Probab. Theory Relat. Fields
DOI 10.1007/s00440-017-0783-z

Intermittency for the stochastic heat equation driven by a rough time fractional Gaussian noise

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Received: 18 February 2016 / Revised: 5 May 2017

Abstract This paper studies the stochastic heat equation driven by time fractional Gaussian noise with Hurst parameter $H \in (0, 1/2)$. We establish the Feynman–Kac representation of the solution and use this representation to obtain matching lower and upper bounds for the $L^p(\Omega)$ moments of the solution.

Keywords Stochastic heat equation · Feynman–Kac integral · Feynman–Kac formula · Time fractional Gaussian noise · Fractional calculus · Moment bounds · Lyapunov exponents · Intermittency

Mathematics Subject Classification Primary 60H15; Secondary 60G60 · 35R60

1 Introduction

As pointed out by Zel′dovich et al. [20, p. 237], *intermittency* is a universal phenomenon provided that a random field is of multiplicative type. Intermittency is characterized by enormous growth rates of moments of the random field and it has been intensively studied in the past two decades for stochastic partial differential equations of various kinds; see, e.g., [1–7,9]. These growth rates both depend on the noise

Hu is partially supported by a Grant from the Simons Foundation #209206; Nualart is partially supported by the NSF Grant DMS1512891 and the ARO Grant FED0070445.
Kalbasi is supported by a fellowship from the Swiss National Science Foundation.

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structures [1,9] and also on the partial differential operators [5,7]. In the literature, the noise is either white in time [2–7] or more regular than the white noise [1,9]. Little is known about the intermittency for the case when the noise in time is rougher than the white noise. This latter fact motivates this current investigation. In particular, we will study in this paper the intermittency property for the following stochastic heat equation subject to a noise which is rougher than the white noise in time,

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) = \frac{1}{2}\Delta u(t,x) + u(t,x)\frac{\partial}{\partial t}W(t,x), & t>0, x\in\mathbb{R}^d, \\ u(0,x) = u_0(x), \end{cases} \tag{1.1}$$

where u_0 is a bounded measurable function. $W = \{W(t,x),\ t \geq 0, x \in \mathbb{R}^d\}$ is a Gaussian random field, which is fractional Brownian motion of Hurst parameter $H \in (0, 1/2)$ in time and has correlation in space given by $Q(x, y)$:

$$\mathbb{E}\left[W(t,x)W(s,y)\right] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right) Q(x,y).$$

We assume that $Q(x, y)$ satisfies the following two conditions:

(H1) There exist some constants $\alpha \in (0, 1]$ and $C_0 > 0$ such that

$$Q(x,x) + Q(y,y) - 2Q(x,y) \leq C_0|x-y|^{2\alpha}, \quad \text{for all } x \text{ and } y \in \mathbb{R}^d. \tag{H1}$$

(H2) There exist some constants $\beta \in [0, 1)$ and $C_2 > 0$ such that for all $M > 0$,

$$Q(x,y) \geq C_2 M^{2\beta}, \qquad \text{for all } x,\ y \in \mathbb{R}^d \text{ with } \min_{i=1,\ldots,d}(|x_i| \wedge |y_i|) > M, \tag{H2}$$

where $a \wedge b := \min(a, b)$.

It is known that *Feynman–Kac formula/representation* for the solution is a powerful tool for studying the moments of the solution; see [3,6,9]. Hence, the first challenging problem in this paper is to establish the following Feynman–Kac formula for the solution to (1.1):

$$u(t,x) = \mathbb{E}^B\left[u_0(B_t^x)\exp\int_0^t W(\mathrm{d}s, B_{t-s}^x)\right], \tag{1.2}$$

where $B = \left\{B_t^x = B_t + x,\ t \geq 0, x \in \mathbb{R}^d\right\}$ is a d-dimensional Brownian motion starting from $x \in \mathbb{R}^d$, independent of W, and the expectation is with respect to the Brownian motion. Hu et al. [11] established this representation (1.2) for the case where $H \in (1/4, 1/2)$. In this paper we will improve their results by allowing the Hurst parameter H to be any value in $(0, 1/2)$. More precisely, we will show that, for any $H \in (0, 1/2)$, if condition (H1) holds and $2H + \alpha > 1$, then the solution to (1.1) is given by (1.2).

We interpret the solution (1.2) in the weak form, where the integral, instead of being a pathwise integral, is a nonlinear stochastic integral in the Stratonovich sense; see

Definition 3.5 below for the precise definition and see also the recent work by Hu and Lê [10] for the case $H > 1/2$. We remark that for $H > 1/2$ the uniqueness of the solution can be proved using Young's integral; see, e.g., Section 5.2 of [9]. For the current case, this might be done using the rough path analysis, but we will not pursue this property in this paper.

Using this representation (1.2), we are able to show that for some nonnegative constants $\overline{C}$, $\underline{C}$, $\overline{C}_x$ and $\underline{C}_x$, the solution to (1.1) satisfies the following moment bounds

$$\underline{C}_x \exp\left(\underline{C}k^{\frac{2-\beta}{1-\beta}} t^{\frac{2H+\beta}{1-\beta}}\right) \le \mathbb{E}\left[u(t,x)^k\right] \le \overline{C}_x \exp\left(\overline{C}k^{\frac{2-\alpha}{1-\alpha}} t^{\frac{2H+\alpha}{1-\alpha}}\right) \tag{1.3}$$

for all $t \ge 1$ and $k \in \mathbb{N}$, where we need to assume condition (H2) and $\inf_{x\in\mathbb{R}^d} u_0(x) > 0$ to establish the lower bound. When $\alpha = \beta$ (see Remark 1.1 below for one example), our exponents in (1.3) are sharp in the sense that one can define the *moment Lyapunov exponents*

$$\overline{m}_k(x) := \limsup_{t\to+\infty} t^{-\frac{2H+\alpha}{1-\alpha}} \log \mathbb{E}\left[u(t,x)^k\right] \quad \text{and}$$
$$\underline{m}_k(x) := \liminf_{t\to+\infty} t^{-\frac{2H+\alpha}{1-\alpha}} \log \mathbb{E}\left[u(t,x)^k\right],$$

and establish easily from (1.3) that

$$\underline{C}k^{\frac{2-\alpha}{1-\alpha}} \le \inf_{x\in\mathbb{R}^d} \underline{m}_k(x) \le \sup_{x\in\mathbb{R}^d} \overline{m}_k(x) \le \overline{C}k^{\frac{2-\alpha}{1-\alpha}}, \quad \text{for all } k \ge 2. \tag{1.4}$$

Therefore, this solution is *fully intermittent* [3, Definition III.1.1].

Remark 1.1 If $d = 1$ and $Q(x, y)$ is the covariance of a fractional Brownian motion $\{B_x^\Theta,\ x \in \mathbb{R}\}$ with Hurst parameter $\Theta \in (0, 1)$, i.e.,

$$Q(x,y) = \mathbb{E}\left[B_x^\Theta B_y^\Theta\right] = \frac{1}{2}\left(|x|^{2\Theta} + |y|^{2\Theta} - |x-y|^{2\Theta}\right),$$

then it is easy to see that both conditions (H1) and (H2) are satisfied with $\alpha = \beta = \Theta$ and (1.3) becomes

$$\underline{C}_x \exp\left(\underline{C}\, k^{\frac{2-\Theta}{1-\Theta}} t^{\frac{2H+\Theta}{1-\Theta}}\right) \le \mathbb{E}\left[u(t,x)^k\right] \le \overline{C}_x \exp\left(\overline{C}\, k^{\frac{2-\Theta}{1-\Theta}} t^{\frac{2H+\Theta}{1-\Theta}}\right). \tag{1.5}$$

The proofs of both upper and lower bounds in (1.3) use the Feynman–Kac representation (1.2). While the proof of the upper bound follows from some standard arguments, the proof of the lower bond is more involved and it is very different from that of the upper bound. We will use a representation of Q as the covariance function of a Gaussian process and then apply an moment inequality, by Mémin et al. [15], for the related stochastic integral; see Lemma 4.3 below. Note that, since this inequality holds only in one direction, this technique is valid only for establishing the lower bound.

There is an extensive literature on the Feynman–Kac formula for stochastic partial differential equations under various random potentials. We refer interested readers to the references in [11–13]. Hu et al. [13] proved that if the random potential $W = \{W(t,x), t \geq 0, x \in \mathbb{R}^d\}$ is a fractional Brownian sheet with Hurst parameter $(H_0, H_1, \ldots, H_d)$ that satisfies

$$H_i \in (1/2, 1),\ i = 1, \ldots, d, \quad \text{and} \quad 2H_0 + \sum_{i=1}^{d} H_i > d+1, \tag{1.6}$$

then the solution to the following stochastic heat equation

$$\begin{cases} \dfrac{\partial}{\partial t} u(t,x) = \dfrac{1}{2}\Delta u(t,x) + u(t,x) \dfrac{\partial^{d+1}}{\partial t \partial x_1 \cdots \partial x_d} W(t,x), & t > 0,\ x \in \mathbb{R}^d, \\ u(0,x) = u_0(x), \end{cases} \tag{1.7}$$

admits a Feynman–Kac representation

$$u(t,x) = \mathbb{E}^B \left[u_0(B_t^x) \exp\left(\int_0^t \int_{\mathbb{R}^d} \delta\left(B_{t-s}^x - y\right) W(\mathrm{d}s, \mathrm{d}y) \right) \right],$$

where B is a d-dimensional Brownian motion (the same as B in (1.2)), independent of W. In this framework, condition (1.6) implies that $H_0 > 1/2$.

In order to handle the case where $H_0 < 1/2$, one may impose better spatial correlations. When $H_0 \in (1/4, 1/2)$, Hu et al. [11] established the Feynman–Kac representation for (1.2) with a similar spatial covariance $Q(x,y)$ that satisfies a growth condition (see (H3) below) and is locally γ-Hölder continuous with $\gamma > 2 - 4H_0$. Notice that the fact that Q is a covariance function implies that there exists a Gaussian process $Y = \{Y(x), x \in \mathbb{R}^d\}$ such that $Q(x,y) = \mathbb{E}[Y(x)Y(y)]$. Then it is natural to assume some sample path regularity of Y through the following condition

$$\mathbb{E}\left[(Y(x) - Y(y))^2 \right] \leq C_0 |x-y|^{2\alpha}. \tag{H1'}$$

Because Y is Gaussian, (H1') implies that Y is a.s. γ-Hölder continuous for all $\gamma < \alpha$. Clearly the two conditions (H1') and (H1) are equivalent. Then under (H1) (or equivalently (H1')), we are able to establish the Feynman–Kac formula for any $H_0 \in ((1-\alpha)/2, 1/2)$. Note that α can be arbitrarily close to one by choosing Q properly; see Remark 1.1 for an example.

