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Nonlinear stochastic time-fractional slow and fast diffusion equations on $\mathbb{R}^d$

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Abstract

This paper studies the nonlinear stochastic partial differential equation of fractional orders both in space and time variables:

$$\left(\partial^{\beta}+\frac{\nu}{2}(-\Delta)^{\alpha/2}\right)u(t,x)=I_t^{\gamma}\left[\rho(u(t,x))\dot{W}(t,x)\right],\quad t>0,\ x\in\mathbb{R}^d,$$

where $\dot{W}$ is the space–time white noise, $\alpha\in(0,2]$, $\beta\in(0,2)$, $\gamma\geq 0$ and $\nu>0$. Fundamental solutions and their properties, in particular the nonnegativity, are derived. The existence and uniqueness of solution together with the moment bounds of the solution are obtained under Dalang's condition: $d<2\alpha+\frac{\alpha}{\beta}\min(2\gamma-1,0)$. In some cases, the initial data can be measures. When $\beta\in(0,1]$, we prove the sample path regularity of the solution.

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1. Introduction

In this paper, we study the following nonlinear stochastic space–time fractional diffusion equations:

$$\begin{cases} \left(\partial^\beta + \frac{\nu}{2}(-\Delta)^{\alpha/2}\right) u(t,x) = I_t^\gamma \left[\rho\left(u(t,x)\right)\dot{W}(t,x)\right], & t>0,\ x\in\mathbb{R}^d. \\ u(0,\cdot)=\mu & \text{if } \beta\in(1/2,1], \\ u(0,\cdot)=\mu_0, \quad \frac{\partial}{\partial t}u(0,\cdot)=\mu_1 & \text{if } \beta\in(1,2), \end{cases} \tag{1.1}$$

with $\alpha \in (0,2]$ and $\gamma > 0$. In this equation, $\Delta = \sum_{i=1}^d \partial^2/(\partial x_i^2)$ is the Laplacian with respect to the space variables and $(-\Delta)^{\alpha/2}$ is the fractional Laplacian. $\dot{W}$ denotes the space–time white noise. $\nu > 0$ is the diffusion parameter. The initial data μ, μ_0 and μ_1 are assumed to be some measures. ρ is a Lipschitz continuous function. ∂^β denotes the *Caputo fractional differential operator*:

$$\partial^\beta f(t) := \begin{cases} \frac{1}{\Gamma(m-\beta)}\int_0^t \mathrm{d}\tau\, \frac{f^{(m)}(\tau)}{(t-\tau)^{\beta+1-m}} & \text{if } m-1<\beta<m, \\ \frac{\mathrm{d}^m}{\mathrm{d}t^m} f(t) & \text{if } \beta = m\ , \end{cases}$$

and I_t^γ is the *Riemann–Liouville fractional integral* of order $\gamma > 0$:

$$I_t^\gamma f(t) := \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1} f(s)\mathrm{d}s, \quad \text{for } t>0,$$

with the convention $I_t^0 = \mathrm{Id}$ (the identity operator). We refer to [18,31,33] for more details of these fractional differential operators.

This paper is motivated by some special cases of Eq. (1.1) studied in the literature.

(1) When $d = 1$, $\alpha = 2$ and $\gamma = \lceil\beta\rceil - \beta$, where $\lceil\beta\rceil$ denotes the smallest integer not less than β, this equation was studied by the first author [4], who proved the existence of a mild solution for all $\beta \in (0,2)$. The motivation comes from the study of diffusions in the *viscoelasticitic* media (a media between fluids and solids, such as honey or rubber) perturbed by a multiplicative noise. In the absence of noise, we refer to the work [19,25,32]. Notice that the role of the fractional integral operator I_t^γ is to make the noise term more regular. When this smoothing factor disappears (namely $\gamma = 0$), the solution becomes less regular. In this case one can show that the mild solution exists only when $\beta \in (2/3,2]$. On the other hand, motivated by modeling the molecular motion in biological cell, the special case $\alpha = 2$, $\beta \in (0,1)$ and $\gamma = 0$ of Eq. (1.1) was studied in [9,21]. The dimension can be general and the noise is a colored one. We refer to the references in [21] for more biological application when the noise is absent. One may find more interesting motivations in recent work [13,27,28].

(2) The above case is for time fractional (general β). The spatial fractional one (general α, fractional Laplacian) is a classical subject in probability theory. The special case when $\beta = 1$, $\gamma = 0$ of Eq. (1.1) has been studied in literature. We refer to [8,11] and the references therein.

It is worth pointing out the following two famous special cases of our equation.

(3) When $d = 1$, $\beta = 2$, $\alpha = 2$ and $\gamma = 0$, the spde (1.1) reduces to the *stochastic wave equation* (SWE) on $\mathbb{R}$:

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\nu}{2}\frac{\partial^2}{\partial x^2}\right) u(t,x) = \rho(u(t,x))\dot{W}(t,x)\,, \tag{1.2}$$

with the speed of wave propagation $(\nu/2)^{1/2}$.

(4) When $d = 1$, $\beta = 1$, $\alpha = 2$ and $\gamma = 0$, the spde (1.1) reduces to the *stochastic heat equation* (SHE) on $\mathbb{R}$:

$$\left(\frac{\partial}{\partial t} - \frac{\nu}{2}\frac{\partial^2}{\partial x^2}\right) u(t,x) = \rho(u(t,x))\dot{W}(t,x)\,. \tag{1.3}$$

These two special cases (3) and (4) have been studied extensively and there are many references. Among them let us mention [3,5–7,14,15,17,20]. The spde (1.1) for $\beta \in (0,1]$ and $\gamma = 1-\beta$ has been recently studied in [27,28]. When the noise does not depend on time, a similar model with a general elliptic operator has been studied in [21]. Another related equation is the *stochastic fractional heat equation* (SFHE) on $\mathbb{R}$:

$$\left(\frac{\partial}{\partial t} - {}_xD_\delta^\alpha\right) u(t,x) = \rho(u(t,x))\dot{W}(t,x)\,, \tag{1.4}$$

which has been studied recently in [8,12] under the setting of $\alpha \in (1,2]$ and $|\delta| \le 2-\alpha$; see also [17,20]. Here ${}_xD_\delta^\alpha$ is the general (asymmetric) stable operator with $\alpha \in (0,2]$ and the skewness parameter δ such that $|\delta| \le \min(2-\alpha,\alpha)$. In particular, ${}_xD_0^\alpha = -(-\Delta)^{\alpha/2}$.

The goal of this paper is to unify the above mentioned special equations. There are two considerations for our unification: one is that the unified equation should be general enough to cover all the mentioned cases; the second one is that the equation should also be sufficiently specific so that we can solve it. We found that Eq. (1.1) meets both of these goals.

One motivation for us to find such a unified theory is that we may be able to apply some ideas from the study of one special equation to the study of other special equations. However, in this work we shall concentrate on the existence, uniqueness and sample regularity of the solution.

To solve the general equation (1.1) we shall first find its corresponding Green's functions. As proved in the next sections, there is a triplet

$$\left\{\, Z(t,x),\ Z^*(t,x)\,, Y(t,x):\ \ (t,x) \in [0,\infty) \times \mathbb{R}^d \,\right\},$$

depending on the parameters $(\alpha,\beta,\gamma,\nu)$, such that the solution to (1.1) with $\rho(u(t,x))\dot{W}(t,x)$ replaced by a continuous function $f(t,x)$ with compact support is represented by

$$u(t,x) = \begin{cases} (Z(t,\cdot) * \mu)(x) + (Y \star f)\,(t,x), & \text{if } \beta \in (0,1], \\ (Z^*(t,\cdot) * \mu_0)(x) + (Z(t,\cdot) * \mu_1)(x) + (Y \star f)\,(t,x), & \text{if } \beta \in (1,2), \end{cases} \tag{1.5}$$

where "$*$" denotes the convolution in the space variable:

$$(Z(t,\cdot) * \mu)(x) := \int_{\mathbb{R}^d} Z(t,x-y)\mu(\mathrm{d}y), \tag{1.6}$$

and "$\star$" denotes the convolution in both space and time variables:

$$(Y \star f)(t,x) := \int_0^t \int_{\mathbb{R}^d} Y(t-s,x-y) f(s,y)\mathrm{d}s\mathrm{d}y.$$

The difficulty to solve (1.1) comes from the complexity of the Green's functions, which are expressed by using the *Fox H-functions* [22]. This special function is much more complex than the heat kernels associated with the Laplacian or fractional Laplacians. To get a sense of our idea let us explain our approach to consider the problem of estimating the second moment. As we shall see very soon, this boils down to finding an effective estimation of the following space–time convolution

$$\int_0^t \int_{\mathbb{R}^d} Y^2(t-s,x-y)Y^2(s,y)\mathrm{d}s\mathrm{d}y,$$

in a way that one can do it recursively. The new and key idea to achieve this in this paper is to bound the function $(t,x)\mapsto Y^2(t,x)$ from above (resp. below) by some known kernel functions that satisfy the semigroup property, specifically by the heat kernel type or Poisson kernel type functions (see (3.7)). For the upper bound, one may need $x\mapsto Y(1,x)$ to be bounded; for the lower bound, one may need this function to be nonnegative.

If we denote the solution to the homogeneous equation of (1.1) by $J_0(t,x)$, i.e.,

$$J_0(t,x) = \begin{cases} (Z(t,\cdot)*\mu)(x) & \text{if } \beta\in(0,1], \\ (Z^*(t,\cdot)*\mu_0)(x) + (Z(t,\cdot)*\mu_1)(x) & \text{if } \beta\in(1,2), \end{cases} \tag{1.7}$$

then the rigorous meaning of (1.1) is the following stochastic integral equation:

$$\begin{aligned} &u(t,x) = J_0(t,x) + I(t,x), \quad \text{where} \\ &I(t,x) = \iint_{[0,t]\times\mathbb{R}} Y\,(t-s,x-y)\,\rho\,(u(s,y))\,W(\mathrm{d}s,\mathrm{d}y). \end{aligned} \tag{1.8}$$

The stochastic integral in the above equation is in the sense of Walsh [36].

1.1. Existence and uniqueness

To establish the existence and uniqueness of random field solutions to (1.1), the first step is to check *Dalang's condition* [16]:

$$\int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}y\,|Y(s,y)|^2 < \infty, \quad \text{for all } t>0,$$

which is equivalent to the following condition (see Lemma 5.3):

$$d < 2\alpha + \frac{\alpha}{\beta}\min(2\gamma-1,0) =: \Theta, \tag{1.9}$$

which is also equivalent to

$$\beta+\gamma > \frac{1}{2}\left(1+\frac{d\beta}{\alpha}\right) \quad \text{and} \quad d<2\alpha. \tag{1.10}$$

In the following, we will call these two equivalent conditions (1.9) and (1.10) Dalang's condition, which are assumed throughout of the paper.

Note that (1.10) implies that the space dimension should be less than or equal to 3. Among all possible cases in (1.9), the following two special cases have better properties:

$$\gamma = 0 \quad \text{or} \quad \alpha > d = 1, \tag{1.11}$$

$$\alpha > d = 1. \tag{1.12}$$

Clearly, case (1.12) is a special case of (1.11). As shown in Lemma 4.3 and Remark 4.4, under (1.11), the function $x\mapsto Y(1,x)$ is bounded near zero and hence bounded for all $x\in\mathbb{R}^d$. We rely on this property to implement a procedure of bounding $Y^2(t,x)$ from above by the heat-type or Poisson-type kernel functions. If we strengthen the condition from (1.11) to (1.12), then all functions $Z(1, x)$, $Z^*(1, x)$ and $Y(1, x)$ are bounded near zero. Since $Z(1, x)$ and $Z^*(1, x)$ get involved in the equation only through the initial data (see (1.7)), this boundedness property of $Z(1, x)$ and $Z^*(1, x)$ near zero allows the initial data to be unbounded near zero, for example, in this case one may allow the initial data to be the Dirac delta measure (distribution). The differences between these conditions are manifested in full details through the three cases – Cases I, II and III – below.

We prove the existence and uniqueness of random field solutions to (1.8) in the following three cases:

Case I: If we assume only Dalang's condition (1.9), we prove the existence and uniqueness when the initial data are such that

$$\sup_{(s,x)\in[0,t]\times\mathbb{R}^d} |J_0(s, x)| < \infty, \qquad \text{for all } t > 0, \tag{1.13}$$

which is satisfied, for example, when initial data are bounded measurable functions.

Case II: Under both (1.9) and (1.11), we obtain moment formulas that are similar to those in [4,7,8]. The initial data satisfy (1.13).

Case III: Under both (1.9) and (1.12), the initial data can be measures. Let $\mathcal{M}(\mathbb{R})$ be the set of signed (regular) Borel measures on $\mathbb{R}$. For $x \in \mathbb{R}$, define an auxiliary function

$$f_\beta(\eta, x) := \exp\left(-\eta\, |x|^{1+\lfloor\beta\rfloor}\right), \tag{1.14}$$

where $\lfloor\beta\rfloor$ is the largest integer not greater than β. Note that the difference between $\lceil\beta\rceil$ and $\lfloor\beta\rfloor + 1$ for $\beta \in (0, 2)$ is only at $\beta = 1$. The initial data are assumed to be Borel measures such that

$$\begin{cases} \left(|\mu| * f_\beta(\eta, \cdot)\right)(x) < \infty, & \text{for all } \eta > 0 \text{ and } x \in \mathbb{R}, \text{ if } \alpha = 2, \\ \sup_{y\in\mathbb{R}} \int_{\mathbb{R}} |\mu|(\mathrm{d}x) \dfrac{1}{1 + |x-y|^{1+\alpha}} < +\infty, & \text{if } \alpha \in (1, 2), \end{cases} \tag{1.15}$$

where for any Borel measure μ, $\mu = \mu_+ - \mu_-$ is the Jordan decomposition and $|\mu| = \mu_+ + \mu_-$. We use $\mathcal{M}_{\alpha,\beta}(\mathbb{R})$ to denote these measures. In this case, we prove the existence and uniqueness of a solution to (1.4) for all initial data from $\mathcal{M}_{\alpha,\beta}(\mathbb{R})$.

Here are some special cases:

(1) For (1.3), i.e., $\alpha = 2$, $\beta = 1$ and $\gamma = 0$, the set of admissible initial data studied in [7] is $\mathcal{M}_H(\mathbb{R})$, which corresponds to $\mathcal{M}_{2,1}(\mathbb{R})$ in this paper.
(2) Under the condition that $d = 1$, $\alpha = 2$, $\beta \in (0, 2)$, $\gamma = \lceil\beta\rceil - \beta$ (as in [4]), one can easily verify that condition (1.9) is always true. The possible initial data is $\mathcal{M}_T^\beta(\mathbb{R})$, which corresponds to $\mathcal{M}_{2,\beta}(\mathbb{R})$ in this paper.
(3) If $\gamma = 1 - \beta$ and $\beta \in (0, 1)$, then it is ready to see that (1.9) reduces to

$$d < \alpha \min(2, \beta^{-1}),$$

which recovers the condition by Mijena and Nane [27].
(4) If $\gamma = 0$, then (1.9) becomes

$$\frac{d}{\alpha} + \frac{1}{\beta} < 2.$$

Moreover, if $\alpha = 2$ and $d = 1$, then this condition becomes $\beta > 2/3$, which coincides to the condition in [13, Section 5.2].

(5) If $\beta = 1$ and $\gamma = 0$, then Dalang's condition (1.9) reduces to $\alpha > d$. Since $\alpha \in (0, 2]$, we have that $\alpha \in (1, 2]$ and $d = 1$, which recovers the condition in [8].

As in [6–8], we will obtain similar moment formulas expressed using a kernel function $\mathcal{K}(t, x)$ when (1.11) is satisfied. For the SHE and the SWE, this kernel function $\mathcal{K}(t, x)$ has explicit forms. But for the SFHE [8], (1.1) with $d = 1$, $\gamma = \lceil\beta\rceil - \beta$ and $\alpha = 2$ in [4], and the current spde (1.1), we obtain some estimates on it. In particular, we will obtain both upper and lower bounds on $\mathcal{K}(t, x)$.

1.2. Hölder regularity

After establishing the existence and uniqueness of the solution, we will study the sample-path regularity for the slow diffusion equations (i.e., the case when $\beta \in (0, 1]$) with $\gamma \in [0, 1 - \beta]$. Given a subset $D \subseteq [0, \infty) \times \mathbb{R}^d$ and positive constants b_1, b_2, denote by $C_{b_1,b_2}(D)$ the set of functions $v : [0, \infty) \times \mathbb{R}^d \mapsto \mathbb{R}$ with the following property: for each compact set $K \subseteq D$, there is a finite constant C such that for all (t, x) and $(s, y) \in K$,

$$|v(t, x) - v(s, y)| \le C \left(|t - s|^{b_1} + |x - y|^{b_2}\right). \tag{1.16}$$

Denote

$$C_{b_1-,b_2-}(D) := \cap_{\alpha_1 \in (0,b_1)} \cap_{\alpha_2 \in (0,b_2)} C_{\alpha_1,\alpha_2}(D) .$$

We will show that for slow diffusion equations with $\gamma \in [0, 1 - \beta]$, if the initial data has a bounded density, i.e., $\mu(\mathrm{d}x) = f(x)\mathrm{d}x$ with $f \in L^\infty(\mathbb{R}^d)$, then

$$u(\cdot, \cdot) \in C_{\frac{1}{2}(2(\beta+\gamma)-1-d\beta/\alpha)-,\ \frac{1}{2}\min(\Theta-d,2)-}\left((0, \infty) \times \mathbb{R}^d\right), \quad \text{a.s.}, \tag{1.17}$$

where Θ is defined in (1.9).

Example 1.1. When $\gamma = 0$, $\alpha = 2$ and $d = 1$, Dalang's condition (1.9) becomes $\beta > 2/3$. The exponents b_1 and b_2 in (1.16) become

$$b_1 = \frac{3\beta - 2}{4} \qquad \text{and} \qquad b_2 = \frac{3}{2} - \frac{1}{\beta}.$$

Both b_1 and b_2, viewed as functions of $\beta \in (2/3, 1]$, are nondecreasing, i.e., the more derivative one takes in ∂^β, the more regularity of the solution one obtains. So ∂^β plays a regularization role.

Example 1.2. When $\gamma = 1 - \beta$, $\alpha = 2$ and $d = 1$, the exponents in (1.16) reduce to

$$b_1 = \frac{2 - \beta}{4} \qquad \text{and} \qquad b_2 = \left[\left(\frac{1}{\beta} - \frac{1}{2}\right) \wedge 1\right] = \begin{cases} 1 & \text{if } \beta \in (0, 2/3], \\ 1/\beta - 1/2 & \text{if } \beta \in (2/3, 1], \end{cases}$$

which recover the temporal Hölder exponent in Theorem 3.2 of [4] and improve the spatial Hölder exponent obtained in the same reference from $1/2$ to b_2. It is clear that $1/2 \le b_2 \le 1$ for $\beta \in (0, 1]$ and when $\beta = 1$, $b_2 = 1/2$, which recovers the classical results; see [5,36]. Contrary to the previous Example 1.1, with the presence of the smoothing operator I_t^γ both exponents b_1 and b_2, viewed as functions of β, are nonincreasing, i.e., the larger is β, the less regularity of the solution we have. This means that regularization factor $I_t^{1-\beta}$ (noticing the negative sign in the exponent β) plays a more important role than the regularization factor ∂^β.

1.3. Moment Lyapunov exponents and intermittency

When the initial data are spatially homogeneous (i.e., the initial data are constants), so is the solution $u(t,x)$, and then the *moment Lyapunov exponents*

$$\overline{m}_p := \limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}\left[|u(t,x)|^p\right], \tag{1.18}$$

$$\underline{m}_p := \liminf_{t\to\infty} \frac{1}{t} \log \mathbb{E}\left[|u(t,x)|^p\right], \tag{1.19}$$

do not depend on the spatial variable. In this case, a solution is called *fully intermittent* if $m_1 = 0$ and $\underline{m}_2 > 0$ (see [2, Definition III.1.1, on p. 55]). As for the weak intermittency, there are various definitions. For convenience of stating our results, we will call the solution *weakly intermittent of type I* if $\underline{m}_2 > 0$, and *weakly intermittent of type II* if $\overline{m}_2 > 0$. Clearly, the weak intermittency of type I is stronger than the weak intermittency of type II, but weaker than the full intermittency by missing $m_1 = 0$. The weak intermittency of type II is used in [20].

