P\left(B_t^1\in\mathrm{d}y,L_t^0\in\mathrm{d}v\right)=\frac{|y|+v}{\sqrt{2\pi t^3}}\exp\left(-\frac{(|y|+v)^2}{2t}\right)1_{\{v\ge 0\}}\mathrm{d}y\mathrm{d}v. \tag{2.13}$$

More generally, we have the following theorem.

Theorem 2.8 *The joint distribution of* (B_t,L_t^a) *for* $t>0$ *and* $a\in\mathbb{R}$ *is*

$$\begin{aligned}P\left(B_t^1\in\mathrm{d}y,L_t^a\in\mathrm{d}v\right)=1_{\{v\ge 0\}}\Bigg[&\frac{|a|+|y-a|+v}{(2\pi t)^3}\exp\left(-\frac{(|a|+|y-a|+v)^2}{2t}\right)\\&+\frac{1}{\sqrt{2\pi t}}\left(e^{-\frac{y^2}{2t}}-e^{-\frac{(2a-y)^2}{2t}}\right)1_{\{sign(a)y\le|a|\}}\delta_0(v)\Bigg]\mathrm{d}y\mathrm{d}v,\end{aligned} \tag{2.14}$$

where $sign(a)=1$ *if* $a\ge 0$ *and* -1 *if* $a<0$.

Corollary 2.9 *The law of* L_t^a *for* $t>0$ *and* $a\in\mathbb{R}$ *is*

$$P\left(L_t^a\in\mathrm{d}v\right)=\left(\frac{\sqrt{2}}{\sqrt{\pi t}}\exp\left(-\frac{(v+|a|)^2}{2t}\right)+\left[2\Phi\left(\frac{|a|}{\sqrt{t}}\right)-1\right]\delta_0(v)\right)1_{\{v\ge 0\}}\mathrm{d}v.$$

Proof Integrate the right hand of (2.14) for dy over $\mathbb{R}$. For the first term, use the integration-by-parts formula. For the second term, use the definition of $\Phi(\cdot)$. □

2.2 Feynman-Kac Formulas for Measure-Valued Initial Data

Let H be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and let $W=\{W(h),\ h\in H\}$ be a zero mean Gaussian process with covariance function $\mathbb{E}(W(h)W(g))=\langle h,g\rangle$. For any square integrable random variable $F\in L^2(\Omega)$, let

$$F=\mathbb{E}[F]+\sum_{n=1}^{\infty}I_n(f_n)$$

be its chaos expansion, where $f_n \in H^{\hat{\otimes} n}$ (symmetric tensor product) and I_n denotes the multiple stochastic integral. Let L be the generator of the *Ornstein-Uhlenbeck* semigroup, i.e., $LF = -\sum_{n=1}^{\infty} nI_n(f_n)$. For any $s \in \mathbb{R}$, denote by $\mathbb{D}^{s,p}(H)$ the completion of H-valued polynomial random variables with respect to the norm

$$||F||_{s,p} = \left|\left|(I-L)^{s/2}F\right|\right|_{L^p(\Omega;H)}.$$

We refer [11] for more details.

In our framework, $H = L^2(\mathbb{R}_+;\mathbb{R}^n)$ and for any $h \in H$, $W(h) = \sum_{i=1}^n \int_0^\infty h_s^i dB_s^i$, where $\{B_t^i, t \geq 0\}$, $i = 1\ldots,n$, are n independent standard Brownian motions on $\mathbb{R}$.

Theorem 2.10 *For any $\mu_i \in \mathcal{M}_H(\mathbb{R})$, $x_i \in \mathbb{R}$, $h_i \in H$, $i = 1,\ldots,n$. It holds that*

$$\prod_{i=1}^n \mu_i(W(h_i)+x_i) \in \mathbb{D}^{-\alpha,p}(\mathbb{R}) \qquad \textit{if } \alpha + n/p > n,\ \alpha > 0 \textit{ and } p > 1.$$

Theorem 2.11 *Let $\widetilde{B}^i = \{\widetilde{B}_t^i, t \geq 0\}$, $i \in I$, be one-dimensional Brownian motions belonging to Gaussian space spanned by W, where I is a finite set with m elements. Let $L_t^{i,x}$, $i \in I$, be the local time of $\widetilde{B}^i$. Suppose that the function $f : \mathbb{R}_+^m \mapsto \mathbb{R}$ satisfies that $\frac{\partial^2}{\partial x_i^2} f \geq 0$ and*

$$\max_{i=1,\ldots,m} \sup_{\epsilon_1,\ldots,\epsilon_m \in (-1,1)} \left|\left| f_i\left(L_t^{1,x_1+\epsilon_1}+\epsilon_1,\ldots,L_t^{m,x_m+\epsilon_m}+\epsilon_m\right)\right|\right|_p < +\infty, \quad (2.15)$$

for all $t \geq 0$, $x_i \in \mathbb{R}$, and $p \geq 2$, where $f_i = \frac{\partial}{\partial x_i} f$. Then

$$f\left(L_t^{1,x_1},\ldots,L_t^{m,x_m}\right) \in \mathbb{D}^{\alpha,p}(\mathbb{R}) \qquad \textit{if } p > 1 \textit{ and } \alpha < 1/2.$$

In particular, one can choose $f(z_1,\ldots,z_m) = \exp\left(\lambda^2 \sum_{j=1}^m z_j\right)$, $\lambda \in \mathbb{R}$.

Choosing $h_i = \mathbf{1}_{[0,t]}$, $1 \leq i \leq n$, Theorem 2.10 implies that

$$\prod_{i=1}^n u_0\left(x_i + B_t^i\right) \in \mathbb{D}^{-\alpha,p}(\mathbb{R}) \qquad \text{if } \alpha + n/p > n, \alpha > 0 \text{ and } p > 1, \qquad (2.16)$$

for $u_0 \in \mathcal{M}_H(\mathbb{R})$. On the other hand, choosing $I = \{(j,k) : 1 \leq j < k \leq n\}$ and $\widetilde{B}^{(j,k)} = B^j - B^k$, Theorem 2.11 implies that the random variable Y_n defined in (1.8) belongs to $\mathbb{D}^{\alpha,q}(\mathbb{R})$ if $q > 1$ and $\alpha < 1/2$. Therefore, by choosing p close to 1 such that $n(1-1/p) < 1/2$, and choosing q such that $1/p + 1/q = 1$, one can see that the Feynman-Kac formula (1.7) is well defined as the duality relationship (1.8), for any $u_0 \in \mathcal{M}_H(\mathbb{R})$. This can be proved using the fact that $u_0 * G_1(\epsilon,\cdot)$ converges to u_0, as ϵ tends to zero, in the topology of $\mathbb{D}^{-\alpha,p}(\mathbb{R})$.