The above representation of Q using Y implies a growth condition of Q, which is listed below for the convenience of later reference,

(H3) There exists a constant $C_1 > 0$ such that for all $M > 0$,

$$|Q(x,y)| \leq C_1 (1+M)^{2\alpha}, \quad \text{for all } x, y \in \mathbb{R}^d \text{ with } |x|, |y| \leq M. \tag{H3}$$

When the space $\mathbb{R}^d$ is replaced by $\mathbb{Z}^d$ in (1.1), the Brownian motion B in (1.2) should be replaced by a locally constant random walk. Kalbasi and Mountford [14] recently studied this case and established the Feynman–Kac formula for any $H_0 \in (0, 1)$.

It is interesting, even formally, to compare the exponents obtained in this work with the previous ones. Hu et al. [9] recently studied (1.7) with the noise having the following covariance form

$$\mathbb{E}\left[\dot{W}(t,x)\dot{W}(s,y)\right] = \gamma(t-s)\Lambda(x-y), \tag{1.8}$$

where $\dot{W} := \frac{\partial^{d+1} W}{\partial t \partial x_1 \cdots \partial x_d}$; see also a closely related work by Balan and Conus [1]. Under the condition that for some constants $c_0, C_0, c_1, C_1, \kappa \in (0,1)$ and $\sigma \in (0,2)$,

$$c_0|t|^{-\kappa} \le \gamma(t) \le C_0|t|^{-\kappa} \quad \text{and} \quad c_1|x|^{-\sigma} \le \Lambda(x) \le C_1|x|^{-\sigma}, \tag{1.9}$$

it is proved in [9] that

$$\underline{C}\exp\left(\underline{C}\, k^{\frac{4-\sigma}{2-\sigma}} t^{\frac{4-2\kappa-\sigma}{2-\sigma}}\right) \le \mathbb{E}\left[u(t,x)^k\right] \le \overline{C}\exp\left(\overline{C}\, k^{\frac{4-\sigma}{2-\sigma}} t^{\frac{4-2\kappa-\sigma}{2-\sigma}}\right). \tag{1.10}$$

The noises for both equations (1.1) and (1.7) (with noise (1.8)) are similar in time. Our noise formally corresponds to the case $\kappa = 2-2H$, where the condition $\kappa \in (0,1)$ imposed in [9], implies that $H \in (1/2, 1)$. However, after substituting κ by $2-2H$ in the exponents of (1.10) and comparing the following two exponents,

$$\underbrace{k^{\frac{2-\alpha}{1-\alpha}}\; t^{\frac{2H+\alpha}{1-\alpha}} \text{ in (1.3)}}_{(1-\alpha)/2<H<1/2} \qquad \text{and} \qquad \underbrace{k^{\frac{2-\sigma/2}{1-\sigma/2}}\; t^{\frac{2H-\sigma/2}{1-\sigma/2}} \text{ in (1.10)},}_{1/2<H<1}$$

we immediately see a mismatch of the sign in the exponent of t if one takes $\sigma = 2\alpha$. In both cases, no matter whether $H > 1/2$ (the positive correlation case) or $H < 1/2$ (the negative correlation case), the larger value the parameter H has, the more correlations the noise produces, and the larger exponent of t we have and hence the larger moment we obtain. If we formally take $H = 1/2$ in both cases, we have $t^{(1+\alpha)/(1-\alpha)}$ for (1.3) and t for (1.10), where the latter case is what one usually expects for a noise that is white in time. Nevertheless, this mismatch of the sign makes the exponent of t in our case always bigger than one. This discrepancy is due to the different natures of these two noises in space. Our noise in space is nonhomogeneous and the function $x \mapsto Q(x,x)$ is finite at the origin but has a growth rate at infinity. On the other hand, the noise with Λ in (1.8) is homogeneous and it is singular at the origin but decreases to zero at infinity. Nevertheless, in both cases, the exponents of k depend only on the spatial correlations. Moreover, when $\sigma = 1$ (noise is white in space for (1.7)) and $\Theta = 1/2$ in (1.5) (the case when Q is a correlation function of a Brownian motion; see Remark 1.1), both exponents of k are equal to 3.

Throughout this paper, denote $\alpha_H = 2H(2H-1)$, which is negative for $H \in (0, 1/2)$. For $t,\ s \in \mathbb{R}$, denote

$$R_H(t,s) := \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right). \tag{1.11}$$

Let $||\cdot||_\kappa$ be the κ-Hölder norm and $C^\kappa([0,T])$ be the set of κ-Hölder continuous functions on $[0,T]$.

This paper is organized as follows: In Sect. 2, we define the stochastic integral in (1.2) through approximation and derive some properties of this stochastic integral. In Sect. 3, we first make sense of expression (1.2) by showing that the stochastic integral in (1.2) has exponential moments. As a consequence, we derive the upper bound of (1.3). Then we validate that (1.2) is a weak solution to (1.1). The lower bound in (1.3) is proved in Sect. 4. Finally, some technical lemmas are proved or listed in "Appendix".

2 Stochastic integral with respect to W

In this section, we introduce the stochastic integral with respect to W that appears in (1.2) and prove some useful properties. The integral is defined through an approximation scheme, which requires an extension of the noise W from $t \geq 0$ to $t \in \mathbb{R}$, i.e., $W = \{W(t,x),\ t \in \mathbb{R},\ x \in \mathbb{R}^d\}$ is a mean zero Gaussian process with the following covariance

$$\mathbb{E}\left[W(t,x)W(s,y)\right] = R_H(t,s)Q(x,y), \quad \text{for all } t, s \in \mathbb{R} \text{ and } x, y \in \mathbb{R}^d.$$

Definition 2.1 Given a continuous function $\phi : [0,T] \mapsto \mathbb{R}^d$, define

$$\int_0^t W(\mathrm{d}s, \phi_s) := \lim_{\epsilon \to 0} \int_0^t \dot{W}^\epsilon(s, \phi_s)\mathrm{d}s,$$

if the limit exists in $L^2(\Omega)$, where

$$\dot{W}^\epsilon(s,x) = (2\epsilon)^{-1}\left(W(s+\epsilon, x) - W(s-\epsilon, x)\right). \tag{2.1}$$

The aim of this section is to prove the following Theorem 2.2 and Proposition 2.4. Denote

$$\widehat{Q}(u,v,\phi,\psi) = \frac{1}{2}\left[Q(\phi_u,\psi_u) + Q(\phi_v,\psi_v) - Q(\phi_u,\psi_v) - Q(\phi_v,\psi_u)\right].$$

Theorem 2.2 *Assume that Q satisfies condition* (H1). *Then for all $0 < t \leq T$ and $\phi, \psi \in C^\kappa([0,T])$ with $\alpha\kappa + H > 1/2$, the stochastic integral $I_t(\phi) := \int_0^t W(\mathrm{d}s, \phi_s)$ exists and*

$$\mathbb{E}\left[I_t(\phi)I_t(\psi)\right] = H\int_0^t \theta^{2H-1}\left[Q(\phi_\theta,\psi_\theta)+Q(\phi_{t-\theta},\psi_{t-\theta})\right]\mathrm{d}\theta$$
$$-\alpha_H\int_0^t\int_0^\theta r^{2H-2}\widehat{Q}(\theta,\theta-r,\phi,\psi)\mathrm{d}r\mathrm{d}\theta. \tag{2.2}$$

Moreover,

$$|\mathbb{E}\left[I_t(\phi)I_t(\psi)\right]| \le H\int_0^t \theta^{2H-1}\left[Q(\phi_\theta,\psi_\theta)+Q(\phi_{t-\theta},\psi_{t-\theta})\right]\mathrm{d}\theta$$
$$+\frac{|\alpha_H|C_0}{2}\int_0^t\int_0^\theta r^{2H-2}|\phi_\theta-\phi_{\theta-r}|^\alpha|\psi_\theta-\psi_{\theta-r}|^\alpha\mathrm{d}r\mathrm{d}\theta \tag{2.3}$$
$$\le C_{\phi,\psi}t^{2(H+\alpha\kappa)}+C^*_{\phi,\psi}t^{2H}, \tag{2.4}$$

where

$$C_{\phi,\psi} := \frac{H(1-2H)C_0\,||\phi||_\kappa^\alpha\,||\psi||_\kappa^\alpha}{2(H+\alpha\kappa)(2(H+\alpha\kappa)-1)} \quad \textit{and} \quad C^*_{\phi,\psi} := C_1\left(1+||\phi||_\infty\vee||\psi||_\infty\right)^{2\alpha},$$

and the constants C_0 and C_1 are defined in (H1) *and* (H3)*, respectively.*

Remark 2.3 By symmetry,

$$\int_0^t\int_0^\theta r^{2H-2}\widehat{Q}(\theta,\theta-r,\phi,\psi)\mathrm{d}r\mathrm{d}\theta=\frac{1}{2}\int_0^t\int_0^t|u-v|^{2H-2}\widehat{Q}(u,v,\phi,\psi)\mathrm{d}u\mathrm{d}v. \tag{2.5}$$

Therefore, (2.2) can be equivalently written as

$$\mathbb{E}[I_t(\phi)I_t(\psi)] = H\int_0^t Q(\phi_s,\psi_s)\left[s^{2H-1}+(t-s)^{2H-1}\right]\mathrm{d}s$$
$$+\frac{|\alpha_H|}{2}\int_0^t\int_0^t|u-v|^{2H-2}\widehat{Q}(u,v,\phi,\psi)\mathrm{d}u\mathrm{d}v, \tag{2.6}$$

and similarly,

$$\int_0^t\int_0^\theta r^{2H-2}|\phi_\theta-\phi_{\theta-r}|^\alpha|\psi_\theta-\psi_{\theta-r}|^\alpha\mathrm{d}r\mathrm{d}\theta$$
$$=\frac{1}{2}\int_0^t\int_0^t|u-v|^{2H-2}|\phi_u-\phi_v|^\alpha|\psi_u-\psi_v|^\alpha\mathrm{d}u\mathrm{d}v. \tag{2.7}$$

Proposition 2.4 *Suppose $\phi \in C^\kappa([0,T])$ with $\alpha\kappa + H > 1/2$. Then for all $0 \le s < t \le T$,*

$$\mathbb{E}\left[\left(\int_0^t W(\mathrm{d}r,\phi_r) - \int_0^s W(\mathrm{d}r,\phi_r)\right)^2\right] \le C'\left(1+||\phi||_\infty\right)^{2\alpha}(t-s)^{2H} + C''\,||\phi||_\kappa^{2\alpha}\,(t-s)^{2(H+\alpha\kappa)}, \tag{2.8}$$

where the constants C' and C'' depend on H, T, α and κ. As a consequence, the process $X_t = \int_0^t W(\mathrm{d}r,\phi_r)$ is almost surely $(H-\epsilon)$-Hölder continuous for any $\epsilon > 0$.