The full intermittency for the SHE and the SFHE are established in [1] and [12], respectively. The weak intermittency of type I and II for SWE are proved in [6] and [14, Theorem 2.3], respectively. We will establish the weak intermittency of type II for both slow and fast diffusion equations. For some slow diffusion equations, we will prove the weak intermittency of type I. Moreover, we show that

$$\overline{m}_p \le C\, p^{1+\frac{1}{2(\beta+\gamma)-1-d\beta/\alpha}}\,. \tag{1.20}$$

It reduces to the following special cases:

(1) The SHE case, i.e., $\beta = 1$, $\alpha = 2$, $\gamma = 0$ and $d = 1$: $\overline{m}_p \le C\, p^3$. See [1,7,20];
(2) The SWE case, i.e., $\beta = 2$, $\alpha = 2$, $\gamma = 0$ and $d = 1$: $\overline{m}_p \le C\, p^{3/2}$. See [6];
(3) The SFHE case, i.e., $\beta = 1$, $\gamma = 0$ and $d = 1$: $\overline{m}_p \le C\, p^{2+1/(\alpha-1)}$. See [8];
(4) The time-fractional diffusion equation case as in [4] that $\alpha = 2$, $\gamma = \lceil\beta\rceil - \beta$ and $d = 1$: $\overline{m}_p \le C\, p^{\frac{4\lceil\beta\rceil-\beta}{4\lceil\beta\rceil-2-\beta}}$;
(5) The time-fractional spde as in Theorems 3.11 and 3.13 of [9] with $d = 1$ and $\kappa = 1$[3]:

$$\begin{cases} \overline{m}_p \le C\, p^{\frac{2\alpha-\beta}{2\alpha-\beta-\alpha}} & \text{when } \gamma = 0, \\ \overline{m}_p \le C\, p^{\frac{2\alpha\lceil\beta\rceil-\beta}{2\alpha\lceil\beta\rceil-\beta-\alpha}} & \text{when } \gamma = \lceil\beta\rceil - \beta. \end{cases}$$

In general, meaningful lower moment bounds are usually harder to obtain than the upper bounds. Much more effort is required, especially in our general framework. One of the key ingredients that we need for the lower bounds is the nonnegativity of Y, which we are able to prove in this paper for the following cases:

$$\begin{cases} \text{Case I:} & \alpha \in (0,2],\ \beta \in (0,1),\ d \in \mathbb{N},\ \gamma \ge 0, \\ \text{Case II:} & \alpha \in (0,2],\ \beta = 1,\ d \in \mathbb{N}, \gamma \in \{0\} \cup (1,\infty), \\ \text{Case III:} & 1 < \beta < \alpha \le 2,\ d \le 3,\ \gamma \ge 0, \\ \text{Case IV:} & 1 < \beta = \alpha < 2,\ d \le 3,\ \gamma \ge (d+3)/2 - \beta; \end{cases} \tag{1.21}$$

see Theorem 4.6. These results, which generalize those obtained by Mainardi et al. [26], Pskhu [32], and Chen et al. [9], have their own interest. Based on this nonnegativity property,

[3] In [9], the constant κ is the exponent for the Riesz kernel, and the case $\kappa = 1$ corresponds to the space–time white noise.

we can carry out the procedure of bounding Y^2 from below by some reference kernel functions, similar to that for the upper moment bounds, to derive some satisfactory lower moment bounds.

1.4. Some comments

After introducing our main results we elaborate in more details about the smoothing effect of the fractional operator I_t^{γ}, which can be seen from the following three aspects: Firstly, for the existence and uniqueness of solution, we can see immediately that the larger is γ, the larger is the domain of α and β, for which Dalang's condition (1.9) is satisfied. Secondly, for the Hölder regularity (1.17), the exponents both in time and space are nondecreasing functions of γ, i.e., the large is γ, the more regular is the solution. Lastly, from the upper bound of the moment Lyapunov exponents in (1.20), we see that the larger is γ, the smaller is the upper bound, hence the less intermittent is the solution.

We would like to highlight some contributions of this paper here. Firstly, to the best of our knowledge, this is the first time that the SPDE (1.1) of this generality has been ever studied. This equation not only contains some very interesting known equations but also covers new ones. For example, while in the literature γ is set to be either 0 or $\lceil \beta \rceil - \beta$, in this paper, γ can be any positive real values. Secondly, from the methodology point of view, we give a way to handle equations that do not satisfy the semigroup property (i.e., the case when $\beta \neq 1$) by comparing with known semigroup kernel functions, through which we obtain upper and lower moment bounds. It turns out this is a very robust, though not necessarily sharp, method that may be applied to other SPDE's, for example, the same SPDE as in this paper with a more general Gaussian noise. Thirdly, the nonnegativity of the fundamental solution is established for all cases in (1.21), which generalizes recent results by Pskhu [32] and Chen et al. [9]. The proof is very technical where we have used some ideas from Pskhu [32]. Lastly, even in such generality, we are still able to prove many fine properties of the solution, beyond the existence and uniqueness of the solution, such as sample path regularity for the slow diffusion case, upper and lower moment bounds, intermittency, etc.

Finally, we list several open problems for future investigation.

(1) The Hölder regularity is proved for $\beta \in (0, 1]$ (slow diffusion case) with the constraint $\gamma \in [0, 1-\beta]$. It is interesting to study the case $\gamma > 1-\beta$. A more challenging problem is the regularity when $\beta \in (1, 2]$ (fast diffusion case). It seems that almost none is known except for the case of the stochastic wave equation (i.e., $\alpha = \beta = 2$, $d = 1$, $\gamma = 0$); see e.g. Theorem 4.2.1 of [3].

(2) Sample path comparison principle plays a vital role in the study of the stochastic heat equation; see [12,29]. One would expect this principle continues to be true for the slow diffusion case $\beta \in (0, 1]$. A related and more ambitious problem is to establish the existence and smoothness of the density of the random variable $u(t, x)$ with both t and x fixed; see [10] for a special case.

(3) Even though the PDE part of the SPDE studied in this paper takes a very general form, the noise is very special, that is, $\dot{W}$ is the space–time white noise. One may investigate this same SPDE (possibly for the linear case $\rho(u) = \lambda u$) with a general colored Gaussian noise $\dot{W}$.

This paper is structured as follows. In Section 2 we first give some notation and preliminaries. The main results are stated in Section 3. The fundamental solutions are studied in Section 4. The proof of the two existence and uniqueness theorems are given in Section 5. Finally, in the Appendix, we prove some properties of the Fox H-functions.

2. Some preliminaries and notation

Let $W = \left\{W_t(A) : A \in \mathcal{B}_b\left(\mathbb{R}^d\right), t \geq 0\right\}$ be a space–time white noise defined on a complete probability space $(\Omega, \mathcal{F}, P)$, where $\mathcal{B}_b\left(\mathbb{R}^d\right)$ is the collection of Borel sets with finite Lebesgue measure. Let

$$\mathcal{F}_t = \sigma\left(W_s(A) : 0 \leq s \leq t, A \in \mathcal{B}_b\left(\mathbb{R}^d\right)\right) \vee \mathcal{N}, \quad t \geq 0,$$

be the natural filtration augmented by the σ-field $\mathcal{N}$ generated by all P-null sets in $\mathcal{F}$. We use $||\cdot||_p$ to denote the $L^p(\Omega)$-norm ($p \geq 1$). In this setup, W becomes a worthy martingale measure in the sense of Walsh [36], and $\iint_{[0,t]\times\mathbb{R}^d} X(s, y)W(\mathrm{d}s, \mathrm{d}y)$ is well-defined for a suitable class of random fields $\left\{X(s, y),\ (s, y) \in [0, \infty) \times \mathbb{R}^d\right\}$.

Recall that the rigorous meaning of the spde (1.1) is in the integral form (1.8).

Definition 2.1. A process $u = \left\{u(t, x),\ (t, x) \in (0, \infty) \times \mathbb{R}^d\right\}$ is called a *random field solution* to (1.1) if

(1) u is adapted, i.e., for all $(t, x) \in (0, \infty) \times \mathbb{R}^d$, $u(t, x)$ is $\mathcal{F}_t$-measurable;
(2) u is jointly measurable with respect to $\mathcal{B}\left((0, \infty) \times \mathbb{R}^d\right) \times \mathcal{F}$;
(3) for all $(t, x) \in (0, \infty) \times \mathbb{R}^d$, the following space–time convolution is finite:

$$\left(Y^2 \star ||\rho(u)||_2^2\right)(t, x) := \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}y\, Y^2(t-s, x-y)\, ||\rho(u(s, y))||_2^2 < +\infty;$$

(4) the function $(t, x) \mapsto I(t, x)$ mapping $(0, \infty) \times \mathbb{R}^d$ into $L^2(\Omega)$ is continuous;
(5) u satisfies (1.8) a.s., for all $(t, x) \in (0, \infty) \times \mathbb{R}^d$.

Assume that the function $\rho : \mathbb{R} \mapsto \mathbb{R}$ is globally Lipschitz continuous with Lipschitz constant $\mathrm{Lip}_\rho > 0$. We need some growth conditions on ρ[4]: assume that for some constants $L_\rho > 0$ and $\overline{\varsigma} \geq 0$,

$$|\rho(x)|^2 \leq L_\rho^2\left(\overline{\varsigma}^2 + x^2\right), \qquad \text{for all } x \in \mathbb{R}. \tag{2.1}$$

Sometimes we need a lower bound on $\rho(x)$: assume that for some constants $l_\rho > 0$ and $\underline{\varsigma} \geq 0$,

$$|\rho(x)|^2 \geq l_\rho^2\left(\underline{\varsigma}^2 + x^2\right), \qquad \text{for all } x \in \mathbb{R}\,. \tag{2.2}$$

For all $(t, x) \in (0, \infty) \times \mathbb{R}^d$, $n \in \mathbb{N}$ and $\lambda \in \mathbb{R}$, define

$$\begin{aligned}
&\mathcal{L}_0(t, x) := Y^2(t, x)\\
&\mathcal{L}_n(t, x) := (\mathcal{L}_0 \star \ldots \star \mathcal{L}_0)(t, x), \quad \text{for } n \geq 1, (n \text{ convolutions}),
\end{aligned} \tag{2.3}$$

$$\mathcal{K}(t, x; \lambda) := \sum_{n=0}^{\infty} \lambda^{2(n+1)} \mathcal{L}_n(t, x)\,. \tag{2.4}$$

We will use the following conventions to the kernel functions $\mathcal{K}(t, x; \lambda)$:

$$\begin{aligned}
&\mathcal{K}(t, x) := \mathcal{K}(t, x; \lambda), \qquad \overline{\mathcal{K}}(t, x) := \mathcal{K}\left(t, x; L_\rho\right),\\
&\underline{\mathcal{K}}(t, x) := \mathcal{K}\left(t, x; l_\rho\right), \quad \widehat{\mathcal{K}}_p(t, x) := \mathcal{K}\left(t, x; 4\sqrt{p}L_\rho\right), \quad \text{for } p \geq 2\,.
\end{aligned} \tag{2.5}$$

Throughout the paper, denote

$$\sigma := 2(1-\beta-\gamma) + \beta d/\alpha. \tag{2.6}$$

[4] This is a consequence of the Lipschitz continuity of ρ.

Note that

$$(1.9) \quad \Rightarrow \quad d < 2\alpha + \frac{\alpha}{\beta}(2\gamma - 1) \quad \Leftrightarrow \quad 2(\beta+\gamma) - 1 - d\beta/\alpha > 0 \quad \Leftrightarrow \quad \sigma < 1. \tag{2.7}$$

Let ${}_tD_+^\alpha$ denote the *Riemann–Liouville fractional derivative* of order $\alpha \in \mathbb{R}$ (see, e.g., [31, (2.79) and (2.88)]):

$$ {}_tD_+^\alpha f(t) := \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \dfrac{\mathrm{d}^m}{\mathrm{d}t^m} \displaystyle\int_0^t \mathrm{d}\tau \, \frac{f(\tau)}{(t-\tau)^{\alpha+1-m}} & \text{if } m-1 < \alpha < m \text{ and } \alpha \geq 0, \\ \dfrac{\mathrm{d}^m}{\mathrm{d}t^m} f(t) & \text{if } \alpha = m \geq 0, \\ \dfrac{1}{\Gamma(-\alpha)} \displaystyle\int_0^t s^{-\alpha-1} f(s)\mathrm{d}s & \text{if } \alpha < 0. \end{cases} \tag{2.8}$$

We will need the *two-parameter Mittag-Leffler function*

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \qquad \alpha > 0,\ \beta > 0\,, \tag{2.9}$$

which is a generalization of exponential function, $E_{1,1}(z) = e^z$; see, e.g., [31, Section 1.2]. A function is called *completely monotonic* if $(-1)^n f^{(n)}(x) \geq 0$ for $n = 0, 1, 2, \ldots$; see [37, Definition 4]. An important fact [34] concerning the Mittag-Leffler function is that

$$x \in [0, \infty) \mapsto E_{\alpha,\beta}(-x) \text{ is completely monotonic} \quad \Longleftrightarrow \quad 0 < \alpha \leq 1 \wedge \beta. \tag{2.10}$$

By [22, (2.9.27)], the above Mittag-Leffler function is a special case of the Fox H-function:

$$E_{\alpha,\beta}(z) = H_{1,2}^{1,1}\left(-z \,\middle|\, \begin{matrix} (0,1) \\ (0,1),\ (1-\beta,\alpha) \end{matrix}\right).$$

3. Main results

The first two theorems are about the existence, uniqueness and moment estimates of the solutions to (1.1). The second one, in particular, possesses the same form as the one in [4, Theorem 3.1]. See also similar results for other equations, e.g., SHE [7, Theorem 2.4], SWE [6, Theorem 2.3], and SFHE [8, Theorem 3.1].

Theorem 3.1 (*Existence, Uniqueness and Moments (I)*)**.** *Under (1.9), the spde (1.1) has a unique (in the sense of versions) random field solution $\{u(t,x) : (t,x) \in (0,\infty) \times \mathbb{R}^d\}$ if the initial data are such that*

$$\widehat{C}_t := \sup_{(s,x)\in[0,t]\times\mathbb{R}^d} |J_0(s,x)| < +\infty. \tag{3.1}$$

Moreover, the following statements are true:

(1) $(t,x) \mapsto u(t,x)$ is $L^p(\Omega)$-continuous for all $p \geq 2$;
(2) For all even integers $p \geq 2$, all $t > 0$ and $x, y \in \mathbb{R}^d$,

$$||u(t,x)||_p^2 \leq 2J_0^2(t,x) + \left[\overline{\varsigma}^2 + 2\widehat{C}_t^2\right] \exp\left(C\lambda^{\frac{2}{1-\sigma}} t\right), \tag{3.2}$$

where C is some universal constant not depending on p, and σ is defined in (2.6).

This theorem is proved in Section 5.6. Note that if the initial data are bounded functions, then (3.1) is satisfied.

Theorem 3.2 (*Existence, Uniqueness and Moments (II)*). *If Dalang's condition (1.9) is satisfied, then the spde (1.1) has a unique (in the sense of versions) random field solution* $\{u(t,x) : (t,x) \in (0,\infty)\times\mathbb{R}^d\}$ *starting from either initial data that satisfy (3.1) under condition (1.11) or any Borel measures from* $\mathcal{M}_{\alpha,\beta}(\mathbb{R})$ *under condition (1.12). Moreover, the following statements are true:*

(1) $(t,x)\mapsto u(t,x)$ *is* $L^p(\Omega)$*-continuous for all* $p \ge 2$*;*
(2) For all even integers $p \ge 2$*, all* $t > 0$ *and* $x, y \in \mathbb{R}^d$*,*

$$||u(t,x)||_p^2 \le \begin{cases} J_0^2(t,x) + \left(\left[\overline{\varsigma}^2 + J_0^2\right] \star \overline{\mathcal{K}}\right)(t,x), & \text{if } p = 2\,, \\ 2J_0^2(t,x) + \left(\left[\overline{\varsigma}^2 + 2J_0^2\right] \star \widehat{\mathcal{K}}_p\right)(t,x), & \text{if } p > 2\,; \end{cases} \tag{3.3}$$

(3) If ρ *satisfies (2.2), then under (1.9), (1.11) and the first two cases of (1.21), for all* $t > 0$ *and* $x, y \in \mathbb{R}^d$*, it holds that*

$$||u(t,x)||_2^2 \ge J_0^2(t,x) + \left(\left(\underline{\varsigma}^2 + J_0^2\right) \star \underline{\mathcal{K}}\right)(t,x)\,. \tag{3.4}$$

The proof of this theorem is given in Section 5.5.

The following theorem gives the Hölder continuity of the solution for slow diffusion equations.

Theorem 3.3. *Recall that the constants* σ *and* Θ *are defined in (2.6) and (1.9), respectively. If* $\beta \in (0,1]$*,* $\gamma \in [0, 1-\beta]$ *and (3.1) holds, then under (1.9),*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} ||u(t,x)||_p^2 < +\infty, \quad \text{for all } T \ge 0 \text{ and } p \ge 2. \tag{3.5}$$

Moreover, we have

$$I(\cdot,\cdot) \in C_{\frac{1}{2}(1-\sigma)-,\ \frac{1}{2}\min(\Theta-d,2)-}\left([0,\infty)\times\mathbb{R}^d\right)\ , \quad a.s.\,, \tag{3.6}$$

and (1.17) holds.

Proof. The bound (3.5) is due to (3.1) and (3.3). The proof of (3.6) is straightforward under (3.5) and Proposition 5.4 (see [5, Remark 4.6]). □

In order to use the moment bounds in (3.3) and (3.4), we need some good estimate on the kernel function $\mathcal{K}(t,x)$. Following [4], define the following *reference kernel functions*:

$$\mathcal{G}_{\alpha,\beta}(t,x) := \begin{cases} c_\beta \left(4\nu\pi t^\beta\right)^{-d/2} \exp\left(-\dfrac{1}{4\nu}\left(t^{-\beta/2}|x|\right)^{\lfloor\beta\rfloor+1}\right), & \text{if } \alpha = 2, \\ \dfrac{c_d\ t^{\beta/\alpha}}{\left(t^{2\beta/\alpha}+|x|^2\right)^{(d+1)/2}}, & \text{if } \alpha \in (0,2), \end{cases} \tag{3.7}$$

for $\beta \in (0,2)$ where $|x|^2 = x_1^2 + \cdots + x_d^2$, $c_\beta = 1$ if $\beta \in [1,2)$ and $c_\beta = 2^{-(1+d)}\nu^{-d/2}\Gamma(d/2)/\Gamma(d)$ if $\beta \in (0,1)$, and $c_d = \pi^{-(d+1)/2}\Gamma((d+1)/2)$. Define also

$$\underline{\mathcal{G}}_{\alpha,\beta}(t,x) := \begin{cases} \left(\nu\pi t^\beta\right)^{-d/2} \exp\left(-\dfrac{|x|^2}{\nu t^\beta}\right) & \text{if } \alpha = 2, \\ \dfrac{c_d\ t^{\beta/\alpha}}{\left(t^{2\beta/\alpha}+|x|^2\right)^{(d+1)/2}}, & \text{if } \alpha \in (0,2). \end{cases} \tag{3.8}$$

These reference kernels are nonnegative and the constants c_β and c_d are chosen such that the integration of these kernels on $\mathbb{R}^d$ is equal to one.

Theorem 3.4. *Fix $\lambda \in \mathbb{R}$.*

(1) Under (1.9) and (1.11), there are two nonnegative constants C and Υ depending on α, β, γ, and ν, such that, for all $(t,x) \in (0,\infty) \times \mathbb{R}^d$,

$$\mathcal{K}(t,x;\lambda) \le \frac{C}{t^\sigma}\, \mathcal{G}_{\alpha,\beta}(t,x) \left(1 + t^\sigma \exp\left(\lambda^{\frac{2}{1-\sigma}}\ \Upsilon\, t\right)\right), \tag{3.9}$$

where σ is defined in (2.6);

(2) Under (1.9), (1.11) and the first two cases in (1.21), there are two nonnegative constants $\underline{C}$ and $\underline{\Upsilon}$ depending on α, β, γ, and ν, such that, for all $(t,x) \in (0,\infty) \times \mathbb{R}^d$,

$$\mathcal{K}(t,x;\lambda) \ge \underline{C}\, \underline{\mathcal{G}}_{\alpha,\beta}(t,x) \exp\left(\lambda^{\frac{2}{1-\sigma}}\ \underline{\Upsilon}\, t\right). \tag{3.10}$$

Proof. This theorem is due to Propositions 5.10, 5.11 and [4, Proposition 5.2]. □

The last set of results are the weak intermittency.