3 Proof of Theorem 2.1

Proof of Theorem 2.1 The proof consists of the following two steps:

Step 1. We first prove the second moment (2.5). Write J_0^2 in the form of double integrals

$$(J_0^2 \star \mathcal{K})(t,x) = \int_0^t \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y\, \mathcal{K}(t-s,x-y) \iint_{\mathbb{R}^2} \mu(\mathrm{d}z_1)\mu(\mathrm{d}z_2) G_\nu(s,y-z_1)G_\nu(s,y-z_2).$$

By [6, Lemma 5.4] and the notation (2.6),

$$G_\nu(s, y - z_1)G_\nu(s, y - z_2) = G_{\nu/2}(s, y - \bar{z})G_\nu(2s, \Delta z). \tag{3.1}$$

Denote

$$H(t) = \frac{1}{\sqrt{4\pi\nu t}} + \frac{\lambda^2}{2\nu}\, e^{\frac{\lambda^4 t}{4\nu}}\, \Phi\left(\lambda^2\sqrt{\frac{t}{2\nu}}\right). \tag{3.2}$$

By Fubini's theorem and the semigroup property of the heat kernel,

$$\begin{aligned}(J_0^2 \star \mathcal{K})(t, x) &= \lambda^2 \iint_{\mathbb{R}^2} \mu(\mathrm{d}z_1)\mu(\mathrm{d}z_2)\int_0^t \mathrm{d}s\; G_\nu(2s, \Delta z) \int_{\mathbb{R}} \mathrm{d}y\; \mathcal{K}(t-s, x-y)G_{\nu/2}(s, y-\bar{z}) \\ &= \lambda^2 \iint_{\mathbb{R}^2} \mu(\mathrm{d}z_1)\mu(\mathrm{d}z_2)G_{\nu/2}(t, x - \bar{z})\int_0^t \mathrm{d}s\; G_\nu(2s, \Delta z)H(t - s).\end{aligned}$$

Now we will use the Laplace transform to evaluate the ds-integral. By [9, (27) on p. 146],

$$\mathcal{L}[G_{2\nu}(\cdot, x)](z) = \frac{\exp\left(-\nu^{-1/2}\, |x|\, \sqrt{z}\right)}{2\sqrt{\nu z}}. \tag{3.3}$$

By (1) on p. 137 and (5) on p. 176 of [9],

$$\begin{aligned}\mathcal{L}[H](z) &= \frac{\lambda^2}{4\nu z - \lambda^4} + \frac{1}{2\sqrt{\nu z}} + \frac{\lambda^4}{2\sqrt{\nu z}\left(4\nu z - \lambda^4\right)} \\ &= \frac{1}{2}\left(\frac{1}{2\sqrt{\nu z} - \lambda^2} - \frac{1}{2\sqrt{\nu z} + \lambda^2}\right) + \frac{1}{2\sqrt{\nu z}} \\ &\quad + \frac{\lambda^2}{4\sqrt{\nu z}}\left(\frac{1}{2\sqrt{\nu z} - \lambda^2} - \frac{1}{2\sqrt{\nu z} + \lambda^2}\right).\end{aligned}$$

Hence,

$$\mathcal{L}[H](z)\mathcal{L}[G_{2\nu}(\cdot, x)](z) = f_{1,-}(z, x) - f_{1,+}(z, x) + f_2(z, x) + f_{3,-}(z, x) - f_{3,+}(z, x),$$

where

$$\begin{aligned}f_{1,\pm}(z, x) &= \frac{\exp\left(-\nu^{-1/2}|x|\sqrt{z}\right)}{4\sqrt{\nu z}\left(2\sqrt{\nu z} \pm \lambda^2\right)}, \\ f_2(z, x) &= \frac{\exp\left(-\nu^{-1/2}|x|\sqrt{z}\right)}{4\nu z}, \\ f_{3,\pm}(z, x) &= \frac{\lambda^2 \exp\left(-\nu^{-1/2}|x|\sqrt{z}\right)}{8\nu z\left(2\sqrt{\nu z} \pm \lambda^2\right)}.\end{aligned}$$

Apply [9, (16) on p. 247] with $\alpha = \nu^{-1/2}|x|$ and $\beta = \pm\frac{\lambda^2}{\sqrt{4\nu}}$,

$$\mathcal{L}^{-1}[f_{1,\pm}(\cdot, x)](t) = \frac{1}{8\nu} e^{\pm\frac{\lambda^2|x|}{2\nu} + \frac{\lambda^4 t}{4\nu}}\, \mathrm{erfc}\left(\frac{|x|}{\sqrt{4\nu t}} \pm \frac{\lambda^2\sqrt{t}}{\sqrt{4\nu}}\right). \tag{3.4}$$

Apply [9, (3) on p. 245] with $\alpha = \nu^{-1}|x|^2$,

$$\mathcal{L}^{-1}[f_2(\cdot, x)](t) = \frac{1}{4\nu}\mathrm{erfc}\left(\frac{|x|}{\sqrt{4\nu t}}\right).$$

Apply [9, (14) on p. 246] with $\alpha = \nu^{-1/2}|x|$ and $\beta = \pm\frac{\lambda^2}{\sqrt{4\nu}}$,

$$\mathcal{L}^{-1}[f_{3,\pm}(\cdot, x)](t) = \pm\frac{1}{8\nu}\left(\mathrm{erfc}\left(\frac{|x|}{\sqrt{4\nu t}}\right) - e^{\pm\frac{\lambda^2|x|}{2\nu} + \frac{\lambda^4 t}{4\nu}}\, \mathrm{erfc}\left(\frac{|x|}{\sqrt{4\nu t}} \pm \frac{\lambda^2\sqrt{t}}{\sqrt{4\nu}}\right)\right).$$

Therefore, the ds-integral is equal to

$$\begin{aligned}\int_0^t \mathrm{d}s\; G_{2\nu}(s,\Delta z)H(t-s) &= \frac{1}{4\nu}e^{-\frac{\lambda^2|\Delta z|}{2\nu}+\frac{\lambda^4 t}{4\nu}}\operatorname{erfc}\left(\frac{|\Delta z|}{\sqrt{4\nu t}}-\frac{\lambda^2\sqrt{t}}{\sqrt{4\nu}}\right)\\ &= \frac{1}{2\nu}e^{-\frac{\lambda^2|\Delta z|}{2\nu}+\frac{\lambda^4 t}{4\nu}}\Phi\left(\frac{\lambda^2\sqrt{t}}{\sqrt{2\nu}}-\frac{|\Delta z|}{\sqrt{2\nu t}}\right).\end{aligned}$$

Finally, by (3.1),

$$J_0^2(t,x) = \iint_{\mathbb{R}^2}\mu(\mathrm{d}z_1)\mu(\mathrm{d}z_1)G_{\nu/2}(t,x-\bar{z})G_{2\nu}(t,\Delta z).$$

This proves the formula (2.5), i.e.,

$$\begin{aligned}||u(t,x)||_2^2 = \iint_{\mathbb{R}^2}&\mu(\mathrm{d}z_1)\mu(\mathrm{d}z_2)\;G_{\nu/2}(t,x-\bar{z})\\ &\times\left(G_{2\nu}(t,\Delta z)+\frac{\lambda^2}{2\nu}e^{-\frac{\lambda^2|\Delta z|}{2\nu}+\frac{\lambda^4 t}{4\nu}}\Phi\left(\frac{\lambda^2\sqrt{t}}{\sqrt{2\nu}}-\frac{|\Delta z|}{\sqrt{2\nu t}}\right)\right).\end{aligned} \tag{3.5}$$

