

# A boundedness trichotomy for the stochastic heat equation<sup>1</sup>

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**Abstract.** We consider the stochastic heat equation with a multiplicative white noise forcing term under standard “intermittency conditions.” The main finding of this paper is that, under mild regularity hypotheses, the a.s.-boundedness of the solution  $x \mapsto u(t, x)$  can be characterized generically by the decay rate, at  $\pm\infty$ , of the initial function  $u_0$ . More specifically, we prove that there are 3 generic boundedness regimes, depending on the numerical value of  $\Lambda := \lim_{|x| \rightarrow \infty} |\log u_0(x)| / (\log |x|)^{2/3}$ .

**Résumé.** Nous nous intéressons à l'équation de la chaleur stochastique avec un bruit blanc multiplicatif sous des «conditions d'intermittence» standard. Le résultat principal de cet article est que, sous des hypothèses de régularité raisonnables, le caractère presque sûrement borné de la solution  $x \mapsto u(t, x)$  est entièrement déterminé par la vitesse de décroissance en  $\pm\infty$  de la condition initiale  $u_0$ . Plus précisément, nous démontrons qu'il existe trois régimes distincts selon la valeur de  $\Lambda := \lim_{|x| \rightarrow \infty} |\log u_0(x)| / (\log |x|)^{2/3}$ .

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## 1. Introduction

It has been recently shown [3] that a large family of parabolic stochastic PDEs are chaotic in the sense that small changes in their initial value can lead to drastic changes in the global structure of the solution. In this paper we describe some of the quantitative aspects of the nature of that chaos.

Consider the solution  $u = \{u(t, x)\}_{t>0, x \in \mathbb{R}}$  of the stochastic initial-value problem<sup>2</sup>

$$\begin{cases} \dot{u}(t, x) = \frac{1}{2}u''(t, x) + \sigma(u(t, x))\xi(t, x), & [t > 0, x \in \mathbb{R}], \\ \text{subject to } u(0, x) = u_0(x), & [x \in \mathbb{R}], \end{cases} \quad (1.1)$$

where  $\xi$  denotes space–time white noise; that is, a centered Gaussian random field with covariance functional

$$\text{Cov}[\xi(t, x), \xi(s, y)] = \delta_0(s - t)\delta_0(x - y) \quad [s, t \geq 0, x, y \in \mathbb{R}].$$

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<sup>2</sup>As usual,  $\dot{u}(t, x) := \partial u(t, x) / \partial t$  and  $u''(t, x) := \partial^2 u(t, x) / \partial t \partial x$ .

Alternatively, we can construct  $\xi$  as  $\xi(t, x) = \partial_{x,t}^2 B(t, x)$ , in the sense of distributions, where  $B := \{B(t, x)\}$  is a mean-zero continuous Gaussian process with covariance

$$\text{Cov}[B(t, x), B(s, y)] = \min(s, t) \times \min(|x|, |y|) \times \mathbb{1}_{(0, \infty)}(xy),$$

for all  $s, t \geq 0$  and  $x, y \in \mathbb{R}$ . The random field  $B$  is sometimes referred to as *space–time Brownian sheet*.

Some of the commonly-used assumptions on the initial value  $u_0$  and the nonlinearity  $\sigma$  are that:

- (a)  $u_0 \in L^\infty(\mathbb{R})$  is non random;  $u_0(x) \geq 0$  for almost all  $x \in \mathbb{R}$ ; and  $u_0 > 0$  on a set of positive Lebesgue measure; and
- (b)  $\sigma : \mathbb{R} \rightarrow \mathbb{R}$  is Lipschitz continuous and nonrandom.

These conditions will be in place from now on. Under these conditions, it is well known [4,9,13] that (1.1) admits a continuous predictable solution  $u$  that is uniquely defined via the a priori condition,

$$\sup_{t \in [0, T]} \sup_{x \in \mathbb{R}} E(|u(t, x)|^k) < \infty \quad \text{for all } T > 0 \text{ and } k \geq 2.$$

Throughout, we will suppose, in addition, that the nonlinearity  $\sigma$  satisfies

$$\sigma(0) = 0 \quad \text{and} \quad L_\sigma := \inf_{w \in \mathbb{R}} |\sigma(w)/w| > 0. \tag{1.2}$$

The first condition in (1.2) implies that there exists a P-null set off which

$$u(t, x) > 0 \quad \text{for all } t > 0 \text{ and } x \in \mathbb{R};$$

see Mueller [10,11]. And the condition on the positivity of  $L_\sigma$  is an “intermittency condition,” and implies among other things that the moments of  $u(t, x)$  grow exponentially with time [6]. The intermittency condition arose earlier in the work of Shiga [12] on interacting infinite systems of Itô-type diffusion processes.

Together, the two conditions in (1.2) suffice to ensure that the solution  $u$  to (1.1) is “chaotic” in the sense that its global behavior, at all times, depends strongly on its initial state  $u_0$ . To be more concrete, we know for example that if  $\inf_{x \in \mathbb{R}} u_0(x) \geq \varepsilon$  for a constant  $\varepsilon > 0$ , then (1.2) implies that

$$P\left\{\sup_{x \in \mathbb{R}} u(t, x) = \infty\right\} = 1 \quad \text{for all } t > 0;$$

see [3]. And by contrast,  $\sup_{x \in \mathbb{R}} u(t, x) < \infty$  a.s. for all  $t > 0$  for example when  $u_0$  is Lipschitz continuous with compact support; see [7]. Based on these results, one can imagine that if and when  $u_0(x) \rightarrow 0$  as  $|x| \rightarrow \infty$ , then  $\sup_{x \in \mathbb{R}} u(t, x)$  can be finite or infinite for some or even all  $t$ , depending on the nature of the decay of  $u_0$  at  $\pm\infty$ . The goal of this article is to describe precisely the amount of decay  $u_0$  needs in order to ensure that  $u(t, \cdot)$  is a bounded function almost surely. Because we are interested in almost-sure finiteness of the global maximum of the solution, this undertaking is different in style, as well as in methodology, from results that describe stochastic PDEs for which the spatial maximum of the solution is in  $L^k(P)$  for some  $1 \leq k < \infty$  [5,7].

We will make additional simplifying assumptions on the function  $u_0$  in order to make our derivations as non-technical as possible, yet good enough to describe the new phenomenon that we plan to present. In view of this, we will assume throughout that

$$\lim_{z \rightarrow \infty} u_0(z) = 0, \quad u_0(x) = u_0(-x), \quad \text{and} \quad u_0(x) \geq u_0(y) \quad \text{if } 0 \leq x \leq y. \tag{1.3}$$

Finally, we assume that the following limit exists:

$$\Lambda := \lim_{|x| \rightarrow \infty} \frac{|\log u_0(x)|}{(\log |x|)^{2/3}}. \tag{1.4}$$

The existence of this limit is a mild condition, since  $\Lambda$  can be any number in the *closed* interval  $[0, \infty]$ .