Theorem 3.5 (*Weak Intermittency*)**.** *Suppose that (1.9) holds and the initial data satisfy (3.1).*

(1) If ρ satisfies (2.1), then for some finite constant $C > 0$,

$$\overline{m}_p \le C L_\rho^{\frac{2}{2(\beta+\gamma)-1-d\beta/\alpha}}\, p^{1+\frac{1}{2(\beta+\gamma)-1-d\beta/\alpha}}, \quad \textit{for all } p \ge 2 \textit{ even.}$$

(2) Suppose that the initial data are uniformly bounded from below, i.e., $\mu(\mathrm{d}x) = f(x)\mathrm{d}x$ and $f(x) \ge c > 0$ for all $x \in \mathbb{R}^d$. If ρ satisfies (2.2) with $|c| + |\underline{\varsigma}| \ne 0$, then under (1.9), (1.11) and the first two cases in (1.21), there is some finite constant $C' > 0$ such that

$$\underline{m}_p \ge C' l_\rho^{\frac{2}{2(\beta+\gamma)-1-d\beta/\alpha}}\, p, \quad \textit{for all } p \ge 2.$$

Proof. By (3.3), (3.9) and (5.11),

$$||u(t,x)||_p^2 \le \widehat{C}_t^2 + C\, t^{-\sigma} \left(\overline{\varsigma}^2 + 2\widehat{C}_t^2\right) \left(1 + t^\sigma \exp\left(\Upsilon L_\rho^{\frac{2}{1-\sigma}}\, p^{\frac{1}{1-\sigma}}\, t\right)\right).$$

Then increase the power by a factor $p/2$. As for the lower bound, it holds that

$$||u(t,x)||_p^2 \ge ||u(t,x)||_2^2 \ge c^2 + \underline{C}\left(\underline{\varsigma}^2 + c^2\right) \exp\left(l_\rho^{\frac{2}{1-\sigma}}\, \underline{\Upsilon} t\right),$$

thanks to (3.4) and (3.10). □

4. Fundamental solutions

Theorem 4.1. *For $\alpha \in (0,2]$, $\beta \in (0,2)$ and $\gamma \ge 0$, the solution to*

$$\begin{cases} \left(\partial^\beta + \frac{\nu}{2}(-\Delta)^{\alpha/2}\right) u(t,x) = I_t^\gamma\left[f(t,x)\right], & t > 0,\ x \in \mathbb{R}^d, \\ \left.\frac{\partial^k}{\partial t^k} u(t,x)\right|_{t=0} = u_k(x), & 0 \le k \le \lceil \beta \rceil - 1,\ \ x \in \mathbb{R}^d, \end{cases} \tag{4.1}$$

is

$$u(t,x) = J_0(t,x) + \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}y\ f(s,y)\, {}_tD_+^{\lceil\beta\rceil-\beta-\gamma} Z(t-s, x-y), \tag{4.2}$$

where

$$J_0(t,x) := \sum_{k=0}^{\lceil\beta\rceil-1} \int_{\mathbb{R}^d} u_{\lceil\beta\rceil-1-k}(y)\partial^k Z(t,x-y)\mathrm{d}y \tag{4.3}$$

is the solution to the homogeneous equation and

$$Z_{\alpha,\beta,d}(t,x) := \pi^{-d/2} t^{\lceil\beta\rceil-1}|x|^{-d} H_{2,3}^{2,1}\left(\frac{|x|^\alpha}{2^{\alpha-1}\nu t^\beta}\,\middle|\, \begin{matrix}(1,1),\ (\lceil\beta\rceil,\beta)\\ (d/2,\alpha/2),\ (1,1),\ (1,\alpha/2)\end{matrix}\right), \tag{4.4}$$

$$\begin{aligned} Y_{\alpha,\beta,\gamma,d}(t,x) &:= {}_tD_+^{\lceil\beta\rceil-\beta-\gamma} Z_{\alpha,\beta,d}(t,x) = \pi^{-d/2}|x|^{-d} t^{\beta+\gamma-1} \\ &\quad\times H_{2,3}^{2,1}\left(\frac{|x|^\alpha}{2^{\alpha-1}\nu t^\beta}\,\middle|\, \begin{matrix}(1,1),\ (\beta+\gamma,\beta)\\ (d/2,\alpha/2),\ (1,1),\ (1,\alpha/2)\end{matrix}\right) \end{aligned} \tag{4.5}$$

and, if $\beta \in (1,2)$,

$$Z^*_{\alpha,\beta,d}(t,x) := \frac{\partial}{\partial t} Z_{\alpha,\beta,d}(t,x) = \pi^{-d/2}|x|^{-d} H_{2,3}^{2,1}\left(\frac{|x|^\alpha}{2^{\alpha-1}\nu t^\beta}\,\middle|\, \begin{matrix}(1,1),\ (1,\beta)\\ (d/2,\alpha/2),\ (1,1),\ (1,\alpha/2)\end{matrix}\right). \tag{4.6}$$

Moreover,

$$\mathcal{F}Z_{\alpha,\beta,d}(t,\cdot)(\xi) = t^{\lceil\beta\rceil-1} E_{\beta,\lceil\beta\rceil}(-2^{-1}\nu t^\beta|\xi|^\alpha), \tag{4.7}$$

$$\mathcal{F}Y_{\alpha,\beta,\gamma,d}(t,\cdot)(\xi) = t^{\beta+\gamma-1} E_{\beta,\beta+\gamma}(-2^{-1}\nu t^\beta|\xi|^\alpha), \tag{4.8}$$

$$\mathcal{F}Z^*_{\alpha,\beta,d}(t,\cdot)(\xi) = E_{\beta}(-2^{-1}\nu t^\beta|\xi|^\alpha), \quad \textit{if } \beta \in (1,2). \tag{4.9}$$

This theorem is proved in Section 4.2. For convenience, we will use the following notation

$$Y(t,x) := Y_{\alpha,\beta,\gamma,d}(t,x) = {}_tD_+^{\lceil\beta\rceil-\beta-\gamma} Z(t,x), \tag{4.10}$$

$$Z^*(t,x) := Z^*_{\alpha,\beta,d}(t,x) = \frac{\partial}{\partial t} Z(t,x), \quad \text{if } \beta \in (1,2). \tag{4.11}$$

A direct consequence of expression (4.5) is the following scaling property

$$Y(t,x) = t^{\beta+\gamma-1-d\beta/\alpha} Y\left(1, t^{-\beta/\alpha}x\right). \tag{4.12}$$

Remark 4.2. By choosing $\alpha = 2$, $d = 1$ and β arbitrarily close to 2, one can see that the first condition in (1.10) suggests the condition $\gamma > -1$. However, when $\gamma \in (-1,0)$, one needs to specify another initial condition, namely, $I_t^{1-\gamma} f(t,x)\Big|_{t=0}$. For example (Example 4.1 in [31, p. 138]), the differential equation

$${}_tD_+^{1/2} g(t) + g(t) = 0, \qquad (t>0); \qquad I_t^{1/2} g(t)\Big|_{t=0} = C,$$

is solved by $g(t) = C\left(\frac{1}{\sqrt{\pi t}} - e^t \operatorname{erfc}(\sqrt{t})\right)$. This initial condition is obscure when the driving term f becomes the multiplicative noise $\rho(u(t,x))\dot{W}(t,x)$. Hence, throughout this paper, we assume $\gamma \geq 0$.

This following lemma gives the asymptotics of fundamental solutions $Z_{\alpha,\beta,d}(1,x)$, $Y_{\alpha,\beta,\gamma,d}(1,x)$, and $Z^*_{\alpha,\beta,d}(1,x)$ at $x = 0$ by choosing suitable values for η:

$$\eta = \begin{cases} \lceil\beta\rceil & \text{in case of } Z, \\ \beta+\gamma & \text{in case of } Y, \\ 1 & \text{in case of } Z^*, \text{ when } \beta \in (1,2). \end{cases}$$

Lemma 4.3. *Suppose $\alpha \in (0,2]$, $\beta \in (0,2)$, $\eta \in \mathbb{R}$, and $d \in \mathbb{N}$. Let*

$$g(x) = x^{-d} H^{2,1}_{2,3}\left(x^{\alpha} \,\middle|\, \begin{matrix} (1,1),\ (\eta,\beta) \\ (d/2,\alpha/2),\ (1,1),\ (1,\alpha/2) \end{matrix}\right), \quad x > 0.$$

Then as $x \to 0_+$, the following holds:

$$g(x) = \begin{cases} \dfrac{\Gamma(d-\alpha)/2}{\Gamma(\eta-\beta)\Gamma(\alpha/2)}\, x^{\alpha-d} + O(x^{\min(2\alpha-d,0)}) & \text{if } \eta \neq \beta \text{ and } d > \alpha: \quad \text{Case 1,} \\ -\dfrac{\alpha}{\Gamma(\eta-\beta)\Gamma(1+d/2)}\, \log x + O(1) & \text{if } \eta \neq \beta \text{ and } d = \alpha: \quad \text{Case 2,} \\ \dfrac{2}{\alpha}\, \dfrac{\Gamma(1-d/\alpha)\Gamma(d/\alpha)}{\Gamma(\eta - d\beta/\alpha)\Gamma(d/2)} + O(x^{\alpha-d}) & \text{if } \eta \neq \beta \text{ and } d < \alpha: \quad \text{Case 3,} \\ \dfrac{2\Gamma(d/\alpha)}{\alpha\Gamma(d/2)} + O(x^2) & \text{if } \beta = \eta = 1: \quad \text{Case 4,} \\ -\dfrac{\Gamma((d-2\alpha)/2)}{\Gamma(-\beta)\Gamma(\alpha)}\, x^{2\alpha-d} + O(x^{\min(3\alpha-d,0)}) & \text{if } \beta = \eta \neq 1 \text{ and } d/\alpha > 2: \quad \text{Case 5,} \\ \dfrac{2\alpha}{\Gamma(-\beta)\Gamma(1+d/2)}\, \log x + O(x^{\alpha}) & \text{if } \beta = \eta \neq 1 \text{ and } d/\alpha = 2: \quad \text{Case 6,} \\ \dfrac{2}{\alpha}\, \dfrac{\Gamma(1-d/\alpha)\Gamma(d/\alpha)}{\Gamma(\beta(1-d/\alpha))\Gamma(d/2)} + O(x^{2\alpha-d}) & \text{if } \beta = \eta \neq 1 \text{ and } d/\alpha \in (1,2): \quad \text{Case 7,} \\ -\dfrac{\beta}{\Gamma(1+d/2)} + O(x^{\alpha}) & \text{if } \beta = \eta \neq 1 \text{ and } d/\alpha = 1: \quad \text{Case 8,} \\ \dfrac{2}{\alpha}\, \dfrac{\Gamma(1-d/\alpha)\Gamma(d/\alpha)}{\Gamma(\beta(1-d/\alpha))\Gamma(d/2)} + O(x^{\alpha}) & \text{if } \beta = \eta \neq 1 \text{ and } d/\alpha < 1: \quad \text{Case 9,} \end{cases}$$

where all the coefficients of the leading terms are finite and nonvanishing.

The calculations in the proof of this lemma is quite lengthy. We postpone it to Appendix A.1.

Remark 4.4. Since Dalang's condition (1.9) implies $d < 2\alpha$, the cases 5 and 6 are void under (1.9). Combining the rest seven cases in Lemma 4.3, we have that

$$\lim_{x\to 0} Y_{\alpha,\beta,\gamma,d}(1,x) = \begin{cases} +\infty & \text{if } \gamma > 0 \text{ and } \alpha \le d < 2\alpha: & \text{Cases 1–2,} \\ C_1 & \text{if } \gamma > 0 \text{ and } \alpha > d = 1: & \text{Case 3,} \\ C_2 & \text{if } \gamma = 0,\ \beta = 1 \text{ and } \alpha \neq d: & \text{Case 4,} \\ C_3 & \text{if } \gamma = 0,\ \beta \neq 1 \text{ and } d < 2\alpha: & \text{Cases 7–9,} \end{cases} \tag{4.13}$$

and

$$\lim_{x\to 0} Z_{\alpha,\beta,d}(1,x) = \begin{cases} +\infty & \text{if } \beta \neq 1 \text{ and } \alpha \le d < 2\alpha: & \text{Cases 1–2,} \\ C_4 & \text{if } \beta \neq 1 \text{ and } \alpha > d = 1: & \text{Case 3,} \\ C_2 & \text{if } \beta = 1 \text{ and } \alpha \neq d: & \text{Case 4,} \end{cases} \tag{4.14}$$

and when $\beta \in (1,2)$,

$$\lim_{x\to 0} Z^*_{\alpha,\beta,d}(1,x) = \begin{cases} +\infty & \text{if } \alpha \le d < 2\alpha: & \text{Cases 1–2,} \\ C_5 & \text{if } \alpha > d = 1: & \text{Case 3,} \end{cases} \tag{4.15}$$

where the constants $C_i \in \mathbb{R} \setminus \{0\}$, $i = 1, \ldots, 5$, only depend on α, β, γ and d. Combining all these cases, we see that under (1.11), $Y(1,x)$ is bounded at $x = 0$, and under (1.12), all functions $Z(1,x)$, $Z^*(1,x)$ and $Y(1,x)$ are bounded at $x = 0$.

Lemma 4.5. *$Y_{\alpha,\beta,\gamma,d}(1,x)$ has the following asymptotic property as $|x|\to\infty$:*

$$Y_{\alpha,\beta,\gamma,d}(1,x)\sim\begin{cases}A_\alpha|x|^{-(d+\alpha)} & \text{if } \alpha\neq 2,\\ A_2|x|^a e^{-b|x|^c} & \text{if } \alpha=2,\end{cases} \tag{4.16}$$

where the nonnegative constants are

$$A_\alpha=\begin{cases}-\pi^{-d/2}\nu 2^{\alpha-1}\frac{\Gamma((d+\alpha)/2)}{\Gamma(2\beta+\gamma)\Gamma(-\alpha/2)} & \text{if } \alpha\neq 2,\\ \pi^{-d/2}(2-\beta)^{d/2-(\beta+\gamma)}\beta^{\frac{\beta(d+2-2(\beta+\gamma))}{2(2-\beta)}}(2\nu)^{\frac{2(\beta+\gamma)-(d+2)}{2(2-\beta)}} & \text{if } \alpha=2,\end{cases} \tag{4.17}$$

and

$$a=\frac{d(\beta-1)-2(\beta+\gamma-1)}{2-\beta},\qquad b=(2-\beta)\beta^{\frac{\beta}{2-\beta}}(2\nu)^{\frac{1}{\beta-2}},\qquad \text{and}\qquad c=\frac{2}{2-\beta}. \tag{4.18}$$

Moreover, the asymptotic properties for $Z(1,x)$ and $Z^*(1,x)$ are the same as that for $Y(1,x)$ except that the argument γ in both (4.17) and (4.18) *should be replaced by $\lceil\beta\rceil-\beta$ and 1, respectively.*

These asymptotics are obtained from [22, Sections 1.5 and 1.7]. We leave the details for interested readers.

Theorem 4.6. *Suppose that $\alpha\in(0,2]$, $\beta\in(0,2)$, and $\gamma\geq 0$. The functions $Z(t,x):=Z_{\alpha,\beta,d}(t,x)$, $Y(t,x):=Y_{\alpha,\beta,\gamma,d}(t,x)$ and $Z^*(t,x):=Z^*_{\alpha,\beta,d}(t,x)$, defined in Theorem 4.1, satisfy the following properties:*

(1) For all $d\in\mathbb{N}$ and $\beta\in(0,1)$, both functions Z and Y are nonnegative. When $\beta=1$, Z is nonnegative, and Y is nonnegative if either $\gamma=0$ or $\gamma>1$;
(2) All functions Z, Z^* and Y are nonnegative if $d\leq 3$ and $1<\beta<\alpha\leq 2$. When $1<\beta=\alpha<2$, Y is nonnegative if $\gamma>(d+3)/2-\beta$;
(3) When $d\geq 4$, $Y_{\alpha,\beta,0,d}(t,x)$ assumes both positive and negative values for all $\alpha\in(0,2]$ and $\beta\in(1,2)$.

This theorem is proved in Section 4.3. It generalizes the results by Mainardi et al. [26] from one-space dimension to higher space-dimension. Moreover, in [26] only Z when $\beta\in(0,1]$ and Z^* when $\beta\in(1,2)$ are studied. When $\beta\in(1,2)$, it also generalizes the results by Pskhu [32] from $\alpha=2$ and $\gamma=0$ to general $\alpha\in(0,2]$ and $\gamma>-1$.

4.1. Some special cases

In this part, we list some special cases.

Example 4.7. When $\gamma=0$ or $\gamma=\lceil\beta\rceil-\beta$, the expressions for Z, Y and Z^* in Theorem 4.1 recover the results in [9].

Example 4.8. When $\alpha=2$, by [22, Property 2.2], we see that

$$Z_{2,\beta,d}(t,x)=\pi^{-d/2}t^{\lceil\beta\rceil-1}|x|^{-d}H^{2,0}_{1,2}\left(\frac{|x|^2}{2\nu t^\beta}\,\middle|\,\begin{matrix}(\lceil\beta\rceil,\beta)\\ (d/2,1),\ (1,1)\end{matrix}\right), \tag{4.19}$$

and

$$Y_{2,\beta,\gamma,d}(t,x)=\pi^{-d/2}t^{\beta+\gamma-1}|x|^{-d}H^{2,0}_{1,2}\left(\frac{|x|^2}{2\nu t^\beta}\,\middle|\,\begin{matrix}(\beta+\gamma,\beta)\\ (d/2,1),\ (1,1)\end{matrix}\right), \tag{4.20}$$

and, when $\beta \in (1, 2)$,

$$Z^*_{2,\beta,d}(t, x) = \pi^{-d/2}|x|^{-d} H^{2,0}_{1,2}\left(\frac{x^2}{2\nu t^\beta} \,\middle|\, \begin{matrix}(1, \beta)\\(d/2, 1),\ (1, 1)\end{matrix}\right). \tag{4.21}$$

In particular, for $\beta \in (0, 1)$ and $\gamma = 0$, the expressions for Z and Y recover those in [19,24]. For $Z_{2,\beta,d}$, see also [23, Chapter 6]. When $\beta \in (1, 2)$, $\gamma = 0$ and $\nu = 2$, the expression for Y recovers the result in [32].

Example 4.9. When $\alpha = 2$ and $d = 1$, using Lemma A.2 and (A.8), we see that

$$Z_{2,\beta,1}(t, x) = |x|^{-1} t^{\lceil\beta\rceil-1} H^{1,0}_{1,1}\left(\frac{2x^2}{\nu t^\beta} \,\middle|\, \begin{matrix}(\lceil\beta\rceil, \beta)\\(1, 2)\end{matrix}\right) = \frac{t^{\lceil\beta\rceil-1-\beta/2}}{\sqrt{2\nu}} M_{\beta/2,\lceil\beta\rceil}\left(\frac{|x|}{\sqrt{\nu/2}\, t^{\beta/2}}\right), \tag{4.22}$$

and

$$Y_{2,\beta,\gamma,1}(t, x) = |x|^{-1} t^{\beta+\gamma-1} H^{1,0}_{1,1}\left(\frac{2x^2}{\nu t^\beta} \,\middle|\, \begin{matrix}(\beta+\gamma, \beta)\\(1, 2)\end{matrix}\right) = \frac{t^{\beta/2+\gamma-1}}{\sqrt{2\nu}} M_{\beta/2,\beta+\gamma}\left(\frac{|x|}{\sqrt{\nu/2}\, t^{\beta/2}}\right), \tag{4.23}$$

and, when $\beta \in (1, 2)$,

$$Z^*_{2,\beta,1}(t, x) = |x|^{-1} H^{1,0}_{1,1}\left(\frac{2x^2}{\nu t^\beta} \,\middle|\, \begin{matrix}(1, \beta)\\(1, 2)\end{matrix}\right) = \frac{t^{-\beta/2}}{\sqrt{2\nu}} M_{\beta/2,1}\left(\frac{|x|}{\sqrt{\nu/2}\, t^{\beta/2}}\right), \tag{4.24}$$

where $M_{\lambda,\mu}(z)$ is the *two-parameter Mainardi functions* (see [4]) of order $\lambda \in [0, 1)$,

$$M_{\lambda,\mu}(z) := \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{n!\, \Gamma(\mu - (n+1)\lambda)}, \qquad \text{for } \mu \text{ and } z \in \mathbb{C}. \tag{4.25}$$

For example, $M_{1/2,1}(z) = \frac{1}{\sqrt{\pi}} \exp\left(-z^2/4\right)$. The *one-parameter Mainardi functions* $M_\lambda(z)$ are used by Mainardi, et al. in [25,26].