Step 2. Now let us consider the two-point correlation function (2.1). Fix $t > 0$ and $x_1, x_2 \in \mathbb{R}$. Apply (1.6), (3.1) and (3.5), and then integrate over dy using the semigroup property:

$$\begin{aligned}\mathcal{I}(t,x_1,t,x_2;\nu,\lambda) &= \lambda^2\int_0^t \mathrm{d}r\int_{\mathbb{R}}\mathrm{d}y\;||u(r,y)||_2^2\,G_{\nu/2}(t-r,\bar{x}-y)G_{2\nu}(t-r,\Delta x)\\ &= \lambda^2\int_0^t \mathrm{d}r\int_{\mathbb{R}}\mathrm{d}y\;G_{\nu/2}(t-r,\bar{x}-y)G_{2\nu}(t-r,\Delta x)\iint_{\mathbb{R}^2}\mu(\mathrm{d}z_1)\mu(\mathrm{d}z_1)\\ &\quad\times G_{\nu/2}(r,y-\bar{z})\left(G_{2\nu}(r,\Delta z)+\frac{\lambda^2}{2\nu}e^{-\frac{\lambda^2|\Delta z|}{2\nu}+\frac{\lambda^4 r}{4\nu}}\Phi\left(\frac{\lambda^2\sqrt{r}}{\sqrt{2\nu}}-\frac{|\Delta z|}{\sqrt{2\nu r}}\right)\right)\\ &= \lambda^2\iint_{\mathbb{R}^2}\mu(\mathrm{d}z_1)\mu(\mathrm{d}z_2)\;G_{\nu/2}(t,\bar{x}-\bar{z})\int_0^t \mathrm{d}r\;G_{2\nu}(t-r,\Delta x)\widetilde{H}(r,\Delta z),\end{aligned}$$

where

$$\widetilde{H}(t,x) := G_{2\nu}(t,x)+\tfrac{\lambda^2}{2\nu}e^{-\frac{\lambda^2|x|}{2\nu}+\frac{\lambda^4 t}{4\nu}}\;\Phi\left(\lambda^2\sqrt{\tfrac{t}{2\nu}}-\tfrac{|x|}{\sqrt{2\nu t}}\right).$$

Here, $\widetilde{H}(t,0) = H(t)$; see (3.2). Notice that

$$\frac{1}{4\nu}e^{-\frac{\lambda^2|x|}{2\nu}+\frac{\lambda^4 t}{4\nu}}\Phi\left(\lambda^2\sqrt{\frac{t}{2\nu}}-\frac{|x|}{\sqrt{2\nu t}}\right)=\frac{1}{8\nu}e^{-\frac{\lambda^2|x|}{2\nu}+\frac{\lambda^4 t}{4\nu}}\operatorname{erfc}\left(-\lambda^2\sqrt{\frac{t}{4\nu}}+\frac{|x|}{\sqrt{4\nu t}}\right).$$

Hence,

$$\widetilde{H}(t,x) = G_{2\nu}(t,x)+2\lambda^2\mathcal{L}^{-1}[f_{1,-}(\cdot,x)](t).$$

Together with (3.3), we have that,

$$\mathcal{L}[\widetilde{H}(\cdot,x)](z) = \tfrac{\exp\left(-\nu^{-1/2}|x|\sqrt{z}\right)}{\sqrt{4\nu z}-\lambda^2},$$

and

$$\mathcal{L}[G_{2\nu}(\cdot,x)](z)\mathcal{L}[\widetilde{H}(\cdot,x')](z) = \tfrac{\exp\left(-\nu^{-1/2}(|x|+|x'|)\sqrt{z}\right)}{\sqrt{4\nu z}\left(\sqrt{4\nu z}-\lambda^2\right)} = 2f_{1,-}(z,|x|+|x'|),$$

By (3.4),

$$\int_0^t \mathrm{d}r\, G_{2\nu}(t-r,\Delta x)\widetilde{H}(r,\Delta z) = \frac{1}{2\nu} e^{-\frac{\lambda^2(|\Delta x|+|\Delta z|)}{2\nu}+\frac{\lambda^4 t}{4\nu}} \Phi\left(\frac{\lambda^2\sqrt{t}}{\sqrt{2\nu}} - \frac{|\Delta x|+|\Delta z|}{\sqrt{2\nu t}}\right).$$

Therefore,

$$\mathbb{E}\left[u(t,x_1)u(t,x_2)\right] = J_0(t,x_1)J_0(t,x_2) + \iint_{\mathbb{R}^2} \mu(\mathrm{d}z_1)\mu(\mathrm{d}z_2)\, G_{\nu/2}(t,\bar{x}-\bar{z}) \times \frac{\lambda^2}{2\nu} e^{-\frac{\lambda^2(|\Delta x|+|\Delta z|)}{2\nu}+\frac{\lambda^4 t}{4\nu}} \Phi\left(\frac{\lambda^2\sqrt{t}}{\sqrt{2\nu}} - \frac{|\Delta x|+|\Delta z|}{\sqrt{2\nu t}}\right).$$

This completes the proof of Theorem 2.1.

□

4 Proof of Theorem 2.8

Proof of Theorem 2.8 The case when $a=0$ is covered in (2.13). Let $g:\mathbb{R}\times\mathbb{R}_+\mapsto\mathbb{R}$ be a smooth and bounded function. Denote the joint density of (B_t, L_t^a) by $f_{a,t}(x,y)$. Then

$$\mathbb{E}\left[g(B_t,L_t^a)\right] = \int_0^\infty \mathrm{d}y \int_{\mathbb{R}} \mathrm{d}x\, g(x,y) f_{a,t}(x,y).$$

First assume that $a>0$. By the reflection principle (see [13, p. 107]), the density of $T_a := \inf(s\geq 0, B_s=a)$ is

$$f_{T_a}(s) = \frac{a}{\sqrt{2\pi s^3}} \exp\left(-\frac{a^2}{2s}\right), \qquad \text{for } s\geq 0 \text{ and } a>0.$$

Let $\{\mathcal{F}_t\}_{t\geq 0}$ be the standard Brownian filtration. By the strong Markov property,

$$\begin{aligned}
\mathbb{E}\left[g(B_t,L_t^a)\right] &= \mathbb{E}\left[\mathbb{E}\left(g(B_t,L_t^a)|\mathcal{F}_{T_a}\right)1_{\{T_a\leq t\}}\right] + \mathbb{E}\left[\mathbb{E}\left(g(B_t,L_t^a)|\mathcal{F}_{T_a}\right)1_{\{T_a>t\}}\right] \\
&= \int_0^t \mathrm{d}s\, f_{T_a}(s) E\left[g(\hat{B}_{t-s}+a, \hat{L}^0_{t-s})\right] + \mathbb{E}\left(g(B_t,0)1_{\{T_a>t\}}\right) \\
&=: I_1 + I_2,
\end{aligned}$$

where $\hat{B}_t$ is a standard Brownian motion and $\hat{L}_t^0$ is its local time at the level 0. We first compute I_1. By (2.13),

$$\begin{aligned}
I_1 &= \iint_{\mathbb{R}\times\mathbb{R}_+} \mathrm{d}y\mathrm{d}v\, g(y+a,v) \int_0^t \mathrm{d}s \frac{a}{\sqrt{2\pi s^3}} \exp\left(-\frac{a^2}{2s}\right) \frac{|y|+v}{\sqrt{2\pi(t-s)^3}} \exp\left(-\frac{(|y|+v)^2}{2(t-s)}\right) \\
&= \iint_{\mathbb{R}\times\mathbb{R}_+} \mathrm{d}y\mathrm{d}v\, g(y+a,v) \frac{a+|y|+v}{\sqrt{2\pi t^3}} \exp\left(-\frac{(a+|y|+v)^2}{2t}\right),
\end{aligned}$$

where in the last step, we have used the fact that the densities f_{T_a} form a convolution semigroup, namely, $f_{T_a} * f_{T_b} = f_{T_{a+b}}$ (see [13, p.107] or Lemma 4.1 below for a short proof).