The proofs of Theorem 2.2 and Proposition 2.4 require some lemmas. Denote

$$I_{t,\epsilon}(\phi) = \int_0^t \dot{W}^\epsilon(s,\phi_s)\mathrm{d}s.$$

By (3.2) of [11], for $\phi, \psi \in C([0,T])$,

$$\begin{aligned}\mathbb{E}\left[I_{t,\epsilon}(\phi)I_{t,\delta}(\psi)\right] &= \int_0^t\int_0^t Q(\phi_u,\psi_v)V_{\epsilon,\delta}^{2H}(u-v)\mathrm{d}u\mathrm{d}v\\ &= \frac{1}{2}\int_0^t\int_0^\theta \left[Q(\phi_\theta,\psi_{\theta-r}) + Q(\phi_{\theta-r},\psi_\theta)\right]V_{\epsilon,\delta}^{2H}(r)\mathrm{d}r\mathrm{d}\theta,\end{aligned} \tag{2.9}$$

where

$$V_{\epsilon,\delta}^{2H}(r) = \frac{1}{4\epsilon\delta}\left(|r+\epsilon+\delta|^{2H} + |r-\epsilon-\delta|^{2H} - |r-\epsilon+\delta|^{2H} - |r+\epsilon-\delta|^{2H}\right). \tag{2.10}$$

Lemma 2.5 *There is some constant $C_H > 0$ such that for all $r > 0$, $\epsilon \ge \delta > 0$,*

$$V_{\epsilon,\delta}^{2H}(r)1_{[4\epsilon,+\infty)}(r) \le C_H\, r^{2H-2}.$$

Proof Because $r \ge 4\epsilon \ge 2(\epsilon+\delta)$, we see that $r \pm \epsilon \pm \delta > 0$ and

$$V_{\epsilon,\delta}^{2H}(r) = \frac{1}{4}\int_{-1}^1\int_{-1}^1 \alpha_H(r+\eta\epsilon+\xi\delta)^{2H-2}\mathrm{d}\xi\mathrm{d}\eta.$$

Because

$$\begin{aligned}(r+\eta\epsilon+\xi\delta)^{2H-2} &= \left(1+\frac{\eta\epsilon+\xi\delta}{r}\right)^{2H-2} r^{2H-2} \le \left(1-\frac{\epsilon+\delta}{r}\right)^{2H-2} r^{2H-2}\\ &\le \left(1-\frac{1}{2}\right)^{2H-2} r^{2H-2} = 2^{2-2H}r^{2H-2},\end{aligned}$$

we have that $V_{\epsilon,\delta}^{2H}(r) \le 2^{3-2H}H(2H-1)r^{2H-2}$. This proves Lemma 2.5. □

Lemma 2.6 *If $\psi : [0, T] \mapsto \mathbb{R}$ is a bounded function, then either for $\widehat{\psi}(t,\theta) = \psi(\theta)$ or for $\widehat{\psi}(t,\theta) = \psi(t-\theta)$, we have that*

$$\left|\int_0^t \mathrm{d}\theta\, \widehat{\psi}(t,\theta) \int_0^\theta \mathrm{d}r\, V_{\epsilon,\delta}^{2H}(r) - 2H \int_0^t \widehat{\psi}(t,\theta)\theta^{2H-1}\mathrm{d}\theta\right| \le 4\,||\psi||_\infty\,(\epsilon+\delta)^{2H}. \tag{2.11}$$

Proof The case $\widehat{\psi}(t,\theta) = \psi(\theta)$ is proved by Hu, Lu and Nualart in [11, Lemma 3.2]. Their arguments can be easily extended to the case $\widehat{\psi}(t,\theta) = \psi(t-\theta)$. □

Lemma 2.7 *For some constant $C_0 > 0$ and some $\alpha \in (0, 1]$, (H1) holds if and only if*

$$|Q(x,u) + Q(y,w) - Q(x,w) - Q(y,u)| \le C_0|x-y|^\alpha|u-w|^\alpha, \tag{2.12}$$

for all $x, y, w, u \in \mathbb{R}^d$.

Proof Since Q is a covariance function, one can find a process $\{Y_x,\ x \in \mathbb{R}^d\}$ such that $Q(x,y) = E[Y_xY_y]$. Then the left-hand side of (H1) is equal to $\mathbb{E}[(Y_x - Y_y)^2]$ and the left-hand side of (2.12) is equal to $|\mathbb{E}[(Y_x-Y_y)(Y_u-Y_w)]|$. With this representation, the equivalence between (H1) and (2.12) is clear by the Cauchy–Schwarz inequality. □

Proof of Theorem 2.2 Throughout the proof, we use C to denote a generic constant which may vary from line to line. Notice that

$$\begin{aligned}\frac{1}{2}\left[Q(\phi_\theta,\psi_{\theta-r}) + Q(\phi_{\theta-r},\psi_\theta)\right] &= -\widehat{Q}(\theta,\theta-r,\phi,\psi)\\ &\quad + \frac{1}{2}\left[Q(\phi_\theta,\psi_\theta) + Q(\phi_{\theta-r},\psi_{\theta-r})\right].\end{aligned} \tag{2.13}$$

By (2.12) and by the Hölder continuity of ϕ and ψ, we see that

$$\left|\widehat{Q}(\theta,\theta-r,\phi,\psi)\right| \le \frac{C_0}{2}|\phi_\theta-\phi_{\theta-r}|^\alpha|\psi_\theta-\psi_{\theta-r}|^\alpha \le \frac{C_0}{2}\,||\phi||_\kappa^\alpha\,||\psi||_\kappa^\alpha\ r^{2\alpha\kappa}, \tag{2.14}$$

for all $0 \le r \le \theta \le T$. Hence, using (2.9) and (2.13),

$$\begin{aligned}&\Bigg|\mathbb{E}\left[I_{t,\epsilon}(\phi)I_{t,\delta}(\psi)\right] + \alpha_H\int_0^t\int_0^\theta r^{2H-2}\widehat{Q}(\theta,\theta-r,\phi,\psi)\mathrm{d}r\mathrm{d}\theta\\ &\quad - H\int_0^t \theta^{2H-1}\left[Q(\phi_\theta,\psi_\theta) + Q(\phi_{t-\theta},\psi_{t-\theta})\right]\mathrm{d}\theta\Bigg|\\ &\le \left|\int_0^t\int_0^\theta \widehat{Q}(\theta,\theta-r,\phi,\psi)\left(V_{\epsilon,\delta}^{2H}(r) - \alpha_H r^{2H-2}\right)\mathrm{d}r\mathrm{d}\theta\right|\\ &\quad + \frac{1}{2}\left|\int_0^t\int_0^\theta Q(\phi_\theta,\psi_\theta)V_{\epsilon,\delta}^{2H}(r)\mathrm{d}r\mathrm{d}\theta - 2H\int_0^t \theta^{2H-1}Q(\phi_\theta,\psi_\theta)\mathrm{d}\theta\right|\end{aligned}$$

$$+\frac{1}{2}\left|\int_0^t\int_0^\theta Q(\phi_{\theta-r},\psi_{\theta-r})V_{\epsilon,\delta}^{2H}(r)\mathrm{d}r\mathrm{d}\theta-2H\int_0^t\theta^{2H-1}Q(\phi_{t-\theta},\psi_{t-\theta})\mathrm{d}\theta\right|$$
$$=: I_1+\frac{I_2}{2}+\frac{I_3}{2}.$$

We claim that

$$\lim_{\epsilon,\delta\to 0} I_i=0,\quad i=1,2,3. \tag{2.15}$$

Therefore, we have that

$$\lim_{\epsilon,\delta\to 0}\mathbb{E}\left[I_{t,\epsilon}(\phi)I_{t,\delta}(\psi)\right]=H\int_0^t\theta^{2H-1}\left[Q(\phi_\theta,\psi_\theta)+Q(\phi_{t-\theta},\psi_{t-\theta})\right]\mathrm{d}\theta$$
$$-\alpha_H\int_0^t\int_0^\theta r^{2H-2}\widehat{Q}(\theta,\theta-r,\phi,\psi)\mathrm{d}r\mathrm{d}\theta. \tag{2.16}$$

When $\psi=\phi$, this implies that $\{I_{t,\epsilon_n}(\phi),\ n\geq 1\}$ is a Cauchy sequence in $L^2(\Omega)$ for any sequence $\epsilon_n\downarrow 0$. Therefore, $\lim_{\epsilon\to 0}I_{t,\epsilon}(\phi)$ exists in $L^2(\Omega)$ and is denoted by $I_t(\phi):=\int_0^t W(\mathrm{d}s,\phi_s)$. Formula (2.2) is a consequence of (2.16). As for moment bound (2.4), by (H3) and (2.14),

$$|\mathbb{E}[I_t(\phi)I_t(\psi)]|\leq\frac{|\alpha_H|C_0}{2}\,||\phi||_\kappa^\alpha\,||\psi||_\kappa^\alpha\int_0^t\int_0^\theta r^{2\alpha\kappa+2H-2}\mathrm{d}r\mathrm{d}\theta$$
$$+\ C_1\left(1+||\phi||_\infty\vee||\psi||_\infty\right)^{2\alpha}(2H)\int_0^t\theta^{2H-1}\mathrm{d}\theta$$
$$=\frac{C_0|\alpha_H|\,||\phi||_\kappa^\alpha\,||\psi||_\kappa^\alpha\,t^{2(H+\alpha\kappa)}}{4(H+\alpha\kappa)(2(H+\alpha\kappa)-1)}+C_1(1+||\phi||_\infty\vee||\psi||_\infty)^{2\alpha}t^{2H}.$$

Therefore, it remains to prove (2.15), which will be done in the following two steps. □

Step 1. We first prove (2.15) for I_1. Notice that I_1 can be decomposed as

$$I_1\leq\left|\int_0^t\int_{4\epsilon}^\theta\widehat{Q}(\theta,\theta-r,\phi,\psi)\left(V_{\epsilon,\delta}^{2H}(r)-\alpha_H r^{2H-2}\right)\mathrm{d}r\mathrm{d}\theta\right|$$
$$+\left|\int_0^t\int_0^{4\epsilon}\widehat{Q}(\theta,\theta-r,\phi,\psi)\left(V_{\epsilon,\delta}^{2H}(r)-\alpha_H r^{2H-2}\right)\mathrm{d}r\mathrm{d}\theta\right|$$
$$=: I_{1,1}+I_{1,2}.$$