Example 4.10. In [26], the fundamental solutions $Z_{\alpha,\beta,d}(t, x)$ for $\beta \in (0, 1]$ and $Z^*_{\alpha,\beta,d}(t, x)$ for $\beta \in (1, 2]$ have been studied for all $\alpha \in (0, 2)$ and $d = 1$. From the Mellin–Barnes integral representation (6.6) of [26], we can see that the reduced Green function of [26] can be expressed using the Fox H-function:

$$K^\theta_{\alpha,\beta}(x) = \frac{1}{|x|} H^{2,1}_{3,3}\left(|x|^\alpha \,\middle|\, \begin{matrix}(1, 1),\ (1, \beta),\ (1, \frac{\alpha-\theta}{2})\\(1, 1),\ (1, \alpha),\ (1, \frac{\alpha-\theta}{2})\end{matrix}\right), \qquad x \in \mathbb{R}, \tag{4.26}$$

where α and β have the same meaning as this paper and θ is the skewness: $|\theta| \le \min(\alpha, 2-\alpha)$. For the symmetric α-stable case, i.e., $\theta = 0$, this expression can be simplified using Lemma A.2. Hence,

$$K^0_{\alpha,\beta}(x) = \frac{1}{\sqrt{\pi}|x|} H^{2,1}_{2,3}\left((|x|/2)^\alpha \,\middle|\, \begin{matrix}(1, 1),\ (1, \beta)\\(1/2, \alpha/2),\ (1, 1),\ (1, \alpha/2)\end{matrix}\right), \qquad x \in \mathbb{R}. \tag{4.27}$$

Therefore, their fundamental solution [26, (1.3)]

$$G^0_{\alpha,\beta}(x, t) = t^{-\beta/\alpha} K^0_{\alpha,\beta}(t^{-\beta/\alpha} x) = \frac{1}{\sqrt{\pi}|x|} H^{2,1}_{2,3}\left(\frac{|x|^\alpha}{2^\alpha t^\beta} \,\middle|\, \begin{matrix}(1, 1),\ (1, \beta)\\(1/2, \alpha/2),\ (1, 1),\ (1, \alpha/2)\end{matrix}\right)$$

corresponds, in the case when $\nu = 2$, to our $Z_{\alpha,\beta,1}(t, x)$ when $\beta \in (0, 1]$ and $Z^*_{\alpha,\beta,1}(t, x)$ when $\beta \in (1, 2)$.

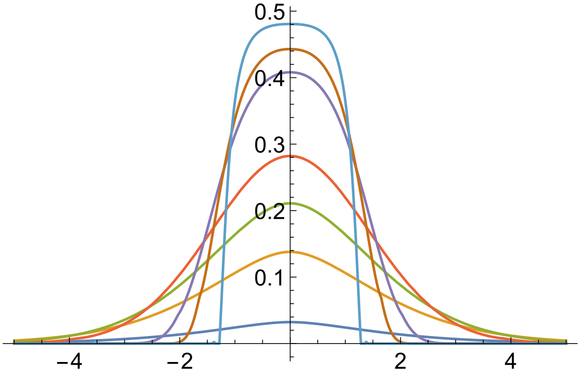
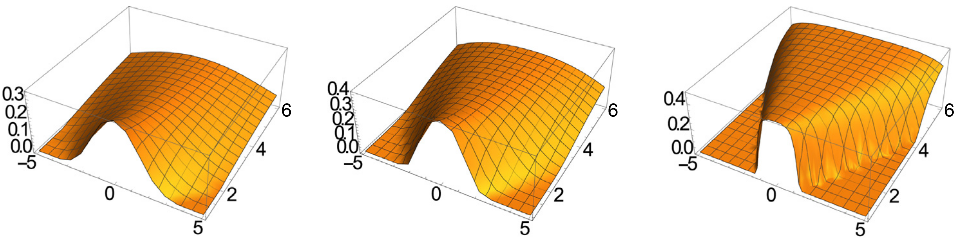


Fig. 1. Some graphs of the function $Y_{2,\beta,0,1}(1, x)$ with $\nu = 2$, and $\beta = 15/8, 5/3, 3/2, 1, 3/4, 1/2$, and $1/8$ from top to bottom.



(a) $\beta = 6/5$.

(b) $\beta = 3/2$.

(c) $\beta = 15/8$.

Fig. 2. Graphs of the Green functions $Y_{2,\beta,0,1}(t, x)$ for $1 < \beta < 2$ for $1 \le t \le 6$ and $|x| \le 5$.

Here we draw some graphs[5] of these Green functions $Y(t, x)$: see Figs. 1 and 2. As β approaches 2, the graphs of $Y(t, x)$ become closer to the wave kernel $\frac{1}{2}1_{|x|\le\nu t/2}$.

4.2. *Proof of* Theorem 4.1

Proof of Theorem 4.1. Eqs.(4.4)–(4.9) have been proved in [9] when $\gamma = 0$. Let $\widehat{f}$ and $\widetilde{g}$ denote the Fourier transform in the space variable and the Laplace transform in the time variable, respectively. Apply the Fourier transform to (4.1) to obtain

$$\begin{cases} \partial^\beta \widehat{u}(t,\xi) + \dfrac{\nu}{2}|\xi|^\alpha \widehat{u}(t,\xi) = I_t^\gamma\left[\widehat{f}(t,\xi)\right], & \xi \in \mathbb{R}^d \\ \dfrac{\partial^k}{\partial t^k}\widehat{u}(t,\xi)\Big|_{t=0} = \widehat{u}_k(\xi), & 0 \le k \le \lceil\beta\rceil - 1, \xi \in \mathbb{R}^d . \end{cases}$$

Apply the Laplace transform on the Caputo derivative using [18, Theorem 7.1]:

$$\mathcal{L}\left[\partial^\beta\, \widehat{u}(t,\xi)\right](s) = s^\beta\, \widetilde{\widehat{u}}(s,\xi) - \sum_{k=0}^{\lceil\beta\rceil-1} s^{\beta-1-k}\, \widehat{u}_k(\xi).$$

[5] The graphs are produced by truncating the infinite sum in (4.25) by the first 24 terms. In Fig. 2, due to the bad approximations for small t when truncating the infinite sum, the graphs are produced for t staying away from 0.

On the other hand, it is known that (see, e.g., [33, (7.14)]),

$$\mathcal{L}I_t^{\gamma}\left[\widehat{f}(t,\xi)\right] = s^{-\gamma}\widetilde{\widehat{f}}(s,\xi), \quad \mathrm{Re}(\gamma) > 0.$$

Thus,

$$\widetilde{\widehat{u}}(s,\xi) = \left(s^{\beta} + \frac{\nu}{2}|\xi|^{\alpha}\right)^{-1}\left[\sum_{k=0}^{\lceil\beta\rceil-1} s^{\beta-1-k}\,\widehat{u}_k(\xi) + s^{-\gamma}\widetilde{\widehat{f}}(s,\xi)\right].$$

Notice that (see [31, (1.80)])

$$\mathcal{L}\left[t^{\beta-1}E_{\alpha,\beta}(-\lambda t^{\alpha})\right](s) = \frac{s^{\alpha-\beta}}{s^{\alpha}+\lambda}, \quad \text{for } \mathrm{Re}(s) > |\lambda|^{1/\alpha}.$$

Hence,

$$\widehat{u}(t,\xi) = \sum_{k=0}^{\lceil\beta\rceil-1} t^k E_{\beta,k+1}\left(-\frac{\nu}{2}|\xi|^{\alpha}t^{\beta}\right)\widehat{u}_k(\xi) + \int_0^t \mathrm{d}\tau\; \tau^{\beta+\gamma-1}E_{\beta,\beta+\gamma}\left(-\frac{\nu}{2}|\xi|^{\alpha}\tau^{\beta}\right)\widehat{f}(\tau,\xi),$$

from which (4.7)–(4.9) are proved. The expressions for Z and Z^* in (4.4) and (4.6), respectively, are proved in [9]. By the fact that (see [31, (1.82)])

$${}_tD_+^{\gamma}\left(t^{\beta-1}E_{\alpha,\beta}(\lambda t^{\alpha})\right) = t^{\beta-\gamma-1}E_{\alpha,\beta-\gamma}(\lambda t^{\alpha}), \quad \gamma \in \mathbb{R}.$$

Recall that ${}_tD_+^{\alpha}$ is the Riemann–Liouville fractional derivative of order $\alpha \in \mathbb{R}$ (see (2.8)). Hence, we see that

$$Y(t,x) = {}_tD_+^{\theta}Z(t,x), \quad \text{with } \theta := \lceil\beta\rceil - \beta - \gamma,$$

which can be evaluated using [22, Theorem 2.8] in the same way as in [9] for the case $\gamma = 0$. This completes the proof of Theorem 4.1. □

4.3. Nonnegativity of the fundamental solutions (proof of Theorem 4.6)

We first prove some lemmas.

Lemma 4.11. *The following Fox H-functions are nonnegative:*

(1) for all $\theta \in (0,1)$,

$$H_{2,2}^{1,1}\left(x\,\middle|\,\begin{matrix}(0,1),\ (0,\theta)\\ (0,1),\ (0,\theta)\end{matrix}\right) = \frac{1}{\pi}\frac{x^{1/\theta}}{1+2x^{1/\theta}\cos(\pi\theta)+x^{2/\theta}} > 0, \quad \textit{for } x > 0; \tag{4.28}$$

(2) for all $\mu > 0$ *and* $0 < \theta \le \min(1,\mu)$,

$$\mathbb{R} \ni x \mapsto H_{1,1}^{1,0}\left(|x|\,\middle|\,\begin{matrix}(\mu,\theta)\\ (1,1)\end{matrix}\right) \ge 0. \tag{4.29}$$

(3) for all $d \in \mathbb{N}$ *and* $\alpha \in (0,2]$,

$$\mathbb{R} \ni x \mapsto H_{1,2}^{1,1}\left(|x|\,\middle|\,\begin{matrix}(1,1)\\ (d/2,\alpha/2),\ (1,\alpha/2)\end{matrix}\right) > 0. \tag{4.30}$$

Proof. (2) and (3) are covered by Lemma 4.5 and Theorem 3.3 of [9], respectively. As for (1), expression (4.28) can be found in [26, (4.38)] for the *neutral-fractional diffusions*. For completeness, we give a proof here. Because the parameters Δ and δ, defined in (A.2) and (A.5), of this Fox H-function are equal to 0 and 1, respectively, Theorem 1.3 implies that for $x \in (0, 1)$,

$$\begin{aligned} H^{1,1}_{2,2}\left(x \,\middle|\, \begin{matrix}(0,1),\ (0,\theta)\\(0,1),\ (0,\theta)\end{matrix}\right) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\Gamma(k+1)}{\Gamma(-k\theta)\Gamma(1+k\theta)} x^{k/\theta} \\ &= \sum_{k=0}^{\infty} -(-1)^k \sin(\pi k\theta) x^{k/\theta} \qquad (4.31)\\ &= \operatorname{Im}\left(\sum_{k=0}^{\infty} -(-1)^k e^{k\theta i} x^{k/\theta}\right) \\ &= -\operatorname{Im} \frac{1}{1+e^{\theta i} x^{1/\theta}} \\ &= \frac{1}{\pi} \frac{x^{1/\theta}}{1+2x^{1/\theta}\cos(\pi\theta)+x^{2/\theta}}, \end{aligned}$$

where we have applied [30, (5.5.3)] in (4.31). Similarly, when $x > 1$, Theorem 1.4 of [23] implies that

$$\begin{aligned} H^{1,1}_{2,2}\left(x \,\middle|\, \begin{matrix}(0,1),\ (0,\theta)\\(0,1),\ (0,\theta)\end{matrix}\right) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\Gamma(k+1)}{\Gamma(-(1+k)\theta)\Gamma(1+(1+k)\theta)} x^{-(1+k)/\theta} \\ &= \sum_{k=0}^{\infty} (-1)^k \sin(\pi(1+k)\theta) x^{-(1+k)/\theta} \\ &= \operatorname{Im}\left(\sum_{k=0}^{\infty} -(-1)^k e^{k\theta i} x^{-k/\theta}\right) \\ &= -\operatorname{Im} \frac{1}{1+e^{\theta i} x^{-1/\theta}} \\ &= \frac{1}{\pi} \frac{x^{1/\theta}}{1+2x^{1/\theta}\cos(\pi\theta)+x^{2/\theta}}. \end{aligned}$$

Finally, the existence of this Fox H-function at $x = 1$ is not covered by Theorem 1.1 of [22] because $\Delta = 0$ and $\mu = 0$ (see (A.4) for the definition of the parameter μ). In fact, as one can see that the series in (4.31) with $x = 1$ diverges. Nevertheless, we may define that

$$H^{1,1}_{2,2}\left(1 \,\middle|\, \begin{matrix}(0,1),\ (0,\theta)\\(0,1),\ (0,\theta)\end{matrix}\right) := \lim_{x\to 1} \frac{1}{\pi} \frac{x^{1/\theta}}{1+2x^{1/\theta}\cos(\pi\theta)+x^{2/\theta}} = \frac{1}{2\pi} \frac{1}{1+\cos(\pi\theta)} > 0.$$

This completes the proof of Lemma 4.11. □

Lemma 4.12. *For* $\mu \in (0, 2]$, $\theta \in (0, 2]$ *and* $d \geq 1$, *the function*

$$f_{d,\mu,\theta}(x) := x^{-d} H^{2,0}_{1,2}\left(x^2 \,\middle|\, \begin{matrix}(\mu,\theta)\\(d/2,1),\ (1,1)\end{matrix}\right), \quad x > 0,$$

has the following properties:

(a) $\frac{\mathrm{d}}{\mathrm{d}x} f_{d,\mu,\theta}(x) = -2x f_{d+2,\mu,\theta}(x)$.

(b) $f_{d,\mu,\theta}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty \mathrm{d}z \frac{z}{\sqrt{z^2 - x^2}} f_{d+1,\mu,\theta}(z)$ *for* $x > 0$.

(c) $f_{d,\mu,\theta}(x) \geq 0$ *for all* $x > 0$ *if* $\theta \leq 2\min(1, \mu)$ *and* $d \leq 3$.

Proof. (a) Apply [22, Property 2.8] with $k = 1$, $w = -d$, $c = 1$ and $\sigma = 2$ to get

$$\frac{\mathrm{d}}{\mathrm{d}x} f_{d,\mu,\theta}(x) = x^{-d-1} H^{2,1}_{2,3}\left(x^2 \,\middle|\, \begin{matrix} (d,2),\ (\mu,\theta) \\ (d/2,1),\ (1,1),\ (1+d,2) \end{matrix}\right).$$

By the recurrence relation of the Gamma function, we see that

$$\frac{\Gamma(1-d-2s)}{\Gamma(-d-2s)}\,\Gamma(d/2+s) = -2(s+d/2)\Gamma(d/2+s) = -2\Gamma(1+d/2+s).$$

By the definition of the Fox H-function, the above expression can be simplified as

$$\frac{\mathrm{d}}{\mathrm{d}x} f_{d,\mu,\theta}(x) = -2x^{-d-1} H^{2,1}_{1,2}\left(x^2 \,\middle|\, \begin{matrix} (\mu,\theta) \\ ((d+2)/2,1),\ (1,1) \end{matrix}\right) = -2x f_{d+2,\mu,\theta}(x).$$

(b) By the definition of the Fox H-function,

$$f_{d+1,\mu,\theta}(x) = x^{-(d+1)} \frac{1}{2\pi i} \int_{L_{i\gamma\infty}} \frac{\Gamma((d+1)/2+s)\Gamma(1+s)}{\Gamma(1-\mu-\theta s)} x^{-2s} \mathrm{d}s, \quad \text{for any } \gamma > -1, \tag{4.32}$$

where the contour $L_{i\gamma\infty}$ is defined in Definition A.1. Assuming that we can switch the integrals, which can be made rigorous by writing f in the series form and applying Fubini's theorem, we see that

$$\begin{aligned} \int_x^\infty \mathrm{d}z \,\frac{z}{\sqrt{z^2-x^2}} f_{d+1,\mu,\theta}(z) = {} & \frac{1}{2\pi i} \int_{L_{i\gamma\infty}} \mathrm{d}s \,\frac{\Gamma((d+1)/2+s)\Gamma(1+s)}{\Gamma(1-\mu-\theta s)} \\ & \times \int_x^\infty \mathrm{d}z \,\frac{z^{-2s-d}}{\sqrt{z^2-x^2}}. \end{aligned}$$

By change of variable $(z/x)^2 - 1 = y$ and Euler's Beta integral (see, e.g., [30, 5.12.3 on p.142]), we see that

$$\int_x^\infty \mathrm{d}z \,\frac{z^{-2s-d}}{\sqrt{z^2-x^2}} = \frac{x^{-2s-d}}{2} \int_0^\infty y^{\frac{1}{2}-1}(1+y)^{-\frac{1}{2}-\frac{2s+d}{2}} \mathrm{d}y = \frac{x^{-2s-d}}{2} \frac{\sqrt{\pi}\,\Gamma(d/2+s)}{\Gamma((d+1)/2+s)}.$$

Note that the above integral is convergent provided that $\mathrm{Re}(2s+d) > 0$, which is satisfied by choosing, e.g., $\gamma = \mathrm{Re}(s) = 0$ in (4.32). Therefore,

$$\begin{aligned} \int_x^\infty \mathrm{d}z \,\frac{z}{\sqrt{z^2-x^2}} f_{d+1,\mu,\theta}(z) &= \frac{\sqrt{\pi}}{2} x^{-d} \frac{1}{2\pi i} \int_{L_{i\gamma\infty}} \frac{\Gamma(d/2+s)\Gamma(1+s)}{\Gamma(1-\mu-\theta s)} x^{-2s} \mathrm{d}s \\ &= \frac{\sqrt{\pi}}{2} f_{d,\mu,\theta}(x). \end{aligned}$$

(c) By the recurrence in (b), we only need to prove the case $d = 3$. Apply Lemma A.2, Properties 2.4 and 2.5 in [23] to obtain

$$\begin{aligned} f_{3,\mu,\theta}(x) &= \frac{\sqrt{\pi}}{4} x^{-3} H_{1,1}^{1,0}\left(4x^2 \,\middle|\, \begin{matrix} (\mu,\theta) \\ (2,2) \end{matrix}\right) \\ &= \frac{\sqrt{\pi}}{8} x^{-3} H_{1,1}^{1,0}\left(2x \,\middle|\, \begin{matrix} (\mu,\theta/2) \\ (2,1) \end{matrix}\right) \\ &= \frac{\sqrt{\pi}}{16} x^{-4} H_{1,1}^{1,0}\left(2x \,\middle|\, \begin{matrix} (\mu-\theta/2,\theta/2) \\ (1,1) \end{matrix}\right). \end{aligned}$$

Then (c) is proved by an application of part (2) of Lemma 4.11. □

Proof of Theorem 4.6. By comparing the Fox H-functions in (4.4), (4.5), and (4.6), We only need to consider the following Fox H-function:

$$g(x) = H_{2,3}^{2,1}\left(x \,\middle|\, \begin{matrix} (1,1),\ (\eta,\beta) \\ (d/2,\alpha/2),\ (1,1),\ (1,\alpha/2) \end{matrix}\right), \quad x > 0.$$

The parameter η takes the following values

$$\eta = \begin{cases} \lceil \beta \rceil & \text{in case of } Z, \\ \beta+\gamma & \text{in case of } Y, \\ 1 & \text{in case of } Z^*. \end{cases}$$

(1) If $\beta = 1$ and $\gamma = 0$, then $Z = Y$ and by Property 2.2 of [23],

$$g(x) = H_{1,2}^{2,0}\left(x \,\middle|\, \begin{matrix} (1,1) \\ (d/2,\alpha/2),\ (1,\alpha/2) \end{matrix}\right), \qquad x > 0$$

which is positive by part (3) of Lemma 4.11. If $\beta < 1$, then we can apply Theorem A.5 to obtain that

$$g(x) = \int_0^\infty H_{1,2}^{1,1}\left(t \,\middle|\, \begin{matrix} (1,1) \\ (d/2,\alpha/2),\ (1,\alpha/2) \end{matrix}\right) H_{1,1}^{1,0}\left(\frac{x}{t} \,\middle|\, \begin{matrix} (\eta,\beta) \\ (1,1) \end{matrix}\right) \frac{\mathrm{d}t}{t}. \tag{4.33}$$

In fact, conditions (A.12) are satisfied because

$$A_1 = 0, \quad B_1 = d/\alpha, \quad A_2 = 1, \quad B_2 = \infty.$$

Moreover, $a_1^* = 1$ and $\beta \in (0,1)$ implies that $a_2^* = 1-\beta > 0$. Hence, condition (1) of Theorem A.5 is satisfied. This proves (4.33). If $\beta = 1$ and $\gamma > 0$, then $a_2^* = \Delta_2 = 0$. In view of condition (3) of Theorem A.5, relation (4.33) is still true if $\mathrm{Re}(\mu_2) > -1$ with $\mu_2 = 1-\eta$, i.e., $\gamma > 1$. The two Fox H-functions in (4.33) are nonnegative by parts (2) and (3) of Lemma 4.11.