Throughout, define

$$M(t) := \sup_{x \in \mathbb{R}} u(t, x) \quad [t > 0].$$

The following trichotomy is the main finding of this paper.

**Theorem 1.1.** *Suppose that  $\sigma : \mathbb{R} \rightarrow \mathbb{R}$  is a Lipschitz continuous function that satisfies (1.2). Suppose also that  $u_0 \in L^\infty(\mathbb{R})$  is nonrandom and satisfies (1.3), and that the Lebesgue measure of  $u_0^{-1}(0, \infty)$  is positive. Finally, suppose that the limit in (1.4) exists in  $[0, \infty]$ . Then, the following statements hold:*

1. *If  $\Lambda = \infty$ , then  $\mathbb{P}\{M(t) < \infty \text{ for all } t > 0\} = 1$ ;*
2. *If  $\Lambda = 0$ , then  $\mathbb{P}\{M(t) = \infty \text{ for all } t > 0\} = 1$ ;<sup>3</sup>*
3. *If  $0 < \Lambda < \infty$ , then there exists a random variable  $\mathcal{T}$  and two nonrandom constants  $t_1, t_2 \in (0, \infty)$  such that:*  
 (i)  $t_1 < \mathcal{T} < t_2$  a.s.; and (ii)

$$\mathbb{P}\{M(t) < \infty \forall t < \mathcal{T} \text{ and } M(t) = \infty \forall t > \mathcal{T}\} = 1.$$

From now on we find it more convenient to write the solution to (1.1), using more standard probability notation, as

$$u_t(x) := u(t, x) \quad \text{for all } t > 0 \text{ and } x \in \mathbb{R}.$$

In particular,  $u_t$  does not refer to the time derivative of  $u$ .

We also denote the Lipschitz constant of  $\sigma$  by

$$\text{Lip}_\sigma := \sup_{-\infty < x \neq y < \infty} \left| \frac{\sigma(x) - \sigma(y)}{x - y} \right|.$$

## 2. Tail probabilities via insensitivity analysis

One of the first problems that we need to address is itself related to matters of chaos, and more specifically to the problem of how sensitive the solution of (1.1) is to “small” changes in the initial function. A suitable solution to this sensitivity problem has a number of interesting consequences. In the present context, we will use sensitivity analysis to derive sharp estimates for the tail of the distribution of the solution  $u_t(x)$  to (1.1).

We will have to interpret our sensitivity problem in a rather specific way, which we would like to describe in terms of an adversarial game between a player (Player 1) and Mother Nature (Player 2).

In this game, both players know the values of the external noise  $\xi$ . Player 2 knows also the initial function  $u_0$ , and hence the solution  $u_t(x)$  at all space–time points  $(t, x)$ . Player 1, on the other hand, knows the values of  $u_0(x)$  only for  $x$  in some pre-determined interval  $[a - r, a + r]$ . Player 1 guesses that the initial function is  $v_0$ , in some fashion or another, where  $v_0(x) = u_0(x)$  for all  $x \in [a - r, a + r]$ .

Let  $v_t(x)$  denote the solution to (1.1) with initial values  $v_0$ ; the function  $v$  is Player 1’s guess for the solution to (1.1). The following shows that if  $t \ll r$ , then the solution appears essentially the same to both Players 1 and 2 near the middle portion of the spatial interval  $[a - r, a + r]$ . Consequently, it follows that the values of the solution to (1.1) in the middle portion of  $[a - r, a + r]$  are insensitive to basically all possible changes to the initial value outside of  $[a - r, a + r]$ .

**Theorem 2.1.** *Choose and fix two parameters  $a \in \mathbb{R}$  and  $r > 0$ . Let  $u$  and  $v$  denote the solutions to (1.1) with respective initial values  $u_0$  and  $v_0$ , where  $u_0, v_0 \in L^\infty(\mathbb{R})$  are nonrandom and  $u_0(x) = v_0(x)$  a.e. on  $[a - r, a + r]$ . Then, for all  $t > 0$ ,*

$$\sup_{|x-a| \leq r/4} \mathbb{E}(|u_t(x) - v_t(x)|^2) \leq C\ell \|u_0 - v_0\|_{L^\infty(\mathbb{R})}^2 \exp\left(-\frac{r^2}{16t} + \frac{\text{Lip}_\sigma^4 t}{4}\right), \tag{2.1}$$

<sup>3</sup>Of course,  $t = 0$  is different from  $t > 0$  since  $M(0) < \infty$  in all cases.

where  $\ell := 1 + \text{Lip}_\sigma^4$  and  $C := 96[1 \vee \text{Lip}_\sigma^{-4}]$ .

It might help to emphasize that  $t \ll r$  if and only if the exponential on the right-hand side of (2.1) is a small quantity.

The proof of Theorem 2.1 relies on two technical lemmas. The first lemma is an elementary fact about the linear 1-dimensional heat equation. Define for every  $t > 0$  and  $x \in \mathbb{R}$ ,

$$p_t(x) := \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2t}\right).$$

**Lemma 2.2.** *Suppose  $h \in L^\infty(\mathbb{R})$  is a nonrandom function that is equal to zero a.e. in an interval  $[a - r, a + r]$ . Then, for all  $t > 0$ ,*

$$\sup_{x:|x-a|\leq r/2} |(p_t * h)(x)| \leq 2\|h\|_{L^\infty(\mathbb{R})} \cdot e^{-r^2/(8t)}.$$

**Proof.** By Minkowski’s inequality,

$$|(p_t * h)(x)| \leq \|h\|_{L^\infty(\mathbb{R})} \cdot \int_{|w+x-a|>r} p_t(w) \, dw.$$

If  $|w + x - a| > r$  and  $|x - a| \leq r/2$ , then certainly  $|w| > r/2$ . The lemma follows from the simple bound,  $\int_{|w|>r/2} p_t(w) \, dw \leq 2 \exp\{-r^2/(8t)\}$ . □

The statement of the second lemma requires the introduction of some notation. Let “ $\odot$ ” denote space–time convolution. That is, for all measurable space–time functions  $f$  and  $g$ ,

$$(f \odot g)_t(x) := \int_0^t ds \int_{-\infty}^\infty dy f_{t-s}(x - y) g_s(y),$$

pointwise, whenever the (Lebesgue) integral is absolutely convergent. For every  $\alpha > 0$ , consider the space–time kernel  $\mathcal{K}^{(\alpha)}$ , defined as

$$\mathcal{K}_t^{(\alpha)}(x) := \frac{\alpha^2}{2} p_{t/2}(x) \left[ \frac{1}{\sqrt{\pi t}} + \alpha^2 \exp\left(\frac{\alpha^4 t}{4}\right) \Phi\left(\alpha^2 \sqrt{\frac{t}{2}}\right) \right], \tag{2.2}$$

for all  $t > 0$  and  $x \in \mathbb{R}$ , where  $\Phi(x) := (2\pi)^{-1/2} \int_{-\infty}^x \exp(-y^2/2) \, dy$  denotes the cumulative distribution function of a standard normal law on the line. The following is the second technical lemma of this section.