Notice that for $r>0$,

$$\lim_{\epsilon,\delta\to 0}V_{\epsilon,\delta}^{2H}(r)=\alpha_H r^{2H-2}.$$

Because $H+\alpha\kappa>1/2$, by Lemma 2.5 and (2.14), we can apply dominated convergence theorem to see that

$$\lim_{\epsilon,\delta\to 0} I_{1,1} = 0.$$

As for $I_{1,2}$, we see that

$$\begin{aligned}
I_{1,2} &= \left| \int_0^t \mathrm{d}\theta \int_0^{4\epsilon} \widehat{Q}(\theta, \theta - r, \phi, \psi) \left(V_{\epsilon,\delta}^{2H}(r) - \alpha_H r^{2H-2} \right) \mathrm{d}r \right| \\
&\leq \frac{C}{\epsilon} \int_0^t \mathrm{d}\theta \int_0^{4\epsilon} \mathrm{d}r \, \left| \widehat{Q}(\theta, \theta - r, \phi, \psi) \right| \\
&\quad \times \int_{-1}^1 \mathrm{d}y \, \left[|r - \epsilon + \delta y|^{2H-1} + |r + \epsilon + \delta y|^{2H-1} + \epsilon r^{2H-2} \right].
\end{aligned}$$

Then by (2.14),

$$\begin{aligned}
I_{1,2} \leq & \frac{C \, ||\phi||_\kappa^\alpha \, ||\psi||_\kappa^\alpha \, t}{\epsilon} \int_0^{4\epsilon} \mathrm{d}r \, r^{2\alpha\kappa} \int_{-1}^1 \mathrm{d}y \\
& \times \left[|r - \epsilon + \delta y|^{2H-1} + |r + \epsilon + \delta y|^{2H-1} + \epsilon r^{2H-2} \right] \\
\leq & C \, ||\phi||_\kappa^\alpha \, ||\psi||_\kappa^\alpha \, t \, \epsilon^{2\alpha\kappa - 1} \int_{-1}^1 \mathrm{d}y \int_0^{4\epsilon} \mathrm{d}r \\
& \times \left[|r - \epsilon + \delta y|^{2H-1} + |r + \epsilon + \delta y|^{2H-1} + \epsilon r^{2H-2} \right].
\end{aligned}$$

Because $\epsilon > \delta > 0$ and $y \in [-1, 1]$, we have that

$$\begin{aligned}
\int_0^{4\epsilon} \mathrm{d}r \left[|r + \epsilon + \delta y|^{2H-1} + \epsilon r^{2H-2} \right] &= \int_0^{4\epsilon} \mathrm{d}r \left[(r + \epsilon + \delta y)^{2H-1} + \epsilon r^{2H-2} \right] \\
&\leq C[(5\epsilon + \delta y)^{2H} + \epsilon^{2H}] \leq C \, \epsilon^{2H};
\end{aligned}$$

and because $\epsilon - \delta y \in [0, 4\epsilon]$, we see that

$$\begin{aligned}
\int_0^{4\epsilon} \mathrm{d}r |r - \epsilon + \delta y|^{2H-1} &= \int_0^{\epsilon - \delta y} \mathrm{d}r \, (\epsilon - \delta y - r)^{2H-1} + \int_{\epsilon - \delta y}^{4\epsilon} \mathrm{d}r \, (r - \epsilon + \delta y)^{2H-1} \\
&= \frac{1}{2H} \left[(\epsilon - \delta y)^{2H} + (3\epsilon + \delta y)^{2H} \right] \leq C \, \epsilon^{2H}.
\end{aligned}$$

Hence,

$$I_{1,2} \leq C \, ||\phi||_\kappa^\alpha \, ||\psi||_\kappa^\alpha \; t \, \epsilon^{2\alpha\kappa - 1 + 2H}.$$

Therefore, the condition $\alpha\kappa + H \geq 1/2$ implies

$$\lim_{\epsilon,\delta\to 0} I_{1,2} = 0.$$

Step 2. Now we prove (2.15) for I_2 and I_3. The case for I_2 is true due to Lemma 2.6. As for I_3, notice that

$$\begin{aligned}\int_0^t \mathrm{d}\theta \int_0^\theta \mathrm{d}r\ Q(\phi_{\theta-r},\psi_{\theta-r})V_{\epsilon,\delta}^{2H}(r) &= \int_0^t \mathrm{d}r\ V_{\epsilon,\delta}^{2H}(r)\int_r^t \mathrm{d}\theta\ Q(\phi_{\theta-r},\psi_{\theta-r})\\ &= \int_0^t \mathrm{d}r\ V_{\epsilon,\delta}^{2H}(r)\int_0^{t-r} \mathrm{d}s\ Q(\phi_s,\psi_s)\\ &= \int_0^t \mathrm{d}s\ Q(\phi_s,\psi_s)\int_0^{t-s} \mathrm{d}r\ V_{\epsilon,\delta}^{2H}(r)\\ &= \int_0^t \mathrm{d}\theta\ Q(\phi_{t-\theta},\psi_{t-\theta})\int_0^\theta \mathrm{d}r\ V_{\epsilon,\delta}^{2H}(r).\end{aligned}$$

Hence, one can apply Lemma 2.6 to prove (2.15) for I_3. This completes the proof of Theorem 2.2. ◻

Proof of Proposition 2.4 We only need to prove that

$$\mathbb{E}\left[\left(\int_0^t \dot{W}^\epsilon(\mathrm{d}r,\phi_r)-\int_0^s \dot{W}^\epsilon(\mathrm{d}r,\phi_r)\right)^2\right] \le C'\left(1+||\phi||_\infty\right)^{2\alpha}(t-s)^{2H} + C''\,||\phi||_\kappa^{2\alpha}\,(t-s)^{2(H+\alpha\kappa)}. \quad (2.17)$$

Then (2.8) follows from (2.17), Theorem 2.2, and Fatou's lemma. By the arguments in the proof of [11, Proposition 3.6] and by denoting $\hat{\phi}_t=\phi_{t+s}$, we see that

$$\begin{aligned}\mathbb{E}\left[\left(\int_0^t \dot{W}^\epsilon(\mathrm{d}r,\phi_r)-\int_0^s \dot{W}^\epsilon(\mathrm{d}r,\phi_r)\right)^2\right] &= \int_0^{t-s}\mathrm{d}\theta\int_0^\theta \mathrm{d}r\ Q(\phi_{s+\theta},\phi_{s+\theta-r})V_{\epsilon,\epsilon}^{2H}(r)\\ &= \int_0^{t-s}\mathrm{d}\theta\int_0^\theta \mathrm{d}r\ Q(\hat{\phi}_\theta,\hat{\phi}_{\theta-r})V_{\epsilon,\epsilon}^{2H}(r)\\ &= \int_0^{t-s}\int_0^{t-s} Q(\hat{\phi}_u,\hat{\phi}_v)V_{\epsilon,\epsilon}^{2H}(u-v)\mathrm{d}u\mathrm{d}v\\ &= \mathbb{E}\left[I_{t-s,\epsilon}^2(\hat{\phi})\right],\end{aligned}$$

where the last equality is due to (2.9). Finally, after passing to the limit using (2.16) and then applying the bound in (2.4), we complete the proof of Proposition 2.4. ◻

3 Feynman–Kac formula and upper bound of moments

In this section, we will establish the Feynman–Kac representation of the solution to (1.1) and obtain a upper bound of its moments.

3.1 Feynman–Kac integral and its moment bound

The goal of this part is to prove the upper bound in (1.3).

Theorem 3.1 *Suppose that Q satisfies condition (H1) with $2H+\alpha>1$ and u_0 is bounded. Then for all $t>0$ and $x\in\mathbb{R}^d$, the random variable $\int_0^t W(\mathrm{d}s, B_{t-s}^x)$ is exponentially integrable and the random field $u(t,x)$ given by (1.2) is in $L^p(\Omega)$ for all $p\geq 1$. Moreover, for some constants $C=C(d,H,\alpha,||u_0||_\infty)>0$ and $C_x=C_x(d,H,\alpha,||u_0||_\infty,x)>0$,*

$$\mathbb{E}\left[|u(t,x)|^k\right]\leq C_x\exp\left(Ck^{\frac{2-\alpha}{1-\alpha}}t^{\frac{2H+\alpha}{1-\alpha}}\right),$$

for all $t\geq 1$, $x\in\mathbb{R}^d$ and $k\in\mathbb{N}$.

Notice that since the trajectories of the Brownian motion $\{B_t^x, t\geq 0\}$ are Hölder continuous of order κ for any $\kappa<1/2$, the stochastic integral $\int_0^t W\left(\mathrm{d}s, B_{t-s}^x\right)$ is well defined due to Theorem 2.2. We first prove some lemmas.

Lemma 3.2 *Suppose that $\alpha\in(0,1]$ and $2H+\alpha>1$. Let*

$$U=\int_0^1\int_0^1|B_u-B_v|^{2\alpha}|u-v|^{2H-2}\mathrm{d}u\mathrm{d}v, \tag{3.1}$$

where B_t is a standard Brownian motion on $\mathbb{R}^d$. Then for some constant $C_{\alpha,d,H}>0$,

$$\mathbb{E}\left[e^{\lambda U}\right]\leq C_{\alpha,d,H}\exp\left(C_{\alpha,d,H}\lambda^{\frac{1}{1-\alpha}}\right),\quad \text{for all } \lambda\geq 0.$$

Proof Notice that

$$\begin{aligned}\mathbb{E}\left[|B_u-B_v|^{2\alpha n}\right]&=\mathbb{E}\left[|B_{u-v}|^{2\alpha n}\right]=|u-v|^{\alpha n}\mathbb{E}\left[|B_1|^{2\alpha n}\right]\\&=C_d2^{\alpha n}\Gamma(d/2+n\alpha)|u-v|^{\alpha n}.\end{aligned}$$

By Minkovski's inequality,

$$\begin{aligned}\mathbb{E}[U^n]&\leq\left[\int_0^1\int_0^1\mathbb{E}\left[|B_u-B_v|^{2\alpha n}\right]^{1/n}|u-v|^{2H-2}\mathrm{d}u\mathrm{d}v\right]^n\\&=\left[\int_0^1\int_0^1\left[C_d2^{\alpha n}|u-v|^{\alpha n}\Gamma(d/2+\alpha n)\right]^{1/n}|u-v|^{2H-2}\mathrm{d}u\mathrm{d}v\right]^n\\&=C_d2^{\alpha n}\Upsilon^n\Gamma(d/2+\alpha n),\end{aligned}$$

where

$$\Upsilon:=\int_0^1\int_0^1|u-v|^{2H-2+\alpha}\mathrm{d}u\mathrm{d}v=\frac{2}{(2H+\alpha-1)(2H+\alpha)}.$$

By Lemma 4.4, we see that for some constant $C_{\alpha,d}>0$,

$$\frac{\Gamma(d/2+\alpha n)}{n!}\leq\frac{C_{\alpha,d}}{\Gamma((3-d)/2+(1-\alpha)n)},\quad \text{for all } n\in\mathbb{N}.$$