As for I_2, let $M_t = \sup_{s\leq t} B_s$. Then, by the joint law of (B_t, M_t) (see, e.g., [13, Ex. 3.14 p. 110]),

$$I_2 = \mathbb{E}\left[g(B_t,0)1_{\{M_t<a\}}\right] = \int_0^a \mathrm{d}m \int_{-\infty}^m \mathrm{d}y\, g(y,0) \frac{\sqrt{2}(2m-y)}{\sqrt{\pi t^3}} \exp\left(-\frac{(2m-y)^2}{2t}\right)$$

$$= \int_{-\infty}^{a} \mathrm{d}y\, g(y,0) \int_{y_+}^{a} \mathrm{d}m \frac{\sqrt{2}(2m-y)}{\sqrt{\pi t^3}} \exp\left(-\frac{(2m-y)^2}{2t}\right)$$
$$= \int_{-\infty}^{a} \mathrm{d}y\, g(y,0) \frac{1}{\sqrt{2\pi t}} \left(e^{-\frac{(2y_+-y)^2}{2t}} - e^{-\frac{(2a-y)^2}{2t}}\right),$$

where $y_+ = \max(y, 0)$. Note that $2y_+ - y = |y|$. Combining these two parts, we see that

$$f_a(y,v) = \Bigg[\quad \frac{a+|y-a|+v}{(2\pi t)^3} \exp\left(-\frac{(a+|y-a|+v)^2}{2t}\right)$$
$$+ \frac{1}{\sqrt{2\pi t}} \left(e^{-\frac{y^2}{2t}} - e^{-\frac{(2a-y)^2}{2t}}\right) 1_{\{y\le a\}} \delta_0(v)\Bigg] 1_{\{v\ge 0\}}.$$

If $a < 0$, by symmetry, let $\tilde{B}_t = -B_t$ and $\tilde{L}_t^a$ be its local time. Clearly, the density of $(\tilde{B}_t, \tilde{L}_t^{-a})$ is $f_{-a}(y, v)$. Therefore, the law of (B_t, L_t^a) is $f_{|a|}(-y, v)$. Then use the fact that $|-y-|a|| = |y-a|$ and $|2|a|+y| = |2a-y|$. This completes the proof of Theorem 2.8. □

Lemma 4.1 *For $t > 0$ and $a, b \neq 0$, the following integral is true*

$$\int_0^t |ab|(s(t-s))^{-3/2} \exp\left(-\frac{a^2}{2s} - \frac{b^2}{2(t-s)}\right) \mathrm{d}s = \frac{|a|+|b|}{\sqrt{2\pi t^3}} \exp\left(-\frac{(|a|+|b|)^2}{2t}\right).$$

Proof Denote the integral by $I(t)$ and let $g_a(t) = |a|\, t^{-3/2} e^{a^2/(2t)}$. By [9, (28) on p. 146],

$$\mathcal{L}\left[g_a\right](z) = \sqrt{2\pi} e^{-\sqrt{2}|a|\sqrt{z}}\,. \tag{4.1}$$

So, $\mathcal{L}[I](z) = \mathcal{L}\left[g_a\right](z)\mathcal{L}\left[g_b\right](z) = 2\pi e^{-\sqrt{2}(|a|+|b|)\sqrt{z}}$. Then use (4.1) for the inversion. □

5 An Alternative Proof of Theorem 2.1

If the initial data μ is such that $\mu(\mathrm{d}x) = u_0(x)\mathrm{d}x$, where u_0 is a bounded measurable function, then we can use the Feynman-Kac representation (1.7) with $n = 2$ and Theorem 2.8 to give an alternative proof of Theorem 2.1.

An alternative proof of Theorem 2.1 For initial data specified above, $\mathbb{E}\left(u(t,x_1)u(t,x_2)\right)$ admits the Feynman-Kac representation (1.7). Let

$$W_{2\nu t}^1 := B_{\nu t}^1 - B_{\nu t}^2 \quad \text{and} \quad W_{2\nu t}^2 := B_{\nu t}^1 + B_{\nu t}^2.$$

Note that W_t^1 and W_t^2 are two independent standard Brownian motions. Then,

$$\mathbb{E}\left(u(t,x_1)u(t,x_2)\right) = \mathbb{E}\Bigg(u_0\left(x_1 + 2^{-1}[W_{2\nu t}^1 + W_{2\nu t}^2]\right) u_0\left(x_2 + 2^{-1}[W_{2\nu t}^2 - W_{2\nu t}^1]\right)$$
$$\times \exp\left(\lambda^2 \int_0^t \delta_{x_2-x_1}\left(W_{2\nu s}^1\right) \mathrm{d}s\right)\Bigg)$$
$$= \int_{\mathbb{R}} \mathrm{d}z\, G_{2\nu}(t,z) \mathbb{E}\Bigg(u_0\left(x_1 + 2^{-1}[W_{2\nu t}^1 + z]\right) u_0\left(x_2 + 2^{-1}[z - W_{2\nu t}^1]\right)$$

$$\times \exp\left(\lambda^2 \int_0^t \delta_{x_2-x_1}\left(W^1_{2\nu s}\right) \mathrm{d}s\right)\Bigg).$$

Notice that

$$\exp\left(\lambda^2 \int_0^t \delta_{x_2-x_1}\left(W^1_{2\nu s}\right) \mathrm{d}s\right) = \exp\left(\frac{\lambda^2}{2\nu}\int_0^{2\nu t} \delta_{x_2-x_1}\left(W^1_s\right) \mathrm{d}s\right) = \exp\left(\frac{\lambda^2}{2\nu} L^{x_2-x_1}_{2\nu t}\right),$$

where L^a_t be the local time of W^1_t at the level a. Then the expectation in the above integrand becomes

$$\mathbb{E}\Bigg(u_0\left(x_1+2^{-1}[W^1_{2\nu t}+z]\right) u_0\left(x_2+2^{-1}[z-W^1_{2\nu t}]\right) \exp\left(\frac{\lambda^2}{2\nu} L^{x_2-x_1}_{2\nu t}\right)\Bigg). \tag{5.1}$$