(2) In this case, we have that $d \le 3$. When $\alpha = 2$, by Property 2.2 of [23] and Lemma 4.12,

$$g(x) = H_{1,2}^{2,0}\left(x \,\middle|\, \begin{matrix} (\eta,\beta) \\ (d/2,1),\ (1,1) \end{matrix}\right) = x^{d/2} f_{d,\eta,\beta}(\sqrt{x}) \ge 0, \quad x > 0,$$

because $\beta < 2 \le 2\min(1,\eta)$. If $\alpha \ne 2$, by Property 2.2 of [23], we see that

$$g(x) = H_{3,4}^{3,1}\left(x \,\middle|\, \begin{matrix} (1,1),\ (\eta,\beta),\ (1,\alpha/2) \\ (d/2,\alpha/2),\ (1,\alpha/2),\ (1,1),\ (1,\alpha/2) \end{matrix}\right), \quad x > 0,$$

As in the previous case, by Theorem A.5, we see that

$$\int_0^\infty H^{2,0}_{1,2}\left(t \,\middle|\, \begin{matrix}(\eta,\beta)\\ (d/2,\alpha/2),\ (1,\alpha/2)\end{matrix}\right) H^{1,1}_{2,2}\left(\frac{x}{t} \,\middle|\, \begin{matrix}(1,1),\ (1,\alpha/2)\\ (1,1),\ (1,\alpha/2)\end{matrix}\right)\frac{\mathrm{d}t}{t}$$
$$= H^{3,1}_{3,4}\left(x \,\middle|\, \begin{matrix}(1,1),\ (\eta,\beta),\ (1,\alpha/2)\\ (d/2,\alpha/2),\ (1,\alpha/2),\ (1,1),\ (1,\alpha/2)\end{matrix}\right). \tag{4.34}$$

Note that condition (A.12) is satisfied because in this case,

$$A_1 = \infty, \quad B_1 = \min(d,2)/\alpha, \quad A_2 = 1, \quad B_2 = 0.$$

When $\alpha < \beta$, then

$$a_1^* = \alpha - \beta > 0, \quad a_2^* = 2-\alpha > 0,$$

and condition (1) in Theorem A.5 is satisfied. When $1 < \beta = \alpha < 2$, then

$$a_2^* = 2-\alpha > 0, \quad a_1^* = \Delta_1 = 0, \quad \mathrm{Re}(\mu_1) = 1 + \frac{d}{2} - \eta - \frac{1}{2}.$$

Hence, in view of condition (2) of Theorem A.5, the integral (4.34) is still true if $1 + d/2 - \eta - 1/2 < -1$, i.e., $\gamma > (d+3)/2 - \beta$.

Now, by Property 2.4 of [23], the first Fox H-function in (4.34) is equal to

$$\frac{2}{\alpha} H^{2,0}_{1,2}\left(t^{2/\alpha} \,\middle|\, \begin{matrix}(\eta,2\beta/\alpha)\\ (d/2,1),(1,1)\end{matrix}\right) = \frac{2}{\alpha} t^{d/\alpha} f_{d,\eta,2\beta/\alpha}(t^{1/\alpha}).$$

By Lemma 4.12(c), we see that under the condition that $\frac{2\beta}{\alpha} \le 2\min(1,\eta)$, the first Fox H-function in (4.34) is nonnegative. This condition is satisfied if $1 \le \beta \le \alpha \le 2$. By Property 2.3 in [22], the second Fox H-function in (4.34) is equal to

$$H^{1,1}_{2,2}\left(\frac{t}{x} \,\middle|\, \begin{matrix}(0,1),\ (0,\alpha/2)\\ (0,1),\ (0,\alpha/2)\end{matrix}\right).$$

Thanks to Lemma 4.11(1), this function is strictly positive for $t/x \neq 0$ when $\alpha \in (0,2)$.

(3) Now we consider the case when $d \ge 4$. The case $\alpha = 2$ is covered by Lemma 25 of [32]. In the following, we assume that $\alpha \in (0,2)$. By the scaling property, we may only consider the case $t = 1$. Hence, it suffices to study the following function

$$g(x) = x^{-d} H^{2,1}_{2,3}\left(x^\alpha \,\middle|\, \begin{matrix}(1,1),\ (\beta,\beta)\\ (d/2,\alpha/2),\ (1,1),\ (1,\alpha/2)\end{matrix}\right), \quad x > 0.$$

Because $a^* = 2 - \beta > 0$, we can apply Theorem 1.7 of [23] to obtain that

$$g(x) = -\frac{\Gamma((d+\alpha)/2)}{\Gamma(2\beta)\Gamma(-\alpha/2)} x^{-d-1} + O(x^{-(d+1)}), \quad \text{as } x \to \infty.$$

The condition $\alpha \in (0,2)$ implies that $\Gamma(-\alpha/2) < 0$. Thus, the coefficient of x^{-d-1} is positive. Hence, g can assume positive values.

Now we consider the behavior of $g(x)$ around zero. Because $\beta > 1$ and $2\alpha < 4 \le d$, we can apply the case 6 of Lemma 4.3:

$$g(x) = -\frac{\Gamma((d-2\alpha)/2)}{\Gamma(-\beta)\Gamma(\alpha)} x^{2\alpha-d} + O(x^{\min(3\alpha-d,0)}), \quad \text{as } x \to 0_+.$$

The coefficient of $x^{2\alpha-d}$ is negative because $\Gamma(-\beta) > 0$ for $\beta \in (1,2)$. Therefore, $g(x)$ can assume negative values. This completes the proof of Theorem 4.6. □

5. Proofs of Theorems 3.1 and 3.2

The proofs of Theorems 3.1 and 3.2 will follow the same arguments as the proof of [7, Theorem 1.2], which requires some lemmas and propositions.

5.1. Dalang's condition

Lemma 5.1. *Suppose that $\theta > 1/2$ and $\beta \in (0,2)$. The following statements are true:*

(a) There is some nonnegative constant $C_{\beta,\theta}$ such that for all $t > 0$ and $\lambda > 0$,

$$\int_0^t w^{2(\theta-1)} E_{\beta,\theta}^2(-\lambda w^\beta)\mathrm{d}w \le C_{\beta,\theta}\, \frac{t^{2\theta-1}}{1+(t\lambda^{1/\beta})^{\min(2\beta,2\theta-1)}}, \tag{5.1}$$

(b) If $\beta \le \min(1,\theta)$, then for some nonnegative constant $C'_{\beta,\theta}$,

$$\int_0^t w^{2(\theta-1)} E_{\beta,\theta}^2(-\lambda w^\beta)\mathrm{d}w \ge C'_{\beta,\theta}\, \frac{t^{2\theta-1}}{1+(t\lambda^{1/\beta})^{\min(2\beta,2\theta-1)}},$$

for all $t > 0$ and $\lambda > 0$.

Remark 5.2. When $\theta = \beta = 1$, then $E_1(x) = e^x$ and thus $\int_0^t e^{-2\lambda w}\mathrm{d}w = (2\lambda)^{-1}(1-e^{-2t\lambda})$ and (5.1) is clear for this case.

Proof of Lemma 5.1. (a) In this case, by the asymptotic property of the Mittag-Leffler function (see [31, Theorem 1.6]), for some nonnegative constants C_i's,

$$\int_0^t w^{2(\theta-1)} E_{\beta,\theta}^2(-\lambda w^\beta)\mathrm{d}w \le C_1 \int_0^t \frac{w^{2(\theta-1)}}{(1+\lambda^{1/\beta}w)^{2\beta}}\mathrm{d}w \tag{5.2}$$

$$= C_2\, t^{2\theta-1}\int_0^1 \frac{u^{2(\theta-1)}}{(1+\lambda^{1/\beta}tu)^{2\beta}}\mathrm{d}u$$

$$= C_3\, t^{2\theta-1}\, {}_2F_1(2\beta, 2\theta-1, 2\theta; -t\lambda^{1/\beta}) \tag{5.3}$$

$$= C_4\, t^{2\theta-1} H_{2,2}^{1,2}\left(t\lambda^{1/\beta}\,\middle|\, \begin{matrix}(1-2\beta,1),\ (2(1-\theta),1)\\ (0,1),\ (1-2\theta,1)\end{matrix}\right), \tag{5.4}$$

where in (5.3) we have applied [30, 15.6.1] under the condition that $\theta > 1/2$, and (5.4) is due to [22, (2.9.15)]. Notice that $\Delta = 0$ for the above Fox H-function, which allows us to apply Theorems 1.7 and 1.11 of [22]. In particular, by [22, Theorem 1.11], we know that

$$H_{2,2}^{1,2}\left(x\,\middle|\, \begin{matrix}(1-2\beta,1),\ (2(1-\theta),1)\\ (0,1),\ (1-2\theta,1)\end{matrix}\right) \sim O(1), \quad \text{as } x \to 0.$$

When $\theta \ne \beta$, by [22, Theorem 1.7],

$$H_{2,2}^{1,2}\left(x\,\middle|\, \begin{matrix}(1-2\beta,1),\ (2(1-\theta),1)\\ (0,1),\ (1-2\theta,1)\end{matrix}\right) \sim O\left(x^{-\min(2\beta,2\theta-1)}\right), \quad \text{as } x \to \infty.$$

In particular, when $\theta = \beta$, by Property 2.2 and (2.9.5) of [22],

$$H_{2,2}^{1,2}\left(x\,\middle|\, \begin{matrix}(1-2\beta,1),\ (2(1-\theta),1)\\ (0,1),\ (1-2\theta,1)\end{matrix}\right) = H_{1,1}^{1,1}\left(x\,\middle|\, \begin{matrix}(2(1-\theta),1)\\ (0,1)\end{matrix}\right) = \Gamma(2\theta-1)(1+x)^{1-2\theta}.$$

(b) When $\beta < \min(1,\theta)$, by (2.10), we know that $E_{\beta,\theta}(-|x|)$ is nonnegative, hence, for another nonnegative constant C', one can reverse the inequality (5.2). This completes the proof of Lemma 5.1. □

Lemma 5.3 (*Dalang's Condition*)**.** *Let $Y(t,x) = Y_{\alpha,\beta,\gamma,d}(t,x)$ with $\alpha \in (0,2]$, $\beta \in (0,2)$, and $\gamma > 0$. The following two conditions are equivalent:*

$$\text{(i)}\ \ d < 2\alpha + \frac{\alpha}{\beta}\min(2\gamma-1,0)$$

$$\iff \qquad \text{(ii)}\ \ \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}y\, Y(s,y)^2 < \infty, \quad \textit{for all } t > 0.$$

Proof. (i)⇒(ii): Fix an arbitrary $t > 0$. By the Plancherel theorem and (4.8), we only need to prove that

$$\int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}\xi\ s^{2(\beta+\gamma-1)} E^2_{\beta,\beta+\gamma}(-s^\beta|\xi|^\alpha) < +\infty. \tag{5.5}$$

Notice that $d > 0$ and Condition (i) together imply that $\beta + \gamma > 1/2$; see also (2.7). Thus, we can integrate ds first using Lemma 5.1(a). Then it reduces to prove that

$$\int_{\mathbb{R}^d} \frac{1}{1 + |\xi|^{2\alpha+\frac{\alpha}{\beta}\min(2\gamma-1,0)}} \mathrm{d}\xi < +\infty,$$

which is guaranteed by (i).

(ii)⇒(i): The case $\beta \in (0,1]$ can be proved in the same way as above by an application of Lemma 5.1(b). The case $\beta \in (1,2)$ is trickier. Fix $t > 0$. Denote the integral in (5.5) by $I(t)$. Then by change of variables,

$$I(t) = C\int_0^t \mathrm{d}s\ s^{2(\beta+\gamma-1)-\frac{\beta d}{\alpha}} \int_0^\infty \mathrm{d}y\ y^{\frac{d}{\alpha}-1} E^2_{\beta,\beta+\gamma}(-y).$$

Note that the double integral is decoupled. The integrability of ds at zero implies that $2(\beta+\gamma) - 1 - \frac{\beta d}{\alpha} > 0$, which is equivalent to

$$d < 2\alpha + \frac{\alpha}{\beta}(2\gamma - 1). \tag{5.6}$$

By the asymptotics of $\mathbb{E}_{\beta,\beta+\gamma}(-y)$ at $+\infty$ (see, e.g., Theorem 1.3 in [31]; note that the condition $\beta \in (1,2)$ is used here), we see that there exist $y_0 > 0$ and some constant $C > 0$ such that

$$E^2_{\beta,\beta+\gamma}(-y) \ge \frac{C}{1+y^2} \qquad \text{for all } y \ge y_0.$$

Hence, the integrability of dy at zero and infinity implies the following conditions:

$$d/\alpha - 1 > -1 \qquad \text{and} \qquad (d/\alpha - 1) - 2 < -1. \tag{5.7}$$

Combining (5.6) and (5.7) gives (i). This completes the proof of Lemma 5.3. □

5.2. *Some continuity results on Y*

This part contains some continuity results on Y. All the results proved in this part will be used in the proof of Theorem 3.2. In particular, Proposition 5.4 will be used to prove the Hölder continuity (Theorem 3.3).

Proposition 5.4. *Suppose $\alpha \in (0,2]$, $\beta \in (0,2)$, $\gamma \ge 0$, and (1.9) holds. Then $Y(t,x) = Y_{\alpha,\beta,\gamma,d}(t,x)$ satisfies the following two properties:*

(i) For all $0 < \theta < (\Theta - d) \wedge 2$ and $T > 0$, there is some nonnegative constant $C = C(\alpha, \beta, \gamma, \nu, \theta, T, d)$ such that for all $t \in (0, T]$ and $x, y \in \mathbb{R}^d$,

$$\iint_{[0,\infty)\times\mathbb{R}^d} \mathrm{d}r\mathrm{d}z\ \left(Y\left(t-r, x-z\right) - Y\left(t-r, y-z\right)\right)^2 \le C\left|x-y\right|^{\theta}, \tag{5.8}$$

(ii) If $\beta \le 1$ and $\gamma \le \lceil\beta\rceil - \beta$, then there is some nonnegative constant $C = C(\alpha, \beta, \gamma, \nu, d)$ such that for all $s, t \in (0, \infty)$ with $s \le t$, and $x \in \mathbb{R}^d$,

$$\int_0^s \mathrm{d}r \int_{\mathbb{R}^d} \mathrm{d}z \left(Y\left(t-r, x-z\right) - Y\left(s-r, x-z\right)\right)^2 \le C(t-s)^{2(\beta+\gamma)-1-d\beta/\alpha}, \tag{5.9}$$

and

$$\int_s^t \mathrm{d}r \int_{\mathbb{R}^d} \mathrm{d}z\ Y^2\left(t-r, x-z\right) \le C(t-s)^{2(\beta+\gamma)-1-d\beta/\alpha}. \tag{5.10}$$

Proof. (i) Fix $t > 0$. By Plancherel's theorem and (4.8), the left hand side of (5.8) is equal to

$$\begin{aligned}
&\frac{1}{(2\pi)^d}\int_0^t \mathrm{d}r\ (t-r)^{2(\beta+\gamma-1)} \int_{\mathbb{R}^d} \mathrm{d}\xi\ E_{\beta,\beta+\gamma}^2\left(-2^{-1}\nu(t-r)^{\beta}|\xi|^{\alpha}\right)\ \left|e^{-i\xi\cdot x} - e^{-i\xi\cdot y}\right|^2\\
=&\frac{2}{(2\pi)^d}\int_{\mathbb{R}^d} \mathrm{d}\xi\ \left(1-\cos(\xi\cdot(x-y))\right)\int_0^t \mathrm{d}r\ (t-r)^{2(\beta+\gamma-1)}\ E_{\beta,\beta+\gamma}^2\left(-2^{-1}\nu(t-r)^{\beta}|\xi|^{\alpha}\right)\\
\le&\frac{2}{(2\pi)^d}\int_{\mathbb{R}^d} \mathrm{d}\xi\ \left(1-\cos(\xi\cdot(x-y))\right)\frac{C_{\beta,\gamma,T}}{1+|\xi|^{2\alpha+\frac{\alpha}{\beta}\min(0,2\gamma-1)}}
\end{aligned}$$

where we have applied Lemma 5.1 in the last step (see also the proof of Lemma 5.3). Denote $\Theta := 2\alpha + \frac{\alpha}{\beta}\min(0, 2\gamma - 1)$. Because $1 - \cos(x) \le 2 \wedge \left(x^2/2\right)$ for all $x \in \mathbb{R}$, we only need to bound

$$\begin{aligned}
\int_{\mathbb{R}^d} \mathrm{d}\xi\ \frac{2 \wedge \left[|x-y|\ |\xi|/\sqrt{2}\right]^2}{1+|\xi|^{\Theta}} \le C'\Bigg(&\ |x-y|^{-d}\int_0^{\sqrt{2}} \frac{u^{d+1}}{(1+|x-y|^{-1}u)^{\Theta}}\mathrm{d}u\\
&+|x-y|^{\Theta-d}\int_{\sqrt{2}}^{\infty}\frac{2}{u^{\Theta+1-d}}\mathrm{d}u\Bigg)
\end{aligned}$$

The second integral on the right hand side of the above inequality is finite provided that $\Theta > d$, which is Dalang's condition. By [30, 15.6.1], for some constant $C > 0$,

$$\int_0^{\sqrt{2}} \frac{u^{d+1}}{(1+|x-y|^{-1}u)^{\Theta}}\mathrm{d}u = C\ {}_2F_1(\Theta, 2+d, 3+d; -\sqrt{2}|x-y|^{-1}),$$

which is true under the condition that $d + 3 > d + 2 > 0$. By [22, 2.9.15],

$$\int_0^{\sqrt{2}} \frac{u^{d+1}}{(1+|x-y|^{-1}u)^{\Theta}}\mathrm{d}u = C'' H_{2,2}^{1,2}\left(\sqrt{2}|x-y|^{-1}\ \middle|\ \begin{matrix}(-1-d,1),\ (1-\Theta,1)\\(0,1),\ (-2-d,1)\end{matrix}\right).$$

Since $\Delta = 0$, by [22, Theorem 1.7], for all $\theta \in (0, \min(\Theta, 2 + d))$,

$$\int_0^{\sqrt{2}} \frac{u^{d+1}}{(1+|x-y|^{-1}u)^{\Theta}}\mathrm{d}u = O(|x-y|^{\theta-d}), \quad \text{as } |x-y| \to 0.$$

Combining these cases, we have proved (i).