Notice that $d \geq 1$ implies that $(3-d)/2 \leq 1$. Hence, by Lemma 4.6, we see that for some constant $C_{\alpha,d,H} \geq 1$,

$$
\begin{aligned}
\mathbb{E}[e^{\lambda U}] &= \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}[U^n] \leq \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} C_d \, 2^{\alpha n} \Upsilon^n \Gamma(d/2 + \alpha n) \\
&\leq C_{\alpha,d} \sum_{n=0}^{\infty} \frac{\lambda^n 2^{\alpha n} \Upsilon^n}{\Gamma((3-d)/2 + (1-\alpha)n)} \\
&\leq C_{\alpha,d,H} \exp\left(C_{\alpha,d,H} \lambda^{\frac{1}{1-\alpha}} [2^\alpha \Upsilon]^{\frac{1}{1-\alpha}} \right),
\end{aligned}
$$

for $\lambda \geq 0$. This proves Lemma 3.2. $\square$

Lemma 3.3 *Let B_t be a standard Brownian motion on $\mathbb{R}^d$ and $W = \sup_{s \in [0,1]} |B_s|$. Then for all $\alpha \in [0,1)$, there exists some constant $C_{\alpha,d} > 0$ such that*

$$
\mathbb{E}\left[e^{\lambda (1+W)^{2\alpha}} \right] \leq C_{\alpha,d} \exp\left(C_{\alpha,d} \, \lambda^{\frac{1}{1-\alpha}} \right), \quad \textit{for all } \lambda \geq 0.
$$

Proof By Fernique's theorem (see, e.g., [8, Theorem 4.14]), for some $\beta_d > 0$ it holds that

$$
\mathbb{E} \exp\left(\beta W^2 \right) < \infty \quad \text{for all } \beta < \beta_d.
$$

Apply the inequality $ab \leq p^{-1} a^p + q^{-1} b^q$ where $a, b \geq 0$ and $1/p + 1/q = 1$ to see that

$$
\mathbb{E}\left[e^{\lambda (1+W)^{2\alpha}} \right] \leq \mathbb{E}\left[e^{p^{-1} \left(\frac{\lambda}{a}\right)^p + q^{-1} a^q (1+W)^{2\alpha q}} \right].
$$

Then the lemma is proved by choosing $q = 1/\alpha$, $p = 1/(1-\alpha)$ and a sufficiently small such that $q^{-1} a^q < \beta_d$. $\square$

Proof of Theorem 3.1 Let $u(t,x)$ be the random field given by (1.2). Without of loss of generality, we may assume that $u_0(x) \equiv 1$. Notice that

$$
\begin{aligned}
\mathbb{E}\left[u(t,x)^k \right] &= \mathbb{E}^W \mathbb{E}^B \exp\left\{ \sum_{j=1}^{k} \int_0^t W\left(\mathrm{d}s, B_{t-s}^{j,x} \right) \right\} \\
&= \mathbb{E}^B \exp\left\{ \frac{1}{2} \mathbb{E}^W \left[\left| \sum_{j=1}^{k} \int_0^t W\left(\mathrm{d}s, B_{t-s}^{j,x} \right) \right|^2 \right] \right\} \\
&= \mathbb{E}^B \exp\left\{ \frac{1}{2} \sum_{i,j=1}^{k} \mathbb{E}^W \left[\int_0^t W\left(\mathrm{d}s, B_{t-s}^{i,x} \right) \int_0^t W\left(\mathrm{d}s, B_{t-s}^{j,x} \right) \right] \right\},
\end{aligned}
\tag{3.2}
$$

where $\{B_t^{j,x}, t \geq 0\}$, $1 \leq j \leq k$, are independent Brownian motions on $\mathbb{R}^d$ starting from x. By (2.3),

$$\mathbb{E}\left[u(t,x)^k\right] \leq \mathbb{E}^B \exp\left\{\sum_{i,j=1}^k \frac{C_0|\alpha_H|}{4}\int_0^t\int_0^t \left|B_u^{i,x}-B_v^{i,x}\right|^\alpha \right. \\ \times \left|B_u^{j,x}-B_v^{j,x}\right|^\alpha |u-v|^{2H-2}\mathrm{d}u\mathrm{d}v \\ \left. +\frac{H}{2}\sum_{i,j=1}^k\int_0^t \theta^{2H-1}\left[Q\left(B_\theta^{i,x},B_\theta^{j,x}\right)+Q\left(B_{t-\theta}^{i,x},B_{t-\theta}^{j,x}\right)\right]\mathrm{d}\theta\right\}.$$

Then by Cauchy–Schwarz inequality,

$$\mathbb{E}\left[u(t,x)^k\right] \leq \mathbb{E}^B\left[\exp\left\{\sum_{i,j=1}^k \frac{C_0|\alpha_H|}{2}\int_0^t\int_0^t \left|B_u^{i,x}-B_v^{i,x}\right|^\alpha \left|B_u^{j,x}\right.\right.\right. \\ \left.\left.\left. -B_v^{j,x}\right|^\alpha |u-v|^{2H-2}\mathrm{d}u\mathrm{d}v\right\}\right]^{1/2} \\ \times \mathbb{E}^B\left[\exp\left\{H\sum_{i,j=1}^k\int_0^t \theta^{2H-1}\left[Q\left(B_\theta^{i,x},B_\theta^{j,x}\right)\right.\right.\right. \\ \left.\left.\left. +Q\left(B_{t-\theta}^{i,x},B_{t-\theta}^{j,x}\right)\right]\mathrm{d}\theta\right\}\right]^{1/2} \\ =: \left(\mathbb{E}^B\left[I_1\right]\mathbb{E}^B\left[I_2\right]\right)^{1/2}.$$

Step 1. We first consider $\mathbb{E}^B[I_1]$:

$$\mathbb{E}^B[I_1] \leq \mathbb{E}^B\exp\left\{\sum_{i,j=1}^k\frac{C_0|\alpha_H|}{4}\int_0^t\int_0^t\left[\left|B_u^{i,x}-B_v^{i,x}\right|^{2\alpha}+\left|B_u^{j,x}-B_v^{j,x}\right|^{2\alpha}\right]|u \right. \\ \left. -v|^{2H-2}\mathrm{d}u\mathrm{d}v\right\} \\ = \mathbb{E}^B\exp\left\{2^{-1}C_0k\sum_{i=1}^k|\alpha_H|\int_0^t\int_0^t\left|B_u^{i,x}-B_v^{i,x}\right|^{2\alpha}|u-v|^{2H-2}\mathrm{d}u\mathrm{d}v\right\} \\ = \left[\mathbb{E}^B\exp\left(2^{-1}C_0k|\alpha_H|\int_0^t\int_0^t|B_u-B_v|^{2\alpha}|u-v|^{2H-2}\mathrm{d}u\mathrm{d}v\right)\right]^k,$$

where $\{B_t, t \geq 0\}$ in the last line is a standard Brownian montion on $\mathbb{R}^d$. By change of variables $u = tu'$ and $v = tv'$ and by the scaling property of Brownian motions,

$$\int_0^t \int_0^t |B_u - B_v|^{2\alpha} |u - v|^{2H-2} \mathrm{d}u\mathrm{d}v = t^{2H} \int_0^1 \int_0^1 |B_{tu'} - B_{tv'}|^{2\alpha} |u' - v'|^{2H-2} \mathrm{d}u'\mathrm{d}v'$$
$$\overset{\text{in law}}{=} t^{2H+\alpha} \int_0^1 \int_0^1 |B_{u'} - B_{v'}|^{2\alpha} |u' - v'|^{2H-2} \mathrm{d}u'\mathrm{d}v'.$$

Hence,

$$\mathbb{E}^B \exp\left(2^{-1} C_0 k |\alpha_H| \int_0^t \int_0^t |B_u - B_v|^{2\alpha} |u - v|^{2H-2} \mathrm{d}u\mathrm{d}v\right)$$
$$= \mathbb{E}^B \exp\left(2^{-1} C_0 k |\alpha_H|\, t^{2H+\alpha} U\right),$$

where U is defined in (3.1). Then apply Lemma 3.2 to $\mathbb{E}^B \exp\left(2^{-1} C_0 k |\alpha_H|\, t^{2H+\alpha} U\right)$ to see that for some constant $C_{\alpha,d,H} > 0$,

$$\mathbb{E}^B [I_1]^{1/2} \leq C_{\alpha,d,H} \exp\left(C_{\alpha,d,H} k^{\frac{2-\alpha}{1-\alpha}} t^{\frac{2H+\alpha}{1-\alpha}}\right), \quad \text{for all } t \geq 0.$$

Step 2. Now we study $E^B[I_2]$. Set $\left|\left|B^{i,x}\right|\right|_{\infty,t} = \sup_{0 \leq s \leq t} |B_s^{i,x}|$. By condition (H3),

$$\mathbb{E}[I_2] \leq \mathbb{E}^B \exp\left(HC_1 \sum_{i,j=1}^k \left[\left(1 + \left|\left|B^{i,,x}\right|\right|_{\infty,t}\right)^{2\alpha}\right.\right.$$
$$\left.\left. + \left(1 + \left|\left|B^{j,x}\right|\right|_{\infty,t}\right)^{2\alpha}\right] \int_0^t \theta^{2H-1} \mathrm{d}\theta\right)$$
$$\leq \mathbb{E}^B \exp\left(C_1 k\, t^{2H} \sum_{i=1}^k \left(1 + \left|\left|B^{i,x}\right|\right|_{\infty,t}\right)^{2\alpha}\right)$$
$$= \left[\mathbb{E}^B \exp\left(C_1 k\, t^{2H} \left(1 + |x| + ||B||_{\infty,t}\right)^{2\alpha}\right)\right]^k,$$

where $\{B_t, t \geq 0\}$ in the last line is a standard Brownian motion on $\mathbb{R}^d$. By the scaling property and $t \geq 1$, we see that

$$\mathbb{E}^B \exp\left(C_1 k\, t^{2H} \left(1 + |x| + ||B||_{\infty,t}\right)^{2\alpha}\right)$$
$$\leq \mathbb{E}^B \exp\left(C_1 k\, t^{2H+\alpha} \left(1 + |x| + ||B||_{\infty,1}\right)^{2\alpha}\right)$$
$$\leq \exp\left(C_1' k\, t^{2H+\alpha} |x|^{2\alpha}\right) \mathbb{E}^B \exp\left(C_1' k\, t^{2H+\alpha} \left(1 + ||B||_{\infty,1}\right)^{2\alpha}\right).$$

By Lemma 3.3 with $\lambda = C_1' k t^{2H+\alpha}$, we have that

$$\mathbb{E}^B \exp\left(C_1' k\, t^{2H+\alpha} \left(1 + ||B||_{\infty,1}\right)^{2\alpha}\right) \leq C_{\alpha,d} \exp\left(C_{\alpha,d} k^{\frac{1}{1-\alpha}} t^{\frac{2H+\alpha}{1-\alpha}}\right).$$