By Theorem 2.8, this expectation is equal to

$$\begin{aligned}
&\int_{\mathbb{R}} \mathrm{d}y \quad u_0\left(x_1+\frac{z+y}{2}\right) u_0\left(x_2+\frac{z-y}{2}\right)\\
&\quad\times \int_0^\infty \mathrm{d}v\, \frac{|y-\Delta x|+|\Delta x|+v}{4\sqrt{\pi\nu^3 t^3}} \exp\left(-\frac{(|y-\Delta x|+v+|\Delta x|)^2}{4\nu t}+\frac{\lambda^2 v}{2\nu}\right) \\
&\quad+ \int_{sign(\Delta x)y\le|\Delta x|} \mathrm{d}y\, u_0(x_1+(z+y)/2)u_0(x_2+(z-y)/2)\left[G_{2\nu}(t,y)-G_{2\nu}(t,2\Delta x-y)\right].
\end{aligned} \tag{5.2}$$

By integration by parts, the dv-integration in (5.2) is equal to

$$\begin{aligned}
&\frac{1}{\sqrt{4\pi\nu t}} e^{\frac{\lambda^2 v}{2\nu}} e^{-\frac{(|y-\Delta x|+|\Delta x|+v)^2}{4\nu t}}\Bigg|_{v=\infty}^{v=0} + \frac{\lambda^2}{2\nu}\int_0^\infty \mathrm{d}v\, \frac{1}{\sqrt{4\pi\nu t}} e^{\frac{\lambda^2 v}{2\nu}} e^{-\frac{(|y-\Delta x|+|\Delta x|+v)^2}{4\nu t}}\\
&= G_{2\nu}(t,|y-\Delta x|+|\Delta x|) + \frac{\lambda^2}{2\nu} e^{\frac{\lambda^4 t}{4\nu}-\frac{\lambda^2(|y-\Delta x|+|\Delta x|)}{2\nu}}\Phi\left(\lambda^2\sqrt{\frac{t}{2\nu}}-\frac{|y-\Delta x|+|\Delta x|}{\sqrt{2\nu t}}\right).
\end{aligned}$$

Denote the dy-integral in (5.2) by I. By symmetry, we will only consider the case where $\Delta x>0$. In this case, the dy-integral is from $-\infty$ to Δx. By the change of variables $y'=2\Delta x-y$,

$$\begin{aligned}
&\int_{-\infty}^{\Delta x} \mathrm{d}y u_0(x_1+(z+y)/2)u_0(x_2+(z-y)/2)G_{2\nu}(t,2\Delta x-y)\\
&\quad= \int_{\Delta x}^\infty \mathrm{d}y'\, u_0(x_1+(z+y')/2)u_0(x_2+(z-y')/2)G_{2\nu}(t,y').
\end{aligned} \tag{5.3}$$

Hence,

$$I = \int_{\mathbb{R}} \mathrm{d}y\, u_0(x_1+(z+y)/2)u_0(x_2+(z-y)/2)\left[G_{2\nu}(t,y)1_{\{y\le\Delta x\}} - G_{2\nu}(t,y)1_{\{y>\Delta x\}}\right].$$

Notice that

$$\begin{aligned}
&G_{2\nu}(t,y)1_{\{y\le\Delta x\}} - G_{2\nu}(t,y)1_{\{y>\Delta x\}} + G_{2\nu}(t,|y-\Delta x|+\Delta x)\\
&= \left[G_{2\nu}(t,y)+G_{2\nu}(t,2\Delta x-y)\right]1_{\{y\le\Delta x\}}.
\end{aligned}$$

Therefore, by (5.3), we know that

$$\begin{aligned}
&\int_{\mathbb{R}} \quad \mathrm{d}y\, u_0(x_1+(z+y)/2)u_0(x_2+(z-y)/2)\\
&\quad\times\left(G_{2\nu}(t,y)1_{\{y\le\Delta x\}} - G_{2\nu}(t,y)1_{\{y>\Delta x\}} + G_{2\nu}(t,|y-\Delta x|+\Delta x)\right)
\end{aligned}$$

$$= \int_{\mathbb{R}} \mathrm{d}y\, u_0(x_1 + (z+y)/2) u_0(x_2 + (z-y)/2) G_{2\nu}(t, y).$$

Combining these calculations, we have that

$$\begin{aligned}\mathbb{E}\left(u(t,x_1)u(t,x_2)\right) = &\iint_{\mathbb{R}^2} \mathrm{d}z\mathrm{d}y\, u_0(x_1 + (z+y)/2) u_0(x_2 + (z-y)/2) G_{2\nu}(t,z)\\ &\times \left(G_{2\nu}(t,y) + \frac{\lambda^2}{2\nu} e^{\frac{\lambda^4 t}{4\nu} - \frac{\lambda^2(|y-\Delta x| + |\Delta x|)}{2\nu}} \Phi\left(\lambda^2\sqrt{\frac{t}{2\nu}} - \frac{|y-\Delta x| + |\Delta x|}{\sqrt{2\nu t}}\right)\right).\end{aligned}$$

Finally, by change of variables $z_1 = x_1 + (z+y)/2$ and $z_2 = x_2 + (z-y)/2$, we have that

$$\begin{aligned}\mathbb{E}\left(u(t,x_1)u(t,x_2)\right) = &\iint_{\mathbb{R}^2} \mathrm{d}z_1\mathrm{d}z_2\, u_0(z_1)u_0(z_2) 2G_{2\nu}(t, (z_1+z_2) - (x_1+x_2))\\ &\times \Bigg(G_{2\nu}\left(t, \Delta z - \Delta x\right)\\ &+ \frac{\lambda^2}{2\nu}\exp\left(\frac{\lambda^4 t}{4\nu} - \frac{\lambda^2|z_1-z_2| + |\Delta x|}{2\nu}\right)\Phi\left(\lambda^2\sqrt{\frac{t}{2\nu}} - \frac{|z_1-z_2| + |\Delta x|}{\sqrt{2\nu t}}\right)\Bigg).\end{aligned}$$

Notice that $2G_{2\nu}(t, (z_1+z_2) - (x_1+x_2)) = G_{\nu/2}(t, \bar{x} - \bar{z})$. This proves (2.1) (see (2.8) and (2.10)). □

6 Feynman-Kac Formula for Measure-Valued Initial Data

6.1 Initial Data Part (Proof of Theorem 2.10)

We need some lemmas.