**Lemma 2.3.** *Choose and fix a deterministic function  $f \in L^\infty(\mathbb{R})$ , and define a space–time function  $\mathcal{J}$  via  $\mathcal{J}_t(x) := (p_t * f)(x)$  for all  $t > 0$  and  $x \in \mathbb{R}$ . Suppose  $(t, x) \mapsto F_t(x)$  is a measurable space–time function that is bounded in  $x$  and grows at most exponentially in  $t$ , and satisfies*

$$F \leq \mathcal{J}^2 + \alpha^2(F \odot p^2), \tag{2.3}$$

pointwise for a fixed constant  $\alpha > 0$ . Then,

$$F \leq \mathcal{J}^2 + (\mathcal{J}^2 \odot \mathcal{K}^{(\alpha)}) \text{ pointwise.} \tag{2.4}$$

**Proof (sketch).** This is basically the first part of equation (2.21) of Chen and Dalang [2], but is stated here in slightly more general terms. Therefore, we skip the details and merely point out how one can relate Lemma 2.3 to the work of Chen and Dalang [2], deferring the details to the latter reference.

In order to see how one can deduce this lemma from the arguments of Chen and Dalang, let us consider the stochastic heat equation (1.1) with a nonrandom initial value  $f$ , and let  $U$  denote the solution. We can write the

solution in integral form as follows:

$$U_t(x) = \mathcal{J}_t(x) + \int_{(0,t) \times \mathbb{R}} p_{t-s}(y-x) \sigma(U_s(y)) \xi(ds dy).$$

Elementary properties of the stochastic integral imply that

$$\begin{aligned} \mathbb{E}(|U_t(x)|^2) &= \mathcal{J}_t^2(x) + \int_0^t ds \int_{-\infty}^{\infty} dy [p_{t-s}(y-x)]^2 \mathbb{E}(|\sigma(U_s(y))|^2) \\ &\leq \mathcal{J}_t^2(x) + \text{Lip}_\sigma^2 \int_0^t ds \int_{-\infty}^{\infty} dy [p_{t-s}(y-x)]^2 \mathbb{E}(|U_s(y)|^2). \end{aligned}$$

That is, in the special case that  $F_t(x) = \mathbb{E}(|U_t(x)|^2)$  satisfies (2.3) with  $\alpha = \text{Lip}_\sigma$ . In this special case, Theorem 2.4 of Chen and Dalang [2] implies (2.4), and our function  $\mathcal{K}^{(\text{Lip}_\sigma)}$  coincides with their function  $\overline{\mathcal{K}}$ . For general  $F$  and  $\alpha$ , the very same proof works equally well.  $\square$

**Proof of Theorem 2.1.** The proof begins by writing  $u$  and  $v$  in integral form as follows:

$$\begin{aligned} u_t(x) &= (p_t * u_0)(x) + \int_{(0,t) \times \mathbb{R}} p_{t-s}(y-x) \sigma(u_s(y)) \xi(ds dy), \\ v_t(x) &= (p_t * v_0)(x) + \int_{(0,t) \times \mathbb{R}} p_{t-s}(y-x) \sigma(v_s(y)) \xi(ds dy). \end{aligned}$$

Define

$$f(x) := |u_0(x) - v_0(x)| \quad \text{for all } x \in \mathbb{R},$$

and set  $\mathcal{J}_t(x) := (p_t * f)(x)$  for all  $t > 0$  and  $x \in \mathbb{R}$ . Then clearly,

$$\begin{aligned} \mathbb{E}(|u_t(x) - v_t(x)|^2) &\leq |\mathcal{J}_t(x)|^2 + \int_0^t ds \int_{-\infty}^{\infty} dy [p_{t-s}(y-x)]^2 \mathbb{E}(|\sigma(u_s(y)) - \sigma(v_s(y))|^2) \\ &\leq |\mathcal{J}_t(x)|^2 + \text{Lip}_\sigma^2 \int_0^t ds \int_{-\infty}^{\infty} dy [p_{t-s}(y-x)]^2 \mathbb{E}(|u_s(y) - v_s(y)|^2). \end{aligned}$$

In other words, the space–time function,

$$F_t(x) := \mathbb{E}(|u_t(x) - v_t(x)|^2) \quad [t > 0, x \in \mathbb{R}],$$

satisfies (2.3) with  $\alpha = \text{Lip}_\sigma$ . Therefore, (2.4) implies that

$$F \leq \mathcal{J}^2 + (\mathcal{J}^2 \odot \mathcal{K}^{(\text{Lip}_\sigma)}) \quad \text{pointwise.} \tag{2.5}$$

An inspection of the function  $\mathcal{K}^{(\text{Lip}_\sigma)}$  – see (2.2) – shows that

$$\mathcal{K}_t^{(\text{Lip}_\sigma)}(x) \leq \ell p_{t/2}(x) \left[ \frac{1}{\sqrt{t}} + \exp\left(\frac{\text{Lip}_\sigma^4 t}{4}\right) \right],$$

for all  $t > 0$  and  $x \in \mathbb{R}$ . Consequently,

$$(\mathcal{J}^2 \odot \mathcal{K}^{(\text{Lip}_\sigma)})_t(x) \leq \ell \int_0^t (\mathcal{J}_{t-s}^2 * p_{s/2})(x) \left[ \frac{1}{\sqrt{s}} + \exp\left(\frac{\text{Lip}_\sigma^4 s}{4}\right) \right] ds.$$

Set  $B := \|u_0 - v_0\|_{L^\infty(\mathbb{R})}$ , and observe that  $\|\mathcal{J}_{t-s}\|_{L^\infty(\mathbb{R})} \leq B$ . According to Lemma 2.2,

$$\sup_{|y-a| \leq r/2} \mathcal{J}_{t-s}(y) \leq 2B \exp\left(-\frac{r^2}{8(t-s)}\right) \leq 2B e^{-r^2/(8t)}. \tag{2.6}$$