Then, by the fact that $t \geq 1$ and $k \geq 1$, we see that

$$\begin{aligned} C_{\alpha,d} \exp\left(C_{\alpha,d} k^{\frac{1}{1-\alpha}} t^{\frac{2H+\alpha}{1-\alpha}}\right) &= \exp\left(C_{\alpha,d} k^{\frac{1}{1-\alpha}} t^{\frac{2H+\alpha}{1-\alpha}} + \log C_{\alpha,d}\right) \\ &\leq \exp\left(C'_{\alpha,d} k^{\frac{1}{1-\alpha}} t^{\frac{2H+\alpha}{1-\alpha}}\right), \end{aligned}$$

where one can choose $C'_{\alpha,d} = C_{\alpha,d} + \log\left(C_{\alpha,d} \vee 1\right)$. Therefore,

$$\mathbb{E}[I_2] \leq \exp\left(C'_1 k^2\, t^{2H+\alpha} |x|^{2\alpha}\right) \exp\left(C'_{\alpha,d} k^{\frac{2-\alpha}{1-\alpha}} t^{\frac{2H+\alpha}{1-\alpha}}\right).$$

By the inequality $ab \leq p^{-1}a^p + q^{-1}b^q$ with $a = k^2 t^{2H+\alpha}$, $b = |x|^{2\alpha}$, $p = \frac{2-\alpha}{2(1-\alpha)}$ and $q = \frac{2-\alpha}{\alpha}$, we see that

$$\begin{aligned} \exp\left(C'_1 k^2\, t^{2H+\alpha} |x|^{2\alpha}\right) &\leq \exp\left(C''_1 k^{\frac{2-\alpha}{1-\alpha}}\, t^{\frac{(2H+\alpha)(2-\alpha)}{2(1-\alpha)}} + C'''_1 |x|^{2(2-\alpha)}\right) \\ &\leq \exp\left(C'''_1 |x|^{2(2-\alpha)}\right) \exp\left(C''_1 k^{\frac{2-\alpha}{1-\alpha}}\, t^{\frac{2H+\alpha}{1-\alpha}}\right) \\ &=: C'_{\alpha,d,x} \exp\left(C''_1 k^{\frac{2-\alpha}{1-\alpha}}\, t^{\frac{2H+\alpha}{1-\alpha}}\right), \end{aligned}$$

where the second inequality is due to $t \geq 1$. Therefore, for some constants $C_{\alpha,d,x} > 0$ and $C_{\alpha,d} > 0$,

$$\mathbb{E}^B[I_2]^{1/2} \leq C_{\alpha,d,x} \exp\left(C_{\alpha,d} k^{\frac{2-\alpha}{1-\alpha}}\, t^{\frac{2H+\alpha}{1-\alpha}}\right) \qquad \text{for all } t \geq 1.$$

Finally, Theorem 3.1 is proved by combining the results in the above two steps. $\square$

3.2 Validation of the Feynman–Kac formula

In this part we will show that $u(t,x)$ is a weak solution to (1.1).

Definition 3.4 Given a random field $v = \{v(t,x),\ t \geq 0,\ x \in \mathbb{R}^d\}$ such that

$$\int_0^t \int_{\mathbb{R}^d} |v(s,x)| \mathrm{d}x \mathrm{d}s < \infty \quad \text{a.s. for all } t > 0,$$

the *Stratonovich integral* is defined as the following limit in probability if it exists

$$\lim_{\epsilon \to 0} \int_0^t \int_{\mathbb{R}^d} v(s,x) \dot{W}^{\epsilon}(s,x) \mathrm{d}s \mathrm{d}x,$$

where $\dot{W}^{\epsilon}(t,x)$ is defined in (2.1).

Definition 3.5 A random field $u = \{u(t,x),\ t \geq 0, x \in \mathbb{R}^d\}$ is a *weak solution* to (1.1) if for any $\phi \in C_0^\infty(\mathbb{R}^d)$, we have that

$$\begin{aligned}\int_{\mathbb{R}^d} [u(t,x) - u_0(x)]\,\phi(x)\mathrm{d}x = &\int_0^t \int_{\mathbb{R}^d} u(s,x)\Delta\phi(x)\mathrm{d}x\mathrm{d}s \\ &+ \int_0^t \int_{\mathbb{R}^d} u(s,x)\phi(x)W(\mathrm{d}s,x)\mathrm{d}x,\end{aligned} \tag{3.3}$$

almost surely, for all $t > 0$, where the last term is a Stratonovich stochastic integral defined in (3.4).

Theorem 3.6 *Suppose that Q satisfies condition (H1) with $2H + \alpha > 1$ and u_0 is a bounded measurable function. Let $u(t,x)$ be the random field defined in (1.2). Then for any $\phi \in C_0^\infty(\mathbb{R}^d)$, $u(t,x)\phi(x)$ is Stratonovich integrable and $u(t,x)$ is a weak solution to (1.1) in the sense of Definition 3.5.*

With Theorems 2.2 and 3.1, and Proposition 2.4, the proof of Theorem 3.6 follows exactly the same arguments as those of Theorem 5.3 in [11]. We will not repeat the proofs and instead leave them to interested readers.

4 Lower bounds of moments

In this section, we prove the lower bound in (1.3).

Theorem 4.1 *Suppose that Q satisfies condition (H1) with $2H + \alpha > 1$ and $\inf_{x\in\mathbb{R}^d} u_0(x) > 0$. If Q satisfies condition (H2) as well for some $\beta \in [0,1)$, then there exist some constants $C = C(d, H, \alpha, \beta, \inf_{x\in\mathbb{R}^d} u_0(x)) > 0$ and $C_x = C_x(d, H, \alpha, \beta, \inf_{x\in\mathbb{R}^d} u_0(x), x) > 0$, such that for all $t \geq 1$, $x \in \mathbb{R}^d$ and $k \in \mathbb{N}$,*

$$\mathbb{E}\left[u(t,x)^k\right] \geq C_x \exp\left(C k^{\frac{2-\beta}{1-\beta}} t^{\frac{2H+\beta}{1-\beta}}\right).$$

We first remark that if the initial data is $u_0(x) \equiv 1$, then from (3.2) and (2.6), we see that

$$\begin{aligned}\mathbb{E}\left[u(t,x)^k\right] = \mathbb{E}^B \exp\Bigg\{ &\frac{|\alpha_H|}{2} \sum_{i,j=1}^k \int_0^t \int_0^t |u-v|^{2H-2} \widehat{Q}\left(u, v, B^{i,x}, B^{j,x}\right) \mathrm{d}u\mathrm{d}v \\ &+ H \sum_{i,j=1}^k \int_0^t Q\left(B_s^{i,x}, B_s^{i,x}\right)\left[s^{2H-1} + (t-s)^{2H-1}\right]\mathrm{d}s \Bigg\}.\end{aligned}$$

Since the sign of $\widehat{Q}$ can be either positive or negative, it is hard to find a lower bound starting from the above formula. Instead, we will introduce another Gaussian field Y as in Lemma 4.2 below.

Now we need some notation. Fix $a > 0$. Let $\kappa = H - 1/2$. As is proved in [18], the space

$$\mathcal{H}_a = \left\{ f : \exists \phi_f \in L^2(0,a) \text{ such that } f(u) = u^{-\kappa} \left(I_{a-}^{-\kappa} \phi_f(s) \right)(u) \right\} \tag{4.1}$$

with the inner product

$$\langle f, g \rangle_{\mathcal{H}_a} = \frac{\pi\kappa(2\kappa+1)}{\Gamma(1-2\kappa)\sin(\pi\kappa)} \int_0^a s^{-2\kappa} \left(I_{a-}^{\kappa} u^{\kappa} f(u) \right)(s) \left(I_{a-}^{\kappa} u^{\kappa} g(u) \right)(s) \, \mathrm{d}s$$

is a Hilbert space, where I_{a-}^{κ} with $\kappa < 0$ is the right-sided fractional derivative (see [18]). It is known that (see [16, p. 284])

$$C^{\gamma}([0,a]) \subset \mathcal{H}_a \subset L^2(0,a), \quad \text{for all } \gamma > 1/2 - H. \tag{4.2}$$

Lemma 4.2 *There exist a Gaussian process $Y = \{Y(x), x \in \mathbb{R}^d\}$ and an independent fractional Brownian motion $\{\widehat{B}_t, t \in \mathbb{R}\}$ with Hurst parameter H, such that*

(a) *For all $x, y \in \mathbb{R}^d$, $\mathbb{E}[Y(x)Y(y)] = Q(x,y)$.*
(b) *For all $0 < t \leq T$ and $\phi \in C^{\kappa}([0,T])$ with $\alpha\kappa + H > 1/2$, the integral $\int_0^t Y(\phi_s)\mathrm{d}\widehat{B}_s$ is a well-defined Wiener integral for each realization of Y. Moreover,*

$$\int_0^t Y(\phi_s)\mathrm{d}\widehat{B}_s = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_0^t Y(\phi_s) \left(\widehat{B}_{s+\epsilon} - \widehat{B}_{s-\epsilon} \right) \mathrm{d}s, \quad \text{in } L^2(\Omega). \tag{4.3}$$

(c) *For all $0 < t \leq T$ and $\phi, \psi \in C^{\kappa}([0,T])$ with $\alpha\kappa + H > 1/2$,*

$$\mathbb{E}^W \left[\int_0^t W(\mathrm{d}s, \phi_s) \int_0^t W(\mathrm{d}s, \psi_s) \right] = \mathbb{E}^{Y,\widehat{B}} \left[\int_0^t Y(\phi_s)\mathrm{d}\widehat{B}_s \int_0^t Y(\psi_s)\mathrm{d}\widehat{B}_s \right]. \tag{4.4}$$

Proof Since Q is a covariance function, one can find such a Gaussian process Y such that part (a) holds. As for (b), by (H1), we see that

$$\begin{aligned} \mathbb{E}\left[|Y(\phi_t) - Y(\phi_s)|^p \right] &\leq C_p \mathbb{E}\left[|Y(\phi_t) - Y(\phi_s)|^2 \right]^{p/2} \\ &\leq C'_p \, |\phi_t - \phi_s|^{\alpha p} \leq C'_p \, \|\phi\|_{\kappa} \, |t-s|^{\alpha\kappa p}. \end{aligned} \tag{4.5}$$

Hence, $t \mapsto Y(\phi_t)$ is γ-Hölder continuous for all $\gamma < \alpha\kappa$. Since $\alpha\kappa > 1/2 - H$, one can find γ' such that $1/2 - H < \gamma' < \alpha\kappa$. Because Y and $\widehat{B}$ are independent, for each realization of Y, the integral $\int_0^t Y(\phi_s)\mathrm{d}\widehat{B}_s$ is actually a Wiener integral. By (4.2), we see that the integral $\int_0^t Y(\phi_s)\mathrm{d}\widehat{B}_s$ is a well-defined Wiener integral for each realization of Y.