Lemma 6.1 *For any $t > 0$, $s > 0$, $x \in \mathbb{R}$, $y_i \in \mathbb{R}$, $i = 1, \ldots, p$, it holds that*

$$\int_{\mathbb{R}} G_1(s, x+z) \prod_{j=1}^{p} G_1(t, z - y_j)\mathrm{d}z \le (p+1)^{p/2}\sqrt{\frac{t}{ps+t}}\, e^{\frac{px^2}{2(ps+t)}} \prod_{i=1}^{p} G_1((p+1)t, y_i).$$

Proof Denote $\bar{y} = (y_1 + \cdots + y_p)/p$. Notice that

$$\begin{aligned}\prod_{j=1}^{p} G_1(t, z-y_j) &= (2\pi t)^{-p/2}\exp\left(-\frac{(z-\bar{y})^2}{2t/p}\right)\exp\left(-\frac{y_1^2 + \cdots + y_p^2 - p\,\bar{y}^2}{2t}\right)\\ &= (2\pi t)^{-(p-1)/2} p^{-1/2} G_1(t/p, z-\bar{y})\exp\left(-\frac{y_1^2 + \cdots + y_p^2 - p\,\bar{y}^2}{2t}\right).\end{aligned}$$

Hence, by the semigroup property of the heat kernel, the $\mathrm{d}z$ integral is equal to

$$\begin{aligned}\int_{\mathbb{R}} \mathrm{d}z \quad &G_1(s, x+z)\prod_{j=1}^{p} G_1(t, z-y_j)\\ &= (2\pi t)^{-(p-1)/2} p^{-1/2} G_1(s + t/p, x + \bar{y})\exp\left(-\frac{y_1^2 + \cdots + y_p^2 - p\,\bar{y}^2}{2t}\right)\\ &= (2\pi)^{-p/2} t^{-(p-1)/2}(ps+t)^{-1/2}\exp\left(-\frac{p(x+\bar{y})^2}{2(ps+t)} - \frac{y_1^2 + \cdots + y_p^2 - p\,\bar{y}^2}{2t}\right)\end{aligned}$$

$$
\begin{aligned}
&\leq (2\pi)^{-p/2} t^{-(p-1)/2} (ps+t)^{-1/2} \exp\left(-\frac{p\bar{y}^2 - 2px^2}{4(ps+t)} - \frac{y_1^2 + \cdots + y_p^2 - p\,\bar{y}^2}{2t}\right) \\
&= (2\pi)^{-p/2} t^{-(p-1)/2} (ps+t)^{-1/2} e^{\frac{px^2}{2(ps+t)}} \exp\left(\frac{p\bar{y}^2(t+2ps)}{4t(ps+t)} - \frac{y_1^2 + \cdots + y_p^2}{2t}\right) \\
&\leq (2\pi)^{-p/2} t^{-(p-1)/2} (ps+t)^{-1/2} e^{\frac{px^2}{2(ps+t)}} \exp\left(\left(\frac{t+2ps}{4t(ps+t)} - \frac{1}{2t}\right)(y_1^2 + \cdots + y_p^2)\right) \\
&\leq (2\pi)^{-p/2} t^{-(p-1)/2} (ps+t)^{-1/2} e^{\frac{px^2}{2(ps+t)}} \exp\left(-\frac{y_1^2 + \cdots + y_p^2}{2(p+1)t}\right) \\
&= \sqrt{\frac{t}{ps+t}} (p+1)^{p/2} e^{\frac{px^2}{2(ps+t)}} \prod_{i=1}^{p} G_1((p+1)t, y_i).
\end{aligned}
$$

This proves Lemma 6.1. □

Let T_t be the Ornstein Uhlenbeck semigroup of operators on $L^2(\Omega)$ with generator L.

Lemma 6.2 *For any $\mu_i \in \mathcal{M}_H(\mathbb{R})$, $h_i \in H$, and $x_i \in \mathbb{R}$, $i = 1, \ldots, n$, it holds that*

$$
T_t\left(\prod_{i=1}^{n} \mu_i\,(x_i + W_i(h_i))\right) = \prod_{i=1}^{n} \left[\mu_i * G_1(|h_i|^2(1-e^{-2t}), \cdot)\right](x_i + e^{-t} W_i(h_i)),
$$

for $t > 0$, where W_i are i.i.d. zero mean Gaussian processes $W_i = \{W_i(h), h \in H\}$ with covariance function $\mathbb{E}(W_i(h)W_i(g)) = \langle h, g\rangle$.

Proof Fix $\epsilon > 0$. Let $\mu_{i,\epsilon}(x) = (\mu_i * G_1(\epsilon, \cdot))(x)$. By Hölder's inequality,

$$
\mathbb{E}\left[\prod_{i=1}^{n} \mu_{i,\epsilon}(W_i(h_i) + x_i)^2\right] \leq \prod_{i=1}^{n} \mathbb{E}\left[\mu_{i,\epsilon}(W_i(h_i) + x_i)^{2n}\right]^{1/n}.
$$

Notice that for all $p \geq 1$,

$$
\begin{aligned}
\mathbb{E}\left[|\mu_{i,\epsilon}(W_i(h_i) + x_i)|^p\right] &= \int_{\mathbb{R}} \mathrm{d}z\, G_1(|h_i|^2, x+z) \left|\int_{\mathbb{R}} G_1(\epsilon, z-y)\mu_i(\mathrm{d}y)\right|^p \\
&\leq \int_{\mathbb{R}^p} |\mu_i|(\mathrm{d}y_1)\ldots|\mu_i|(\mathrm{d}y_p) \int_{\mathbb{R}} \mathrm{d}z\, G_1(|h_i|^2, x+z) \prod_{j=1}^{p} G_1(\epsilon, z-y_j).
\end{aligned}
$$

By Lemma 6.1 with $t = \epsilon$ and $s = |h_i|^2$,

$$
\mathbb{E}\left[|\mu_{i,\epsilon}(W_i(h_i) + x_i)|^p\right] \leq (p+1)^{p/2} e^{\frac{px^2}{2(p|h_i|^2+\epsilon)}} \left[(|\mu_i| * G_1(\epsilon(p+1), \cdot))(0)\right]^p,
$$

which is finite because $\mu_i \in \mathcal{M}_H(\mathbb{R})$. Hence, $\prod_{i=1}^n \mu_{i,\epsilon}(W_i(h_i) + x_i) \in L^2(\Omega)$, and one can apply Mehler's formula (see, e.g., [11, Section 1.4.1] to obtain that

$$
\begin{aligned}
T_t\left(\prod_{i=1}^{n} \mu_\epsilon\,(x_i + W_i(h_i))\right) &= \mathbb{E}'\left[\prod_{i=1}^{p} \mu_{i,\epsilon}\left(x_i + e^{-t} W_i(h_i) + \sqrt{1-e^{-2t}}\, W_i'(h_i)\right)\right] \\
&= \int_{\mathbb{R}^n} \prod_{i=1}^{n} \mu_{i,\epsilon}\left(x_i + e^{-t} W_i(h_i) + y_i\right) G_1\left(|h_i|^2(1-e^{-2t}), y_i\right) \mathrm{d}y_i
\end{aligned}
$$

$$= \prod_{i=1}^{n} \left[\mu_{i,\epsilon} * G_1(|h_i|^2(1-e^{-2t}), \cdot) \right] (x + e^{-t} W_i(h_i))$$

$$= \prod_{i=1}^{n} \left[\mu_i * G_1(\epsilon + |h_i|^2(1-e^{-2t}), \cdot) \right] (x_i + e^{-t} W_i(h_i)). \quad (6.1)$$

Finally, $T_t \left(\prod_{i=1}^n \mu_i \, (x_i + W_i(h_i)) \right)$ is the $L^2(\Omega)$-limit of $T_t \left(\prod_{i=1}^n \mu_{i,\epsilon} \, (x_i + W_i(h_i)) \right)$, and this limit can be obtained by sending ϵ to zero in (6.1). This proves Lemma 6.2. □