Consequently, one can split up the ensuing integral into regions where  $|y - a| \leq r/2$  and where  $|y - a| > r/2$  in order to see that

$$\begin{aligned} (\mathcal{J}_{t-s}^2 * p_{s/2})(x) &\leq 4B^2 e^{-r^2/(4t)} + B^2 \int_{|y+x-a| > r/2} p_{s/2}(y) \, dy \\ &\leq 4B^2 e^{-r^2/(4t)} + B^2 \int_{|y| > r/4} p_{s/2}(y) \, dy, \end{aligned}$$

uniformly for all  $|x - a| \leq r/4$  and  $0 < s < t$ . This and a simple tail bound together yield

$$\sup_{|x-a| \leq r/4} (\mathcal{J}_{t-s}^2 * p_{s/2})(x) \leq 4B^2 e^{-r^2/(4t)} + 2B^2 e^{-r^2/(16s)} \leq 6B^2 e^{-r^2/(16t)},$$

for all  $0 < s < t$ . Thus, it follows that, uniformly for all  $t > 0$  and all  $x$  that satisfy  $|x - a| \leq r/4$ ,

$$\begin{aligned} (\mathcal{J}^2 \circ \mathcal{K}(\text{Lip}_\sigma))_t(x) &\leq 6\ell B^2 e^{-r^2/(16t)} \cdot \int_0^t \left[ \frac{1}{\sqrt{s}} + \exp\left(\frac{\text{Lip}_\sigma^4 s}{4}\right) \right] \, ds \\ &\leq 6\ell B^2 e^{-r^2/(16t)} \cdot \left[ 2\sqrt{t} + \frac{4}{\text{Lip}_\sigma^4} \exp\left(\frac{\text{Lip}_\sigma^4 t}{4}\right) \right] \\ &\leq 6\ell B^2 \exp\left(-\frac{r^2}{16t} + \frac{\text{Lip}_\sigma^4 t}{4}\right) \cdot \sup_{s>0} \left[ 2\sqrt{s} \exp\left(-\frac{\text{Lip}_\sigma^4 s}{4}\right) + \frac{4}{\text{Lip}_\sigma^4} \right] \\ &\leq 48\ell \left[ 1 \vee \frac{1}{\text{Lip}_\sigma^4} \right] B^2 \exp\left(-\frac{r^2}{16t} + \frac{\text{Lip}_\sigma^4 t}{4}\right); \end{aligned}$$

consult also (2.5). Combine this estimate with (2.6) and (2.5) to finish. □

The two technical Lemmas 2.2 and 2.3 yield the following tail probability bounds.

**Theorem 2.4.** *There exist universal constants  $0 < K, L < \infty$  such that for all  $\varepsilon > 0$ ,*

$$-\frac{L\Lambda^{3/2}}{\sqrt{t}} \leq \liminf_{|x| \rightarrow \infty} \frac{\log \mathbb{P}\{u_t(x) > \varepsilon\}}{\log |x|} \leq \limsup_{|x| \rightarrow \infty} \frac{\log \mathbb{P}\{u_t(x) > \varepsilon\}}{\log |x|} \leq -\frac{K\Lambda^{3/2}}{\sqrt{t}},$$

uniformly for all  $t$  in every fixed compact subset of  $(0, \infty)$ .

This theorem is proved in two parts. In the first part of the proof, the claimed lower bound on  $\liminf_{|x| \rightarrow \infty} (\dots)$  is derived. The corresponding upper bound on  $\limsup_{|x| \rightarrow \infty} (\dots)$  is established afterward in a second part.

**Proof of Theorem 2.4: Part 1.** Let  $u_t^{(0)}(x) := u_0(x)$  and iteratively define

$$u_t^{(n+1)}(x) := (p_t * u_0)(x) + \int_{(0,t) \times \mathbb{R}} p_{t-s}(y-x) \sigma(u_s^{(n)}(y)) \xi(ds \, dy),$$

for all  $n \geq 0, t > 0$ , and  $x \in \mathbb{R}$ . It is well known that  $u_t^{(n)}(x) \rightarrow u_t(x)$  in  $L^2(\mathbb{P})$  as  $n \rightarrow \infty$ , for every  $t > 0$  and  $x \in \mathbb{R}$ ; see Walsh [13]. Since  $u^{(0)}$  and  $p_t * u_0$  are symmetric functions, the distributional symmetry of white noise shows that

$\{u_t^{(n+1)}(x)\}_{x \in \mathbb{R}}$  and  $\{u_t^{(n+1)}(-x)\}_{x \in \mathbb{R}}$  have the same law for all  $n \geq 0$ . Let  $n \rightarrow \infty$  in order to deduce, in particular, that the random variables  $u_t(x)$  and  $u_t(-x)$  have the same distribution for each  $t > 0$  and  $x \in \mathbb{R}$ .

In light of the preceding symmetry property, in order to derive the stated lower bound for  $\mathbb{P}\{u_t(x) > \varepsilon\}$ , it remains to prove that if  $\Lambda < \infty$ , then

$$\liminf_{x \rightarrow \infty} \frac{\log \mathbb{P}\{u_t(x) > \varepsilon\}}{\log x} \geq -\frac{L\Lambda^{3/2}}{\sqrt{t}}. \tag{2.7}$$

The assertion holds trivially when  $\Lambda = \infty$ . Let us consider now the case that  $\Lambda < \infty$ .

Choose and fix an arbitrary number  $a > 0$ , and define  $w = \{w_t(x)\}_{t>0, x \in \mathbb{R}}$  to be the solution to (1.1) with the following respective initial value:

$$w_0(x) := u_0(|x| \vee (3a/2)) \quad \text{for all } x \in \mathbb{R}.$$

The construction of the process  $w$  does not present any problems because  $w_0$  is a nonrandom elements of  $L^\infty(\mathbb{R})$ ; in fact,  $0 \leq w_0 \leq u_0$ . These inequalities have the additional consequence that

$$w_t(x) \leq u_t(x) \quad \text{for all } t > 0 \text{ and } x \in \mathbb{R}, \tag{2.8}$$

thanks to Mueller’s comparison principle [10,11]. Therefore, it remains to find a lower bound for the tails of the distribution of  $w_t(x)$ .