As for (4.3), denote

$$I_{t,\epsilon}(\phi) := \frac{1}{2\epsilon} \int_0^t Y(\phi_s) \left(\widehat{B}_{s+\epsilon} - \widehat{B}_{s-\epsilon} \right) \mathrm{d}s.$$

Then by the same arguments as in the proof of Theorem 2.2, one can show that $I_{t,\epsilon_n}(\phi)$ is a Cauchy sequence in $L^2(\Omega)$ for every sequence $\epsilon_n \downarrow 0$. We omit the details of this proof. Denote the limit, which does not depend on the sequence, by $I_t(\phi)$. In order to show that $I_t(\phi)$ equals to the left-hand side of (4.3), it suffices to show that for any $t_0 \in [0, t]$ and any bounded random variable Z measurable with respect to the process Y, we have that

$$\mathbb{E}\left[\widehat{B}_{t_0} Z \int_0^t Y(\phi_s)\mathrm{d}\widehat{B}_s\right] = \mathbb{E}\left[\widehat{B}_{t_0} Z I_t(\phi)\right]. \tag{4.6}$$

For the right-hand side of (4.6), we can write

$$\begin{aligned}\mathbb{E}\left[\widehat{B}_{t_0} Z I_t(\phi)\right] &= \lim_{\epsilon\to 0}\frac{1}{2\epsilon}\int_0^t \mathbb{E}[ZY(\phi_s)](R_H(t_0, s+\epsilon) - R_H(t_0, s-\epsilon))\mathrm{d}s \\ &= 2H\int_0^t \mathbb{E}[ZY(\phi_s)](s^{2H-1} + |t_0 - s|^{2H-1}\mathrm{sign}(t_0 - s))\mathrm{d}s. \end{aligned}\tag{4.7}$$

On the other hand, by Fubini's theorem, the left-hand side of (4.6) equals to

$$\mathbb{E}^{\widehat{B}}\left[\widehat{B}_{t_0}\int_0^t \mathbb{E}^Y[ZY(\phi_s)]\mathrm{d}\widehat{B}_s\right],$$

which coincides with (4.7), due to the properties of stochastic Y-integrals. In fact, this property holds when $\mathbb{E}[ZY(\phi_s)]$ is a step function and it holds for any element in space $\mathcal{H}_t$ (see 4.1) of integrable functions on $[0, t]$, because $\mathcal{H}_t$ is continuously embedded into $L^{1/H}(0, t)$.

As for (c), because Y and $\widehat{B}$ are independent, by (4.3), we see that

$$\begin{aligned}&\mathbb{E}^{Y,\widehat{B}}\left[\int_0^t Y(\phi_s)\mathrm{d}\widehat{B}_s \int_0^t Y(\psi_s)\mathrm{d}\widehat{B}_s\right] \\ &\quad= \lim_{\epsilon\to 0}\mathbb{E}^{Y,\widehat{B}}\left[\int_0^t Y(\phi_s)\frac{\widehat{B}_{s+\epsilon} - \widehat{B}_{s-\epsilon}}{2\epsilon}\mathrm{d}s \int_0^t Y(\psi_s)\frac{\widehat{B}_{s+\epsilon} - \widehat{B}_{s-\epsilon}}{2\epsilon}\mathrm{d}s\right] \\ &\quad= \lim_{\epsilon\to 0}\int_0^t\int_0^t Q(\phi_u, \psi_v)\, V_{\epsilon,\epsilon}^{2H}(u - v)\mathrm{d}u\mathrm{d}v, \end{aligned}\tag{4.8}$$

where $V_{\epsilon,\delta}^{2H}(\cdot)$ is defined in (2.10). The limit in (4.8) has been calculated in Theorem 2.2 and it is equal to the right-hand side of (2.2) or (2.6). This completes the proof of the lemma. □

Lemma 4.3 *Assume that* $\{\widehat{B}_s, s \geq 0\}$ *is a fractional Brownian motion with* $H \in (0, 1/2)$. *Then there exists a constant* $\theta := \theta(H, r)$ *such that for all* $a > 0$ *and all* $r > 0$, *it holds that*

$$\mathbb{E}\left(\left|\int_0^a f(s)\mathrm{d}\widehat{B}_s\right|^r\right) \geq \theta\, ||f||_{L^{1/H}(0,a)}^r \qquad \textit{for all } f \in \mathcal{H}_a. \tag{4.9}$$

Moreover, if $f(s)$ is a process with values in a separable Hilbert space V, one can view f as a two-parameter process: $f : [0,a] \times D \ni (s,\omega) \mapsto f(s,\omega) \in \mathbb{R}$. If $f(\cdot,\omega) \in \mathcal{H}_a$ for all $\omega \in D$, then,

$$\mathbb{E}\left(\left\|\int_0^a f(s)\mathrm{d}\widehat{B}_s\right\|_V^r\right) \geq \theta\left(\int_0^a \|f(s)\|_V^{1/H}\,\mathrm{d}s\right)^{rH}. \tag{4.10}$$

Proof Because $\int_0^a f(s)\mathrm{d}\widehat{B}_s$ is a centered Gaussian random variable, there exists a finite constant $C_r > 0$ such that

$$\mathbb{E}\left[\left|\int_0^a f(s)\mathrm{d}\widehat{B}_s\right|^r\right] \geq C_r\left(\mathbb{E}\left[\left|\int_0^a f(s)\mathrm{d}\widehat{B}_s\right|^2\right]\right)^{r/2}.$$

Hence, we only need to prove the case where $r = 2$.

We first note that (4.9) is proved in part (i) of Theorem 1.2 in [15] for all f that has bounded variation on $[0,a]$, and in particular, it holds for all simple functions. Now fix $f \in \mathcal{H}_a$. There exist simple functions f_n on $[0,a]$ such that $\|f - f_n\|_{\mathcal{H}_a} \to 0$ as $n \to 0$. Then

$$\begin{aligned}\mathbb{E}\left[\left(\int_0^a f(s)\mathrm{d}\widehat{B}_s\right)^2\right] &= \lim_{n\to\infty}\mathbb{E}\left[\left(\int_0^a f_n(s)\mathrm{d}\widehat{B}_s\right)^2\right]\\ &\geq \lim_{n\to\infty}\theta\left(\int_0^a |f_n(s)|^{1/H}\mathrm{d}s\right)^{2H}.\end{aligned} \tag{4.11}$$

Because (4.9) holds for simple functions, we see that

$$\|f_n - f_m\|_{L^{1/H}(0,a)} \leq \|f_n - f_m\|_{\mathcal{H}_a}.$$

Thus, $\{f_n\}_{n\geq 1}$ is a Cauchy sequence in $L^{1/H}(0,a)$. Hence, by passing to a subsequence when necessary, it implies that $f_n \to f$ almost everywhere. Therefore, (4.9) is proved by applying Fatou's lemma to the right-hand side of (4.11).

Now if $f(s)$ is a process with values in a separable Hilbert space V, let $\{e_i\}_{i\in\mathbb{N}}$ be a set of orthonormal basis of V. Since $f(\cdot,\omega) \in \mathcal{H}_a$ for all $\omega \in D$, we see that $\langle f(s,\cdot), e_i\rangle_V \in \mathcal{H}_a$. Hence, by (4.9),

$$\begin{aligned}\mathbb{E}\left(\left\|\int_0^a f(s)\mathrm{d}\widehat{B}_s\right\|_V^2\right) &= \mathbb{E}\left(\left\|\sum_{i=1}^\infty \int_0^a \langle f(s), e_i\rangle_V\,\mathrm{d}\widehat{B}_s\, e_i\right\|_V^2\right)\\ &= \sum_{i=1}^\infty \mathbb{E}\left(\left|\int_0^a \langle f(s), e_i\rangle_V\,\mathrm{d}\widehat{B}_s\right|^2\right)\\ &\geq \theta\sum_{i=1}^\infty\left(\int_0^a \left|\langle f(s), e_i\rangle_V^2\right|^{\frac{1}{2H}}\,\mathrm{d}s\right)^{2H}\end{aligned}$$

$$
\begin{aligned}
&= \theta \sum_{i=1}^{\infty} \left\| \langle f(s), e_i \rangle_V^2 \right\|_{L^{\frac{1}{2H}}(0,a)} \\
&\geq \theta \left\| \sum_{i=1}^{\infty} \langle f(s), e_i \rangle_V^2 \right\|_{L^{\frac{1}{2H}}(0,a)} \\
&= \theta \left\| \|f(s)\|_V^2 \right\|_{L^{\frac{1}{2H}}(0,a)},
\end{aligned}
$$

where we can apply Minkovski's inequality in the last inequality because $H \in (0, 1/2)$. This completes the proof of Lemma 4.3. $\square$

Proof of Theorem 4.1 Since u_0 is bounded below away from zero, we may assume that $u_0 \equiv 1$. From (3.2) and by Lemma 4.2, we see that

$$
\mathbb{E}\left[u(t,x)^k\right] = \mathbb{E}^B \exp\left\{ \mathbb{E}^{Y,\widehat{B}}\left[\left(\int_0^t \sum_{i=1}^k Y(B_{t-s}^{i,x}) \mathrm{d}\widehat{B}_s \right)^2 \right] \right\}.
$$

Then by (4.5), we see that $s \mapsto \sum_{i=1}^k Y(B_{t-s}^{i,x})$ is γ-Hölder continuous a.s. for all $\gamma < \alpha/2$. Since $\alpha/2 > 1/2 - H$, one can find γ' such that $1/2 - H < \gamma' < \alpha/2$. Hence, by (4.2), $s \mapsto \sum_{i=1}^k Y(B_{t-s}^{i,x})$ is in $\mathcal{H}_t$ for all realizations of Y. Therefore, by Lemma 4.3, for some constant $C_H' > 0$,

$$
\begin{aligned}
\mathbb{E}^{Y,\widehat{B}}\left[\left(\int_0^t \sum_{i=1}^k Y\left(B_{t-s}^{i,x}\right) \mathrm{d}\widehat{B}_s \right)^2 \right] &\geq C_H' \left(\int_0^t \mathbb{E}^Y \left[\left| \sum_{i=1}^k Y\left(B_s^{i,x}\right) \right|^2 \right]^{\frac{1}{2H}} \mathrm{d}s \right)^{2H} \\
&= C_H' I_t,
\end{aligned}
$$

where

$$
I_t := \left(\int_0^t \left[\sum_{i,j=1}^k Q\left(B_s^{i,x}, B_s^{j,x}\right) \right]^{\frac{1}{2H}} \mathrm{d}s \right)^{2H}.
$$