Proof of Theorem 2.10 Without loss of generality, assume that $\mu_i \geq 0$. Following [12], we write $(I-L)^{-\alpha/2}$ in the following form

$$(I-L)^{-\alpha/2} = \Gamma(\alpha/2)^{-1} \int_0^\infty e^{-t} t^{\alpha/2-1} T_t \mathrm{d}t.$$

By Lemma 6.2,

$$\begin{aligned}
&\left\| (I-L)^{-\alpha/2} \prod_{i=1}^n \mu(W_i(h_i) + x_i) \right\|_p \\
&= \left\| \Gamma(\alpha/2)^{-1} \int_0^\infty e^{-t} t^{\alpha/2-1} \prod_{i=1}^n \left[\mu_i * G_1(|h_i|^2(1-e^{-2t}), \cdot) \right] (e^{-t} W_i(h_i) + x_i) \mathrm{d}t \right\|_p \\
&\leq \Gamma(\alpha/2)^{-1} \int_0^\infty e^{-t} t^{\alpha/2-1} \left\| \prod_{i=1}^n \left[\mu_i * G_1(|h_i|^2(1-e^{-2t}), \cdot) \right] (e^{-t} W_i(h_i) + x_i) \right\|_p \mathrm{d}t. \quad (6.2)
\end{aligned}$$

Now

$$\begin{aligned}
&\left\| \prod_{i=1}^n \left[\mu_i * G_1(|h_i|^2(1-e^{-2t}), \cdot) \right] (e^{-t} W_i(h_i) + x_i) \right\|_p^p \\
&= \mathbb{E} \left(\left| \prod_{i=1}^n \int_{\mathbb{R}} \frac{e^{-\frac{(x_i + e^{-t} W_i(h_i) - y)^2}{2|h_i|^2(1-e^{2t})}}}{\sqrt{2\pi |h_i|^2(1-e^{-2t})}} \mu_i(\mathrm{d}y) \right|^p \right) \\
&= \int_{\mathbb{R}^n} \mathrm{d}z_1 \ldots \mathrm{d}z_n \left(\prod_{i=1}^n G_1(S_i, x_i + z_i) \right) \int_{\mathbb{R}^{np}} \prod_{j=1}^p \prod_{i=1}^n G_1(T_i, z_i - y_{ij}) \mu_i(\mathrm{d}y_{ij}) \\
&= \int_{\mathbb{R}^{np}} \left(\prod_{j=1}^p \prod_{i=1}^n \mu_i(\mathrm{d}y_{ij}) \right) \prod_{i=1}^n \int_{\mathbb{R}} \mathrm{d}z_i \; G_1(S_i, x_i + z_i) \prod_{j=1}^p G_1(T_i, z_i - y_{ij})
\end{aligned}$$

where

$$T_i = |h_i|^2(1-e^{-2t}) \quad \text{and} \quad S_i = e^{-2t}|h_i|^2.$$

By Lemma 6.1, the $\mathrm{d}z_i$ integral is bounded by

$$\begin{aligned}
\int_{\mathbb{R}} \mathrm{d}z_i \; G_1(S_i, x_i + z_i) \prod_{j=1}^p G_1(T_i, z_i - y_{ij}) &\leq (p+1)^{p/2} \sqrt{\frac{T_i}{pS_i + T_i}} \, e^{\frac{px_i^2}{2(pS_i + T_i)}} \prod_{j=1}^p G_1((p+1)T_i, y_{ij}) \\
&\leq T_i^{(1-p)/2} \frac{|h_i|^p}{\sqrt{pS_i + T_i}} e^{\frac{px_i^2}{2(pS_i + T_i)}} \prod_{j=1}^p G_1(|h_i|^2(p+1), y_{ij}),
\end{aligned}$$

where we have applied the inequality

$$G_1((p+1)T_i, y_{ij}) \le \frac{|h_i|}{\sqrt{T_i}} G_1((p+1)|h_i|^2, y_{ij}).$$

Hence,

$$\left\|\prod_{i=1}^n \left[\mu_i * G_1(|h_i|^2(1-e^{-2t}), \cdot)\right](e^{-t}W_i(h_i)+x_i))\right\|_p \le \prod_{i=1}^n \frac{T_i^{\frac{1}{2p}-\frac{1}{2}}|h_i| e^{\frac{x_i^2}{2(pS_i+T_i)}}}{(pS_i+T_i)^{1/(2p)}} J_0((p+1)|h_i|^2, 0).$$

By substituting the above upper bound into (6.2), we see that the integral in (6.2) converges provided that

$$\int_{0_+} t^{\frac{\alpha}{2}-1+\frac{n}{2p}-\frac{n}{2}} \mathrm{d}t < \infty,$$

where we have used the fact that

$$pS_i + T_i \ge |h_i|^2 p e^{-2t} \quad \text{and} \quad T_i = 2|h_i|^2(t + O(t^2)).$$

Therefore, $\alpha + n/p > n$. This completes the proof of Lemma 2.10. ☐

6.2 Local Time Part (Proof of Theorem 2.11)

Proof of Theorem 2.11. This is a slight extension of [1, Theorem 1]. The proof consists three steps. Let L_t^x be the local time of the standard one-dimensional Brownian motion.

Step 1. Fix $\epsilon > 0$. Denote

$$F_{\epsilon,x}(y) = \begin{cases} 1 & \text{if } y > x+\epsilon, \\ (y-x+\epsilon)/(2\epsilon) & \text{if } |y-x| \le \epsilon, \\ 0 & \text{if } y < x-\epsilon. \end{cases}$$

Define

$$N_\epsilon(x,t) = \int_0^t F_{\epsilon,x}(B_s)\mathrm{d}B_s \quad \text{and} \quad N(x,t) = \int_0^t 1_{\{B_s > x\}}\mathrm{d}B_s.$$

By Tanaka's formula,

$$L_t^x = (B_t - x)^+ - (-x)^+ - N(x,t).$$

Notice that $L(x,t) \in [0,t]$ is a bounded random variable. On the other hand, let

$$L_{\epsilon,t}^x := (B_t - x)^+ - (-x)^+ - N_\epsilon(x,t).$$

Notice that

$$N(x+\epsilon, t) \le N_\epsilon(x,t) \le N(x-\epsilon,t),$$

Using the fact that for all $x \in \mathbb{R}$ and $y \ge 0$

$$x^+ \le (x+y)^+ \le x^+ + y \quad \text{and} \quad x^+ \ge (x-y)^+ \ge x^+ - y,$$

we see that

$$L_{\epsilon,t}^x \le (B_t - (x+\epsilon)+\epsilon)^+ - (-(x+\epsilon)+\epsilon)^+ - N(x+\epsilon,t) \le L_t^{x+\epsilon} + \epsilon,$$

and

$$L_{\epsilon,t}^x \ge (B_t - (x-\epsilon)-\epsilon)^+ - (-(x-\epsilon)-\epsilon)^+ - N(x-\epsilon,t) \ge L_t^{x-\epsilon} - \epsilon.$$

Hence, it holds that

$$L_t^{x-\epsilon} - \epsilon \le L_{\epsilon,t}^x \le L_t^{x+\epsilon} + \epsilon. \tag{6.3}$$