Define

$$z_0(x) := u_0(3a/2) \quad \text{for all } x \in \mathbb{R},$$

and let  $z := \{z_t(x)\}_{t>0, x \in \mathbb{R}}$  denote the solution to (1.1) with initial value  $z_0$ . By the comparison principle,  $w_t(x) \leq z_t(x)$  for all  $t \geq 0$  and  $x \in \mathbb{R}$ . We now use our susceptibility estimate (Theorem 2.1) in order to prove that there is a similar lower bound near the point  $x = a$ , provided that we introduce a small error. Specifically, we apply Theorem 2.1 with  $r := a/2$  in order to see that

$$\begin{aligned} \sup_{x \in [7a/8, 9a/8]} \mathbb{E}(|w_t(x) - z_t(x)|^2) &\leq C\ell \|w_0 - z_0\|_{L^\infty(\mathbb{R})}^2 \exp\left(-\frac{a^2}{64t} + \frac{\text{Lip}_\sigma^4 t}{4}\right) \\ &\leq C\ell \|u_0\|_{L^\infty(\mathbb{R})}^2 \exp\left(-\frac{a^2}{64t} + \frac{\text{Lip}_\sigma^4 t}{4}\right). \end{aligned} \tag{2.9}$$

Since  $z$  solves (1.1) with constant initial function  $z_0(\cdot) \equiv u_0(3a/2)$ , Theorems 5.5 [page 44] and 6.4 [page 57] of Ref. [9] tell us that there exists a finite universal constant  $A > 2$  such that

$$A^{-k} [u_0(3a/2)]^k e^{k^3 t/A} \leq \mathbb{E}(|z_t(x)|^k) \leq A^k [u_0(3a/2)]^k e^{Ak^3 t}, \tag{2.10}$$

simultaneously for all  $x \in \mathbb{R}$ ,  $t > 0$ , and  $k \in [2, \infty)$ . Actually, the results of [9] imply the lower bound for  $\mathbb{E}(|z_t(x)|^k)$  only in the case that  $\sigma(z) = \text{const} \cdot z$  for all  $z \in \mathbb{R}$ . The general case follows from that fact and the moment comparison theorem of Joseph, Khoshnevisan and Mueller [8].

In any case, we apply the Paley–Zygmund inequality, as in Ref. [9, Chapter 7], in order to see that

$$\begin{aligned} \mathbb{P}\left\{z_t(x) \geq \frac{1}{2} A^{-1} u_0(3a/2) e^{k^2 t/A}\right\} &\geq \mathbb{P}\left\{z_t(x) \geq \frac{1}{2} \|z_t(x)\|_{L^k(\mathbb{P})}\right\} \\ &\geq \frac{1}{4} \cdot \frac{[\mathbb{E}(|z_t(x)|^k)]^2}{\mathbb{E}(|z_t(x)|^{2k})} \\ &\geq \frac{1}{4} A^{-4k} \exp\left(-\left[8A - \frac{2}{A}\right] k^3 t\right), \end{aligned}$$

uniformly for all real number  $x \in \mathbb{R}$ ,  $k \geq 2$ , and  $t > 0$ . Since  $A > 0$ , it follows that  $8A - (2/A) < 8A$ , and hence

$$\mathbb{P}\left\{z_t(x) \geq \frac{1}{2}A^{-1}u_0(3a/2)e^{k^2t/A}\right\} \geq \frac{1}{4}\exp\{-8Ak^3t - 4k \log A\},$$

uniformly for all real number  $x \in \mathbb{R}$ ,  $k \geq 2$ , and  $t > 0$ . Choose and fix an arbitrary number  $\varepsilon > 0$ . We apply the preceding with

$$k := \sqrt{\frac{A}{t} \left| \log \left( \frac{4A\varepsilon}{u_0(3a/2)} \right) \right|};$$

equivalently,  $\frac{1}{2}A^{-1}u_0(3a/2)e^{k^2t/A} = 2\varepsilon$ . Since  $u_0(3a/2) \rightarrow 0$  as  $a \rightarrow \infty$ , it follows readily that  $k \geq 2$  if  $a$  is sufficiently large; how large depends only on  $A$ . Hence,

$$\inf_{x \in \mathbb{R}} \mathbb{P}\{z_t(x) \geq 2\varepsilon\} \geq \exp\left(-\frac{L + o(1)}{\sqrt{t}} |\log u_0(3a/2)|^{3/2}\right),$$

for all  $a$  large, where  $o(1) \rightarrow 0$  as  $a \rightarrow \infty$  and  $L := 8A^{5/2} + A^{1/2}$ . Because  $A$  is a universal constant, so is  $L$ .

The preceding estimate, (2.8), and (2.9) together imply that, as  $a \rightarrow \infty$ ,

$$\begin{aligned} \mathbb{P}\{u_t(x) \geq \varepsilon\} &\geq \mathbb{P}\{w_t(x) \geq \varepsilon\} \\ &\geq \mathbb{P}\{z_t(x) \geq 2\varepsilon\} - \mathbb{P}\{|w_t(x) - z_t(x)| \geq \varepsilon\} \\ &\geq \exp\left(-\frac{L + o(1)}{\sqrt{t}} |\log u_0(3a/2)|^{3/2}\right) - A_1 e^{-a^2/(64t)}, \end{aligned} \tag{2.11}$$

uniformly for all  $x \in [7a/8, 9a/8]$ , where  $A_1 < \infty$  does not depend on  $a$ . The condition  $\Lambda < \infty$  implies that  $a^{-2} |\log u_0(3a/2)|^{3/2} \rightarrow 0$  as  $a \rightarrow \infty$ . Therefore, (2.11) implies (2.7) and hence the theorem.  $\square$

**Proof of Theorem 2.4: Part 2.** In analogy with the proof of Part 1, it suffices to establish the following: If  $\Lambda > 0$ , then

$$\limsup_{x \rightarrow \infty} \frac{\log \mathbb{P}\{u_t(x) > \varepsilon\}}{\log x} \leq -\frac{K \Lambda^{3/2}}{\sqrt{t}}.$$

This is true vacuously when  $\Lambda = 0$ . From here on we assume that  $\Lambda > 0$ .

Choose and fix an arbitrary number  $a > 0$ , and define  $w = \{w_t(x)\}_{t>0, x \in \mathbb{R}}$  to be the solution to (1.1) subject to the following initial value:

$$w_0(x) := u_0(|x| \wedge (a/2)) \quad \text{for all } x \in \mathbb{R}.$$

The process  $w$  is the present analogue of its counterpart – also dubbed as  $w$  – in Part 1 of the proof. As was the case in Part 1, one can construct  $w$  in a standard way because  $w_0$  is a nonrandom elements of  $L^\infty(\mathbb{R})$  [ $0 \leq u_0 \leq w_0$ ]. Furthermore,

$$w_t(x) \geq u_t(x) \quad \text{for all } t > 0 \text{ and } x \in \mathbb{R}, \tag{2.12}$$

thanks to Mueller’s comparison principle [10,11]. Compare with (2.8). Therefore, it remains to find an upper bound for the tails of the distribution of  $w_t(x)$ .