(ii) Denote the left hand side of (5.9) by I. Apply Plancherel's theorem and (4.8),

$$I = C_* \int_0^s \mathrm{d}r \int_{\mathbb{R}^d} \mathrm{d}\xi \,\Big| \quad (t-r)^{\beta+\gamma-1} E_{\beta,\beta+\gamma}\left(-2^{-1}\nu(t-r)^\beta|\xi|^\alpha\right) \\ -(s-r)^{\beta+\gamma-1} E_{\beta,\beta+\gamma}\left(-2^{-1}\nu(s-r)^\beta|\xi|^\alpha\right)\Big|^2.$$

Then by Lemma 5.5,

$$I = C_* C_\sharp \int_0^s \mathrm{d}r \left[(t-r)^{2(\beta+\gamma-1)-d\beta/\alpha} + (s-r)^{2(\beta+\gamma-1)-d\beta/\alpha}\right] \\ -2C_* \int_0^s \mathrm{d}r[(t-r)(s-r)]^{\beta+\gamma-1}\, H(r),$$

where

$$H(r) = \int_{\mathbb{R}^d} E_{\beta,\beta+\gamma}(-2^{-1}\nu(t-r)^\beta|\xi|^\alpha) E_{\beta,\beta+\gamma}(-2^{-1}\nu(s-r)^\beta|\xi|^\alpha)\mathrm{d}\xi.$$

By (2.10) and $t \geq s$,

$$\begin{aligned} H(r) &\geq \int_{\mathbb{R}^d} E^2_{\beta,\beta+\gamma}(-2^{-1}\nu(t-r)^\beta|\xi|^\alpha)\mathrm{d}\xi \\ &= (t-r)^{-2(\beta+\gamma-1)} \int_{\mathbb{R}^d} (t-r)^{2(\beta+\gamma-1)} E^2_{\beta,\beta+\gamma}(-2^{-1}\nu(t-r)^\beta|\xi|^\alpha)\mathrm{d}\xi \\ &= (t-r)^{-2(\beta+\gamma-1)} \int_{\mathbb{R}^d} Y(t-r,y)^2 \mathrm{d}y \\ &= C_\sharp (t-r)^{-d\beta/\alpha}, \end{aligned}$$

where in the last step we have applied Lemma 5.5. Because $\beta+\gamma \leq 1$, we see that

$$\int_0^s \mathrm{d}r[(t-r)(s-r)]^{\beta+\gamma-1}\, H(r) \geq C_\sharp \int_0^s \mathrm{d}r (t-r)^{2(\beta+\gamma)-2-d\beta/\alpha}.$$

Denote $\rho := 2(\beta+\gamma) - 1 - d\beta/\alpha$. Note that $\rho > 0$ is implied by Dalang's condition (1.9). Therefore,

$$I \leq \frac{C_* C_\sharp}{\rho}\left[t^\rho - (t-s)^\rho + s^\rho - 2(t^\rho - (t-s)^\rho)\right] \leq \frac{C_* C_\sharp}{\rho}(t-s)^\rho.$$

This proves (5.9). As for (5.10), by a similar reasoning, we have

$$\int_s^t \mathrm{d}r \int_{\mathbb{R}^d} \mathrm{d}z\; Y^2(t-r, x-z) \leq C C_\sharp \int_s^t \mathrm{d}r\; (t-r)^{2(\beta+\nu-1)-d\beta/\alpha} = \frac{C C_\sharp}{\rho}(t-s)^\rho\;.$$

This completes the proof of Proposition 5.4. $\square$

The following lemma is a slight extension of [27, Lemma 1] from the case where $\gamma = 1-\beta$ to a general γ.

Lemma 5.5. *Assume that $d < 2\alpha$, $\beta \in (0,2)$, and $\gamma \geq 0$. Then*

$$\int_{\mathbb{R}^d} Y^2_{\alpha,\beta,\gamma,d}(t,x)\mathrm{d}x = C_\sharp\, t^{2(\beta+\gamma-1)-d\beta/\alpha},$$

for all $t > 0$, where

$$C_\sharp := \frac{2}{\Gamma(d/2)(2\pi\nu)^{d/2}} \int_0^\infty u^{d-1} E^2_{\beta,\beta+\gamma}(-u^\alpha)\mathrm{d}u. \tag{5.11}$$

Proof. By Plancherel's theorem and (4.8),

$$\begin{aligned}\int_{\mathbb{R}^d} Y(t,x)^2 \mathrm{d}x &= \frac{t^{2(\beta+\gamma-1)}}{(2\pi)^d}\int_{\mathbb{R}^d} E^2_{\beta,\beta+\gamma}(-2^{-1}\nu t^\beta|\xi|^\alpha)\mathrm{d}\xi \\ &= \frac{2\pi^{d/2}}{\Gamma(d/2)}\frac{t^{2(\beta+\gamma-1)}}{(2\pi)^d}\int_0^\infty r^{d-1}E^2_{\beta,\beta+\gamma}(-2^{-1}\nu t^\beta r^\alpha)\mathrm{d}r \\ &= \frac{2\pi^{d/2}}{\Gamma(d/2)}\frac{t^{2(\beta+\gamma-1)-d\beta/\alpha}}{(2\pi)^d}(2/\nu)^{d/2}\int_0^\infty u^{d-1}E^2_{\beta,\beta+\gamma}(-u^\alpha)\mathrm{d}u.\end{aligned}$$

Note that by the asymptotic property of the Mittag-Leffler function ([31, Theorem 1.7]), the last integral is finite if $d < 2\alpha$. □

The corresponding results to the next Proposition for the SHE, the SFHE, and the SWE can be found in [7, Proposition 5.3], [8, Proposition 4.7], and [6, Lemma 3.2], respectively. We need some notation: for $\tau > 0$, $\alpha > 0$ and $(t,x) \in (0,\infty)\times\mathbb{R}^d$, denote

$$B_{t,x,\tau,\alpha} := \left\{(t',x') \in (0,\infty)\times\mathbb{R}^d \,:\, 0 \le t' \le t+\tau,\ |x-x'| \le \alpha\right\}.$$

Proposition 5.6. *Suppose that $\beta \in (0,2)$ and $\gamma \in [0, \lceil\beta\rceil-\beta]$. Then for all $(t,x) \in (0,\infty)\times\mathbb{R}^d$, there exists a constant $A > 0$ such that for all $(t',x') \in B_{t,x,1/2,1}$ and all $s \in [0,t')$ and $y \in \mathbb{R}^d$ with $|y| \ge A$, we have that $Y\left(t'-s, x'-y\right) \le Y(t+1-s, x-y)$.*

Proof. Without loss of generality, assume that $\nu = 2$.

Case I. We first prove the case where $\alpha = 2$. The proof here simplifies the arguments of [4, Proposition 6.1]. Fix $(t,x) \in (0,\infty)\times\mathbb{R}^d$. By the scaling property and the asymptotic property of Y, we have that

$$\begin{aligned}\frac{Y(t+1-s,x-y)}{Y(t'-s,x'-y)} \approx \left(\frac{t'-s}{t+1-s}\right)^{f(\beta)}\frac{|x-y|^a}{|x'-y|^a} \\ \times\exp\left(\frac{b|x'-y|^c}{(t'-s)^{\beta c/2}} - \frac{b|x-y|^c}{(t+1-s)^{\beta c/2}}\right),\end{aligned}$$

as $|y| \to \infty$, where the constants a, b and c are defined in (4.18), and

$$f(\beta) = 1 + \frac{d\beta}{2} - \beta - \nu + \frac{a\beta}{2}.$$

Notice that

$$\frac{t+1-s}{t'-s} = 1 + \frac{t+1-t'}{t'-s} \ge 1 + \frac{t+1-t'}{t'} \ge \frac{t+1}{t+1/2} = 1 + \frac{1}{2t+1} > 1\,. \tag{5.12}$$

If $f(\beta) \le 0$, then

$$\left(\frac{t'-s}{t+1-s}\right)^{f(\beta)} = \left(\frac{t+1-s}{t'-s}\right)^{|f(\beta)|} \ge 1.$$

If $f(\beta) > 0$, then

$$\left(\frac{t'-s}{t+1-s}\right)^{f(\beta)} \ge \left(\frac{t'-s}{t+1}\right)^{|f(\beta)|} = (t+1)^{-|f(\beta)|}\exp\left(|f(\beta)|\ \log(t'-s)\right).$$

The rest arguments are the same as the proof of [4, Proposition 6.1]. We will not repeat here.

Case II. Now we consider the case when $\alpha \in (0, 2)$. By the scaling property and the asymptotic property of Y, we have that

$$\frac{Y(t+1-s, x-y)}{Y(t'-s, x'-y)} \approx \left(\frac{t'-s}{t+1-s}\right)^{1-2\beta-\nu} \left(\frac{|x'-y|}{|x-y|}\right)^{d+\alpha},$$

as $|y| \to \infty$. Because $\beta > 1/2$ and $\gamma \geq 0$, we see that $1 - 2\beta - \gamma < 0$. Hence, by (5.12),

$$\left(\frac{t'-s}{t+1-s}\right)^{1-2\beta-\nu} = \left(\frac{t+1-s}{t'-s}\right)^{2\beta+\nu-1} \geq \left(1+\frac{1}{2t+1}\right)^{2\beta+\nu-1} > 1.$$

On the other hand,

$$\frac{|x'-y|}{|x-y|} \geq \frac{|y|-|x'|}{|x|+|y|} \geq \frac{|y|-(|x'-x|+|x|)}{|x|+|y|} \geq \frac{|y|-1+|x|}{|x|+|y|} \to 1,$$

as $|y| \to \infty$. Therefore, we can choose a large constant A, such that for all $|y| \geq A$ and all $(t', x') \in B_{t,x,1/2,1}$ and $s \in [0, t']$,

$$\frac{Y(t+1-s, x-y)}{Y(t'-s, x'-y)} > 1.$$

This completes the proof of Proposition 5.6. □

Proposition 5.7. *For all $(t, x) \in [0, \infty) \times \mathbb{R}^d$, $1 < \beta < 2$ and $\gamma \in [0, 2-\beta]$, we have*

$$\lim_{(t',x')\to(t,x)} \iint_{[0,\infty)\times\mathbb{R}^d} \mathrm{d}s\mathrm{d}y \, \left(Y\left(t'-s, x'-y\right) - Y\left(t-s, x-y\right)\right)^2 = 0.$$

Proof. This proposition is a consequence of Proposition 5.6. The proof follows the same arguments as the proof of [4, Proposition 6.4]. □

5.3. *Estimations of the kernel function $\mathcal{K}$*

Let $G : [0, \infty) \times \mathbb{R}^d \mapsto \mathbb{R}$ with $d \in \mathbb{N}$, $d \geq 1$ be a Borel measurable function.

Assumption 5.8. The function $G : [0, \infty) \times \mathbb{R}^d \mapsto \mathbb{R}$ has the following properties:

(1) There is a nonnegative function $\mathcal{G}(t, x)$, called *reference kernel function*, and constants $C_0 > 0$, $\sigma < 1$ such that

$$G(t, x)^2 \leq \frac{C_0}{t^\sigma} \mathcal{G}(t, x), \quad \text{for all } (t, x) \in [0, \infty) \times \mathbb{R}^d. \tag{5.13}$$

(2) The reference kernel function $\mathcal{G}(t, x)$ satisfies the following *sub-semigroup property*: for some constant $C_1 > 0$,

$$\int_{\mathbb{R}^d} \mathrm{d}y \, \mathcal{G}(t, x-y) \mathcal{G}(s, y) \leq C_1 \, \mathcal{G}(t+s, x), \quad \text{for all } t, s > 0 \text{ and } x \in \mathbb{R}^d. \tag{5.14}$$

Assumption 5.9. The same as Assumption 5.8 except that the two "≤" in (5.13) and (5.14) are replaced by "≥". We call the property (5.14) with "≤" replaced by "≥" the *super-semigroup property*.

Proposition 5.10. *Under conditions (1.9) and (1.11), the function $Y(t,x)$ satisfies Assumption 5.8 with the reference kernel $\mathcal{G}_{\alpha,\beta}(t,x)$ defined in (3.7), two nonnegative constants C_0 and C_1, depending on $(\alpha,\beta,\gamma,\nu,d)$, and σ defined in (2.6).*

Proof. The proof is similar to that of [4, Proposition 5.8]. We first note that $\sigma < 1$ is implied by Dalang's condition (1.9); see (2.7).

Case I. We first consider the case where $\alpha = 2$. In this case,

$$\mathcal{G}_{\alpha,\beta}(t,x) = \left(4\nu\pi t^{\beta}\right)^{-d/2} \exp\left(-\frac{1}{4\nu}\left(t^{-\beta/2}|x|\right)^{\lfloor\beta\rfloor+1}\right).$$

Notice that

$$\frac{2}{2-\beta} > \lfloor\beta\rfloor + 1, \quad \text{for } \beta \in (0,1)\cup(1,2),$$

and when $\beta = 1$, the constant b defined in (4.18) reduces to $1/(2\nu)$, which is bigger than $1/(4\nu)$. Hence, by (4.16) and (4.13), we see that

$$\sup_{(t,x)\in[0,\infty)\times\mathbb{R}^d} \frac{Y(t,x)^2}{t^{-\sigma}\mathcal{G}_{2,\beta}(t,x)} = \sup_{y\in\mathbb{R}^d} \frac{Y(1,y)^2}{\mathcal{G}_{2,\beta}(1,y)} =: C_0 < \infty.$$

Note that in the application of (4.13) in the above equations we have used the fact that Dalang's condition (1.9) implies $d < 2\alpha$. When $\beta = 1$, we see that

$$\mathcal{G}_{2,1}(t,x) = (4\nu\pi t)^{-d/2} \exp\left(-\frac{|x|^2}{4\nu t}\right),$$

and hence, $C_1 = 1$ and (5.14) becomes equality in this case. When $\beta \in (0,1)$,

$$\mathcal{G}_{2,\beta}(t,x) = \left(4\nu\pi t^{\beta}\right)^{-d/2} \exp\left(-\frac{|x|}{4\nu t^{\beta/2}}\right) \le \prod_{i=1}^{d} \left(4\nu\pi t^{\beta}\right)^{-1/2} \exp\left(-\frac{|x_i|}{4\nu t^{\beta/2}}\right).$$

Then by Lemma 5.10 of [4], for some nonnegative constant C_1,

$$\int_{\mathbb{R}^d} \mathcal{G}_{2,\beta}(t-s,x-y)\mathcal{G}_{2,\beta}(s,y)\mathrm{d}y \le C_1\, \mathcal{G}_{2,\beta}(t,x).$$

When $\beta \in (1,2)$,

$$\mathcal{G}_{2,\beta}(t,x) = \left(4\nu\pi t^{\beta}\right)^{-d/2} \exp\left(-\frac{|x|^2}{4\nu t^{\beta}}\right) = \mathcal{G}_{2,1}(t^{\beta},x).$$

Hence, by the semigroup property for the heat kernel,

$$\int_{\mathbb{R}^d} \mathcal{G}_{2,\beta}(t-s,x-y)\mathcal{G}_{2,\beta}(s,y)\mathrm{d}y = \mathcal{G}_{2,1}((t-s)^{\beta}+s^{\beta},x) \le 2^{d(1-\beta)}\mathcal{G}_{2,\beta}(t,x),$$

where in the last step we have applied the inequalities:

$$2^{1-\beta}t^{\beta} \le (t-s)^{\beta} + s^{\beta} \le t^{\beta}, \qquad \text{for } \beta \in (1,2).$$

Hence, in this case, $C_1 = 2^{(1-\beta)d}$. Therefore, Assumption 5.8 is satisfied.

Case II. We now consider the case where $\alpha \neq 2$. In this case,

$$\mathcal{G}_{\alpha,\beta}(t,x) = \frac{c_n\, t^{\beta/\alpha}}{\left(t^{2\beta/\alpha} + |x|^2\right)^{(d+1)/2}} = G_p(t^{\beta/\alpha},x),$$

where $G_p(t,x)$ is the Poisson kernel (see [35, Theorem 1.14]). By the scaling property and the asymptotic property of $Y(1,x)$ at 0 and ∞ shown in (4.13) and (4.16), for some nonnegative constant C,

$$Y(t,x) \le \frac{C\, t^{\beta+\gamma-1-d\beta/\alpha}}{\left(1+t^{-2\beta/\alpha}|x|^2\right)^{(d+\alpha)/2}} \le \frac{C\, t^{\beta+\gamma-1-d\beta/\alpha}}{\left(1+t^{-2\beta/\alpha}|x|^2\right)^{(d+1)/4}},$$

where the second inequality is due to $(d+\alpha)/2 \ge (d+1)/4$, which is equivalent to $d \ge 1-2\alpha$. Hence,

$$\sup_{(t,x)\in[0,\infty)\times\mathbb{R}^d} \frac{Y(t,x)^2}{t^{-\sigma}\mathcal{G}_{\alpha,\beta}(t,x)} = \sup_{y\in\mathbb{R}^d} \frac{Y(1,y)^2}{\mathcal{G}_{\alpha,\beta}(1,y)} =: C_0 < \infty.$$

Then, it is ready to see that

$$\mathcal{G}_{\alpha,\beta}(t,x) = G_p(t^{\beta/\alpha},x)$$

By the semigroup property of the Poisson kernel, we have that

$$\int_{\mathbb{R}^d} \mathcal{G}_{\alpha,\beta}(t-s,x-y)\mathcal{G}_{\alpha,\beta}(s,y)\mathrm{d}y = G_p\left(s^{\beta/\alpha}+(t-s)^{\beta/\alpha},x\right). \tag{5.15}$$

Then use the inequalities

$$\begin{aligned} t^{\beta/\alpha} &\le s^{\beta/\alpha}+(t-s)^{\beta/\alpha} \le 2^{1-\beta/\alpha}t^{\beta/\alpha} && \text{if } \beta/\alpha \le 1,\\ 2^{1-\beta/\alpha}t^{\beta/\alpha} &\le s^{\beta/\alpha}+(t-s)^{\beta/\alpha} \le t^{\beta/\alpha} && \text{if } \beta/\alpha > 1, \end{aligned} \tag{5.16}$$

to conclude that for some constant $C_1 > 0$,

$$\int_{\mathbb{R}^d} \mathcal{G}_{\alpha,\beta}(t-s,x-y)\mathcal{G}_{\alpha,\beta}(s,y)\mathrm{d}y \le C_1 G_p\left(t^{\beta/\alpha},x\right) = C_1\mathcal{G}_{\alpha,\beta}(t,x).$$

This completes the proof of Proposition 5.10. □

Proposition 5.11. *Under (1.9) and the first two cases of (1.21), the function $Y(t,x)$ satisfies Assumption 5.9 with the reference kernel $\underline{\mathcal{G}}_{\alpha,\beta}(t,x)$ defined in (3.8), two nonnegative constants C_0 and C_1, depending on $(\alpha,\beta,\gamma,\nu,d)$, and σ defined in (2.6).*

Proof. The proof is similar to the proof of the previous proposition. We have several cases.

Case I: When $\alpha = 2$ and $\beta \in (0,1)$, by the scaling property (4.12), and the asymptotics at zero and infinity in (4.13) and (4.16), we see that

$$\sup_{(t,x)\in[0,\infty)\times\mathbb{R}^d} \frac{t^{-\sigma}\underline{\mathcal{G}}_{2,\beta}(t,x)}{Y(t,x)^2} = \sup_{y\in\mathbb{R}^d} \frac{\underline{\mathcal{G}}_{2,\beta}(1,y)}{Y(1,y)^2} =: \frac{1}{C_0} < +\infty. \tag{5.17}$$

By the semigroup property of the heat kernel,

$$\int_{\mathbb{R}^d} \underline{\mathcal{G}}_{\alpha,\beta}(t-s,x-y)\underline{\mathcal{G}}_{\alpha,\beta}(s,y)\mathrm{d}y = \underline{\mathcal{G}}_{\alpha,\beta}\left((s^\beta+(t-s)^\beta)^{1/\beta},x\right) \ge 2^{(\beta-1)d/2}\underline{\mathcal{G}}_{\alpha,\beta}(t,x),$$

where the last inequality is due to

$$t^\beta \le (t-s)^\beta + s^\beta \le 2^{1-\beta}t^\beta, \qquad \text{for } \beta\in(0,1).$$

Case II: When $\alpha < 2$ and $\beta \in (0, 1 \vee \alpha)$, by (4.12) and (4.16),

$$\sup_{(t,x)\in[0,\infty)\times\mathbb{R}^d} \frac{t^{-\sigma}\underline{\mathcal{G}}_{\alpha,\beta}(t,x)}{Y(t,x)^2} = \sup_{y\in\mathbb{R}^d} \frac{\underline{\mathcal{G}}_{\alpha,\beta}(1,y)}{Y(1,y)^2} =: \frac{1}{C_0} < +\infty.$$

The super-semigroup property can be proved in the same way from (5.15) and (5.16). □

5.4. A lemma on the initial data

Lemma 5.12. *For all compact sets $K \subseteq (0,\infty)\times\mathbb{R}^d$,*

$$\sup_{(t,x)\in K} \left(\left[1+J_0^2\right]\star\mathcal{K}\right)(t,x) < \infty,$$

under the following two cases:

(1) Both (1.9) and (1.11) are satisfied and the initial data satisfy (3.1);
(2) Both (1.9) and (1.12) are satisfied and the initial data belong to $\mathcal{M}_{\alpha,\beta}(\mathbb{R})$.