Step 2. Now assume that $\epsilon \in (-1, 1)$. By telescoping sum,

$$
\begin{aligned}
\left|f(z_1,\dots,z_m) - f(z_1',\dots,z_m')\right| \le \quad & \left|f(z_1,z_2,\dots,z_m) - f(z_1',z_2,\dots,z_m)\right| \\
& + \left|f(z_1',z_2,z_3,\dots,z_m) - f(z_1',z_2',z_3,\dots,z_m)\right| \\
& + \dots \\
& + \left|f(z_1',\dots,z_{n-1}',z_m) - f(z_1',\dots,z_{n-1}',z_m')\right|. \quad (6.4)
\end{aligned}
$$

Because $\frac{\partial^2}{\partial x_1^2} f \ge 0$, the first term to the right of (6.4) satisfies that

$$
\left|f(z_1,z_2,\dots,z_m) - f(z_1',z_2,\dots,z_m)\right| \le \left(|f_1(z_1,z_2,\dots,z_m)| + |f_1(z_1',z_2,\dots,z_m)|\right) |z_1 - z_1'|.
$$

Now replace z_i and z_i' by $L^i := L_t^{i,x_i}$ and $L_\epsilon^i := L_{\epsilon,t}^{i,x_i}$, $i = 1,\dots,m$, respectively. Denote the quantity in (2.15) by

$$
C_p := C_p(t, x_1, \dots, x_m).
$$

Let $1/r + 1/q = 1$, $q \ge 2$. By Hölder's inequality, the first term on the right hand side of (6.4) satisfies that

$$
\begin{aligned}
& \left|\left| f(L^1, L^2, \dots, L^m) - f(L_\epsilon^1, L^2, \dots, L^m) \right|\right|_p \\
& \le \left|\left| |f_1(L^1, L^2, \dots, L^m)| + |f_1(L_\epsilon^1, L^2, \dots, L^m)| \right|\right|_{pq} \left|\left| L_t^{1,x_1} - L_{\epsilon,t}^{1,x_1} \right|\right|_{pr} \\
& \le 2 \sup_{\epsilon_1 \in (-1,1)} \left|\left| f_1(L_t^{1,x_1+\epsilon_1} + \epsilon_1, L^2, \dots, L^m) \right|\right|_{pq} \left|\left| L_t^{1,x_1} - L_{\epsilon,t}^{1,x_1} \right|\right|_{pr} \\
& \le 2\, C_{pq} \left|\left| L_t^{x_1} - L_{\epsilon,t}^{x_1} \right|\right|_{pr},
\end{aligned}
$$

where we have used the fact (6.3). This inequality is true for all the n terms on the right hand side of (6.4) under the same replacements. Therefore,

$$
\left|\left| f(L^1,\dots,L^m) - f(L_\epsilon^1,\dots,L_\epsilon^m) \right|\right|_p \le 2\, C_{pq} \sum_{i=1}^m \left|\left| L_t^{x_i} - L_{\epsilon,t}^{x_i} \right|\right|_{pr} \le 2m\widetilde{C}_{pq}\epsilon^{1/2}, \quad (6.5)
$$

where the last inequality is due to (2.8) of [1].

By Hölder's inequality with $1/r + 1/q = 1$, we see that

$$
\begin{aligned}
\mathbb{E}\left[\left|\left| Df(L_\epsilon^1,\dots,L_\epsilon^m) \right|\right|_H^p\right] & \le 2^{p-1} \sum_{i=1}^m \mathbb{E}\left(|f_i(L_\epsilon^1,\dots,L_\epsilon^m)|^p \left|\left| DL_{\epsilon,t}^{i,x} \right|\right|_H^p \right) \\
& \le 2^{p-1} \sum_{i=1}^m \mathbb{E}\left(|f_i(L_\epsilon^1,\dots,L_\epsilon^m)|^{pq} \right)^{1/q} \mathbb{E}\left(\left|\left| DL_{\epsilon,t}^{i,x} \right|\right|_H^{pr} \right)^{1/r} \\
& \le 2^{p-1}\, C_{pq}^p \sum_{i=1}^m \left|\left| DL_{\epsilon,t}^{i,x} \right|\right|_{L^{pr}(\Omega;H)}^p,
\end{aligned}
$$

where we have applied (6.3) and (2.15) as before. Hence, by (2.9) of [1],

$$
\begin{aligned}
\left|\left| Df(L_\epsilon^1,\dots,L_\epsilon^m) \right|\right|_{L^p(\Omega;H)} & \le 2\, mC_{pq} \sum_{i=1}^m \Big(\left|\left| D(B_t - x_i)^+ \right|\right|_{L^{pr}(\Omega;H)} \\
& \qquad + ||DN_\epsilon(x_i,t)||_{L^{pr}(\Omega;H)} \Big) \\
& \le 2\, \widetilde{C}_{pq}\, \epsilon^{-1+\frac{1}{2q'}} \quad \text{for all } q' > 1. \qquad (6.6)
\end{aligned}
$$

Therefore, by the same arguments as the proof of Theorem 1 in [1], we see that (6.5) and (6.6) imply that $f(L^1,\dots,L^m) \in \mathbb{D}^{\alpha,p}(\mathbb{R})$ for all $p > 1$ and $\alpha < 1/2$.

Step 3. In this final step, we need to verify that $f(z_1,\dots,z_m)=\exp\left(\lambda^2\sum_{j=1}^m z_j\right)$ satisfies the condition (2.15). By Hölder's inequality, we have that, for all $\epsilon_j\in(-1,1)$,

$$\begin{aligned}\left\|f_i(L_t^{1,x_1+\epsilon_1}+\epsilon_1,\dots,L_t^{m,x_m+\epsilon_m}+\epsilon_m)\right\|_p &= \lambda^2\left\|\exp\left(\lambda^2\sum_{j=1}^m[L_t^{j,x_j+\epsilon_j}+\epsilon_j]\right)\right\|_p\\ &\le \lambda^2\prod_{j=1}^m\left\|\exp\left(\lambda^2[L_t^{j,x_j+\epsilon_j}+\epsilon_j]\right)\right\|_{mp}\\ &\le \lambda^2 e^{\lambda^2 m}\prod_{j=1}^m\left\|\exp\left(\lambda^2 L_t^{j,x_j+\epsilon_j}\right)\right\|_{mp}.\end{aligned}$$

By Corollary 2.6,

$$\left\|\exp\left(\lambda^2 L_t^{x_j+\epsilon_j}\right)\right\|_{mp}=\mathbb{E}\left[\exp\left(\lambda^2 mp L_t^{x_j+\epsilon_j}\right)\right]^{\frac{1}{mp}}\le\left[2e^{\lambda^4m^2p^2t/2}+1\right]^{\frac{1}{mp}}.$$

Therefore,

$$\max_{1\le i\le m}\sup_{\substack{\epsilon_j\in(-1,1)\\1\le j\le m}}\left\|f_i(L_t^{1,x_1+\epsilon_1}+\epsilon_1,\dots,L_t^{m,x_m+\epsilon_m}+\epsilon_m)\right\|_p\le\lambda^2e^{\lambda^2m}\left[2e^{\lambda^4m^2p^2t/2}+1\right]^{\frac{1}{p}}.$$

This completes the proof of Theorem 2.1.

□

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