Define

$$z_0(x) := u_0(a/2) \quad \text{for all } x \in \mathbb{R},$$

and let  $z := \{z_t(x)\}_{t>0, x \in \mathbb{R}}$  denote the solution to (1.1) with initial value  $z_0$ . By the comparison principle,  $w_t(x) \geq z_t(x)$  for all  $t \geq 0$  and  $x \in \mathbb{R}$ . And now use our susceptibility estimate [Theorem 2.1] in analogy with the proof of Part 1 of the theorem in order to see that

$$\mathbb{E}(|w_t(x) - z_t(x)|^2) \leq C\ell \|u_0\|_{L^\infty(\mathbb{R})}^2 \exp\left(-\frac{a^2}{64t} + \frac{\text{Lip}_\sigma^4 t}{4}\right), \tag{2.13}$$

uniformly for all  $x \in [7a/8, 9a/8]$ ; compare with (2.9). Now we relabel (2.10) [ $a/2 \leftrightarrow 3a/2$ ] to see that, for the same constant  $A$  that appeared in (2.10),

$$\mathbb{E}(|z_t(x)|^k) \leq [Au_0(a/2)]^k e^{Ak^3 t},$$

simultaneously for all  $x \in \mathbb{R}$ ,  $t > 0$ , and  $k \in [2, \infty)$ . Chebyshev's inequality yields

$$\begin{aligned} \mathbb{P}\{z_t(x) \geq \varepsilon/2\} &\leq \inf_{k \geq 2} \left[ \frac{2Au_0(a/2)e^{Ak^2 t}}{\varepsilon} \right]^k \\ &= \exp\left(-\frac{2}{3\sqrt{3}At} \left[ \log\left(\frac{\varepsilon}{2Au_0(a/2)}\right) \right]^{3/2}\right), \end{aligned}$$

uniformly for every real number  $x$ , and for all  $a > 0$  sufficiently large to ensure that the argument of the logarithm is greater than one. This, (2.12), and (2.13) together imply that

$$\begin{aligned} \mathbb{P}\{u_t(x) \geq \varepsilon\} &\leq \mathbb{P}\{w_t(x) \geq \varepsilon\} \leq \mathbb{P}\{z_t(x) \geq \varepsilon/2\} + \mathbb{P}\{|w_t(x) - z_t(x)| \geq \varepsilon/2\} \\ &\leq \exp\left(-\frac{2}{3\sqrt{3}At} \left[ \log\left(\frac{\varepsilon}{2Au_0(a/2)}\right) \right]^{3/2}\right) + A_1 e^{-a^2/(64t)}, \end{aligned}$$

uniformly for all  $x \in [7a/8, 9a/8]$ , where  $A_1$  is a finite constant that does not depend on  $a$ . Part 2 can be deduced easily from this estimate. □

### 3. Proof of Theorem 1.1

We will soon see that, in order to prove Theorem 1.1 it suffices to consider separately the cases that  $\Lambda > 0$  and  $\Lambda < \infty$ . There is, of course, some overlap between the two cases. The two portions require different ideas; let us begin with the case  $\Lambda > 0$ , since the proof is uncomplicated and can be carried out swiftly.

#### 3.1. Part 1 of the proof

Throughout this part of the proof, we assume that

$$\Lambda > 0,$$

keeping in mind that  $\Lambda = \infty$  is permissible, as a particular case.

Choose and fix a [finite] number

$$\lambda \in (0, \Lambda),$$

and introduce two new parameters  $\tau$  and  $T$  as follows:

$$0 < \tau < T := \frac{K^2 \lambda^3}{64},$$

where  $K$  is the universal constant that appeared in the statement of Theorem 2.4. We plan to prove that

$$\lim_{x \rightarrow \infty} \sup_{t \in (\tau, T)} u_t(x) = 0 \quad \text{a.s.} \tag{3.1}$$

Suppose, for the moment, that we have established (3.1). Thanks to symmetry, (3.1) also implies that  $\lim_{x \rightarrow -\infty} \sup_{t \in (\tau, T)} u_t(x) = 0$  a.s. Because  $u$  is almost surely continuous [4,9,13] it follows that

$$\mathbb{P}\left\{\sup_{x \in \mathbb{R}} u_t(x) < \infty \text{ for all } t \in (\tau, T)\right\} = 1. \tag{3.2}$$

If  $\Lambda = \infty$ , then we can choose  $\tau$  as close as we like to 0 and  $T$  as close as we like to  $\infty$  in order to deduce Part 1 of Theorem 1.1 from (3.2). Similarly, if  $\Lambda < \infty$ , then we can deduce half of Part 3 of Theorem 1.1; specifically, we can choose  $T$  arbitrarily close to  $K^2\Lambda^3/64$  to see that  $t_1 := K^2\Lambda^3/64$  can serve as a candidate for the constant  $t_1$  of Theorem 1.1, Part 3. We conclude this subsection by verifying (3.1).

Define

$$x_n := \sqrt{n} \quad \text{for all integers } n \geq 0,$$

and

$$t(j, n) := \frac{jT}{n} \quad \text{for all } j \in \mathbb{J}(n; \tau, T) := \left[\frac{n\tau}{T}, n\right] \cap \mathbb{Z}.$$

Theorem 2.4 ensures that for all  $\varepsilon > 0$  and all sufficiently-large integers  $n \gg 1$ ,

$$\begin{aligned} \mathbb{P}\left\{\max_{j \in \mathbb{J}(n; \tau, T)} u_{t(j, n)}(x_n) \geq \varepsilon\right\} &\leq \sum_{j \in \mathbb{J}(n; \tau, T)} \mathbb{P}\{u_{t(j, n)}(x_n) \geq \varepsilon\} \\ &\leq \text{const} \cdot \sum_{j \in \mathbb{J}(n; \tau, T)} \exp\left(-\frac{K\lambda^{3/2}}{\sqrt{t(j, n)}} \log|x_n|\right) \\ &\leq \text{const} \cdot \exp\left(-\left[\frac{K\lambda^{3/2}}{2\sqrt{T}} - 1\right] \log n\right) = O(n^{-3}). \end{aligned}$$

Therefore, the Borel–Cantelli lemma ensures that

$$\lim_{n \rightarrow \infty} \max_{j \in \mathbb{J}(n; \tau, T)} u_{t(j, n)}(x_n) = 0 \quad \text{a.s.} \tag{3.3}$$

Choose and fix an arbitrary number  $\varrho \in (0, \frac{1}{4})$ , and define  $k := \max(2, 3/\varrho)$ . A standard continuity estimate (see, for example, Walsh [13, p. 319] and Chen and Dalang [1]) shows that

$$A_{k, \varrho, \tau, T} := A := \sup_{x \in \mathbb{R}} \mathbb{E}\left(\sup_{\substack{s, t \in (\tau, T): \\ s \neq t}} \frac{|u_s(x) - u_t(x)|^k}{|s - t|^{k\varrho}}\right) < \infty.$$