Proof. In both cases, the kernel function $\mathcal{K}(t,x)$ has the following upper bound

$$\mathcal{K}(t,x;\lambda) \le C_1 \mathcal{G}_{\alpha,\beta}(t,x)\left(t^{-\sigma}+e^{C_2 t}\right).$$

Part (1) is clear because

$$\left(\left[1+J_0^2\right]\star\mathcal{K}\right)(t,x) \le (1+\widehat{C}_t)(1\star\mathcal{K})(t,x) = C_1(1+\widehat{C}_t)\left(\frac{t^{1-\sigma}}{1-\sigma}+\frac{e^{C_2 t}-1}{C_2}\right),$$

where $\sigma < 1$ (see (2.7)). The proof of part (2) requires more work. The case when $\alpha = 2$ is proved in Lemma 6.7 of [4]. The proof for $\alpha \in (0,2)$ is similar to that of Lemma 4.9 in [8]. Let

$$G_{\alpha,\beta,d}(t,x) = \pi^{-d/2} t^{\eta-1} |x|^{-d} H^{2,1}_{2,3}\left(\frac{|x|^\alpha}{2^{\alpha-1}\nu t^\beta} \,\middle|\, \begin{matrix}(1,1),\ (\eta,\beta)\\ (d/2,\alpha/2),\ (1,1),\ (1,\alpha/2)\end{matrix}\right)$$

with $\eta = \lceil\beta\rceil$ in case of Z and $\eta = 1$ in case of Z^*. Hence, we need only consider $J_0(t,x) = (|\mu| * G(t,\cdot))(x)$. By the asymptotic properties both at infinity and at zero (see Lemma 4.5 and Remark 4.4), we have that for all $t \in (0,T]$,

$$G(t,x) = t^{\eta-1-d\beta/\alpha} G(1, t^{-\beta/\alpha}x) \le \frac{C\, t^{\eta-1-d\beta/\alpha}}{1+|t^{-\beta/\alpha}x|^{d+\alpha}} \le \frac{C\, t^{\eta-1-d\beta/\alpha}(1\vee T)^{d\beta}}{1+|x|^{d+\alpha}}.$$

Thus, for $s \in (0,t]$,

$$J_0(s,y) \le A C s^{\eta-1-d\beta/\alpha}(1\vee t)^{d\beta},$$

where

$$A = \sup_{y\in\mathbb{R}} \int_{\mathbb{R}} |\mu|(\mathrm{d}y)\frac{1}{1+|x-y|^{1+\alpha}}.$$

The rest of the proof follows line-by-line the proof of part (2) of Lemma 4.9 in [8]. This completes the proof of Lemma 5.12. □

5.5. Proof of Theorem 3.2

Proof of Theorem 3.2. The proof is the same as the proof of [4, Theorem 3.1], which in turn follows the same six steps as those in the proof of [7, Theorem 2.4] with some minor changes:

The proof relies on estimates on the kernel function $\mathcal{K}(t, x)$, which is given by Proposition 5.10.

In the Picard iteration scheme, one needs to check the $L^p(\Omega)$-continuity of the stochastic integral. This will guarantee that the integrand in the next step is again in $\mathcal{P}_2$, via [7, Proposition 3.4]. Here, the statement of [7, Proposition 3.4] is still true by replacing in its proof [7, Proposition 3.5] by either Proposition 5.4 for the slow diffusion equations or Proposition 5.7 for the fast diffusion equations, and replacing [7, Proposition 5.3] by Proposition 5.6.

In the first step of the Picard iteration scheme, the following property, which determines the set of the admissible initial data, needs to be verified: for all compact sets $K \subseteq [0, \infty) \times \mathbb{R}^d$,

$$\sup_{(t,x)\in K} \left(\left[1 + J_0^2\right] \star \mathcal{K}\right)(t, x) < +\infty.$$

For the SHE, this property is proved in [7, Lemma 3.9]. Here, Lemma 5.12 gives the desired result with minimal requirements on the initial data. This property, together with the calculation of the upper bound on $\mathcal{K}(t, x)$ in Theorem 3.4, guarantees that all the $L^p(\Omega)$-moments of $u(t, x)$ are finite. This property is also used to establish uniform convergence of the Picard iteration scheme, hence $L^p(\Omega)$-continuity of $(t, x) \mapsto I(t, x)$.

The proof of (3.4) is identical to that of the corresponding property in [7, Theorem 2.4]. This completes the proof of Theorem 3.2. □

5.6. *Proof of* Theorem 3.1

Proof of Theorem 3.1. The proof of Theorem 3.1 is similar to that for Theorem 3.2. Because

$$\widehat{C}_t = \sup_{(s,x)\in[0,t]\times\mathbb{R}^d} |J_0(s, x)| < \infty, \quad \text{for all } t > 0,$$

the Picard iterations in the proof of Theorem 2.4 [7] give the following moment formula

$$||u(t, x)||_p^2 \le 2J(t, x)^2 + \left[\overline{\varsigma}^2 + 2\widehat{C}_t^2\right] \left(1 \star \widehat{\mathcal{K}}_p\right)(t, x).$$

Note that the function $\left(1 \star \mathcal{K}_p\right)(t, x)$ is a function of t only. For convenience, we denote it as

$$H(t; \lambda) := \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}y\, \mathcal{K}(s, y; \lambda). \tag{5.18}$$

Therefore, we need only to prove that $H(t; \lambda)$ is finite, which is proved in Lemma 5.13. This completes the proof of Theorem 3.1. □

Lemma 5.13. *For all $\alpha \in (0, 2]$, $\beta \in (0, 2)$, $\gamma \ge 0$, and $d \in \mathbb{N}$, under Dalang's condition (1.9), we have that*

$$H(t; \lambda) \le \exp\left(C\lambda^{\frac{2}{1-\sigma}}\, t\right),$$

for all $t > 0$ and $\lambda \in \mathbb{R}$, where σ is defined in (2.6) and C is some constant depending on α, β, γ and d.

Proof. By Lemma 5.5,

$$(1 \star \mathcal{L}_0)(t, x) \le C_\sharp \int_0^t \mathrm{d}s\, s^{2(\beta+\gamma-1)-d\beta/\alpha} = \frac{C_\sharp s^\theta}{\theta},$$

where $C_\sharp$ is defined in (5.11) and $\theta = 1-\sigma$. Note that $\theta > 0$ is guaranteed by Dalang's condition (1.9). Now we claim that, for $n \geq 0$,

$$(1 \star \mathcal{L}_n)(t,x) \leq \frac{C_\sharp^{n+1} \Gamma(\theta)^{n+1} t^{(n+1)\theta}}{\Gamma((n+1)\theta+1)}, \tag{5.19}$$

of which the case $n = 0$ is just proved. Assume that (5.19) holds for n. By the above calculations, we see that

$$(1 \star \mathcal{L}_{n+1})(t,x) \leq \frac{C_\sharp^{n+2}\Gamma(\theta)^{n+1}}{\Gamma((n+1)\theta+1)} \int_0^t \mathrm{d}s\,(t-s)^{(n+1)\theta} s^\theta = \frac{C_\sharp^{n+2}\Gamma(\theta)^{n+2} t^{(n+2)\theta}}{\Gamma((n+2)\theta+1)}.$$

Therefore,

$$H(t;\lambda) = \sum_{n=0}^\infty \lambda^{2(n+1)} (1 \star \mathcal{L}_n)(t,x) \leq E_{\theta,\theta+1}\left(C_\sharp \Gamma(\theta)\lambda^2 t^\theta\right).$$

Then apply the asymptotic property of the Mittag-Leffler function (see, e.g., [31, Theorem 1.3]). This completes the proof of Lemma 5.13. □

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Appendix. Some properties of the Fox H-functions

In this section, we follow the notation of [22].

Definition A.1. Let m, n, p, q be integers such that $0 \leq m \leq q, 0 \leq n \leq p$. Let $a_i, b_i \in \mathbb{C}$ be complex numbers and let α_j, β_j be positive numbers, $i = 1, 2, \dots, p$; $j = 1, 2, \dots, q$. Let the set of poles of the gamma functions $\Gamma(b_j + \beta_j s)$ does not intersect with that of the gamma functions $\Gamma(1 - a_i - \alpha_i s)$, namely,

$$\left\{ b_{jl} = \frac{-b_j - l}{\beta_j}, l = 0, 1, \dots \right\} \bigcap \left\{ a_{ik} = \frac{1 - a_i + k}{\alpha_i}, k = 0, 1, \dots \right\} = \emptyset$$

for all $i = 1, 2, \dots, p$ and $j = 1, 2, \dots, q$. Denote

$$\mathcal{H}_{pq}^{mn}(s) := \frac{\prod_{j=1}^m \Gamma(b_j + \beta_j s) \prod_{i=1}^n \Gamma(1 - a_i - \alpha_i s)}{\prod_{i=n+1}^p \Gamma(a_i + \alpha_i s) \prod_{j=m+1}^q \Gamma(1 - b_j - \beta_j s)}.$$

The *Fox H-function*

$$H_{p,q}^{m,n}(z) \equiv H_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix} (a_1,\alpha_1) & \cdots & (a_p,\alpha_p) \\ (b_1,\beta_1) & \cdots & (b_q,\beta_q) \end{matrix} \right]$$

is defined by the following integral

$$H_{pq}^{mn}(z) = \frac{1}{2\pi i} \int_L \mathcal{H}_{pq}^{mn}(s) z^{-s} ds, \quad z \in \mathbb{C}, \tag{A.1}$$

where an empty product in (A.1) means 1, and L in (A.1) is the infinite contour which separates all the points b_{jl} to the left and all the points a_{ik} to the right of L. Moreover, L has one of the following forms:

(1) $L = L_{-\infty}$ is a left loop situated in a horizontal strip starting at point $-\infty + i\phi_1$ and terminating at point $-\infty + i\phi_2$ for some $-\infty < \phi_1 < \phi_2 < \infty$

(2) $L = L_{+\infty}$ is a right loop situated in a horizontal strip starting at point $+\infty + i\phi_1$ and terminating at point $\infty + i\phi_2$ for some $-\infty < \phi_1 < \phi_2 < \infty$

(3) $L = L_{i\gamma\infty}$ is a contour starting at point $\gamma - i\infty$ and terminating at point $\gamma + i\infty$ for some $\gamma \in (-\infty, \infty)$

According to [22, Theorem 1.1], the integral (A.1) exists, for example, when

$$\Delta := \sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i \geq 0 \quad \text{and} \quad L = L_{-\infty}, \tag{A.2}$$

or when

$$a^* := \sum_{i=1}^{n} \alpha_i - \sum_{i=n+1}^{p} \alpha_i + \sum_{j=1}^{m} \beta_j - \sum_{j=m+1}^{q} \beta_j \geq 0 \quad \text{and} \quad L = L_{i\gamma\infty}. \tag{A.3}$$

The following two parameters of the Fox H-functions (A.1) will be used in this paper:

$$\mu = \sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i + \frac{p-q}{2}, \tag{A.4}$$

and

$$\delta = \prod_{i=1}^{p} \alpha_i^{-\alpha_i} \prod_{j=1}^{q} \beta_j^{\beta_j}. \tag{A.5}$$

Lemma A.2. *For $b \in \mathbb{C}$ and $\beta > 0$, there holds the relation*

$$H^{2+m,n}_{p,2+q}\left(z \,\middle|\, \begin{matrix} (a_i,\alpha_i)_{1,p} \\ (b,\beta),(b+1/2,\beta),(b_j,\beta_j)_{1,q} \end{matrix}\right) = 2^{1-2b}\sqrt{\pi}\, H^{1+m,n}_{p,1+q}\left(4^\beta z \,\middle|\, \begin{matrix} (a_i,\alpha_i)_{1,p} \\ (2b,2\beta),(b_j,\beta_j)_{1,q} \end{matrix}\right)$$

and

$$H^{m,n}_{p,2+q}\left(z \,\middle|\, \begin{matrix} (a_i,\alpha_i)_{1,p} \\ (b_j,\beta_j)_{1,q},\ (b,\beta),(b+1/2,\beta) \end{matrix}\right) = 4^{-b}\pi^{-1/2}\, H^{1+m,n}_{p,1+q}\left(4^\beta z \,\middle|\, \begin{matrix} (a_i,\alpha_i)_{1,p} \\ (b_j,\beta_j)_{1,q},\ (2b,2\beta) \end{matrix}\right).$$

Proof. Let

$$\mathcal{H}(s) = \frac{\prod_{j=1}^{m} \Gamma(b_j + \beta_j s) \prod_{i=1}^{n} \Gamma(1 - a_i - \alpha_i s)}{\prod_{i=n+1}^{p} \Gamma(a_i + \alpha_i s) \prod_{j=m+1}^{q} \Gamma(1 - b_j - \beta_j s)}.$$

By the definition of the Fox H-function,

$$H^{2+m,n}_{p,2+q}\left(z \,\middle|\, \begin{matrix} (a_i,\alpha_i)_{1,p} \\ (b,\beta),(b+1/2,\beta),(b_j,\beta_j)_{1,q} \end{matrix}\right) = \frac{1}{2\pi i}\int_{\mathcal{L}} \mathcal{H}(s)\Gamma(b+\beta s)\Gamma(b+1/2+\beta s) z^{-s}\mathrm{d}s.$$

By the duplication rule of the Gamma function [30, 5.5.5 on p. 138]

$$\Gamma(z)\Gamma(z+1/2) = \sqrt{\pi}\,2^{1-2z}\Gamma(2z), \quad 2z \neq 0, -1, -2, \ldots, \tag{A.6}$$

we have that

$$H^{2+m,n}_{p,2+q}\left(z \,\middle|\, \begin{matrix} (a_i,\alpha_i)_{1,p} \\ (b,\beta),(b+1/2,\beta),(b_j,\beta_j)_{1,q} \end{matrix}\right) = 2^{1-2b}\sqrt{\pi}\frac{1}{2\pi i}\int_{\mathcal{L}} \mathcal{H}(s)\Gamma(2b+2\beta s)(4^\beta z)^{-s}\mathrm{d}s.$$

Then apply the definition of the Fox H-function. The second relation can be proved similarly. □

Here are some direct consequences of this lemma:

$$H^{1+m,n}_{1+p,1+q}\left(z \,\middle|\, \begin{matrix}(a_i,\alpha_i)_{1,p},\ (b,\beta)\\ (2b,2\beta),\ (b_j,\beta_j)_{1,q}\end{matrix}\right) = 2^{2b-1}\pi^{-1/2}\, H^{1+m,n}_{p,1+q}\left(4^{\beta}z \,\middle|\, \begin{matrix}(a_i,\alpha_i)_{1,p}\\ (1/2+b,\beta),\ (b_j,\beta_j)_{1,q}\end{matrix}\right),$$

$$H^{1+m,n}_{1+p,1+q}\left(z \,\middle|\, \begin{matrix}(a_i,\alpha_i)_{1,p},\ (1/2+b,\beta)\\ (2b,2\beta),\ (b_j,\beta_j)_{1,q}\end{matrix}\right) = 2^{2b-1}\pi^{-1/2}\, H^{1+m,n}_{p,1+q}\left(4^{\beta}z \,\middle|\, \begin{matrix}(a_i,\alpha_i)_{1,p}\\ (b,\beta),\ (b_j,\beta_j)_{1,q}\end{matrix}\right),$$

$$H^{m,1+n}_{1+p,1+q}\left(z \,\middle|\, \begin{matrix}(1/2+b,\beta),\ (a_i,\alpha_i)_{1,p}\\ (b_j,\beta_j)_{1,q},\ (2b,2\beta)\end{matrix}\right) = 4^{b}\pi^{1/2}\, H^{m,n}_{p,1+q}\left(4^{-\beta}z \,\middle|\, \begin{matrix}(a_i,\alpha_i)_{1,p}\\ (b_j,\beta_j)_{1,q},\ (b,\beta)\end{matrix}\right),$$

$$H^{m,1+n}_{1+p,1+q}\left(z \,\middle|\, \begin{matrix}(1/2+b,\beta),\ (a_i,\alpha_i)_{1,p}\\ (b_j,\beta_j)_{1,q},\ (2b,2\beta)\end{matrix}\right) = 4^{b}\pi^{1/2}\, H^{m,n}_{p,1+q}\left(4^{-\beta}z \,\middle|\, \begin{matrix}(a_i,\alpha_i)_{1,p}\\ (b_j,\beta_j)_{1,q},\ (b,\beta)\end{matrix}\right).$$

Remark A.3. In [4], the Green function $G_\beta(t,x)$, which corresponds to $Y_{2,\beta,\lceil\beta\rceil-\beta,1}(t,x)$, is represented using the two-parameter Mainardi function of order $\lambda \in [0,1)$ (see (4.25)). By the series expansion of the Fox H-function ([22, Theorem 1.3], which requires that $\Delta = 1-\lambda > 0$), one can see that

$$M_{\lambda,\mu}(z) = z^{-1} H^{1,0}_{1,1}\left(z \,\middle|\, \begin{matrix}(\mu,\lambda)\\ (1,1)\end{matrix}\right), \qquad \lambda \in [0,1). \tag{A.7}$$

By Property 2.4 of [22], the above relation can also be written as

$$H^{1,0}_{1,1}\left(z^2 \,\middle|\, \begin{matrix}(\mu,\lambda)\\ (1,2)\end{matrix}\right) = \frac{z}{2} M_{\lambda/2,\mu}(z), \qquad \lambda \in [0,1). \tag{A.8}$$

Remark A.4. Another commonly used special function in this setting, such as in [32], is *Wright's function* [38–40] (see also [25, Appendix F]):

$$\phi(\lambda,\mu;z) := \sum_{k=0}^{\infty} \frac{z^k}{k!\,\Gamma(\mu+\lambda k)}, \qquad \text{for } \lambda > -1,\ \mu \in \mathbb{C}. \tag{A.9}$$

We adopt the notation ϕ that is used by E. M. Wright in his original papers. By (2.9.29) and Property 2.5 of [22],

$$\phi(\lambda,\mu;z) = \begin{cases} z^{-1} H^{1,0}_{0,2}\left(z \,\middle|\, \begin{matrix}\overline{\qquad\qquad\qquad}\\ [-1.0em](1,1),\ (1+\lambda-\mu,\lambda)\end{matrix}\right) & \text{if } \lambda > 0,\\ z^{-1} H^{1,0}_{1,1}\left(z \,\middle|\, \begin{matrix}(\mu-\lambda,-\lambda)\\ (1,1)\end{matrix}\right) & \text{if } \lambda \in (-1,0].\end{cases} \tag{A.10}$$

Comparing (A.7) and (A.10), we see that

$$M_{\lambda,\mu}(z) = \phi(-\lambda,\mu-\lambda;z), \qquad \text{for } \lambda \in (0,1]. \tag{A.11}$$

The following theorem is a simplified version of Theorems 2.9 and 2.10 in [23], which is sufficient for our use in the proof of Theorem 4.6.