Then for any $M > 0$ (to be chosen later), by condition (H2) and by writing $B_s^{i,x} = (B_s^{i,x_1,1}, \ldots, B_s^{i,x_d,d})$,

$$
\begin{aligned}
\mathbb{E}\left[u(t,x)^k\right] &\geq \mathbb{E}^B \exp\left(C_H' I_t\right) \\
&\geq \mathbb{P}\left(\left| B_s^{i,x_j,j} \right| > M,\ \forall s \in [t/2, t], \quad \forall i = 1, \ldots, k,\ \forall j \right. \\
&\left. = 1, \ldots, d \right) \exp\left(C_H k^2 M^{2\beta} t^{2H} \right) \\
&\geq \mathbb{P}\left(\left| B_s^{1,y,1} \right| > M,\ \forall s \in [t/2, t] \right)^{kd} \exp\left(C_H k^2 M^{2\beta} t^{2H} \right),
\end{aligned}
$$

where $C_H = C'_H C_2$ and

$$y = \min_{i=1,\dots,d} |x_i|.$$

In the following, for simplicity, we use B_t to denote the one-dimensional standard Brownian motion starting from the origin. Hence,

$$\mathbb{E}\left[u(t,x)^k\right] \geq \mathbb{P}\left(|B_s + y| > M,\ \forall s \in [t/2, t]\right)^{kd} \exp\left(C_H k^2 M^{2\beta} t^{2H}\right).$$

Assume that $M \geq |y|$. Then

$$\begin{aligned}
&\mathbb{P}\left(|B_s + y| > M,\ \forall s \in [t/2, t]\right) \geq \mathbb{P}\left(|B_s| > 2M,\ \forall s \in [t/2, t]\right) \\
&\geq \mathbb{P}\left(|B_s| > \frac{2M}{\sqrt{t}},\ \forall s \in [1/2, 1]\right) \\
&\geq \mathbb{P}\left(|B_{1/2}| > \frac{4M}{\sqrt{t}},\ |B_s - B_{1/2}| < \frac{2M}{\sqrt{t}},\ \forall s \in [1/2, 1]\right) \\
&= \mathbb{P}\left(|B_{1/2}| > \frac{4M}{\sqrt{t}}\right) \mathbb{P}\left(\sup_{s\in[0,1/2]} |B_s| < \frac{2M}{\sqrt{t}}\right),
\end{aligned}$$

where we have used the scaling property of the Brownian motion. By a standard argument

$$\mathbb{P}\left(B_1 > r\right)^2 \geq \frac{1}{2\pi} \int_0^{\pi/2} \mathrm{d}\theta \int_{\sqrt{2}r}^{\infty} e^{-\frac{s^2}{2}} s \mathrm{d}s = \frac{1}{4} e^{-r^2}, \quad (r > 0)$$

we have that

$$\mathbb{P}\left(|B_{1/2}| > \frac{4M}{\sqrt{t}}\right) \geq \mathbb{P}\left(B_1 > \frac{4\sqrt{2}M}{\sqrt{t}}\right) \geq 2^{-1} \exp\left(-\frac{16M^2}{t}\right).$$

By Chebyshev's inequality and Fernique's theorem, for some $\lambda > 0$,

$$\mathbb{P}\left(\sup_{s\in[0,1/2]} |B_s| < \frac{2M}{\sqrt{t}}\right) = 1 - \mathbb{P}\left(\sup_{s\in[0,1/2]} |B_s| > \frac{2M}{\sqrt{t}}\right) \geq 1 - C_\lambda e^{-4\lambda M^2/t},$$

where $C_\lambda = \mathbb{E} \exp\left(\lambda \sup_{s\in[0,1/2]} |B_s|^2\right) < \infty$. Now assume that $M/\sqrt{t}$ is sufficiently large such that

$$\left(1 - C_\lambda e^{-4\lambda M^2/t}\right)^{kd} \geq 1/2. \tag{4.12}$$

Therefore, provided that $M \geq |y|$ and (4.12) is true, we have that

$$\mathbb{E}\left[u(t,x)^k\right] \geq 2^{-(kd+1)} \exp\left(C_H k^2 M^{2\beta} t^{2H} - \frac{16kM^2 d}{t}\right). \tag{4.13}$$

Now we maximize

$$f(M) = C_H k^2 M^{2\beta} t^{2H} - \frac{16kM^2 d}{t}, \quad \text{for } M \geq 0.$$

By solving $f'(M) = 0$, we see that f is maximized at

$$M_0 = \left((16d)^{-1} \beta\, k\, C_H t^{1+2H}\right)^{\frac{1}{2(1-\beta)}}$$

with

$$\sup_{M\geq 0} f(M) = f(M_0) = (16d)^{\frac{\beta}{\beta-1}} (1-\beta) \beta^{\frac{\beta}{1-\beta}} C_H^{\frac{1}{1-\beta}} k^{\frac{2-\beta}{1-\beta}} t^{\frac{\beta+2H}{1-\beta}}. \tag{4.14}$$

Now we consider three cases. In the first case, we fix an arbitrary $t \geq 1$. By replacing t in the expression of M_0 by 1, we see that there exists some $k_0(x) \in \mathbb{N}$ such that for all $k \geq k_0(x)$, both conditions (4.12) (with M replaced by M_0) and $M_0 \geq |y|$ are satisfied. Hence, the equality in (4.14) is valid. The second case is similar to the first one: Fix an arbitrary $k \in \mathbb{N}$. By replacing k in the expression of M_0 by 1, we see that for some $t_0(x) > 0$, both conditions (4.12) (with M replaced by M_0) and $M_0 \geq |y|$ are satisfied for all $t \geq t_0(x)$. Hence, the equality in (4.14) is valid. Finally, in the third case, we fix $t \in [1, t_0(x)]$ and $1 \leq k \leq k_0(x)$. Set

$$C := (16d)^{\frac{\beta}{\beta-1}} (1-\beta) \beta^{\frac{\beta}{1-\beta}} C_H^{\frac{1}{1-\beta}}.$$

By the arguments leading to (4.13) and replacing M by $|y|$, we see that

$$\begin{aligned}
\mathbb{E}\left[u(t,x)^k\right] &\geq 2^{-1} \left(1 - C_\lambda e^{-4\lambda y^2/t}\right)^{kd} \exp\left(C_H k^2 y^{2\beta} t^{2H} - \frac{16ky^2 d}{t}\right) \\
&\geq 2^{-1} \exp\left(C k^{\frac{2-\beta}{1-\beta}} t^{\frac{2H+\beta}{1-\beta}}\right) \\
&\quad \times \inf_{\substack{1\leq t\leq t_0(x) \\ 1\leq k\leq k_0(x)}} \left\{\left(1 - C_\lambda e^{-4\lambda y^2/t}\right)^{kd} \exp\left(C_H k^2 y^{2\beta} t^{2H} - \frac{16ky^2 d}{t}\right.\right. \\
&\quad \left.\left. - C k^{\frac{2-\beta}{1-\beta}} t^{\frac{2H+\beta}{1-\beta}}\right)\right\} =: C_x \exp\left(C k^{\frac{2-\beta}{1-\beta}} t^{\frac{2H+\beta}{1-\beta}}\right).
\end{aligned}$$

This completes the proof of Theorem 4.1. $\square$

Acknowledgements The authors wish to thank the referees for their useful suggestions to improve the paper.

Appendix

Lemma 4.4 *For all $a, b, u, v, w > 0$, if $u + v \leq w + 1/2$ and $w > 1/2$, then*

$$\sup_{n\in\mathbb{N}} \frac{\Gamma(an+u)\Gamma(bn+v)}{\Gamma((a+b)n+w)} < \infty.$$

Proof We only need to show that

$$\lim_{n\to\infty} \frac{\Gamma(an+u)\Gamma(bn+v)}{\Gamma((a+b)n+w)} < \infty.$$

By Stirling's formula (see [17, 5.11.3 or 5.11.7]), as n is large, we see that

$$\begin{aligned}\frac{\Gamma(an+u)\Gamma(bn+v)}{\Gamma((a+b)n+w)} \approx \sqrt{\pi} \exp\Big\{&(an+u-1/2)\log(an)\\ &+(bn+v-1/2)\log(bn)\\ &-((a+b)n+w-1/2)\log((a+b)n)\Big\}.\end{aligned}$$

Denote the right-hand side of the above quantity by I_n. By the supper-additivity of $f(x) = x\log x$, namely $f(x+y) \geq f(x) + f(y)$ for all $x, y \geq 0$, we see that

$$I_n \leq \sqrt{\pi} \exp\Big\{(u-1/2)\log(an) + (v-1/2)\log(bn) - (w-1/2)\log((a+b)n)\Big\}.$$

Because $w > 1/2$, we can apply the inequality $\log((a+b)n) \geq \frac{1}{2}[\log(an)+\log(bn)]$ to obtain that

$$\begin{aligned}I_n &\leq \sqrt{\pi} \exp\Big\{(u-1/4-w/2)\log a+(v-1/4-w/2)\log b+(u+v-w-1/2)\log n\Big\}\\ &= Cn^{u+v-w-1/2} \leq C, \quad \text{for all } n\in\mathbb{N},\end{aligned}$$

where the last inequality is due to the assumption that $u + v - w - 1/2 \leq 0$. □

Let $E_{\alpha,\beta}(z)$ be the *Mittag–Leffler function*

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)}, \quad \Re\alpha > 0,\ \beta\in\mathbb{C},\ z\in\mathbb{C}.$$

Lemma 4.5 (Theorem 1.3 p. 32 in [19]) *If $0 < \alpha < 2$, β is an arbitrary complex number and μ is an arbitrary real number such that*

$$\pi\alpha/2 < \mu < \pi \wedge (\pi\alpha),$$

then for an arbitrary integer $p \geq 1$ the following expression holds:

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha} z^{(1-\beta)/\alpha} \exp\left(z^{1/\alpha}\right) - \sum_{k=1}^{p} \frac{z^{-k}}{\Gamma(\beta - \alpha k)} + O\left(|z|^{-1-p}\right), \; |z| \to \infty, \; |\arg(z)| \leq \mu \, .$$

Lemma 4.6 *For all $\alpha > 0$ and $\beta \leq 1$, there exists some constant $C = C_{\alpha,\beta} \geq 1$ such that*

$$E_{\alpha,\beta}(z) \leq C \exp\left\{C z^{1/\alpha}\right\}, \quad \textit{for all } z \geq 0.$$

Proof By Lemma 4.5, we see that for some constants $C'_{\alpha,\beta} > 0$ and $C_{\alpha,\beta} \geq 1$,

$$E_{\alpha,\beta}(z) \leq C'_{\alpha,\beta} \left(1 + z^{(1-\beta)/\alpha} \exp\left(z^{1/\alpha}\right)\right) \leq C_{\alpha,\beta} \exp\left(C_{\alpha,\beta} z^{1/\alpha}\right),$$

for all $z \geq 0$. □

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