Therefore, for all  $\varepsilon > 0$  and integers  $n \geq 1$ ,

$$\begin{aligned} \mathbb{P}\left\{\sup_{t \in (\tau, T)} \max_{j \in \mathbb{J}(n; \tau, T)} |u_{t(j, n)}(x_n) - u_t(x_n)| \geq \varepsilon\right\} &\leq \mathbb{P}\left\{\sup_{\substack{s, t \in (\tau, T): \\ |s - t| \leq T/n}} |u_s(x_n) - u_t(x_n)| \geq \varepsilon\right\} \\ &\leq \frac{AT^k}{\varepsilon^k n^{k\varrho}} \\ &= O(n^{-3}) \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Therefore, the Borel–Cantelli lemma and (3.3) together imply that

$$\lim_{n \rightarrow \infty} \sup_{t \in (\tau, T)} u_t(x_n) = 0 \quad \text{a.s.} \tag{3.4}$$

Let us recall also the following standard continuity estimate:

$$B := B_{k,\varrho,\tau,T} := \sup_{x \in \mathbb{R}} \mathbb{E} \left( \sup_{t \in (\tau,T)} \sup_{x < y \leq x+1} \frac{|u_t(y) - u_t(x)|^k}{|y - x|^{2k\varrho}} \right) < \infty;$$

see, for example, Walsh [13, p. 319] and Chen and Dalang [1]. Since  $x_{n+1} - x_n \leq (2n)^{-1/2}$  as  $n \rightarrow \infty$ , it follows that

$$\mathbb{P} \left\{ \sup_{t \in (\tau,T)} \sup_{x_n < y \leq x_{n+1}} |u_t(y) - u_t(x_n)| \geq \varepsilon \right\} \leq \frac{B}{\varepsilon^k (2n)^{k\varrho}} = O(n^{-3}) \quad \text{as } n \rightarrow \infty.$$

Thanks to the Borel–Cantelli lemma, the preceding and (3.4) together imply (3.1) and conclude this subsection.

### 3.2. Part 2 of the proof

We now consider the case that  $\Lambda < \infty$ . Throughout, we choose and fix three arbitrary numbers:

$$\varepsilon > 0; \quad \tau > 4L^2\Lambda^3; \quad \text{and} \quad T > \tau;$$

where  $L$  is the constant of Theorem 2.4. Our plan is to prove that

$$\inf_{t \in (\tau,T)} \sup_{x > 0} u_t(x) = \infty \quad \text{a.s.} \tag{3.5}$$

If  $\Lambda > 0$ , then (3.5) implies that, outside a single P-null set,  $\sup_{x \in \mathbb{R}} u_t(x) = \infty$  for all  $t \geq t_2 := 4L^2\Lambda^3$ . And if  $\Lambda = 0$ , then we choose  $\tau$  as close as we would like to zero in order to see that, outside one P-null set,  $\sup_{x \in \mathbb{R}} u_t(x) = \infty$  for all  $t > 0$ . In other words, (3.5) furnishes proof of the remaining half of Theorem 1.1.

Before we prove (3.5), we need to recall a few facts about parabolic stochastic PDEs. Let

$$u_t^{(n,0)}(x) := (p_t * u_0)(x) \quad \text{for all } t > 0, x \in \mathbb{R}, \text{ and } n \geq 0.$$

Then iteratively define for each fixed  $n \geq 0$ ,

$$u_t^{(n,j+1)}(x) = (p_t * u_0)(x) + \int_{(0,t) \times [x-\sqrt{nt}, x+\sqrt{nt}]} p_{t-s}(y-x) \sigma(u_s^{(n,j)}(y)) \xi(ds dy),$$

for all  $j \geq 0, t > 0$ , and  $x \in \mathbb{R}$ . We recall the following result.

**Lemma 3.1 (Lemma 4.3 of Conus, Khoshnevisan and Joseph [3]).** *There exists a finite constant  $A$  such that for all integers  $n \geq 1$  and real numbers  $t > 0$ ,*

$$\sup_{x \in \mathbb{R}} \mathbb{E} \left( |u_t(x) - u_t^{(n,n)}(x)|^2 \right) \leq \frac{Ae^{At-n}}{n^2}.$$

Actually, Conus, Khoshnevisan and Joseph [3] present a slightly different formulation than the one that appears above; see Ref. [9, Lemma 10.10] for this particular formulation, as well as proof.

**Lemma 3.2 (Lemma 4.4 of Conus, Khoshnevisan and Joseph [3]).** *Choose and hold fixed an integer  $n \geq 1$  and real numbers  $t > 0$  and  $x_1, \dots, x_k \in \mathbb{R}$  that satisfy  $|x_i - x_j| \geq 2n^{3/2}\sqrt{t}$  for all  $1 \leq i \neq j \leq k$ . Then,  $\{u_t^{(n,n)}(x_j)\}_{j=1}^k$  are independent.*

**Proof of Theorem 1.1: Part 2.** Choose and fix some  $\varepsilon > 0$ , and consider the events

$$E_t(x) := \{ \omega \in \Omega : u_t(x)(\omega) < \varepsilon \} \quad \text{for every } t, x > 0.$$

According to Theorem 2.4, for every  $\lambda \in (\mathbf{A}, \tau^{1/3}(4L^2)^{-1/3}]$  we can find a real number  $n(\lambda, \varepsilon) > 1$  such that

$$\mathbb{P}(E_t(x)) \leq 1 - x^{-L\lambda^{3/2}/\sqrt{t}} \leq 1 - x^{-1/2}, \quad (3.6)$$

uniformly for all  $x \geq n(\lambda, \varepsilon)$  and  $t \in (\tau, T)$ . Consider the events

$$E_t^{(n)}(x) := \{\omega : u_t^{(n,n)}(x)(\omega) < 2\varepsilon\} \quad \text{for } x \in \mathbb{R} \text{ and } n \geq 1.$$

Lemma 3.1 ensures the existence of a finite constant  $c = c(\tau, T, \varepsilon)$  such that

$$\sup_{t \in (\tau, T)} \mathbb{P}(E_t(x) \setminus E_t^{(n)}(x)) \leq \sup_{t \in (\tau, T)} \mathbb{P}\{|u_t(x) - u_t^{(n,n)}(x)| \geq \varepsilon\} \leq cn^{-2}e^{-n},$$

for all integers  $n \geq 1$ . Therefore,

$$\mathbb{P}\left(\bigcap_{x \in [n^4, 2n^4]} E_t(x)\right) \leq \mathbb{P}\left(\bigcap_{\ell=n^4}^{2n^4} E_t(\ell)\right) \leq \mathbb{P}\left(\bigcap_{\ell=n^4}^{2n^4} E_t^{(n)}(\ell)\right) + cn^2e^{-n}, \quad (3.7)$$

uniformly for all integers  $n \geq 1$  and real numbers  $t \in (\tau, T)$ . Let  $x_1 := n^4$  and define iteratively

$$x_{j+1} := x_j + \lceil 2n^{3/2}\sqrt{t} \rceil \quad \text{for all } j \geq 1.$$