Theorem A.5. *Let (a_1^*,Δ_1,μ_1) and (a_2^*,Δ_2,μ_2) be the constants (a^*,Δ,μ) defined in (A.3), (A.2) and (A.4) for the following two Fox H-functions:*

$$H^{m,n}_{p,q}\left(x \,\middle|\, \begin{matrix}(a_i,\alpha_i)_{1,p}\\ (b_j,\beta_j)_{1,q}\end{matrix}\right) \quad \textit{and} \quad H^{M,N}_{P,Q}\left(x \,\middle|\, \begin{matrix}(d_i,\delta_i)_{1,P}\\ (c_j,\gamma_j)_{1,Q}\end{matrix}\right),$$

respectively. Denote

$$A_1 = \min_{1\le i\le n} \frac{1-Re(a_i)}{\alpha_i}, \quad B_1 = \min_{1\le j\le m} \frac{Re(b_j)}{\beta_j},$$
$$A_2 = \min_{1\le j\le M} \frac{Re(c_j)}{\gamma_j}, \quad B_2 = \min_{1\le i\le N} \frac{1-Re(d_i)}{\delta_i},$$

with the convention that $\min(\phi) = +\infty$. *If either of the following four conditions holds*

(1) $a_1^* > 0$ *and* $a_2^* > 0$*;*
(2) $a_1^* = \Delta_1 = 0$, $Re(\mu_1) < -1$ *and* $a_2^* > 0$*;*
(3) $a_2^* = \Delta_2 = 0$, $Re(\mu_2) < -1$ *and* $a_1^* > 0$*;*
(4) $a_1^* = \Delta_1 = 0$, $Re(\mu_1) < -1$ *and* $a_2^* = \Delta_2 = 0$, $Re(\mu_2) < -1$,

and if

$$A_1 + B_1 > 0, \quad A_2 + B_2 > 0, \quad A_1 + A_2 > 0, \quad B_1 + B_2 > 0, \tag{A.12}$$

then, for all $z > 0$, $x \in \mathbb{R}$,

$$H^{m+M,n+N}_{p+P,q+Q}\left(zx \,\middle|\, \begin{matrix} (a_i,\alpha_i)_{1,n},\ (d_i,\delta_i)_{1,P},\ (a_i,\alpha_i)_{n+1,p} \\ (b_j,\beta_j)_{1,m},\ (c_j,\gamma_j)_{1,Q},\ (b_j,\beta_j)_{m+1,q} \end{matrix} \right)$$
$$= \int_0^\infty H^{m,n}_{p,q}\left(zt \,\middle|\, \begin{matrix} (a_i,\alpha_i)_{1,p} \\ (b_j,\beta_j)_{1,q} \end{matrix} \right) H^{M,N}_{P,Q}\left(\frac{x}{t} \,\middle|\, \begin{matrix} (d_i,\delta_i)_{1,P} \\ (c_j,\gamma_j)_{1,Q} \end{matrix} \right) \frac{\mathrm{d}t}{t}.$$

Proof. By Property 2.3 of [23],

$$H^{M,N}_{P,Q}\left(\frac{x}{t} \,\middle|\, \begin{matrix} (d_i,\delta_i)_{1,P} \\ (c_j,\gamma_j)_{1,Q} \end{matrix} \right) = H^{N,M}_{Q,P}\left(\frac{t}{x} \,\middle|\, \begin{matrix} (1-c_j,\gamma_j)_{1,Q} \\ (1-d_i,\delta_i)_{1,P} \end{matrix} \right). \tag{A.13}$$

If condition (1) holds, one can apply Theorem 2.9 of [23] with $\eta = 0$, $\sigma = 1$, $w = 1/x$, and with the following replacements: $N \to M$, $M \to N$, $P \to Q$, $Q \to P$, $c_j \to 1 - c_j$, $d_i \to 1 - d_i$. If either of conditions (2)–(4) holds, we apply Theorem 2.10 of [23] in the same way. Note that the parameters μ for both Fox H-functions in (A.13) are equal. □

A.1. Proof of Lemma 4.3

Proof of Lemma 4.3. Let

$$f(x) = H^{2,1}_{2,3}\left(x \,\middle|\, \begin{matrix} (1,1),\ (\eta,\beta) \\ (d/2,\alpha/2),\ (1,1),\ (1,\alpha/2) \end{matrix} \right).$$

Then $g(x) = x^{-d} f(x^\alpha)$. Let

$$H_{d,\alpha,\beta,\eta}(s) := \frac{\Gamma(d/2+\alpha s/2)\Gamma(1+s)\Gamma(-s)}{\Gamma(\eta+\beta s)\Gamma(-\alpha s/2)}.$$

Denote the poles of $\Gamma(1+s)$ and $\Gamma(d/2+\alpha s/2)$ by

$$A := \{-(1+k) : k = 0,1,2,\ldots\} \quad \text{and} \quad B := \left\{ -\frac{2l+d}{\alpha} : l = 0,1,2,\ldots \right\},$$

respectively. According to the definition of Fox H-function, to calculate the asymptotic at zero, we need to calculate the residue of $H_{d,\alpha,\beta,\eta}(s) z^{-s}$ at the rightmost poles in $A \cup B$. Because

$a^* = 2 - \beta > 0$, all the nigh cases are covered by either (1.8.1) or (1.8.2) of [23]. The notation h^*_{jl} below follows from (1.3.5) of [23].

Case 1. Assume that $\eta \neq \beta$ and $d/\alpha > 1$. In this case, the rightmost residue in $A \cup B$ is at $s = -1$ and it is a simple pole. Hence,

$$h^*_{20} = \frac{\Gamma((d-\alpha)/2)}{\Gamma(\eta-\beta)\Gamma(\alpha/2)} > 0,$$

and

$$f(x) = h^*_{20} x + O\left(x^{\min(2,d/\alpha)}\right), \quad \text{as } x \to 0_+.$$

Case 2. Assume that $\eta \neq \beta$ and $d/\alpha = 1$. The rightmost residue in $A \cup B$ is at $s = -1$ and it is of order two. Hence,

$$\begin{aligned}\operatorname*{Res}_{s=-1}(H_{d,d,\beta,\eta}(s)x^{-s}) &= \lim_{s\to-1}\left[(s+1)^2 H_{d,d,\beta,\eta}(s)x^{-s}\right]' \\ &= \lim_{s\to-1}\left(\left[(s+1)^2 H_{d,d,\beta,\eta}(s)\right]' - (s+1)^2 H_{d,d,\beta,\eta}(s)\log x\right)x^{-s} \\ &= Cx - \frac{1}{\Gamma(\eta-\beta)\Gamma(1+d/2)} x\log x,\end{aligned}$$

where we have used the fact that $\Gamma(x)$ has simple poles at $x = -n$, $n = 0, 1, \dots,$ with residue $\frac{(-1)^n}{n!}$. Therefore,

$$f(x) = -\frac{1}{\Gamma(\eta-\beta)\Gamma(1+d/2)} x\log x + O(x), \quad \text{as } x \to 0_+.$$

Case 3. Assume that $\eta \neq \beta$ and $d/\alpha < 1$. The rightmost residue in $A \cup B$ is at $s = -d/\alpha$ and it is a simple pole. Hence,

$$h^*_{10} = \frac{2}{\alpha}\frac{\Gamma(1-d/\alpha)\Gamma(d/\alpha)}{\Gamma(\eta-d\beta/\alpha)\Gamma(d/2)} > 0, \tag{A.14}$$

where the nonnegativity is due to the fact that $\eta \geq \beta > \beta d/\alpha$. Therefore,

$$f(x) = h^*_{10} x^{d/\alpha} + O(x^{\min((d+2)/\alpha,1)}) = h^*_{10} x^{d/\alpha} + O(x), \quad \text{as } x \to 0_+.$$

Case 4. Assume that $\eta = \beta = 1$. By Property 2.2 of [23],

$$f(x) = H^{1,1}_{1,2}\left(x \,\middle|\, \begin{matrix}(1,1)\\ (d/2,\alpha/2),\ (1,\alpha/2)\end{matrix}\right).$$

Hence,

$$h^*_{10} = \frac{2\Gamma(d/\alpha)}{\alpha\Gamma(d/2)} \neq 0,$$

and

$$f(x) = h^*_{10} x^{d/\alpha} + O(x^{(d+2)/\alpha}), \quad \text{as } x \to 0_+.$$

Case 5. Assume that $\eta = \beta \neq 1$ and $d/\alpha > 2$. The rightmost residue in $A \cup B$ is at $s = -1$, but this residue is vanishing because $\lim_{s\to-1} 1/\Gamma(\beta+\beta s) = 0$. The rightmost nonvanishing residue in $A \cup B$ is at $s = -2$ and it is a simple pole. Hence,

$$h^*_{21} = -\frac{\Gamma((d-2\alpha)/2)}{\Gamma(-\beta)\Gamma(\alpha)}.$$

Hence,

$$f(x) = h_{21}^* x^2 + O(x^{\min(3,d/\alpha)}), \quad \text{as } x \to 0_+.$$

Case 6. Assume that $\eta = \beta \neq 1$ and $d/\alpha = 2$. As in Case 6, the rightmost nonvanishing residue in $A \cup B$ is at $s = -2$, and it is of order two. Then

$$\begin{aligned}\operatorname*{Res}_{s=-2}(H_{d,d/2,\beta,\beta}(s)x^{-s}) &= \lim_{s\to-2}\left[(s+2)^2 H_{d,d/2,\beta,\beta}(s)x^{-s}\right]' \\ &= \lim_{s\to-2}\left(\left[(s+2)^2 H_{d,d/2,\beta,\beta}(s)\right]' - (s+2)^2 H_{d,d/2,\beta,\beta}(s)\log x\right)x^{-s} \\ &= Cx^2 + \frac{2}{\Gamma(-\beta)\Gamma(1+d/2)}x^2\log x.\end{aligned}$$

Therefore,

$$f(x) = \frac{2}{\Gamma(-\beta)\Gamma(1+d/2)}x^2\log x + O(x^3), \quad \text{as } x \to 0_+.$$

Case 7. Assume that $\eta = \beta \neq 1$ and $d/\alpha \in (1,2)$. As in Case 6, because $h_{20}^* \equiv 0$, the rightmost nonvanishing residue in $A \cup B$ is at $s = -d/\alpha$, and it is a simple pole. Hence,

$$f(x) = h_{10}^* x^{d/\alpha} + O(x^2), \quad \text{as } x \to 0_+,$$

where h_{10}^* is defined in (A.14) with η replaced by β.

Case 8. Assume that $\eta = \beta \neq 1$ and $d/\alpha = 1$. The rightmost nonvanishing residue in $A \cup B$ is at $s = -1$, and it is of order two. Hence,

$$\begin{aligned}\operatorname*{Res}_{s=-1}(H_{d,d,\beta,\beta}(s)x^{-s}) &= \lim_{s\to-1}\left[(s+1)^2 H_{d,d,\beta,\beta}(s)x^{-s}\right]' \\ &= \lim_{s\to-1}\left[\mathcal{H}_1(s)'\mathcal{H}_2(s) + \mathcal{H}_1(s)\mathcal{H}_2(s)' - \mathcal{H}_1(s)\mathcal{H}_2(s)\log x\right]x^{-s},\end{aligned}$$

where

$$\mathcal{H}_1(s) = (s+1)^2\Gamma((1+s)d/2)\Gamma(1+s) \quad \text{and} \quad \mathcal{H}_2(s) = \frac{\Gamma(-s)}{\Gamma(\beta+\beta s)\Gamma(-ds/2)}.$$

As calculated in the proof of Lemma 7.1 of [9], we have that

$$\begin{aligned}\mathcal{H}_1(-1) &= \lim_{s\to-1}\mathcal{H}_1^*(s) = \frac{2}{d} = \lim_{s\to-1}\frac{(1+s)^2}{((1+s)d/2)(1+s)} = \frac{2}{d}, \\ \mathcal{H}_2(-1) &= \lim_{s\to-1}\mathcal{H}_2^*(s) = 0, \\ \frac{\mathrm{d}}{\mathrm{d}s}\mathcal{H}_2(s)\bigg|_{s=-1} &= \lim_{s\to-1}\frac{\Gamma(-s)}{\Gamma(-ds/2)}\left(\frac{1}{\Gamma(\beta(1+s))}\right)' \\ &= \frac{\Gamma(1)}{\Gamma(d/2)}\lim_{s\to-1} -\frac{\psi(\beta(1+s))}{\Gamma(\beta(1+s))} \\ &= \frac{\beta}{\Gamma(d/2)},\end{aligned}$$

where $\psi(z)$ is the digamma function and the last limit is due to (5.7.6) and (5.7.1) of [30]. Therefore,

$$\operatorname*{Res}_{s=-1}(H_{d,d,\beta,\beta}(s)x^{-s}) = \frac{\beta}{\Gamma(1+d/2)}x,$$

and

$$f(x) = \frac{\beta}{\Gamma(1+d/2)}\,x + O(x^2), \quad \text{as } x \to 0_+.$$

Case 9. Assume that $\eta = \beta \neq 1$ and $d/\alpha < 1$. The first nonvanishing residue in $A \cap B$ is at $s = -d/\alpha$ and it is a simple pole. Hence,

$$f(x) = h_{10}^* x^{d/\alpha} + O(x), \quad \text{as } x \to 0_+,$$

where h_{10}^* is defined in (A.14) with η replaced by β. This completes the whole proof of Lemma 4.3. □

References

[1] L. Bertini, N. Cancrini, The stochastic heat equation: Feynman-Kac formula and intermittence, J. Stat. Phys. 78 (5–6) (1995) 1377–1401.

[2] R.A. Carmona, S.A. Molchanov, Parabolic Anderson problem and intermittency, Mem. Amer. Math. Soc. 108 (518) (1994) viii+125.

[3] L. Chen, Moments, Intermittency, and Growth Indices for Nonlinear Stochastic PDE's with Rough Initial Conditions (Ph.D. thesis), École Polytechnique Fédérale de Lausanne, 2013.

[4] L. Chen, Nonlinear stochastic time-fractional diffusion equations on R: moments, Hölder regularity and intermittency, Trans. Amer. Math. Soc. 369 (12) (2017) 8497–8535.

[5] L. Chen, R.C. Dalang, Hölder-continuity for the nonlinear stochastic heat equation with rough initial conditions, Stoch. Partial Differ. Equ. Anal. Comput. 2 (3) (2014) 316–352.

[6] L. Chen, R.C. Dalang, Moment bounds and asymptotics for the stochastic wave equation, Stochastic Process. Appl. 125 (4) (2015) 1605–1628.

[7] L. Chen, R.C. Dalang, Moments and growth indices for nonlinear stochastic heat equation with rough initial conditions, Ann. Probab. 43 (6) (2015) 3006–3051.

[8] L. Chen, R.C. Dalang, Moments, intermittency, and growth indices for the nonlinear fractional stochastic heat equation, Stoch. Partial Differ. Equ. Anal. Comput. 3 (3) (2015) 360–397.

[9] L. Chen, Y. Hu, G. Hu, J. Huang, Stochastic time-fractional diffusion equations on $\mathbb{R}^d$, Stochastics 89 (1) (2017) 171–206.

[10] L. Chen, Y. Hu, D. Nualart, Regularity and strict positivity of densities for the nonlinear stochastic heat equation. Preprint, arXiv:1611.03909.

[11] X. Chen, Y. Hu, J. Song, X. Song, Temporal asymptotics for fractional parabolic anderson model, Electron. J. Probab. 23 (14) (2018) 1–39.

[12] L. Chen, K. Kim, On comparison principle and strict positivity of solutions to the nonlinear stochastic fractional heat equations, Ann. Inst. Henri Poincaré Probab. Stat. 53 (1) (2017) 358–388.

[13] Z.Q. Chen, K.H. Kim, P. Kim, Fractional time stochastic partial differential equations, Stochastic Process. Appl. 125 (4) (2015) 1470–1499.

[14] D. Conus, M. Joseph, D. Khoshnevisan, S.-Y. Shiu, Intermittency and chaos for a nonlinear stochastic wave equation in dimension 1, in: Malliavin Calculus and Stochastic Analysis, Springer, 2013, pp. 251–279.

[15] D. Conus, M. Joseph, D. Khoshnevisan, S.-Y. Shiu, Initial measures for the stochastic heat equation, Ann. Inst. Henri Poincaré Probab. Stat. 50 (1) (2014) 136–153.

[16] R.C. Dalang, Extending the martingale measure stochastic integral with applications to spatially homogeneous s.p.d.e.'s, Electron. J. Probab. 4 (6) (1999) 29.

[17] L. Debbi, M. Dozzi, On the solutions of nonlinear stochastic fractional partial differential equations in one spatial dimension, Stochastic Process. Appl. 115 (11) (2005) 1764–1781.

[18] K. Diethelm, The Analysis of Fractional Differential Equations, in: Lecture Notes in Mathematics, vol. 2004, Springer-Verlag, Berlin, 2010.

[19] S. Eidelman, A. Kochubei, Cauchy problem for fractional diffusion equations, J. Differential Equations 199 (2) (2004) 211–255.

[20] M. Foondun, D. Khoshnevisan, Intermittence and nonlinear parabolic stochastic partial differential equations, Electron. J. Probab. 14 (21) (2009) 548–568.

[21] G. Hu, Y. Hu, Fractional diffusion in Gaussian noisy environment, Mathematics 3 (2) (2015) 131–152.

[22] A.A. Kilbas, M. Saigo, H-transforms: theory and applications, in: Analytical Methods and Special Functions, Vol. 9, Chapman & Hall/CRC, Boca Raton, FL, 2004.

[23] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, in: North-Holland Mathematics Studies, vol. 204, Elsevier Science B.V., Amsterdam, 2006.

[24] A. Kochubeĭ, Diffusion of fractional order. (Russian), Differ. Uravn. 26 (4) (1990) 660–670, 733–734; translation in Differential Equations 26 (1990), no. 4, 485–492.

[25] F. Mainardi, Fractional Calculus and Waves in Linear Viscoelasticity, Imperial College Press, London, 2010.

[26] F. Mainardi, Y. Luchko, G. Pagnini, The fundamental solution of the space–time fractional diffusion equation, Fract. Calc. Appl. Anal. 4 (2) (2001) 153–192.

[27] J.B. Mijena, E. Nane, Space–time fractional stochastic partial differential equations, Stochastic Process. Appl. 125 (9) (2015) 3301–3326.

[28] J.B. Mijena, E. Nane, Intermittence and time fractional stochastic partial differential equations, Potential Anal. 44 (2) (2016) 295–312.

[29] C. Mueller, On the support of solutions to the heat equation with noise, Stoch. Stoch. Rep. 37 (4) (1991) 225–245.

[30] F.W.J. Olver, D.W. Lozier, R.F. Boisvert, C.W. Clark (Eds.), NIST Handbook of Mathematical Functions, U.S. Department of Commerce National Institute of Standards and Technology, Washington, DC, 2010.

[31] I. Podlubny, Fractional Differential Equations, in: Mathematics in Science and Engineering, vol. 198, Academic Press Inc., San Diego, CA, 1999.

[32] A.V. Pskhu, The fundamental solution of a diffusion-wave equation of fractional order. (Russian), Izv. Ross. Akad. Nauk Ser. Mat. 73 (2) (2009) 141–182, translation in Izv. Math. 73 (2009), no. 2, 351–392.

[33] S.G. Samko, A.A. Kilbas, O.I. Marichev, Fractional integrals and derivatives. Theory and applications, Gordon and Breach Science Publishers, Yverdon, 1993, Edited and with a foreword by S. M. Nikol'skiĭ, Translated from the 1987 Russian original, Revised by the authors.

[34] W.R. Schneider, Completely monotone generalized Mittag-Leffler functions, Expo. Math. 14 (1) (1996) 3–16.

[35] E. Stein, G. Weiss, Introduction to Fourier Analysis on Euclidean Spaces, in: Princeton Mathematical Series, vol. 32, Princeton University Press, Princeton, N. J., 1971.

[36] J.B. Walsh, An introduction to stochastic partial differential equations, in: École d'été de Probabilités de Saint-Flour, XIV—1984, in: Lecture Notes in Math, vol. 1180, Springer, Berlin, 1986, pp. 265–439.

[37] D.V. Widder, The Laplace Transform, in: Princeton Mathematical Series, vol. 6, Princeton University Press, Princeton, N. J., 1941.

[38] E.M. Wright, On the coefficients of power series having exponential singularities, J. Lond. Math. Soc. 8 (1) (1933) 71–79.

[39] E.M. Wright, The asymptotic expansion of the generalized Bessel function, Proc. Lond. Math. Soc. 38 (1934) 257–270.

[40] E.M. Wright, The generalized Bessel function of order greater than one, Quart. J. Math. Oxford Ser. 11 (1940) 36–48.