Let

$$\gamma_n := \max\{j \geq 1 : x_j \leq 2n^4\},$$

and observe that

$$\gamma_n \geq \left\lfloor 1 + \frac{n^{5/2}}{2t^{1/2}} \right\rfloor \geq \frac{n^{5/2}}{2T^{1/2}}, \quad (3.8)$$

uniformly for all  $t \in (\tau, T)$  and  $n$  sufficiently large. Moreover,

$$\begin{aligned} \mathbb{P}\left(\bigcap_{\ell=n^4}^{2n^4} E_t^{(n)}(\ell)\right) &\leq \mathbb{P}\left(\bigcap_{j=1}^{\gamma_n} E_t^{(n)}(x_j)\right) \\ &= \prod_{j=1}^{\gamma_n} \mathbb{P}(E_t^{(n)}(x_j)) \quad [\text{Lemma 3.2}] \\ &\leq \prod_{j=1}^{\gamma_n} [\mathbb{P}(E_t(x_j)) + cn^2e^{-n}] \quad [\text{Lemma 3.1}] \\ &\leq \left[1 - \frac{1}{\sqrt{2n^4}} + cn^2e^{-n}\right]^{\gamma_n}, \end{aligned}$$

uniformly for all  $t \in (\tau, T)$  and  $n$  sufficiently large, owing to (3.6). Since  $1 - y \leq \exp(-y)$  for all  $y \in \mathbb{R}$ , the preceding yields

$$\sup_{\tau < t < T} \mathbb{P}\left(\bigcap_{\ell=n^4}^{2n^4} E_t^{(n)}(\ell)\right) \leq \exp\left(-\frac{n^{1/2}}{4T^{1/2}}\right),$$

for all sufficiently-large integers  $n \gg 1$ . Thanks to (3.8), the preceding and (3.7) together yield

$$\sup_{t \in (\tau, T)} \mathbb{P} \left\{ \sup_{n^4 \leq x \leq 2n^4} u_t(x) < \varepsilon \right\} \leq 2 \exp \left( -\frac{n^{1/2}}{4T^{1/2}} \right), \tag{3.9}$$

for all integers  $n$  sufficiently large. Define  $t(0) := \tau$ , and  $t(j) := \tau + j(T - \tau)/n$  for all  $1 \leq j \leq n$  in order to deduce from (3.9) that, for every sufficiently-large integer  $n$ ,

$$\begin{aligned} & \mathbb{P} \left\{ \inf_{t \in (\tau, T)} \sup_{x \in [n^4, 2n^4]} u_t(x) < \varepsilon \right\} \\ & \leq \mathbb{P} \left\{ \inf_{0 \leq j \leq n} \sup_{x \in [n^4, 2n^4]} u_{t(j)}(x) < 2\varepsilon \right\} + \mathbb{P} \left\{ \sup_{\substack{s, t \in (\tau, T) \\ 0 < t-s < 1/n}} \sup_{x \in [n^4, 2n^4]} |u_t(x) - u_s(x)| > \varepsilon \right\} \\ & \leq 2n \exp \left( -\frac{n^{1/2}}{4T^{1/2}} \right) + \sum_{v=n^4}^{2n^4-1} \mathbb{P} \left\{ \sup_{\substack{s, t \in (\tau, T) \\ 0 < t-s < 1/n}} \sup_{x \in [v, v+1]} |u_t(x) - u_s(x)| > \varepsilon \right\}. \end{aligned} \tag{3.10}$$

A standard modulus of continuity estimate (see, for example, Walsh [13, p. 319] and Chen and Dalang [1]) shows that, for each fixed  $k \geq 1$  and  $\varrho \in (0, \frac{1}{4})$ ,

$$\sup_{v \in \mathbb{R}} \mathbb{E} \left( \sup_{\substack{s, t \in (\tau, T) \\ 0 < t-s < 1/n}} \sup_{x \in [v, v+1]} |u_t(x) - u_s(x)|^k \right) \leq \text{const} \cdot n^{-k\varrho},$$

for all  $n \geq 1$ . Let us apply this with  $\varrho := \frac{1}{8}$  and  $k := 64$ . In this way, we may deduce from (3.10) and Chebyshev's inequality that

$$\mathbb{P} \left\{ \inf_{t \in (\tau, T)} \sup_{x > 0} u_t(x) < \varepsilon \right\} \leq \lim_{n \rightarrow \infty} \mathbb{P} \left\{ \inf_{t \in (\tau, T)} \sup_{x \in [n^4, 2n^4]} u_t(x) < \varepsilon \right\} = 0.$$

Because  $\varepsilon > 0$  is arbitrary, this proves (3.5). □

### 3.3. Part 3 of the proof

We now finish the proof of Part 3. Throughout,  $(\Omega, \mathcal{F}, \mathbb{P})$  denotes the underlying probability space, and we consider only the case that  $0 < \Lambda < \infty$ .

For every integer  $N \geq 1$  consider the stopping time,

$$\mathcal{T}_N := \inf \{ t > 0 : M(t) \geq N \}.$$

Since  $\mathcal{T}_N \leq \mathcal{T}_{N+1}$  for all  $N \geq 1$ , the random variable

$$\mathcal{T} := \lim_{N \rightarrow \infty} \mathcal{T}_N$$

exists. According to Parts 1 and 2 of the proof of Theorem 1.1,

$$0 < t_1 < \mathcal{T} < t_2 < \infty \quad \text{a.s.,}$$

where  $t_1$  and  $t_2$  are non random and depend only on  $\Lambda$ . In addition, if  $t < \mathcal{T}(\omega)$  for some  $\omega \in \Omega$ , then there exists an integer  $N(\omega) > 0$  such that  $t \leq \mathcal{T}_{N(\omega)}(\omega) < \mathcal{T}(\omega)$ . This implies that  $M(t)(\omega) \leq N(\omega) < \infty$ .

On the other hand, if  $t > \mathcal{T}(\omega_1)$  for some  $\omega_1 \in \Omega$ , then there exists some  $N_1(\omega_1) > 0$  such that  $t \geq \mathcal{T}_n(\omega_1)$  for all  $n \geq N_1(\omega_1)$ . It follows that  $M(t)(\omega_1) \geq n$  for all  $n \geq N_1(\omega_1)$ , whence  $M(t)(\omega_1) = \infty$ . This completes the proof of Part 3, and concludes the proof of Theorem 1.1.

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